SOME INEQUALITIES CONCERNING FUNCTIONS OF EXPONENTIAL TYPE

BY

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1. **Introduction and statement of results.** An entire function $f(z)$ is said to be of exponential type α if either $f(z)$ is of order 1 and type $\leqq \alpha$ or it is of order <1. It has been proved by Plancherel and Pólya [9, p. 124] that if $f(x) \in L^p(-\infty, \infty)$ for some $p \geqq 1$, i.e.

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx \tag{1.1}$$

exists, then $f(x) \to 0$ as $x \to \pm\infty$. Hence $f(z)$ is bounded on the real axis. In fact, a more precise statement can be made.

THEOREM A. *If $f(z)$ is an entire function of exponential type α and if* (1.1) *exists, then*

$$|f(x+iy)|^p \leqq A_p \left\{ \int_{-\infty}^{\infty} |f(x)|^p \, dx \right\} \frac{\sinh p\alpha y}{y} \tag{1.2}$$

with

$$A_1 = \pi^{-1}, \quad A_p = 2^k (p\pi)^{-1} < \pi^{-1} \qquad (2^k < p \leqq 2^{k+1}, \quad k = 0, 1, 2, \ldots). \tag{1.3}$$

The above theorem is due to J. Korevaar [7] and the bound in (1.2) is known to be precise for $p=2$. For example, the function

$$f_0(z) = \text{const.} \frac{\sin \alpha(z - \bar{z}_0)}{z - \bar{z}_0} \quad (z_0 = x_0 + iy_0) \tag{1.4}$$

satisfies the conditions of the theorem and

$$|f_0(z_0)|^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} |f_0(x)|^2 \, dx \right\} \frac{\sinh 2\alpha y_0}{y_0}. \tag{1.5}$$

In fact, (1.5) holds only if $f_0(z)$ is [7, p. 59] a constant multiple of

$$(z - \bar{z}_0)^{-1} \sin \alpha(z - \bar{z}_0).$$

The form of the extremal function in the case $p=2$ suggests that for functions which are real for real z we should hope to get better estimates at nonreal points. It is equally clear that for functions which do not vanish in the upper half plane inequality (1.2) can be refined for points in the lower half plane. We prove

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THEOREM 1. *If $f(z)$ is an entire function of exponential type α which is real on the real axis, and if* (1.1) *exists, then for $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$*

$$|f(x+iy)|^p \leqq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(x)|^p\,dx\right\}\int_{-2^k\alpha}^{2^k\alpha} (\cosh yt)^{p/2^k}\,dt. \tag{1.6}$$

In particular, for $p=2$, we have

$$|f(x+iy)|^2 \leqq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(x)|^2\,dx\right\}\frac{1}{2}\left(\frac{\sinh 2\alpha y}{y}+2\alpha\right). \tag{1.7}$$

For large values of $|y|$ the right-hand side of (1.7) is asymptotically

$$\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(x)|^2\,dx\right\}\frac{e^{2\alpha|y|}}{4|y|}$$

which is better by a factor of 1/2 than the corresponding asymptotic bound

$$\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(x)|^2\,dx\right\}\frac{e^{2\alpha|y|}}{2|y|}$$

given by (1.2).

For $y=0$ the two inequalities (1.2) and (1.6) give the same estimate. This is just the thing one should expect (see (1.4)). However, if $f(z)$ is nonnegative on the real axis then the bound can be considerably improved. In fact, we have

THEOREM 2. *If $f(z)$ is an entire function of exponential type α which is nonnegative on the real axis, and if* (1.1) *exists then for $2^{k-1}<p\leqq 2^k$, $k=0, 1, 2, \ldots$*

$$|f(x)|^p \leqq \frac{1}{2\pi}\,2^k\alpha\int_{-\infty}^{\infty} |f(x)|^p\,dx. \tag{1.8}$$

The bound in (1.8) is precise for $p=1$. The function

$$f_1(z) = c\left\{\frac{\sin(\alpha/2)(z-x_1)}{z-x_1}\right\}^2, \qquad c>0,\ x_1 \text{ real}$$

satisfies the conditions of Theorem 2 and

$$|f_1(x_1)| = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} |f_1(x)|\,dx.$$

An entire function $f(z)$ of exponential type is said to be *asymmetric* if it does not vanish in the upper half plane and $h_f(\pi/2)=\limsup_{y\to\infty}\log|f(iy)|/y=0$. Such functions have been studied by R. P. Boas, Jr. [2].

