LOCAL TRIVIALITY OF HUREWICZ FIBER MAPS(1)

BY
SOON-KYU KIM

1. **Introduction.** Let $p : E \to B$ be a Hurewicz fiber map, i.e., the map p has the path lifting property or what is the same the covering homotopy property holds for all spaces.

In [12], Raymond conjectured that if $p : E \to B$ is a Hurewicz fiber map from a manifold E (with empty boundary) onto a weakly locally contractible (wlc) paracompact base B, then the map p is locally trivial. He proved the following two theorems in support of this conjecture.

THEOREM (R1). *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR space E onto a wlc paracompact base B. Suppose that E is a generalized n-manifold (n-gm) over a principal ideal domain L. Then*

(a) *each fiber F_b is a k-gm over L for some integer k,*

(b) *if some component of some fiber is compact, then B is an $(n-k)$-gm over L.*

THEOREM (R2). *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR space E onto a wlc paracompact base B. Suppose that E is a (generalized) manifold (over a principal ideal domain) and some fiber has a connected component which is compact and of dimension $\leqq 2$. Then, the fibering is locally trivial.*

We will refer to these theorems as Theorem (R1) and (R2). We note that, in Theorem (R1) (b), the assumption that when the characteristic of L is 0, then the automorphism group of L is finite or that the fiber has an oriented, connected, compact covering space, is necessary due to the incompleteness of the argument for Corollary (3.5) of Bredon's paper which appeared in Michigan Math. J. **10** (1963).

The main purpose of this paper is to generalize Theorems (R1) and (R2). We consider the case where the total space E has nonempty boundary and the case where the fibers are noncompact. As indicated, [12, p. 43], the conjecture is false if the manifold E has nonempty boundary. However, by imposing an additional "splitting" restriction on the fiber map, $p : E \to B$, where E has nonempty boundary, we obtain results (Theorem (3.5)) similar to those above. In the case where the fibers are noncompact, we compactify the total space along each fiber and get the local triviality of the map p (Theorem (4.1)).

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In some propositions when fibers are compact 2-manifolds with nonempty boundary we have to assume that all fibers are homeomorphic in order to get local triviality of a map. The reason for this is that two different connected compact 2-manifolds with nonempty boundary can have the same homotopy type. However, this difficulty sometimes disappears by a "stabilizing" of a fibering. We study this phenomenon in §2.

The definition of a splitting lifting function will be defined in §3. By a generalized n-manifold (n-gm) we mean what Raymond and Wilder call a locally orientable generalized n-manifold or cohomology n-manifold. (See [13] and [16] for definition.) All other terminologies which are used in this paper are either well known or can be found in the references [2], [5], [8], [12], [14], and [16].

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2. **Stabilizing theorem.** We first state two propositions which are immediate consequences of results from [7] and [12].

(2.1) PROPOSITION. *Let $p : E \to B$ be a Hurewicz fiber map from a connected compact separable metric ANR space E to a wlc and paracompact base B such that all fibers are homeomorphic to a k-manifold M with or without boundary. If $k \leqq 2$, then the fibering (E.B.p.) is locally trivial.*

Note 1. If E is an n-gm over L without boundary, then actually we need only assume that some fiber has $\dim_L \leqq 2$, where $\dim_L M$ means the cohomology dimension of the space M.

Note 2. If $k=3$ and E is locally i-connected (homotopy sense), $i \leqq 2$, and any homotopy cell in M is a 3-cell, then $p : E \to B$ is also a locally trivial fiber map by [6, Theorem (6.1)] and [12, (2.5)].

Proof. By [5], we know that B is a metric ANR. Therefore the fibering has a slicing function by [12, (2.3)]. Then p is a 0-regular map (in the sense of Dyer and Hamstrom) by [12, (2.5)]. Therefore the proposition follows by [7, Corollary 2 of Theorem (3)]. Q.E.D.

(2.2) PROPOSITION. *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR n-gm E over L, with or without boundary, onto a wlc and paracompact base B. Suppose the maximal dimensional fibers are compact and all those fibers are homeomorphic to a k-manifold M with or without boundary, $k \leqq 2$. Then p, restricted to the subspace consisting of all maximal dimensional compact fibers, is a locally trivial fiber map.*

Proof. We know that B is a metric ANR since E is an ANR and B is 0-connected (see [5]). Let $B_1 = \{b \in B \mid p^{-1}(b) \text{ is compact}\}$. Then by (2.7) of [12], B_1 is an open set in B and $P_1 = p|p^{-1}(B_1) : p^{-1}(B_1) \to B_1$ is a proper map since E is locally

compact and B is metric and 0-connected. Let $A=\{b \in B \mid p^{-1}(b)$ is a maximal dimensional fiber$\}$. Then A is an open subset of B by (2.5) of [12]. Hence $C=B_1 \cap A$ is open in B. Therefore, $p_0=p|p^{-1}(C) : p^{-1}(C) \to C$ is a Hurewicz fiber map from a separable metric ANR n-gm $p^{-1}(C)$ onto a paracompact base C. It is easy to see that p_0 is a 0-regular map by (2.5) of [12]. Therefore the local triviality of p_0 follows by Corollary 2 of Theorem 3 of [7]. Q.E.D.

Note. If $k=3$ and if $p^{-1}(C)$ is locally i-connected, $i \leqq 2$, and homotopy cells in M are 3-cells, then p_0 is also a locally trivial fiber map by [6, (6.1)].

The above propositions may be regarded as generalizations of Theorem (R2). The hypotheses of these propositions are very strong and the guiding philosophy of this paper will be to develop reasonable geometric conditions on a fibering which will enable one to relax these stringent conditions. The difficulty in relaxing the conditions of these propositions arises from the fact that two different 2-manifolds with nonempty boundary may have the same homotopy type without being homeomorphic and the fact([2]) that there is a Hurewicz fiber map $p : E \to B$ such that $p^{-1}(b)$, $p^{-1}(b')$ are 2-manifolds of the same homotopy type but not homeomorphic. However, this difficulty sometimes disappears because it is possible, in some sense, to "stabilize" a fibering by taking the product of the total space with an interval and, for this new fibering, deduce that it is locally trivial (Theorem (2.4)).

Suppose that $E_{i,j}$ denotes a connected compact orientable 2-manifold of genus j and with i boundary components. The integers i and j can be any value $\geqq 0$. The nonorientable connected compact 2-manifold of nonorientable genus j (>0) and i boundary components will be denoted by $E^*_{i,j}$.