THEOREM 3. *Let $f(z)$ be an asymmetric entire function of exponential type α. If $f(x)\in L^p(-\infty,\infty)$ then for $y<0$, $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$*

$$|f(x+iy)|^p \leqq \frac{1}{2\pi}\,e^{\alpha p|y|/2}\int_{-2^{k-1}\alpha}^{2^{k-1}\alpha} (\cosh yt)^{p/2^k}\,dt\int_{-\infty}^{\infty} |f(x)|^p\,dx. \tag{1.9}$$

By applying this result to the function $e^{i\alpha z}f(z)$ below we obtain the following

THEOREM 3′. *Let $f(z)$ be an entire function of exponential type α such that $h_f(\pi/2)=\alpha$. If $f(z)$ does not vanish in the upper half plane and if* (1.1) *exists then for $y<0$, $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$*

$$|f(x+iy)|^p \leqq \frac{1}{2\pi}\int_{-2^k\alpha}^{2^k\alpha} (\cosh yt)^{p/2^k}\, dt \int_{-\infty}^{\infty} |f(x)|^p\, dx. \tag{1.6′}$$

Asymmetric entire functions were introduced by the consideration [2, p. 94] that if $P(z)$ is a polynomial of degree n then $P(e^{iz})$ is an entire function $f(z)$ of exponential type n such that $h_f(-\pi/2)=n$ and $f(x)$ is bounded for real x. If $P(z)$ has no zeros in $|z|<1$, then $f(z)$ has no zeros in $y>0$, and moreover (since $P(0)\neq 0$) $h_f(\pi/2)=0$, i.e. $f(z)$ is asymmetric.

Clearly, $P(e^{iz})$ is periodic on the real axis with period 2π. As a consequence $P(e^{iz})$ cannot belong to $L^p(-\infty, \infty)$ for any $p\geqq 1$. However, (1.9) is trivially satisfied for such asymmetric entire functions. Since in this case the right-hand side is $+\infty$ and the left-hand side finite, an inequality like (1.9) is of no value. It is more appropriate to take norms over $(0, 2\pi)$.

THEOREM 4. *Let $f(z)$ be an entire function of exponential type α. If $f(z)$ is periodic on the real axis with period 2π, and does not vanish in the upper half plane, then for $y<0$*

$$|f(x+iy)|^2 \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx\right)\frac{1}{2}\left\{\frac{\sinh (2n+1)y}{\sinh y}+(2n+1)\right\} \tag{1.10}$$

where n is the integral part of α. More generally, for $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$

$$|f(x+iy)|^p \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\, dx\right)\sum_{\nu=-N}^{N} (\cosh \nu y)^{p/2^k}, \qquad N = [2^k\alpha]. \tag{1.11}$$

Since an entire function of exponential type which is periodic on the real axis with period 2π is necessarily a trigonometric polynomial (Lemma 4) we shall not prove Theorem 4 but the following more general

THEOREM 4′. *If the entire function $f(z)=\sum_{\nu=-m}^{n} a_\nu e^{i\nu z}$ does not vanish in the upper half plane, then for $y<0$, $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$*

$$|f(x+iy)|^p \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\, dx\right)e^{((n-m)/2)p|y|}\sum_{\nu=-M}^{N}\left\{\cosh\left(\left(\frac{N-M}{2}-\nu\right)y\right)\right\}^{p/2^k},$$
$$N = 2^k n,\ M = 2^k m.$$

Corresponding to Theorem 1, we have:

THEOREM 5. *Let $f(z)$ be an entire function of exponential type α. If $f(z)$ is periodic on the real axis with period 2π and is real for real z, then for $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$*

$$|f(x+iy)|^p \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\, dx\right)\sum_{\nu=-N}^{N} (\cosh \nu y)^{p/2^k}, \qquad N = [2^k\alpha]. \tag{1.12}$$

The analogue of Theorem 2 for periodic entire functions is the following:

THEOREM 6. *Let $f(z)$ be an entire function of exponential type α. If $f(z)$ is non-negative on the real axis where it is periodic with period 2π, then for $2^{k-1}<p\leqq 2^k$, $k=0, 1, 2, \ldots$*

$$|f(x)|^p \leqq (N+1)\left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right), \qquad N = [2^k\alpha]. \tag{1.13}$$

The function $f_2(z)=(\sum_{\nu=-n}^{n} e^{i\nu z})^2$ satisfies the hypotheses of Theorem 6 with $\alpha=2n$ and for this function equality holds in (1.13) for $p=1$ and certain x.

The following theorem stands in analogy with Theorem A.

THEOREM 7. *If $f(z)=\sum_{\nu=-m}^{n} a_\nu e^{i\nu z}$ then*

$$|f(x+iy)|^p \leqq B_p\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right\}$$

with

$$B_1 = \exp\{(n-m)y/2\}\frac{\sinh\{(n+m+1)y/2\}}{\sinh(y/2)},$$

$$B_p = \exp\{(n-m)py/2\}\,\frac{\sinh\{(n+m+2^{-k})py/2\}}{\sinh(2^{-k-1}py)}$$

$$(2^k < p \leqq 2^{k+1},\ k = 0, 1, 2, \ldots).$$

Thus, in particular, if $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n then for $2^k<p\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$

$$\max_{|z|=1} |P(z)| \leqq \left\{\frac{n2^k+1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p\,d\theta\right\}^{1/p},$$

or

$$\|P\|_\infty \leqq \{n2^k+1\}^{1/p}\|P\|_p,$$

where the norms are taken over $|z|=1$.

We may take norms over an arbitrary piecewise differentiable curve C and seek to maximize $\|P\|_\infty/\|P\|_p$ when P varies inside the class of all polynomials of degree at most n. Here we only consider the case when C is the unit interval $-1\leqq x\leqq 1$, and prove:

THEOREM 8. *If $P(z)$ is a polynomial of degree $n\geqq 1$, then for every $p>1$*

$$\|P\|_\infty = \max_{-1\leqq x\leqq 1} |P(x)| \leqq K(p)n^{2/p}\left(\frac{1}{2}\int_{-1}^{1} |P(t)|^p\,dt\right)^{1/p} = K(p)n^{2/p}\|P\|_p \tag{1.14}$$

where $K(p)$ is a constant which depends only on p, but not on $P(z)$ or on n.

We have a feeling that in (1.14) $n^{2/p}$ cannot be replaced by any function of n tending to ∞ more slowly. For $p=2$ it is definitely so. Mr. G. Labelle has worked out the following precise estimate in this case (see also [4, p. 245]).

THEOREM 8′. *If $P(z)$ is a polynomial of degree n, then*

$$\max_{-1\leqq x\leqq 1} |P(x)| \leqq (n+1)\left(\frac{1}{2}\int_{-1}^{1} |P(t)|^2\,dt\right)^{1/2}.$$

This estimate is sharp but unfortunately the method of proof is limited in scope so much so that it does not give anything if p is other than 2.

The problem of estimating the coefficients of a polynomial $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ in terms of

$$\left(\frac{1}{2}\int_{-1}^{1} |P(t)|^p\,dt\right)^{1/p}, \qquad p \geqq 1$$

is closely connected with the above. For example a_0 is nothing but $P(0)$, a_1 is $P'(0)$ and a_ν is $1/\nu!\,P^{(\nu)}(0)$. We prove

THEOREM 9. *If $P(z)=\sum_{n=0}^{\nu} a_\nu z^\nu$ is a polynomial of degree n then for $0\leqq \nu\leqq n$, $p>1$*

$$|a_\nu| < \frac{1}{2\pi}\left\{4\left(1+\frac{1}{n}\right)e\right\}^{1/p}\left\{\int_{\mathscr{E}_{1+1/n}} \frac{|dz|}{|z|^{(\nu+1)q}}\right\}^{1/q}\left\{\frac{1}{2}\int_{-1}^{1} |P(x)|^p\,dx\right\}^{1/p} \tag{1.15}$$

where $1/p+1/q=1$ and $\mathscr{E}_{1+1/n}$ is the ellipse whose foci lie at ± 1 and the sum of whose semi-axes is $1+1/n$.

AN ESTIMATE FOR $\int_{\mathscr{E}_{1+1/n}}|dz|/|z|^{(\nu+1)q}$. The semi-axes of the ellipse $\mathscr{E}_{1+1/n}$ are

$$A = 1+\frac{1}{2(n^2+n)}, \quad B = \frac{2n+1}{2(n^2+n)}. \tag{1.16}$$

We clearly have

$$\begin{aligned}
\int_{\mathscr{E}_{1+1/n}} \frac{dz}{|z|^{(\nu+1)q}} &\leqq \int_0^{2\pi} \frac{|-A\sin\phi+iB\cos\phi|}{(A^2\cos^2\phi+B^2\sin^2\phi)^{(\nu+1)q/2}}\,d\phi \\
&\leqq 4A\int_0^{\pi/2} \frac{\sin\phi\,d\phi}{(B^2+\cos^2\phi)^{(\nu+1)q/2}}+4B\int_0^{\pi/2} \frac{\cos\phi\,d\phi}{(A^2-\sin^2\phi)^{(\nu+1)q/2}} \\
&= 4A\int_0^1 \frac{dt}{(B^2+t^2)^{(\nu+1)q/2}}+4B\int_0^1 \frac{dt}{(A^2-t^2)^{(\nu+1)q/2}} \\
&< 4A2^{(\nu+1)q/2}\int_0^1 \frac{dt}{(B+t)^{(\nu+1)q}}+4B\int_0^1 \frac{dt}{(A^2-1)^{(\nu+1)q/2}} \\
&< 4A2^{(\nu+1)q/2}\frac{B^{1-(\nu+1)q}}{(\nu+1)q-1}+4B^{1-(\nu+1)q} \\
&= 4\left\{\frac{A}{(\nu+1)q-1}\,2^{(\nu+1)q/2}+1\right\}B^{1-(\nu+1)q}
\end{aligned}$$

where A and B are given by (1.16). There is no doubt that the above estimate can be considerably improved without any difficulty. The fact that $|a_\nu|=O(n^{\nu+1/p})$ is obvious, but the above estimate gives a kind of an upper bound for $|a_\nu|/n^{\nu+1/p}$.