(2.3) LEMMA. *Let E and E' be two orientable (or nonorientable) connected compact 2-manifolds with nonempty boundary. Then E and E' have the same homotopy type if and only if their doubles $D(E)$ and $D(E')$ are homeomorphic. Moreover, in the orientable case, E and E' have the same homotopy type if and only if $E \times I$ and $E' \times I$ are homeomorphic, where I is the unit interval.*

Proof. The first part of the lemma is easy to see since $D(E_{i,j})=E_{0,1+2j-1}$, $D(E^*_{i,j})=E^*_{0,2(i+j-1)}$. Let us prove that $E=E_{i,j}$ and $E'=E_{k,n}$ $(i, k>0)$ have the same homotopy type if and only if $E_{i,j} \times I$, $E_{k,n} \times I$ are homeomorphic. Suppose $E_{i,j} \times I$ and $E_{k,n} \times I$ are homeomorphic. Then it is obvious that $E_{i,j}$ and $E_{k,n}$ have the same homotopy type. Now let $E_{i,j}$ and $E_{k,n}$ be the same homotopy type. Then $i+2j-1 = k+2n-1$ because $E_{i,j}$ and $E_{k,n}$ have the homotopy type of a bouquet of $(i+2j-1)S^1$'s and a bouquet of $(k+2n-1)S^1$'s, respectively. And we know that $\partial(E_{i,j} \times I)=D(E_{i,j})=E_{0,i+2j-1}$ and $\partial(E_{k,n} \times I)=D(E_{k,n})=E_{0,k+2n-1}$. Therefore $E_{i,j} \times I$ and $E_{k,n} \times I$ are connected compact 3-manifolds with boundaries closed

([2]) We learned from the referee that an example of such Hurewicz fiber maps can be easily found via the replacement theorem due to E. Fadell, S. Langston and P. Tulley (to appear).

orientable 2-manifolds each of genus $i+2j-1$. In fact, they are solid tori each of genus $i+2j-1$ (see (6.1) of [11]). Hence $E_{i,j}\times I$ and $E_{k,n}\times I$ are homeomorphic. Q.E.D.

(2.4) THEOREM. *Let $p : E \to B$ be a Hurewicz fiber map from a connected locally compact separable metric ANR E onto a wlc and paracompact base B such that each fiber is a connected compact orientable 2-manifold with nonempty boundary. Then $p' : E\times I \to B$, defined by $p'(x, t)=p(x)$ for each $(x, t)\in E\times I$, is a locally trivial fiber map which is fiber homotopy equivalent to the given fibering (E, B, p). Moreover, if E is an n-gm over L with or without boundary, then the space B is an $(n-2)$-gm over L with or without boundary.*

Proof. Since B is 0-connected, all fibers of the fibering (E, B, p) have the same homotopy type. Therefore all fibers of $(E\times I, B, p')$ are homeomorphic to a connected 3-manifold M with boundary by Lemma (2.3). Since $E\times I$ is a locally compact metric space and E is a metric ANR and each fiber is compact, p' is a proper map by (2.7) of [12]. Since E, being an ANR, is locally i-connected ($i\leqq 2$), $E\times I$ is also locally i-connected ($i\leqq 2$). Therefore p' is homotopically 2-regular by (2.5) of [12]. And we know that the fiber of $p' : E\times I \to B$ has no homotopy cells other than cells by Theorem 1 of [10] since the fiber is imbedded in the 3-sphere S^3. Then the first part of the theorem follows by Theorem (6.1) of [6].

Now B is an $(n-2)$-gm over L because $E\times I$ is an $(n+1)$-gm over L, $p' : E\times I \to B$ is a locally trivial fiber map, and the fiber $p'^{-1}(b)$, $b\in B$, is a 3-gm over L by Theorem 6 of [13]. (In Theorem 6 of [13], the coefficient group L is the integers or a field. But we are told by Raymond that L may be a principal ideal domain since spaces are all ANR's.) Q.E.D.

3. **Hurewicz fiber maps of manifolds with boundary.** Let $p : E\to B$ be a Hurewicz fiber map. Assume E is an n-gm over L with nonempty boundary $\partial(E)$. An effect of assuming the existence of a "splitting lifting function", to be defined in (3.4), is that the restriction of p to the split part is also a Hurewicz fiber map. We shall firstly study some propositions of a fiber map whose restriction to a subset (mostly $\partial(E)$) of E is also a Hurewicz fiber map.

(3.1) PROPOSITION. *Let $p : E\to B$ be a Hurewicz fiber map from an ANR n-gm E over L with nonempty boundary $\partial(E)$ onto a 0-connected metric space B. Assume that $p^{-1}(b)$ is a connected compact orientable k-gm over L with boundary equal to $p^{-1}(b)\cap\partial(E)\neq\varnothing$, and $p^{-1}(b)\cap \operatorname{Int}(E)\neq\varnothing$, for some point $b\in B$. If*

$$p' = p|\partial(E) : \partial(E)\to B$$

is a Hurewicz fiber map, then $p|\operatorname{Int}(E) : \operatorname{Int}(E)\to B$ is onto.

Proof. Suppose there exists a point $b_0\in B$ such that $p^{-1}(b_0)\cap \operatorname{Int}(E) = \varnothing$. Then $p^{-1}(b_0)\subset\partial(E)$, hence $p^{-1}(b_0)=p^{-1}(b_0)\cap\partial(E)=p'^{-1}(b_0)$. Then $p^{-1}(b_0)$ and

$p'^{-1}(b)$ have the same homotopy type since p' is a Hurewicz fiber map and B is 0-connected. Similarly $p^{-1}(b_0)$ and $p^{-1}(b)$ have the same homotopy type. Therefore $p^{-1}(b)$ and $p'^{-1}(b)=\partial(p^{-1}(b))$ have the same homotopy type. By [1, (2.6)] there exists a lifting function λ for the fibering (E, B, p) such that the restriction of λ, to the boundary of E, is a lifting function for the fibering $(\partial(E), B, p')$. Since a homotopy equivalence between two fibers is induced by a lifting function, we may conclude that the homotopy equivalent map from $p'^{-1}(b)$ to $p^{-1}(b)$ is the inclusion map. This contradicts the fact that the inclusion map from the boundary into the manifold itself is never a homotopy equivalence. Q.E.D.

(3.2) PROPOSITION. *Let $p : E \to B$ be a locally trivial fiber map from a connected n-gm E over L with nonempty boundary onto a space B. Assume that the map $p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B$ is onto. Then B is a k-gm over L without boundary, the fiber $p^{-1}(b)$ is an $(n-k)$-gm over L with nonempty boundary and*

$$p' = p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B,$$

as well as $p''=p|\partial(E) : \partial(E) \to B$, are locally trivial and onto B.

Proof. Since $p : E \to B$ is a locally trivial fiber map, the fiber F and the base B are $(n-k)$-gm and k-gm respectively over L by the factorization theorem [13, Theorem 6]. In particular, if $e \in E$, in local coordinates, can be written as $b \times f$, for $b \in B$, and $f \in F$, then $e \in \mathrm{Int}\,(E)$ if and only if $b \in \mathrm{Int}\,(B)$ and $f \in \mathrm{Int}\,(F)$. Restricting to $\mathrm{Int}\,(B)$, $p|p^{-1}(\mathrm{Int}\,(B)) : p^{-1}(\mathrm{Int}\,(B)) \to \mathrm{Int}\,(B)$ is onto and a locally trivial fiber map. However, $\mathrm{Int}\,(E) \subset p^{-1}(\mathrm{Int}\,(B))$, hence $\mathrm{Int}\,(B)=B$. Since E has boundary, some fiber and hence all fibers have boundary.