2. Lemmas.

LEMMA 1. *If* $g(x)\in L^p(-\infty,\infty)$, $1<p\leqq 2$, *then* $g(x)$ *has a Fourier transform* $G(t)$ *defined by* (*limit in the* L^q *metric*)

$$G(t) = \lim_{n\to\infty} (L^q)(2\pi)^{-1/2}\int_{-n}^{n} g(x)e^{itx}\,dx, \tag{2.1}$$

where $p^{-1}+q^{-1}=1$. *The integral of* $|G(t)|^q$ *satisfies the inequality*

$$\left\{(2\pi)^{-1/2}\int_{-\infty}^{\infty}|G(t)|^q\,dt\right\}^{1/q} \leqq \left\{(2\pi)^{-1/2}\int_{-\infty}^{\infty}|g(x)|^p\,dx\right\}^{1/p} \tag{2.2}$$

(*with equality if* $p=q=2$).

For a proof of the above result on Fourier integrals see [10].

DEFINITION. An entire function is said to belong to the class A if [8, p. 479]

$$\sum_{n=1}^{\infty}\left|\operatorname{Im}\frac{1}{z_n}\right| < \infty,$$

where the z_n are all the zeros of this function. For an entire function $f(z)$ of exponential type to belong to the class A, it is necessary and sufficient [1, Theorem 6.3.14] that

$$\int_1^R x^{-2}\log|f(x)f(-x)|\,dx$$

is bounded (or bounded above). In particular, an entire function of exponential type belongs to the class A if it is bounded on the real line.

The following lemma which implies that an entire function of exponential type α, bounded and nonnegative on the real axis, can be expressed as the square of the absolute value of a function of exponential type $\alpha/2$ with its zeros in the closed upper half plane is due to N. I. Ahiezer [8, pp. 437–439].

LEMMA 2. *For an entire function* $f(z)$ *of exponential type* α *to have the representation* $f(z)=\phi(z)\bar{\phi}(z)$ *where* $\phi(z)$ *is an entire function of exponential type* $\alpha/2$ *with zeros in one of the half planes* $\operatorname{Im} z\geqq 0$ *or* $\operatorname{Im} z\leqq 0$, *it is necessary and sufficient that* $f(z)$ *belong to the class A and be nonnegative on the real axis.*

LEMMA 3. *If* $f(z)$ *is an asymmetric entire function of exponential type* α *and*

$$\omega(z) = e^{i\alpha z}\bar{f}(z)$$

then for $\operatorname{Im} z<0$, $|f(z)|\leqq|\omega(z)|$.

Proof. The function $g(z)=f(z)e^{-i\alpha z/2}$ has no zeros for $y=\operatorname{Im} z>0$, and $h_g(\pi/2)=\alpha/2\geqq h_g(-\pi/2)$. A theorem of Levin [1, p. 129] states that if $g(z)$ is an entire function of exponential type having no zeros for $y>0$ then

$$h_g(\theta) = \limsup_{r\to\infty} r^{-1}\log|g(re^{i\theta})| \geqq \limsup_{r\to\infty} r^{-1}\log|g(re^{-i\theta})| = h_g(-\theta)$$

for some θ, $0<\theta<\pi$, if and only if $|g(z)|\geqq|g(\bar z)|$ for $y>0$. Hence for $y>0$

$$|f(z)e^{-i\alpha z/2}| \geqq |f(\bar z)e^{-i\alpha\bar z/2}|.$$

Replacing z by $\bar z$ we conclude that for $y<0$

$$|f(\bar z)e^{-i\alpha\bar z/2}| \geqq |f(z)e^{-i\alpha z/2}|$$

or

$$|\bar f(z)e^{i\alpha z/2}| \geqq |f(z)e^{-i\alpha z/2}|.$$

On multiplying both sides by $|e^{i\alpha z/2}|$ the lemma follows.

Lemma 4. *Let $f(z)$ be an entire function of exponential type α, periodic on the real axis with period 2π. Then $f(z)$ has the form*

$$f(z) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu z}, \qquad n \leqq \alpha.$$

Lemma 4 is a well-known result. For a proof see [3].

Lemma 5 (Hausdorff-Young inequality [11, p. 101]). *Let $1<p\leqq 2$. Suppose that $f(t)\in L^p(0,2\pi)$ and*

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt \qquad (n=0,\pm 1,\pm 2,\dots).$$

Then

$$\left(\sum_{n=-\infty}^{\infty} |c_n|^q\right)^{1/q} \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,dt\right)^{1/p},$$

where $q^{-1}=1-p^{-1}$.

Let I be the unit interval $-1\leqq x\leqq 1$. The function $z=\frac{1}{2}(w+w^{-1})$ maps the complement of I with respect to the extended complex plane conformally into the exterior of the unit circle $|w|=1$ in the w-plane. The image in the z-plane of the circle $|w|=R$ is the ellipse $\mathscr{E}_R$ whose foci lie at ± 1 and the sum of whose semi-axes is R. With this the following lemma becomes an immediate consequence of a result due to E. Hille, G. Szegö and J. D. Tamarkin ([6], see Lemma 2.2 and the remark which follows the proof of the lemma).

Lemma 6. *If $P(z)$ is any polynomial of degree n, then for every $p>0$*

$$\int_{\mathscr{E}_R} |P(z)|^p|dz| \leqq 2R^{np+1}\int_{-1}^{1} |P(x)|^p\,dx.$$

3. Proofs of the theorems.

3.1. **Proof of Theorem 1.** *The case* $1<p\leqq 2$. As $f(x)\in L^p\ (-\infty,\infty)$, $f(x)$ has a Fourier transform $F(t)$. Since $f(z)$ is of exponential type α the Fourier transform vanishes [1, pp. 103–107] almost everywhere for $|t|>\alpha$ and $f(z)$ has the representation

$$f(z) = (2\pi)^{-1/2}\int_{-\alpha}^{\alpha} F(t)e^{-izt}\,dt,$$

$F(t)\in L^q(-\alpha,\alpha)$, $q^{-1}=1-p^{-1}$. The function $f(z)$ being real for real z,

$$F(-t) = \bar{F}(t).$$

Hence

$$\begin{aligned}
(2\pi)^{1/2}|f(x+iy)| &\leqq \int_{-\alpha}^{\alpha} |F(t)|e^{yt}\,dt \\
&= \int_{-\alpha}^{0} |F(t)|e^{yt}\,dt+\int_{0}^{\alpha} |F(t)|e^{yt}\,dt \\
&= \int_{0}^{\alpha} |F(-t)|e^{-yt}\,dt+\int_{0}^{\alpha} |F(t)|e^{yt}\,dt \\
&= \int_{0}^{\alpha} |F(t)|(e^{yt}+e^{-yt})\,dt \\
&= 2\int_{0}^{\alpha} |F(t)|\cosh yt\,dt \\
&\leqq \left\{2\int_{0}^{\alpha} |F(t)|^q\,dt\right\}^{1/q}\left\{2\int_{0}^{\alpha} (\cosh yt)^p\,dt\right\}^{1/p} \\
&= \left\{\int_{-\alpha}^{\alpha} |F(t)|^q\,dt\right\}^{1/q}\left\{\int_{-\alpha}^{\alpha} (\cosh yt)^p\,dt\right\}^{1/p} \\
&\leqq \{(2\pi)^{1/2}\}^{(q^{-1}-p^{-1})}\left\{\int_{-\infty}^{\infty} |f(x)|^p\,dx\right\}^{1/p}\left\{\int_{-\alpha}^{\alpha} (\cosh yt)^p\,dt\right\}^{1/p}
\end{aligned}$$

by (2.2). This gives (1.6) for $1<p\leqq 2$, namely:

$$|f(x+iy)|^p < \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(t)|^p\,dt\right\}\int_{-\alpha}^{\alpha} (\cosh yt)^p\,dt. \tag{3.1}$$

The case $p>2$. Let $2^k<p\leqq 2^{k+1}$, k a positive integer. If the entire function $f(z)$ of exponential type α belongs to L^p on the real axis, then the entire function $g(z)=\{f(z)\}^{2^k}$ of exponential type $2^k\alpha$ belongs to $L^{p'}$ ($p'=p/2^k$) on the real axis, $1<p'\leqq 2$. By (3.1)

$$|g(x+iy)|^{p'} \leqq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |g(t)|^{p'}\,dt\right\}\int_{-2^k\alpha}^{2^k\alpha} (\cosh yt)^{p'}\,dt$$

or

$$|f(x+iy)|^{p} \leqq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} |f(t)|^{p}\,dt\right\}\int_{-2^k\alpha}^{2^k\alpha} (\cosh yt)^{p/2^k}\,dt,$$

which proves Theorem 1 completely.

Proof of Theorem 2. By Lemma 2 there exists a function $\phi(z)$ of exponential type $\alpha/2$ with zeros in $\operatorname{Im} z \leqq 0$ such that $f(z)=\phi(z)\bar{\phi}(z)$. In particular,

$$f(x) = |\phi(x)|^2, \qquad -\infty < x < \infty.$$

Thus, if $f(x) \in L^p(-\infty, \infty)$, $\frac{1}{2}<p\leqq 1$, then $\phi(x) \in L^{2p}(-\infty, \infty)$, $1<2p\leqq 2$. From Theorem 3′ it follows that for $-\infty<x<\infty$

$$|\phi(x)|^{2p} \leqq \frac{1}{2\pi} 2(\alpha/2) \int_{-\infty}^{\infty} |\phi(x)|^{2p}\, dx$$

or

$$|f(x)|^p \leqq \frac{1}{2\pi}(\alpha) \int_{-\infty}^{\infty} |f(x)|^p\, dx.$$

This proves Theorem 2 for $\frac{1}{2}<p\leqq 1$. By applying this special case to the function $\{f(z)\}^{2^k}$ we get the result for $2^{k-1}<p\leqq 2^k$, $k=1, 2, \ldots$.