Now for any point $b \in B$, there exist an open set U of b and a homeomorphism $h_u : U \times p^{-1}(b) \to p^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times p^{-1}(b) \arrow[r, "h_u"] \arrow[dr, "\pi"'] & p^{-1}(U) \arrow[d, "p|p^{-1}(U)"] \\ & U \end{tikzcd}$$

commutes. Since U is a gm without boundary, $\partial(U \times p^{-1}(b)) = U \times \partial(p^{-1}(b))$. Therefore $h_u(U \times \partial(p^{-1}(b))) = p''^{-1}(U)$, and $h_u(U \times \mathrm{Int}\,(p^{-1}(b))) = p'^{-1}(U)$. That is $h_u|U \times p'^{-1}(b)$ and $h_u|U \times p''^{-1}(b)$ are homeomorphisms from $U \times p'^{-1}(b)$ onto $p'^{-1}(U)$ and from $U \times p''^{-1}(b)$ onto $p''^{-1}(U)$, respectively. This proves the local triviality of the maps. Q.E.D.

(3.3) PROPOSITION. *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR n-gm over L with nonempty boundary $\partial(E)$ onto a wlc and*

paracompact base B. Suppose that $p|\partial(E) : \partial(E) \to B$ *is onto and a Hurewicz fiber map, and that* $p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B$ *is also onto. Suppose further that there is a collared neighborhood of* $\partial(E)$, *denoted by* $\partial(E) \times [0, 1)$, *such that for any smaller collared neighborhood,* $\partial(E) \times [0, t)$, $t \in (0, 1)$, *there is a fiber retraction* $r_t : E \to E_t = E - (\partial(E) \times [0, t))$. *Then, there is a lifting function* λ'_t, *for the fibering* (E, B, p), *which is stationary in* $\partial(E) \cup E_t$.

Proof. Let λ be a lifting function for the fibering (E, B, p). Let for each $t \in (0, 1)$, $Z_t = \{(e, \omega) \in E_t \times B^I \mid p_t(e) = \omega(0)\}$ and $q : E_t^I \to Z_t$ be such that $q(\alpha) = (\alpha(0), p_t\alpha)$, $\alpha \in E_t^I$, where $p_t = p|E_t$. Define a function $\lambda_t : Z_t \to E_t^I$ by $\lambda_t(e, \omega)(x) = r_t(\lambda(e, \omega)(x))$, $x \in I$, $(e, \omega) \in Z_t$. Then λ_t is a lifting function for the map $p_t : E_t \to B$. Hence (E_t, B, p_t) is a Hurewicz fibering. Now let $E'_t = \partial(E) \cup E_t$ and $p'_t = p|E'_t$. Then $p'_t : E'_t \to B$ is onto and a Hurewicz fiber map. Since the space E'_t is a closed subset of the ANR E, and the space B is metric, there exists a lifting function λ'_t for the fibering (E, B, p) such that $\lambda'_t(e, \omega) \in (E'_t)^I$ if $e \in E'_t$, $(e, \omega) \in Z = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$ by Theorem (2.6) of [1]. Q.E.D.

(3.4) Definition. Let $p : E \to B$ be a Hurewicz fiber map and E_1 a subspace of E. A lifting function λ for the fibering (E, B, p) is said to *split* into E_1 and $E - E_1$ provided, for each $(e, \omega) \in Z = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$, $\lambda(e, \omega)$ is a path in E_1 if $e \in E_1$ and $\lambda(e, \omega)$ is a path in $E - E_1$ if $e \in E - E_1$. Such a lifting function is called a *splitting lifting function.*

We note that if the total space E of a fibering (E, B, p) is a gm over L with nonempty boundary $\partial(E)$, and if $p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B$ is not onto, then there is no lifting function which splits into $\partial(E)$ and $\mathrm{Int}\,(E)$ (see Propositions (3.1) and (3.3)).

Now we are ready to state and prove the main theorem of this section.

(3.5) Theorem. *Let* $p : E \to B$ *be a Hurewicz fiber map from a connected separable metric ANR n-gm E over L with nonempty boundary* $\partial(E)$, *which is also an ANR, onto a wlc and paracompact base B. Suppose there exists a lifting function* λ *for the fibering* (E, B, p) *such that* λ *splits into* $\partial(E)$ *and* $\mathrm{Int}\,(E)$. *Then*

(1) *the restriction of p to* $\partial(E)$, $p' = p|\partial(E) : \partial(E) \to B$, *and the restriction of p to* $\mathrm{Int}\,(E)$, $p'' = p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B$, *are onto,*

(2) *if there is a point* $b' \in B$ *such that (each component of* $p^{-1}(b')) \cap$ *(each component of* $\partial(E)$*) is connected, then each fiber* $p^{-1}(b)$, $b \in B$, *is a gm over L with boundary* $p^{-1}(b) \cap \partial(E)$, *moreover, if* $p^{-1}(b) \cap \partial(E)$ *contains a compact component for some* $b \in B$, *then B is a gm over L. (Here we assume, for technical reasons, that when the characteristic of L is* 0, *then the automorphism group of L is finite or that the fiber has an oriented, connected, compact covering space. This is guaranteed of course if* $L = Z$.)

(3) *if there is a point* $b \in B$ *such that* $p^{-1}(b) \cap \partial(E)$ *has a compact component of* $\dim_L \leqq 2$, *then* $p' = p|\partial(E)$ *is a local trivial fiber map,*

(4) *furthermore, if* $p^{-1}(b')$ *is a compact space of* $\dim_L \leqq 2$ *and (each component of*

$p^{-1}(b')) \cap$ (each component of $\partial(E)$) is connected for a point $b' \in B$, then the fibering (E, B, p) itself is locally trivial. Hence $p' = p|\partial(E)$, and $p'' = p|\mathrm{Int}\,(E)$ become locally trivial fiber maps.

Proof. (1) Let C be a component of $\partial(E)$. Suppose $C \cap p^{-1}(b) = \varnothing$ for some point $b \in B$. We know that there exists a point $b_0 \in B$ such that $p^{-1}(b_0) \cap C \neq \varnothing$. Let $e \in C \cap p^{-1}(b_0)$. Then there is a path α from b_0 to b since B is 0-connected, and $(e, \alpha) \in Z = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$. Therefore the lifted path $\lambda(e, \alpha)$ in E is such that $\lambda(e, \alpha)(0) = e$ and $\lambda(e, \alpha)(1) \in p^{-1}(b)$. But $p^{-1}(b) \cap C = \varnothing$. Hence λ does not split into C and $E - C$. This contradicts the assumption that λ splits into $\partial(E)$ and $\mathrm{Int}\,(E)$. Therefore $p|C : C \to B$ is onto and a Hurewicz fiber map. The proof of ontoness of p'' is similar.