Proof of Theorem 3. It is clear that if $f(z)$ is an asymmetric entire function of exponential type α belonging to L^p on the real line, with $1<p\leqq 2$, then it has the representation

$$f(z) = (2\pi)^{-1/2} \int_{-\alpha}^{0} F(t) e^{-izt}\, dt,$$

$F(t) \in L^q(-\alpha, 0)$. By Lemma 3 it follows that for $y<0$

$$\begin{aligned} |f(x+iy)| &\leqq \tfrac{1}{2}\{|\omega(x+iy)|+|f(x+iy)|\} \\ &\leqq \tfrac{1}{2}(2\pi)^{-1/2}\left\{\int_{-\alpha}^{0} |F(t)| e^{-y(\alpha+t)}\, dt + \int_{-\alpha}^{0} |F(t)| e^{yt}\, dt\right\} \\ &= (2\pi)^{-1/2} e^{-y\alpha/2} \int_{-\alpha}^{0} |F(t)| \cosh y(\alpha/2+t)\, dt \end{aligned}$$

or

$$\begin{aligned} (2\pi)^{1/2}|f(x+iy)| &\leqq e^{-y\alpha/2}\left\{\int_{-\alpha}^{0} |F(t)|^q\, dt\right\}^{1/q}\left\{\int_{-\alpha}^{0} \cosh^p y(\alpha/2+t)\, dt\right\}^{1/p} \\ &\leqq e^{-y\alpha/2}\{(2\pi)^{1/2}\}^{q^{-1}-p^{-1}}\left\{\int_{-\infty}^{\infty} |f(x)|^p\, dx\right\}^{1/p} \\ &\qquad \times\left\{\int_{-\alpha/2}^{\alpha/2} (\cosh yt)^p\, dt\right\}^{1/p} \end{aligned}$$

by (2.2). Hence for $y<0$ and $1<p\leqq 2$,

$$|f(x+iy)| \leqq e^{\alpha|y|/2}\{(2\pi)^{1/2}\}^{-2/p}\left\{\int_{-\infty}^{\infty} |f(x)|^p\, dx\right\}^{1/p}\left\{\int_{-\alpha/2}^{\alpha/2} (\cosh yt)^p\, dt\right\}^{1/p}$$

or

$$|f(x+iy)|^p \leqq \frac{1}{2\pi} e^{\alpha p|y|/2} \int_{-\alpha/2}^{\alpha/2} (\cosh yt)^p\, dt \int_{-\infty}^{\infty} |f(x)|^p\, dx.$$

This is (1.9) for $1<p\leqq 2$. If $2^k<p\leqq 2^{k+1}$ where k is a positive integer then we may consider $\{f(z)\}^{2^k}$.

Proof of Theorem 4′. We can write $e^{imz}f(z)$ in the form $e^{imz}f(z)\equiv P(e^{iz})$ where $P(z)=\sum_{k=0}^{m+n} c_k z^k$ is a polynomial of degree $m+n$ having no zeros in $|z|<1$. If

$$Q(z) = z^{m+n}\overline{P(1/\bar{z})}$$

then the function $Q(z)/P(z)$ is analytic in $|z|\leqq 1$ and $|Q(z)/P(z)|=1$ for $|z|=1$. As a consequence $|Q(z)|\leqq|P(z)|$ for $|z|\leqq 1$. Since

$$z^{m+n}\overline{Q(1/\bar{z})} = P(z)$$

it follows that $|P(z)|\leqq|Q(z)|$ for $|z|\geqq 1$. Hence for every $y<0$

$$\begin{aligned}|P(e^{-y}e^{i\theta})| &\leqq \tfrac{1}{2}\{|P(e^{-y}e^{i\theta})|+|Q(e^{-y}e^{i\theta})|\}\\ &= \frac{1}{2}\left\{\left|\sum_{k=0}^{m+n} c_k e^{-ky}e^{ik\theta}\right| + \left|\sum_{k=0}^{m+n} \bar{c}_k e^{-(m+n-k)y}e^{i(m+n-k)\theta}\right|\right\}\\ &\leqq \frac{1}{2}\left\{\sum_{k=0}^{m+n} |c_k|e^{-ky}+\sum_{k=0}^{m+n} |c_k|e^{-(m+n-k)y}\right\}\\ &= e^{-(m+n)y/2}\sum_{k=0}^{m+n}|c_k|\cosh\left(\frac{m+n}{2}-k\right)y.\end{aligned}$$

Consequently, for $y<0$ and $1<p\leqq 2$ we get

$$\begin{aligned}\max_{0\leqq x<2\pi}|f(x+iy)|^p &= e^{-(n-m)py/2}\max_{0\leqq\theta<2\pi}|e^{-i((m+n)/2)(\theta+iy)}P(e^{i(\theta+iy)})|^p\\ &\leqq e^{-(n-m)py/2}\left(\sum_{k=0}^{m+n}|c_k|^q\right)^{p/q}\sum_{k=0}^{m+n}\cosh^p\left(\frac{m+n}{2}-k\right)y\\ &\leqq e^{-(n-m)py/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right)\sum_{k=0}^{m+n}\cosh^p\left(\frac{m+n}{2}-k\right)y,\\ &\qquad\text{by Lemma 5}\\ &= e^{-(n-m)py/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right)\sum_{\nu=-m}^{n}\cosh^p\left(\frac{n-m}{2}-\nu\right)y.\end{aligned}$$

This proves Theorem 4′ for $1<p\leqq 2$. The case $p>2$ can be treated as in Theorem 1.

Proof of Theorem 5. Since $f(z)$ is periodic on the real axis with period 2π it has the representation

$$f(z) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu z}, \qquad n = [\alpha].$$

The fact that it is real valued implies $a_{-\nu}=\bar{a}_\nu$. Hence

$$\begin{aligned}|f(x+iy)| &= \left|\sum_{\nu=-n}^{n} a_\nu e^{i\nu(x+iy)}\right| \leqq \sum_{\nu=-n}^{n}|a_\nu|e^{-\nu y}\\ &= |a_0|+\sum_{\nu=1}^{n}|a_\nu|(e^{-\nu y}+e^{\nu y})\\ &= |a_0|+2\sum_{\nu=1}^{n}|a_\nu|\cosh\nu y = \sum_{\nu=-n}^{n}|a_\nu|\cosh\nu y.\end{aligned}$$

It follows that if $1<p\leqq 2$, then

$$|f(x+iy)|^p \leqq \left(\sum_{\nu=-n}^{n} |a_\nu|^q\right)^{p/q} \sum_{\nu=-n}^{n} \cosh^p \nu y$$
$$\leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right) \sum_{\nu=-n}^{n} \cosh^p \nu y.$$

The remaining details of the proof are omitted.

Proof of Theorem 6. By a well-known theorem of Fejér and Riesz the function $f(z)$ which by Lemma 4 is a trigonometric polynomial of degree $n=[\alpha]$ can be expressed in the form

$$f(x) = |S(x)|^2 \qquad (-\infty < x < \infty),$$

where $S(x)=\sum_0^n b_k e^{ikx}$. The trigonometric polynomial $S(x)$ can be chosen so that all its zeros are in the closed lower half plane. From Theorem 4′ it follows that for $1<p\leqq 2$

$$|S(x)|^p \leqq (N+1)\left(\frac{1}{2\pi}\int_0^{2\pi} |S(x)|^p\,dx\right).$$

This gives (1.13) for $\frac{1}{2}<p\leqq 1$. To obtain the general result we may consider $\{f(z)\}^{2^k}$.

Proof of Theorem 7. *The case* $1<p\leqq 2$. We have

$$|f(x+iy)|^p = \left|\sum_{\nu=-m}^{n} a_\nu e^{i\nu(x+iy)}\right|^p$$
$$\leqq \left(\sum_{\nu=-m}^{n} |a_\nu|^q\right)^{p/q} \sum_{\nu=-m}^{n} e^{\nu p y} \tag{3.2}$$
$$\leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right) e^{(n-m)py/2}\frac{\sinh((n+m+1)/2)py}{\sinh(py/2)}.$$

Hence the theorem is proved for $1<p\leqq 2$. By making $p\downarrow 1$ in (3.2) we get the result for $p=1$. The case $p>2$ can be dealt with by considering $\{f(z)\}^{2^k}$.

Proof of Theorem 8. It follows from Lemma 6 that if $R=1+1/n$ then for every $p>0$

$$\int_{\mathscr{E}_R} |P(z)|^p|dz| < 2\left(1+\frac{1}{n}\right)e^p \int_{-1}^{1} |P(x)|^p\,dx. \tag{3.3}$$

A theorem of Gabriel [5] states that if Γ is any convex closed curve in a complex z-plane and γ any convex curve inside Γ, and if $\psi(z)$ is regular inside and on Γ, then

$$\int_\gamma |\psi(z)|^p|dz| \leqq G\int_\Gamma |\psi(z)|^p|dz|. \tag{3.4}$$

Here p is any number $\geqq 0$ and G an absolute constant.

Let us denote by C_1 the circle $|z-1| = 1/(2(n^2+n))$. We take $\mathscr{E}_{1+1/n}$ as Γ and the circle C_1 as γ. From (3.3) and (3.4) we get

$$\int_{C_1} |P(z)|^p |dz| < 2\left(1+\frac{1}{n}\right) e^p G_1 \int_{-1}^{1} |P(x)|^p \, dx, \quad p > 0 \tag{3.5}$$

where G_1 depends on p but not on $P(z)$ or on n.