(2) Let C be a component of $\partial(E)$ and $p_c = p|C$. By Theorem (R1) $p_c^{-1}(b)$ is a k'-gm for any point $b \in B$, for some k', and B is an $((n-1)-k')$-gm without asserting anything about local orientability. Since this is true for any component of $\partial(E)$, $p'^{-1}(b) = p^{-1}(b) \cap \partial(E)$ is the disjoint union of k'-gm's. Hence $p'^{-1}(b)$ is a k'-gm over L for each $b \in B$. Since λ splits into $\partial(E)$ and $\mathrm{Int}\,(E)$, $p|\mathrm{Int}\,(E) : \mathrm{Int}\,(E) \to B$ is a Hurewicz fiber map and the map is onto by (1). By Theorem (R1), $p^{-1}(b) \cap \mathrm{Int}\,(E)$ is a k-gm over L because $\mathrm{Int}\,(E)$ is also a connected separable metric ANR n-gm over L, and B is an $(n-k)$-gm over L without asserting anything about local orientability. Since the dimension of the base B is $n-k$ and $n-1-k'$ is also the dimension of B, $k = k'+1$. Therefore we know that $p^{-1}(b) \cap \mathrm{Int}\,(E)$ is a k-gm over L and $p^{-1}(b) \cap \partial(E)$ is a $(k-1)$-gm over L, and $p^{-1}(b) \cap \partial(E)$ is a closed subset of $p^{-1}(b)$ because $\partial(E)$ is closed in E. We note that all fibers are the same homotopy type since B is 0-connected. Now we will show that $p^{-1}(b) \cap \partial(E)$ is the boundary of $p^{-1}(b)$ for each $b \in B$, i.e., $p^{-1}(b)$ is a k-gm over L with boundary $p^{-1}(b) \cap \partial(E)$. Let C again be a component of $\partial(E)$, then the set $\mathrm{Int}\,(E) \cup C$, denoted by E', is a subspace of E and an n-gm over L with boundary C and the restriction $p|E' : E' \to B$ is also a Hurewicz fiber map and the lifting function λ', which is the restriction of λ, splits into $\mathrm{Int}\,(E)$ and C. Let us consider the double of E', $D(E') = E_1 \cup C \cup E_2$ with the identification by means of the inclusion map $j : C \subset E_1'$ or E_2', where E_1 and E_2 are copies of $\mathrm{Int}\,(E)$ and E_1' and E_2' are $E_1 \cup C$ and $E_2 \cup C$, respectively. Then $D(E')$ is an n-gm over L without boundary and C separates $D(E')$. The map $p_1 : D(E') \to B$, defined by $p_1|E_1' = p|E_1'$, $p_1|E_2' = p|E_2'$, is a Hurewicz fiber map. Therefore $p_1^{-1}(b)$ is a gm over L by Theorem (R1), and $p_1^{-1}(b) = (p_1^{-1}(b) \cap E_1) \cup (p_1^{-1}(b) \cap C) \cup (p_1^{-1}(b) \cap E_2)$ and $p_1^{-1}(b) \cap E_i$, $i = 1, 2$, is a k-gm over L. Hence the dimension of $p_1^{-1}(b)$ is also k. Let F be a component of $p^{-1}(b)$. Then the component F_1 of $p_1^{-1}(b)$, which contains F, is a (locally orientable) gm over L, and $F_1 \cap C$ is connected (by assumption) and locally orientable. By Theorem 2 of [13], $F_1 \cap E_1$ $(= F \cap \mathrm{Int}\,(E))$ has $F_1 \cap C$ $(= F \cap C)$ as its frontier, and onto $F \cap \mathrm{Int}\,(E)$, $F \cap C$ fits as a generalized manifold, i.e., for any component C of $\partial(E)$, $F \cap C$ is boundary of F. Therefore $F \cap C$ is boundary

of $p^{-1}(b)$. Repeating this argument for other components of $p^{-1}(b)$, and for other components of $\partial(E)$, we conclude that $p^{-1}(b) \cap \partial(E)$ is the boundary of $p^{-1}(b)$. This proves the first part of (2). The fact that B is an $(n-k)$-gm over L is obvious by our assumption and Theorem (R1) since $p_1 : D(E') \to B$ is a Hurewicz fiber map.

(3) Let C be a component of $\partial(E)$. Then the fibering $(C, B, p|C)$ is locally trivial by Theorem (R2). This is true for any component of $\partial(E)$. Hence the fibering $(\partial(E), B, p')$ is locally trivial.

(4) By the assumption and (2), $p^{-1}(b')$ is a compact k-gm over L of dimension $\leqq 2$ with boundary $\partial(p^{-1}(b')) = p^{-1}(b') \cap \partial(E)$. Now for any other point $b \in B$, $b \neq b'$, $\partial(p^{-1}(b))$ and $\partial(p^{-1}(b'))$ are closed subsets of $p^{-1}(b)$ and $p^{-1}(b')$, respectively. Since $p' : \partial(E) \to B$ is a locally trivial fiber map by (3), $\partial(p^{-1}(b))$ and $\partial(p^{-1}(b'))$ are homeomorphic. Hence $\partial(p^{-1}(b))$ and $\partial(p^{-1}(b'))$ are compact sets of dimension $\leqq k-1$, $k \leqq 2$. Since $p : E \to B$ is a Hurewicz fiber map and B is 0-connected, $p^{-1}(b)$ and $p^{-1}(b')$ have the same homotopy type. Let $\phi : p^{-1}(b) \to p^{-1}(b')$ be a homotopy equivalence, which is induced by the splitting lifting function λ, i.e., defined by $\phi(x) = \lambda(e, \omega)(1)$, $x \in p^{-1}(b)$, ω is a path from b to b' such that $\omega(0) = p(x)$. Let F be a component of $p^{-1}(b)$, which contains some components of $\partial(p^{-1}(b))$. Then $\phi(F)$ is connected. Let F' be a component of $p^{-1}(b')$, which contains $\phi(F)$. Then F and F' have the same homotopy type. Since λ splits into Int (E) and $\partial(E)$, we have $\phi(\partial(E)) \subset \partial(p^{-1}(b')) = p^{-1}(b') \cap \partial(E)$ and the different components of $\partial(E)$ go to the different components of $\partial(p^{-1}(b'))$. Hence $\phi(\partial(E)) \subset F' \cap \partial(E)$, i.e., $\phi(\partial(F)) \subset \partial(F')$. Since there is one-to-one correspondence between the components of $\partial(p^{-1}(b))$ and $\partial(p^{-1}(b'))$ as well as between components of $p^{-1}(b)$ and $p^{-1}(b')$, the union of all components of $\partial(F')$, which contains a component of $\phi(\partial(F))$, is the only boundary of F'. But $\partial(F)$ and $\partial(F')$ are homeomorphic since corresponding components of $\partial(p^{-1}(b))$ and $\partial(p^{-1}(b'))$ are homeomorphic. Therefore F and F' are homeomorphic since they are k ($\leqq 2$)-manifolds and they have the same homotopy type and their boundaries are the same. If there is a component of $p^{-1}(b)$, which has no boundary, then the corresponding component, in $p^{-1}(b')$, has no boundary. Hence they are homeomorphic. Thus corresponding components of $p^{-1}(b)$ and $p^{-1}(b')$ are homeomorphic. Therefore $p^{-1}(b)$ and $p^{-1}(b')$ are homeomorphic. By (2.10) of [12] p can be factored as $qp' : E \xrightarrow{p'} B' \xrightarrow{q} B$, where p' is a regular fiber map, B' is a 0-connected space, and q is a covering map such that $p'^{-1}(b')$ is a path component of $p^{-1}(q(b'))$ for each $b' \in B'$. Therefore each $p'^{-1}(b')$ is compact. Hence by (2.7) of [12], p' is proper. Since p' is a regular fiber map and B is locally 0-connected, p' is open by (2.6) of [12]. By (2.5) of [12] p is uniformly locally contractible. Hence p' is a 0-regular map. Therefore, by Corollary 2 of Theorem 3 of [7], p' is a locally trivial fiber map. This completes the proof of the local triviality of p. The local triviality of p' and p'' follows by (1) and Proposition (3.2). Q.E.D.