By Cauchy's integral formula

$$P(1) = \frac{1}{2\pi i} \int_{C_1} \frac{P(z)}{z-1} \, dz.$$

Hence if $p > 1$ then by Hölder's inequality

$$|P(1)| \leqq \frac{1}{2\pi} \left\{ \int_{C_1} |P(z)|^p |dz| \right\}^{1/p} \left\{ \int_{C_1} \frac{|dz|}{|z-1|^q} \right\}^{1/q}$$

where $1/p + 1/q = 1$. We use (3.5) to deduce

$$\begin{aligned} |P(1)| &< \frac{1}{2\pi} \left\{ 2\left(1+\frac{1}{n}\right) e^p G_1 \int_{-1}^{1} |P(x)|^p \, dx \right\}^{1/p} \left\{ \int_{C_1} \frac{|dz|}{|z-1|^q} \right\}^{1/q} \\ &= \left\{ \frac{1}{\pi}\left(1+\frac{1}{n}\right) e^p G_1 \int_{-1}^{1} |P(x)|^p \, dx \right\}^{1/p} \{2(n^2+n)\}^{1/p} \\ &< K_1(p) n^{2/p} \left(\int_{-1}^{1} |P(x)|^p \, dx \right)^{1/p}. \end{aligned} \tag{3.6}$$

where $K_1(p)$ is a constant which depends only on p but not on $P(z)$ or on n.

Let $\frac{1}{2} < a \leqq 1$. Applying (3.6) to $P(az)$ we get

$$\begin{aligned} |P(a)| &< K_1(p) n^{2/p} \left(\frac{1}{a} \int_{-a}^{a} |P(x)|^p \, dx \right)^{1/p} \\ &< 2^{1/p} K_1(p) n^{2/p} \left(\int_{-a}^{a} |P(x)|^p \, dx \right)^{1/p} \\ &\leqq 2^{1/p} K_1(p) n^{2/p} \left(\int_{-1}^{1} |P(x)|^p \, dx \right)^{1/p}. \end{aligned} \tag{3.7}$$

Now let us suppose that $0 \leqq a \leqq \frac{1}{2}$. An elementary discussion gives the following expression for the shortest distance $D = D(a)$ of a from $\mathscr{E}_{1+1/n}$:

$$D(a) = \frac{2n+1}{2(n^2+n)} (1-a^2)^{1/2} > \frac{2n+1}{4(n^2+n)} \sqrt{3}.$$

If C_a denotes the circle with centre a and radius D then Cauchy's integral formula and Hölder's inequality yield

$$\begin{aligned} |P(a)| &\leqq \frac{1}{2\pi} \int_{C_a} \left| \frac{P(z)}{z-a} \right| |dz| \\ &\leqq \frac{1}{2\pi} \left\{ \int_{C_a} |P(z)|^p |dz| \right\}^{1/p} \left\{ \int_{C_a} \frac{|dz|}{|z-a|^q} \right\}^{1/q} \end{aligned}$$

where $1/p+1/q=1$. On using Gabriel's result (loc. cit) and (3.3) we get

$$|P(a)| < \frac{1}{2\pi}\left\{2\left(1+\frac{1}{n}\right)e^p G_2 \int_{-1}^{1} |P(x)|^p\, dx\right\}^{1/p} (2\pi)^{1/q} D^{-1/p}$$

where G_2 depends on p but not on $P(z)$ or on n. Hence for $0 \leqq a \leqq \frac{1}{2}$

$$|P(a)| < K_2(p) n^{1/p} \left(\int_{-1}^{1} |P(x)|^p\, dx\right)^{1/p}. \tag{3.8}$$

Just like G_2 the constant $K_2(p)$ depends only on p.

By considering $P(-z)$ we conclude that (3.7), (3.8) hold also for $-1 \leqq a < -\frac{1}{2}$, $-\frac{1}{2} \leqq a \leqq 0$ respectively. Hence the desired result follows from (3.7) and (3.8).

Proof of Theorem 9. Using Cauchy's integral formula and Hölder's inequality in succession we get

$$\begin{aligned} |a_\nu| &= \left|\frac{P^{(\nu)}(0)}{\nu!}\right| = \left|\frac{1}{2\pi i}\int_{\mathscr{E}_{1+1/n}} \frac{P(z)\, dz}{z^{\nu+1}}\right| \\ &\leqq \frac{1}{2\pi}\left\{\int_{\mathscr{E}_{1+1/n}} |P(z)|^p |dz|\right\}^{1/p} \left\{\int_{\mathscr{E}_{1+1/n}} \frac{|dz|}{|z|^{(\nu+1)q}}\right\}^{1/q}, \end{aligned}$$

$$p > 1, \quad 1/p+1/q = 1.$$

Theorem 9 follows if we now use (3.3).

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