(3.6) Remarks. If λ does not split into $\partial(E)$ and Int (E), then $p|\partial(E) : \partial(E) \to B$

may not be a locally trivial map. In fact, there is a simple example for which the restriction is not even a Hurewicz fiber map. The example, due to E. Fadell, is the Hurewicz fibering (E, B, p) where B is the unit interval and E is a solid isoceles triangle with base B and p is the vertical projection map. Let (E, B, p) be a Hurewicz fibering with E a manifold with nonempty boundary and let (E_1, B, p_1) and (E_2, B, p_2) be two copies of (E, B, p). Let $D(E)$ be the double of E which is obtained from the disjoint union of E_1 and E_2 by identifying the boundaries by means of the inclusions. Then the above example shows that the map $p' : D(E) \to B$, defined by $p'|E_1 = p_1$ and $p'|E_2 = p_2$, may not be a Hurewicz fiber map. We do not know whether or not the fibering (E, B, p) becomes locally trivial if $p' : D(E) \to B$ is a locally trivial fiber map. If the answer to this question is affirmative, then part (4) of Theorem (3.5) follows directly by doubling the total space E.

4. **Hurewicz fiber maps of noncompact manifolds.** In this section, for the second generalization of Theorem (R2), we assume that all fibers of a Hurewicz fiber map $p : E \to B$ are connected noncompact manifolds. The total space E is a gm over a principal ideal domain L, and E may have nonempty boundary.

We prove the following theorem:

(4.1) THEOREM. *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR n-gm E over a principal ideal domain L onto a wlc and paracompact finite-dimensional base B. Suppose that all fibers are homeomorphic to a space M where M is either the real line R^1 or a connected 2-manifold with finitely generated homology groups and exactly one end. Then the fibering (E, B, p) is locally trivial.*

Note. If M is the real line or the plane and if E has no boundary, we actually need only assume that some fiber is homeomorphic to M because all fibers are 1 or 2-manifolds and have the same homotopy type.

Proof. Under the assumptions of the theorem, B is 0-connected and is a separable metric ANR. Furthermore B is locally compact since E is locally compact and the map p is an open map.

(a) *Compactification.* Let $\bar{E}$ be the disjoint union of E and the product space $B \times \{0\}$. We define $\bar{p} : \bar{E} \to B$ by $\bar{p}(e) = p(e)$ for each $e \in E$, and $\bar{p}(e) = b$ for each $e = (b, 0) \in B \times \{0\}$. We give a topology $\bar{\mathscr{U}}$ on $\bar{E}$ in the following way:

Let $\mathscr{U}$ be the collection of all open sets of E. Let W be an open set of B such that p admits a cross-section f on W, and there exists a closed subset W_0 of $p^{-1}(W)$ such that the closure of W_0 in E is compact and $W_0 \supset f(W)$, where $W_0 \cap p^{-1}(b)$ is compact in $p^{-1}(b)$ for each $b \in W$. We claim that subsets W and W_0 exist. Let W' be an open subset of B, which is uniformly contractible in B. For a point $b_0 \in W'$, let

$$\phi_{b_0} : p^{-1}(W') \rightleftarrows W' \times p^{-1}(b_0) : \psi_{b_0}$$

be a fiber homotopy equivalence. Here a regular lifting function is used for defining the maps ϕ_{b_0} and ψ_{b_0}. Then the fiber homotopy equivalences ϕ_{b_0} and ψ_{b_0} are both

the "identity" on $F=p^{-1}(b_0)=b_0\times p^{-1}(b_0)$ (see [12, (2.2)]). Let W be an open subset of B with compact closure such that $b_0\in W\subset \mathrm{cl}\,(W)\subset W'$, where $\mathrm{cl}\,(W)$ denotes the closure of W. Let x be a point in F. Then the projection

$$\pi : W'\times F\to W'$$

admits a cross-section g on W' such that $g(W')\ni x$. We can take a compact set C in $W'\times F$ such that $C\supset g(\mathrm{cl}\,(W))$ and $C\cap(b\times F)$ is compact for each $b\in \mathrm{cl}\,(W)$. Then $\psi_{b_0}(C)$ is a compact subset of $p^{-1}(W')$ and $\psi_{b_0}(C)\supset\psi_{b_0}g(\mathrm{cl}\,(W))\ni x$, where x is in the space $p^{-1}(W')$. Here $\psi_{b_0}g$, denoted by f, is a cross-section on $\mathrm{cl}\,(W)$ of the fibering (E, B, p). Since $\psi_{b_0}(C)$ is closed in E, $\psi_{b_0}(C)\cap p^{-1}(W)$, denoted by W_0, is closed in $p^{-1}(W)$ and $\mathrm{cl}\,(W_0)$ is compact in E. $W_0\supset f(W)$ and $W_0\cap p^{-1}(b)$ is compact in $p^{-1}(b)$ since it is a continuous image of a compact set $C\cap(b\times F)$ for each $b\in W$.

Thus we can find infinitely many such pairs (W, W_0), $W\ni b$, for each point $b\in B$. The collection of all such pairs for all $b\in B$ will be denoted by φ.

Let $\mathscr{V}'$ be the collection of sets of the form $\bar{p}^{-1}(W)-W_0$ for each $(W, W_0)\in\varphi$. Let $\mathscr{V}$ be the collection of all subsets V of $\bar{E}$ such that $V\cap E$ is open in E and $V\cap(\bar{E}-E)\neq\varnothing$, and for each $x\in V\cap(\bar{E}-E)$ there exists an element $V'\in\mathscr{V}'$ such that $x\in V'\subset V$.

We show that any element V' of $\mathscr{V}'$ is an element of $\mathscr{V}$. Let V' be an element of $\mathscr{V}'$. Then $V'=\bar{p}^{-1}(W)-W_0$ for some pair $(W, W_0)\in\varphi$. Therefore $V'\cap E=p^{-1}(W)-W_0$ is open in E since $p^{-1}(W)$ is open in E and W_0 is closed in $p^{-1}(W)$, that is, $V'\cap E\in\mathscr{U}$. Also $V'\cap(\bar{E}-E)\neq\varnothing$. If $x\in V'\cap(\bar{E}-E)$, then $\bar{p}(x)\in W$. Let W' be an open set such that $\bar{p}(x)\in W'\subset W$, and let $W_0\cap p^{-1}(W')=W_0'$. Then $(W', W_0')\in\varphi$ and $V''=\bar{p}^{-1}(W')-W_0'\subset V'$. Thus there exists an element $V''\in\mathscr{V}'$ such that $x\in V''\subset V'$. Hence V' is an element of $\mathscr{V}$.

The collection $\bar{\mathscr{U}}$, consisting of all elements of $\mathscr{U}$ and $\mathscr{V}$, gives a topology for $\bar{E}$. We leave this detail to the reader since it is not too difficult.

(b) *Checking necessary conditions.*

(i) The function $\bar{p} : \bar{E}\to B$, which is defined in (a), is continuous and open. This is easy to check.

(ii) The subspace $B\times\{0\}$ of $\bar{E}$ is homeomorphic to B. This is trivial by (i).

(iii) The space $\bar{E}$ is a Hausdorff space.

We consider the only case where two points x and x' lie in $\bar{p}^{-1}(b)$ for some $b\in B$ since all other cases are trivial. Then one of two points, say x, must be in $p^{-1}(b)$. Then there exists an element $(W, W_0)\in\varphi$ such that $b\in W$, and W_0 contains an open neighborhood U of x. This is obvious by the construction of pairs of φ in (a), the fact that the homotopy equivalence between $p^{-1}(W)$ and $W\times p^{-1}(b)$ is the "identity" on $p^{-1}(b)$ and that a regular lifting function is used for the definition of the homotopy equivalence. Then $U\ni x$ and $(\bar{p}^{-1}(W)-W_0)\ni x'$ are disjoint open sets.

(iv) The space $\bar{E}$ is locally compact.

Since E is locally compact, it is enough to consider points in $\bar{E}-E$. Let $x \in \bar{E}-E$ and U be an open set of $\bar{E}$ containing x. We shall find a compact neighborhood of x in U. Since U is an element of $\mathscr{V}$, there exists a pair (W', W_0') in φ, such that $x \in (\bar{p}^{-1}(W')-W_0') \subset U$. Let W be an open set, containing $\bar{p}(x)$, such that the closure of W, denoted by C, is compact and is contained in W'. Then $\mathrm{cl}\,(\bar{p}^{-1}(C)-W_0')$, denoted by D, is a neighborhood of x in U. We claim that D is compact. Let $\{V\}$ be an open covering of D. Let $\{V\}'$ be the collection of all elements of $\{V\}$, which meet $\bar{E}-E$. Then $\{\{V\}' \cap (\bar{E}-E)\}$, which is the collection of $V \cap (\bar{p}^{-1}(C)-p^{-1}(C))$ for each $V \in \{V\}'$, is an open covering of $\bar{p}^{-1}(C)-p^{-1}(C)$. Since $\bar{p}^{-1}(C)-p^{-1}(C)$ is homeomorphic to C, by (ii), $\bar{p}^{-1}(C)-p^{-1}(C)$ is compact. Therefore there exists a finite number of elements $V_1, \ldots, V_n$ in $\{V\}'$ such that

$$\bigcup_{i=1}^{n} (V_i \cap (\bar{p}^{-1}(C)-p^{-1}(C))) \supset \bar{p}^{-1}(C)-p^{-1}(C).$$

If $D-(\bigcup_{i=1}^{n} V_i)$ is compact, then D is compact. We now prove the compactness of $D-(\bigcup_{i=1}^{n} V_i)$. Since V_i, $i=1, 2, \ldots, n$, is open in D, it is open in $\bar{E}$ if it is in $\bar{p}^{-1}(W)$. Furthermore if V_i meets $\bar{p}^{-1}(\mathrm{cl}\,(W)-p^{-1}(W))$, then there is an open set V_i' in $\bar{E}$ such that $V_i = V_i' \cap D$. Therefore, for any point $x \in \bar{p}^{-1}(C)-p^{-1}(C)$, there exists a pair $(W^x, W_0^x) \in \varphi$ such that $p^{-1}(\mathrm{cl}\,(W^x))-\mathrm{cl}\,(W_0^x)$, denoted by V_x, or $V_x \cap D$, is in $\bigcup_{i=1}^{n} V_i$, here $\mathrm{cl}\,(W_0^x)$ is compact. Then $\{V_x \cap (\bar{p}^{-1}(C)-p^{-1}(C))\}$ is an open covering of $\bar{p}^{-1}(C)-p^{-1}(C)$. Therefore, there is a finite number of elements $V_{x_1}, \ldots, V_{x_m}$, such that $\bigcup_{j=1}^{m} V_{x_j}$ covers $\bar{p}^{-1}(C)-p^{-1}(C)$, and $\bigcup_{j=1}^{m} (V_{x_j} \cap D) \subset \bigcup_{i=1}^{n} V_i$. Therefore $D-(\bigcup_{i=1}^{n} V_i) \subset \bigcup_{j=1}^{m} \mathrm{cl}\,(W^{x_{j_0}})$. Since $D-(\bigcup_{i=1}^{n} V_i)$ is a closed subset of the compact set $\bigcup_{j=1}^{m} \mathrm{cl}\,(W^{x_{j_0}})$, $D-(\bigcup_{i=1}^{n} V_i)$ is compact.

(v) The space $\bar{E}$ is a topologically complete metric space.

By a Simirnov's theorem in [15], $\bar{E}$ is a metric space. Then this property follows by Corollary (2.4) of [3, p. 294].

(vi) The fiber $\bar{p}^{-1}(b)$, as a subspace of $\bar{E}$, is the one point compactification of $p^{-1}(b)$, $b \in B$. Hence $\bar{p}^{-1}(b)$ is an 1-sphere or a compact connected 2-manifold according to the dimension of the fiber $p^{-1}(b)$. We note that the added ideal point, $b \times \{0\}$, to each $p^{-1}(b)$ is an interior point of $\bar{p}^{-1}(b)$.

This is obvious by the construction of the topology for $\bar{E}$.

(vii) The map $\bar{p} : \bar{E} \to B$ is a proper map.

Let B' be a compact subset of B. For each point $b \in B'$, take a uniformly contractible open set W_b' in B, containing b. Let W_b be a compact neighborhood of b in W_b'. Then $\{W_b\}$ is a covering of B'. Therefore, there exists a finite number of sets $W_{b_1}, \ldots, W_{b_n}$ such that $\bigcup_{i=1}^{n} W_{b_i} \supset B'$. It is easy to see that $\bar{p}^{-1}(W_{b_i})$ is compact by an argument similar to that used in (iv). Therefore $\bar{p}^{-1}(B') = \bigcup_{i=1}^{n} \bar{p}^{-1}(W_{b_i})$ is compact.

(viii) The map $\bar{p} : \bar{E} \to B$ is a 0-regular map.

Since we have shown that the map p is an open and proper map, we need only show that for each x, $x \in \bar{p}^{-1}(b)$, $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that if $y, z \in S(x, \delta)$

$\cap\, \bar{p}^{-1}(b')$, $b' \in B$, then there exists an arc from y to z in $S(x, \varepsilon) \cap \bar{p}^{-1}(b')$. Since $p : E \to B$ has the above property by (2.5) of [12], we check only for the points in $\bar{E} - E$. Let $x \in \bar{p}^{-1}(b) \cap (\bar{E} - E)$, $\varepsilon > 0$ be given. Since $S(x, \varepsilon)$ is open in $\bar{E}$, there exists an element $V \in \mathscr{V}$ such that $x \in V \subset S(x, \varepsilon)$, where $V = \bar{p}^{-1}(W) - W_0$ for a pair $(W, W_0) \in \varphi$. Suppose $V \cap \bar{p}^{-1}(b) = \bar{p}^{-1}(b) - (W_0 \cap p^{-1}(b))$ is not 0-connected for some point $b \in W$. Let A_b be the union of all bounded components of $p^{-1}(b) - (W_0 \cap p^{-1}(b))$. Then $\bar{p}^{-1}(b) - ((W_0 \cap p^{-1}(b)) \cup A_b)$ is 0-connected. Let W' be an open subset, with compact closure, such that $W' \subset \mathrm{cl}\,(W') \subset W$. Let $A = \bigcup_{b \in W'} A_b$ and $W_0' = A \cup (W_0 \cap p^{-1}(W'))$. Then $(W', W_0') \in \varphi$, $V' = (\bar{p}^{-1}(W') - W_0) \subset V$ and $V' \cap \bar{p}^{-1}(b)$ is 0-connected for each $b \in W'$. Take a $\delta > 0$ so that $S(x, \delta) \subset V'$. Therefore, if $S(x, \delta) \cap \bar{p}^{-1}(b') \neq \varnothing$, then $b' \in W'$. Hence, if $y, z \in S(x, \delta) \cap \bar{p}^{-1}(b')$, $b' \in B$, then there exists an arc from y to z in $V' \cap \bar{p}^{-1}(b')$ along the fiber $\bar{p}^{-1}(b')$. Therefore this arc is in $S(x, \varepsilon) \cap \bar{p}^{-1}(b')$.

(c) *The proof of the local triviality of the fibering* (E, B, p).

(i) The map $\bar{p} : \bar{E} \to B$ is locally trivial.

Since $\bar{E}$ is a topologically complete metric space, there exists a complete metric space E' such that E' and $\bar{E}$ are homeomorphic. Let h be the homeomorphism between E' and $\bar{E}$. We define a function $p' : E' \to B$ to be $\bar{p}h$. Then p' is a 0-regular map. Therefore p' is a locally trivial map by the Corollary 2 of Theorem 3 of [7]. Therefore, for each $b \in B$, there exist an open set $U \ni b$ and a homeomorphism h_u' from $U \times p'^{-1}(b)$ onto $p'^{-1}(U)$ such that the diagram

$$\begin{tikzcd}
U \times p'^{-1}(b) \arrow[r, "h_u'"] \arrow[dr, "\pi"'] & p'^{-1}(U) \arrow[d, "p'|p'^{-1}(U)"] \\
& U
\end{tikzcd}$$

commutes, where π is the projection map.

In the following diagram, all triangle diagrams are commutative:

$$\begin{tikzcd}
U \times h^{-1}(\bar{p}^{-1}(b)) \arrow[r, "h_u'"] \arrow[d, "\mathrm{id.} \times h^{-1}"] \arrow[dr, "\pi"] & h^{-1}(\bar{p}^{-1}(U)) \arrow[r, "h"] \arrow[d, "p|p'^{-1}(U)"] & \bar{p}^{-1}(U) \arrow[dl, "p|\bar{p}^{-1}(U)"] \\
U \times \bar{p}^{-1}(b) \arrow[r, "\pi"] & U &
\end{tikzcd}$$

Therefore, the diagram

$$\begin{tikzcd} U\times\bar{p}^{-1}(b) \arrow[r, "hh'_u(\mathrm{id.}\times h^{-1})"] \arrow[dr, "\pi"'] & \bar{p}^{-1}(U) \arrow[d, "\bar{p}|\bar{p}^{-1}(U)"] \\ & U \end{tikzcd}$$

commutes, and $hh'_u(\mathrm{id.}\times h^{-1})$ is a homeomorphism. This means that $\bar{p}:\bar{E}\to B$ is locally trivial.

(ii) The fibering (E, B, p) is locally trivial.

We know that, for each $b\in B$, there exist an open set $U\ni b$ in B and a homeomorphism $h: U\times F\to\bar{p}^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U\times F \arrow[r, "h"] \arrow[dr, "\pi"'] & \bar{p}^{-1}(U) \arrow[d, "\bar{p}|\bar{p}^{-1}(U)"] \\ & U \end{tikzcd}$$

commutes, where π is the projection map, $F=\bar{p}^{-1}(b)$. If $h(U\times(F-p^{-1}(b)))=\bar{p}^{-1}(U)-p^{-1}(U)$, then the proof is completed. Suppose this is not the case. Let $f: U\to U\times F$ be a cross-section, which is defined by $f(b')=(b', x_0)$, $b'\in U$, where $x_0=F-p^{-1}(b)$. And let $g: U\to\bar{p}^{-1}(U)$ be a cross-section, which is defined by $g(b')=\bar{p}^{-1}(b')-p^{-1}(b')$, $b'\in U$. Then $h^{-1}g: U\to U\times F$ is a cross-section, which is different from f. We shall prove that, for some open set V in U, there is a fiber preserving homeomorphism (a homeomorphism which maps a fiber into itself)

$$\phi: V\times F\to V\times F \text{ such that } \phi(h^{-1}g(V))=f(V).$$

Since $f(b)=(b, x_0)$ and $h^{-1}g(b)$ are interior points of F, we can find an arc α from $f(b)$ to $h^{-1}g(b)$ in Int (F). Take a disk D in Int (F) such that Int (D) contains the arc α. There exists an open set $V\ni b$ with compact closure such that cl $(V)\subset U$ and $h^{-1}g(\mathrm{cl}\,(V))\subset\mathrm{cl}\,(V)\times\mathrm{Int}\,(D)$, $f(\mathrm{cl}\,(V))\subset\mathrm{cl}\,(V)\times\mathrm{Int}\,(D)$. Let us consider cl $(V)\times D$.

There is a fiber preserving homeomorphism

$$\phi_1: ((\mathrm{cl}\,(V)\times\partial(D))\cup h^{-1}g(\mathrm{cl}\,(V)), h^{-1}g(\mathrm{cl}\,(V)))$$
$$\to((\mathrm{cl}\,(V)\times\partial(D))\cup f(\mathrm{cl}\,(V)), f(\mathrm{cl}\,(V))),$$

which is defined by $\phi_1|\mathrm{cl}\,(V)\times\partial(D)=$ identity and $\phi_1(y)=f(\pi(y))$, $y\in h^{-1}g(\mathrm{cl}\,(V))$,

where π is the projection map from $\mathrm{cl}\,(V)\times F$ onto $\mathrm{cl}\,(V)$. Therefore the diagram

$$\begin{tikzcd}
(\mathrm{cl}\,(V)\times\partial(D))\cup h^{-1}g(\mathrm{cl}\,(V)) \arrow[r, "\phi_1"] \arrow[dr, "\pi"'] & \mathrm{cl}\,(V)\times\partial(D)\cup f(\mathrm{cl}\,(V)) \arrow[d, "\pi"] \\
& \mathrm{cl}\,(V)
\end{tikzcd}$$

commutes. Then, there exists a homeomorphism $\phi_1' : \mathrm{cl}\,(V)\times D\to\mathrm{cl}\,(V)\times D$, which is the extension of ϕ_1 and is such that the diagram

$$\begin{tikzcd}
\mathrm{cl}\,(V)\times D \arrow[r, "\phi_1'"] \arrow[dr, "\pi"'] & \mathrm{cl}\,(V)\times D \arrow[d, "\pi"] \\
& \mathrm{cl}\,(V)
\end{tikzcd}$$

commutes, by Theorem 2 of [4]. In Theorem 2 of [4], the homeomorphism ϕ_1 did not send a cross-section $h^{-1}g(\mathrm{cl}\,(V))$ to another cross-section $f(\mathrm{cl}\,(V))$, and one of the π's was a completely regular map. Since π is completely regular, and the group of homeomorphisms of D^n, n-disk, for any n, fixing $\partial(D^n)$ and an interior point pointwise is locally contractible (proved by J. W. Alexander in 1923), the proof of the existence of the extension ϕ_1' is exactly the same as the one of Theorem 2 of [4].

Then, a map $\phi' : \mathrm{cl}\,(V)\times F\to\mathrm{cl}\,(V)\times F$, defined by $\phi'|\mathrm{cl}\,(V)\times F-(\mathrm{cl}\,(V)\times\mathrm{Int}\,(D))=\text{identity}$ and $\phi'|\mathrm{cl}\,(V)\times D=\phi_1'$, is a homeomorphism, and ϕ' preserves the fibers. Since

$$\phi_1'(h^{-1}g(\mathrm{cl}\,(V))) = f(\mathrm{cl}\,(V)), \qquad \phi'(h^{-1}g(\mathrm{cl}\,(V))) = f(\mathrm{cl}\,(V)).$$

Therefore, $\phi = \phi'|V\times F : V\times F\to V\times F$ is a fiber preserving homeomorphism such that $\phi(h^{-1}g(V))=f(V)$.

Since $V\subset U, h'=h|V\times F : V\times F\to\bar{p}^{-1}(V)$ is a homeomorphism and the diagram

$$\begin{tikzcd}
V\times F \arrow[r, "h'"] \arrow[dr, "\pi"'] & \bar{p}^{-1}(V) \arrow[d, "\bar{p}|\bar{p}^{-1}(V)"] \\
& V
\end{tikzcd}$$

commutes. Therefore, $h'\phi^{-1} : (V\times F, f(V)) \to (V\times F, h^{-1}g(V)) \to (\bar{p}^{-1}(V), g(V)) = (\bar{p}^{-1}(V), \bar{p}^{-1}(V)-p^{-1}(V))$ is a homeomorphism. Hence

$$\begin{aligned} h_v &= h'\phi^{-1}|V\times F - V\times(\bar{p}^{-1}(b)-p^{-1}(b)) : V\times F - V\times(\bar{p}^{-1}(b)-p^{-1}(b)) \\ &= V\times p^{-1}(b) \to \bar{p}^{-1}(V)-(\bar{p}^{-1}(V)-p^{-1}(V)) = p^{-1}(V) \end{aligned}$$

is a homeomorphism. That means that for any point $b \in B$, there exist an open set $V \ni b$, and a homeomorphism $h_v : V\times p^{-1}(b) \to p^{-1}(V)$ such that the diagram

$$\begin{tikzcd} V\times p^{-1}(b) \arrow[r, "h_v"] \arrow[dr, "\pi"'] & p^{-1}(V) \arrow[d, "p|p^{-1}(V)"] \\ & V \end{tikzcd}$$

commutes. This completes the proof of the theorem. Q.E.D.

Note. Let $h : M \to M'$ be a proper map which induces an isomorphism of the ordinary homology groups. Then if M and M' are 2-manifolds with finitely generated homology groups they are homeomorphic. This will follow from extending the map to end point compactification and the observation that the extension induces an isomorphism of the homology groups. Therefore, if we knew the maps carrying one fiber to the other were proper homotopy equivalences then the fibers (which are manifolds) would be homeomorphic.

(4.2) Remark. Under the assumption of Theorem (4.1), we conjecture that if all fibers are homeomorphic to the Euclidean 3-spaces or to a connected 2-manifold which has a finite number of ends and a finite number of genera, then the fibering is locally trivial. We believe that the same method employed in the proof of Theorem (4.1) can be applied to this conjecture.

(4.3) Corollary. *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR n-gm over L with nonempty boundary onto a wlc and paracompact base B which is also a gm over L with nonempty boundary. Suppose $p^{-1}(\partial(B)) = \partial(E)$ and there exists a point $b \in B$ such that $p^{-1}(b)$ has a compact component of* $\dim \leqq 2$. *Then the fibering (E, B, p) is locally trivial. (This is actually a corollary of Theorem* (R2).)

Proof. Take doubles of E and B and define $p' : D(E) \to D(B)$ by $p'|E = p$. Then the corollary follows by Theorem (R2). Q.E.D.

(4.4) Corollary. *Let $p : E \to B$ be a Hurewicz fiber map from a connected separable metric ANR n-gm E over L with nonempty boundary onto a wlc and paracompact connected k-gm B over L with nonempty boundary such that $p^{-1}(\partial(B))$*

$=\partial(E)$. *Then any one of the following implies that the fibering* (E, B, p) *is locally trivial*:

(1) *There exists a point* $b \in B$ *such that* $p^{-1}(b)$ *is the real line.*

(2) *There exists a point* $b \in B$ *such that* $p^{-1}(b)$ *is the plane.*

(3) *For each point* $b \in B$, $p^{-1}(b)$ *is homeomorphic to some connected 2-manifold with finitely generated homology groups and exactly one end.*

Proof. By Theorem (4.1), $p' : D(E) \to D(B)$ is a locally trivial fiber map. Therefore the fibering (E, B, p) itself is locally trivial. Q.E.D.

(4.5) REMARK. We learned by the referee that Gerald S. Ungar has proven the following theorem in his paper *Conditions for a mapping to have the slicing structure property* (to appear).

THEOREM. *Let p be an n-regular closed map from a complete metric space E onto a paracompact locally equiconnected space B. If* $\dim (E \times B) \leqq n$, *then p has the slicing structure property. In particular p is a Hurewicz fiber map.*

Clearly, this result together with a theorem of L. F. McAuley and P. A. Tulley (which appeared in *Topology Seminar Wisconsin*, 1965, p. 232—Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N. J.) will allow many of our results to be stated in terms of Serre fiber maps. A Serre fiber map is a map which has the covering homotopy property for each polyhedron.

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THE UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICHIGAN
UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS