NOETHER-LASKER DECOMPOSITION OF COHERENT ANALYTIC SUBSHEAVES

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In this paper we develop the theory of Noether-Lasker decomposition of coherent analytic subsheaves as an analogue of the algebraic Noether-Lasker decomposition of ideals in Noetherian rings. The decomposition can be described as follows: Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$ in the sense of Grauert. For every point x of X, $\mathscr{S}_x$ as an $\mathscr{O}_x$-submodule of $\mathscr{T}_x$ has a Noether-Lasker decomposition into primary $\mathscr{O}_x$-submodules of $\mathscr{T}_x$. The radicals of these primary submodules are prime ideals of $\mathscr{O}_x$ which define subvariety-germs of X at x. These subvariety germs are pieced together to form global irreducible subvarieties of X which we call *associated subvarieties* of $\mathscr{S}$. A coherent subsheaf of $\mathscr{T}$ which has only one associated subvariety is called *primary*. We prove that every coherent analytic subsheaf can be represented as the intersection of "locally finite" primary subsheaves. This representation is what we call the Noether-Lasker decomposition of the coherent analytic subsheaf. If $(X, \mathscr{O})$ is Stein, then a coherent analytic proper subsheaf $\mathscr{S}$ of a coherent analytic sheaf $\mathscr{T}$ is primary if and only if $\Gamma(X, \mathscr{S})$ is a primary submodule of the $\Gamma(X, \mathscr{O})$-module $\Gamma(X, \mathscr{T})$.

The Noether-Lasker decomposition of subsheaves is derived from the gap-sheaf theory of Thimm [4]. In part I of this paper we give an exposition of Thimm's theory of gap-sheaves by sheaf-theoretical methods. In part II of this paper we establish the Noether-Lasker decomposition of coherent analytic subsheaves.

Notations. All complex spaces in this paper are in the sense of Grauert [1, §1]. Suppose $(X, \mathscr{O})$ is a complex space. A *holomorphic function* on $(X, \mathscr{O})$ is an element of $\Gamma(X, \mathscr{O})$. A holomorphic function f *vanishes* at a point x of X if the germ of f at x is not a unit in $\mathscr{O}_x$. A *subvariety* in X is a set which locally is the set of points where a finite number of locally defined holomorphic functions vanish. The *ideal-sheaf* of a subvariety Y, denoted by Id Y, is the sheaf of germs of holomorphic functions vanishing at every point of Y. A complex space $(Z, \mathscr{H})$ is a *subspace* of $(X, \mathscr{O})$ if Z is a subvariety of X and there exists a coherent ideal-sheaf $\mathscr{I}$ on X such that $\mathscr{H}=(\mathscr{O}/\mathscr{I})|Z$ and $\{z \mid z\in X,\ \mathscr{I}_z\neq\mathscr{O}_z\}=Z$. A *module-sheaf* on $(X, \mathscr{O})$ is an analytic subsheaf of $\mathscr{O}^p$ for some p. If $\mathscr{A}$ is an ideal-sheaf on $(X, \mathscr{O})$, then $\sqrt{\mathscr{A}}$ is the ideal-sheaf defined by $(\sqrt{\mathscr{A}})_x=\sqrt{\mathscr{A}_x}$, where $\sqrt{\mathscr{A}_x}$ is the radical of the ideal $\mathscr{A}_x$ in $\mathscr{O}_x$.

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Suppose that $\mathscr{R}$ and $\mathscr{S}$ are analytic subsheaves of an analytic sheaf $\mathscr{T}$ and $\mathscr{A}$ is an ideal-sheaf on a complex space $(X, \mathscr{O})$. Denote by $\mathscr{R}:\mathscr{S}$ the ideal-sheaf defined as follows: for $x \in X$, $f \in (\mathscr{R}:\mathscr{S})_x$ if and only if $f \in \mathscr{O}_x$ and $f\mathscr{S}_x \subset \mathscr{R}_x$. Denote by $(\mathscr{S}:\mathscr{A})_{\mathscr{T}}$ or simply by $\mathscr{S}:\mathscr{A}$ the subsheaf of $\mathscr{T}$ defined as follows: for $x \in X$, $s \in (\mathscr{S}:\mathscr{A})_x$ if and only if $s \in \mathscr{T}_x$ and $\mathscr{A}_x s \subset \mathscr{S}_x$. If $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{A}$ are coherent, then $\mathscr{R}:\mathscr{S}$ and $\mathscr{S}:\mathscr{A}$ are coherent. If $s \in \Gamma(X, \mathscr{T})$ and $f \in \Gamma(X, \mathscr{O})$, then $\mathscr{R}:s$ denotes $\mathscr{R}:\mathscr{O}s$ and $(\mathscr{S}:f)_{\mathscr{T}}$ or simply $\mathscr{S}:f$ denotes $(\mathscr{S}:\mathscr{O}f)_{\mathscr{T}}$.

Suppose X and Y are two complex spaces, $\mathscr{T}$ is an analytic sheaf on X, and $\pi: X \to Y$ is a holomorphic map (i.e. a morphism of ringed spaces). Then $R^0\pi(\mathscr{T})$ denotes the zeroth direct image of $\mathscr{T}$ under π.

Suppose $x=(x_1, \ldots, x_n) \in \boldsymbol{C}^n$ and $r_1, \ldots, r_n$ are positive numbers. Then $\Delta(x; r_1, \ldots, r_n)$ denotes the polydisc $\{(z_1, \ldots, z_n) \in \boldsymbol{C}^n \mid |z_i - x_i| < r_i,\ 1 \leqq i \leqq n\}$. Suppose $\mathscr{F}$ is a sheaf on a topological space E, $x \in E$, U is an open neighborhood of x in E, and $s \in \Gamma(U, \mathscr{F})$. Then s_x denotes the germ of s at x and $\mathscr{F}_x$ denotes the stalk of $\mathscr{F}$ at x. If f is a (complex-valued) function on E, then f_x denotes the germ of f at x.

Suppose R is a Noetherian ring and E is an R-submodule of a finitely generated R-module F. Then $P(E, F)$ denotes the set of all associated nonunit prime ideals in the Noether-Lasker decomposition of E as a submodule of F.

I. Gap-sheaves.

Definition 1. Suppose $\mathscr{S}$ is an analytic subsheaf of an analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$ and ρ is a nonnegative integer. The ρth *gap-sheaf of $\mathscr{S}$ in $\mathscr{T}$*, denoted by $\mathscr{S}_{[\rho]\mathscr{T}}$ or simply $\mathscr{S}_{[\rho]}$, is the analytic subsheaf of $\mathscr{T}$ defined as follows: for $x \in X$, $s \in (\mathscr{S}_{[\rho]})_x$ if and only if there exist an open neighborhood U of x in X, a subvariety A in U of dimension $\leqq \rho$, and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t_y \in \mathscr{S}_y$ for $y \in U - A$. Denote by $E(\mathscr{S}, \mathscr{T})$ the set $\{x \mid x \in X, \mathscr{S}_x \neq \mathscr{T}_x\}$ and $E^\rho(\mathscr{S}, \mathscr{T})$ denotes $E(\mathscr{S}, \mathscr{S}_{[\rho]})$.

Definition 2. Suppose $\mathscr{S}$ is an analytic subsheaf of an analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$ and A is a subvariety of X. Then the *gap-sheaf of $\mathscr{S}$ in $\mathscr{T}$ with respect to A*, denoted by $\mathscr{S}[A]_{\mathscr{T}}$ or simply $\mathscr{S}[A]$, is the analytic subsheaf of $\mathscr{T}$ defined as follows: for $x \in X$, $s \in \mathscr{S}[A]_x$ if and only if there exist an open neighborhood U of x in X and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t_y \in \mathscr{S}_y$ for $y \in U - A$.

Theorem 1. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$ and A is a subvariety of X. Then*

$$\mathscr{S}[A] = \bigcup_{n=1}^{\infty} (\mathscr{S}:\mathscr{A}^n)_{\mathscr{T}},$$

where $\mathscr{A}$ is the ideal-sheaf of A, and hence is coherent.

Proof. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} (\mathscr{S}:\mathscr{A}^n)_{\mathscr{T}} \subset \mathscr{T}$. $\mathscr{F}$ is coherent, because it is the union of an increasing sequence of coherent subsheaves of a coherent sheaf [1, Satz 8, §2].

Suppose $s \in \mathscr{S}[A]_x$ for some $x \in X$. Then there exist an open neighborhood U of

x in X and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t_y \in \mathscr{S}_y$ for $y \in U - A$. Let $\mathscr{B} = (\mathscr{S}|U):t$. $E(\mathscr{B}, \mathscr{O}|U) \subset A \cap U$. By Hilbert Nullstellensatz [2, III.A.7] $\mathscr{A}_x^n \subset \mathscr{B}_x$ for some n. $s\mathscr{A}_x^n \subset s\mathscr{B}_x \subset \mathscr{S}_x$. $s \in \mathscr{F}$.

Suppose $s \in \mathscr{F}_x$. $s \in (\mathscr{S}:\mathscr{A}^n)_x$ for some n. There is an open neighborhood U of x and $t \in \Gamma(U, \mathscr{T})$ such that $t_x = s$ and $t(\mathscr{A}^n|U) \subset \mathscr{S}|U$. For $y \in U - A$, $\mathscr{A}_y^n = \mathscr{O}_y$. Hence $t_y \in \mathscr{S}_y$. $\mathscr{F} = \mathscr{S}[A]$. Q.E.D.

The following lemma is a particular case of [1, Hauptsatz I, §6] and it can be proved in a very elementary way.

LEMMA 1. *Suppose X and Y are complex spaces, $\mathscr{F}$ is a coherent sheaf on X, and $\pi: X \to Y$ is a proper nowhere degenerate holomorphic map, then $R^0\pi(\mathscr{F})$ is coherent.*

THEOREM 2. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{H})$ and ρ is a nonnegative integer. Then $E^\rho(\mathscr{S}, \mathscr{T})$ is locally contained in a subvariety of dimension $\leqq \rho$, i.e. for every $x \in X$ there exist an open neighborhood U of x in X and a subvariety A in U of dimension $\leqq \rho$ such that $E^\rho(\mathscr{S}, \mathscr{T}) \cap U \subset A$.*

Proof. Since the theorem is local in nature, we can suppose without loss of generality that $(X, \mathscr{H})$ is a subspace of an open subset G of C^n. Let $\mathscr{O}$ be the structure sheaf of G and $\tilde{\mathscr{S}}$ and $\tilde{\mathscr{T}}$ be the trivial extensions of $\mathscr{S}$ and $\mathscr{T}$ on G respectively. We can further suppose without loss of generality that we have a sheaf-epimorphism $\lambda : \mathscr{O}^p \to \tilde{\mathscr{T}}$ on G. Let $\mathscr{M} = \lambda^{-1}(\tilde{\mathscr{S}})$. Then $E^\rho(\mathscr{S}, \mathscr{T}) = E^\rho(\mathscr{M}, \mathscr{O}^p)$. Hence we need only prove that

(1) for every coherent module-sheaf $\mathscr{M} \subset \mathscr{O}^p$ on an open subset G of C^n, $E^\rho(\mathscr{M}, \mathscr{O}^p)$ is locally contained in a subvariety of dimension $\leqq \rho$.

We fix ρ and prove (1) by induction on n. For $n \leqq \rho$, (1) is trivially true. Now suppose (1) is true when n is replaced by $n-1$. We are going to prove by induction on p that (1) is true when n is unreplaced.

(a) $p=1$. Because of the local nature of (1) we can suppose that G is connected. If $\mathscr{M}=0$, then (1) is trivial. So we can suppose $\mathscr{M} \neq 0$. $E(\mathscr{M}, \mathscr{O})$ is a proper subvariety of G. Take $x \in G$. We want to prove that $E^\rho(\mathscr{M}, \mathscr{O})$ is locally contained in a subvariety of dimension $\leqq \rho$ at x. If $x \notin E(\mathscr{M}, \mathscr{O})$, then it is obviously true. So we suppose $x \in E(\mathscr{M}, \mathscr{O})$. There is a nonzero holomorphic function φ on some open neighborhood U of x such that φ vanishes on $E(\mathscr{M}, \mathscr{O}) \cap U$. Without loss of generality we can suppose that U is a polydisc $\Delta(x; r_1, \ldots, r_n)$ and the projection $\pi: Y \to \Delta(\pi(x); r_1, \ldots, r_{n-1})$ defined by $\pi(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1})$, where $Y = \{y \mid y \in U, \varphi(y) = 0\}$, makes Y an analytic cover over $\Delta(\pi(x); r_1, \ldots, r_{n-1})$ [2, III.B.3].

By Hilbert Nullstellensatz, after shrinking U we can suppose without loss of generality that $\varphi^m \in \Gamma(U, \mathscr{M})$ for some m. $\mathscr{Q} = (\mathscr{M}/\mathscr{O}\varphi^m)|Y$ is a coherent analytic sheaf on the complex space $(Y, \mathscr{K})$, where $\mathscr{K}=(\mathcal{O}/\mathcal{O}\varphi^m)|Y$. $U \cap E^\rho(\mathscr{M}, \mathcal{O}) = E^\rho(\mathscr{Q}, \mathscr{K})$. Let $\mathscr{A}=R^0\pi(\mathscr{Q})$ and $\mathscr{B}=R^0\pi(\mathscr{K})$.

By Lemma 1, $\mathscr{A}$ is a coherent analytic subsheaf of the coherent analytic sheaf $\mathscr{B}$ on $\Delta(\pi(x); r_1, \ldots, r_{n-1})$.

If $y \in E^\rho(\mathscr{Q}, \mathscr{K})$, then there exist a subvariety A of dimension $\leqq \rho$ in an open neighborhood B of y in Y and $t \in \Gamma(B, \mathscr{K})$ such that $t_z \in \mathscr{Q}_z$ for $z \in B-A$ and $t_y \notin \mathscr{Q}_y$. Let $\pi^{-1}(\pi(y))=\{y^0, \ldots, y^l\}$, where $y^0=y$, and B^i be disjoint open neighborhoods of y^i, $0 \leqq i \leqq l$, in Y, such that $B^0 \subset B$. Since φ is proper, there is an open neighborhood C of $\pi(y)$ in $\Delta(\pi(x); r_1, \ldots, r_{n-1})$ such that $\pi^{-1}(C) \subset \bigcup_{i=0}^l B^i$. Define $t^* \in \Gamma(\pi^{-1}(C), \mathscr{K})$ as follows: $t^*|\pi^{-1}(C) \cap B^0 = t|\pi^{-1}(C) \cap B^0$ and $t^*|\pi^{-1}(C) \cap B^i=0$ for $1 \leqq i \leqq l$. t^* induces $t' \in \Gamma(C, \mathscr{B})$. $t'_z \in \mathscr{A}_z$ for $z \in C-\pi(A)$ and $t'_{\pi(y)} \notin \mathscr{A}_{\pi(y)}$. Since $C \cap \pi(A)$ is a subvariety of dimension $\leqq \rho$ in C, $\pi(y) \in E^\rho(\mathscr{A}, \mathscr{B})$. Since y is an arbitrary point in $E^\rho(\mathscr{Q}, \mathscr{K})$, $E^\rho(\mathscr{Q}, \mathscr{K}) \subset \pi^{-1}(E^\rho(\mathscr{A}, \mathscr{B}))$. Let $0 < s_i < r_i$, $1 \leqq i \leqq n-1$, and $\mathscr{R}$ be the structure sheaf of $D=\Delta(\pi(x); s_1, \ldots, s_{n-1})$. Then there is a sheaf-epimorphism $\eta: \mathscr{R}^q \to \mathscr{B}|D$.

$E^\rho(\mathscr{A}, \mathscr{B}) \cap D = E^\rho(\eta^{-1}(\mathscr{A}|D), \mathscr{R}^q)$. By induction hypothesis $E^\rho(\eta^{-1}(\mathscr{A}|D), \mathscr{R}^q)$ is locally contained in a subvariety of dimension $\leqq \rho$. There exists a subvariety Z of dimension $\leqq \rho$ in a polydisc $W=\Delta(\pi(x); t_1, \ldots, t_{n-1}) \subset D$ such that $E^\rho(\mathscr{A}, \mathscr{B}) \cap W \subset Z$.

$$E^\rho(\mathscr{M}, \mathcal{O}) \cap \Delta(x; t_1, \ldots, t_{n-1}, r_n) = E^\rho(\mathscr{Q}, \mathscr{K}) \cap \Delta(x; t_1, \ldots, t_{n-1}, r_n) \subset \pi^{-1}(E^\rho(\mathscr{A}, \mathscr{B}) \cap W) \subset \pi^{-1}(Z).$$

$\pi^{-1}(Z)$ is a subvariety of dimension $\leqq \rho$ in $\Delta(x; t_1, \ldots, t_{n-1}, r_n)$. The case $p=1$ is proved.

(b) The case of a general $p \geqq 1$. $\mathcal{O}^p = \mathcal{O}^{p-1} \oplus \mathcal{O}$. Let $\alpha: \mathcal{O}^p \to \mathcal{O}^{p-1}$ be the projection onto the first summand and $\beta: \mathcal{O} \to \mathcal{O}^p$ be the injection from the second summand. Let $\mathscr{N}=\alpha(\mathscr{M})$ and $\mathscr{P}=\mathscr{M} \cap \beta(\mathcal{O})$. Take $x \in G$. Then by induction hypothesis and by (a) there exist subvarieties Z_1 and Z_2 of dimensions $\leqq \rho$ in an open neighborhood U of x such that $U \cap E^\rho(\mathscr{N}, \mathcal{O}^{p-1}) \subset Z_1$ and $U \cap E^\rho(\mathscr{P}, \mathcal{O}) \subset Z_2$. It is readily checked that $U \cap E^\rho(\mathscr{M}, \mathcal{O}^p) \subset Z_1 \cup Z_2$. Q.E.D.

THEOREM 3. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent sheaf $\mathscr{T}$ on a complex space $(X, \mathcal{O})$ and ρ is a nonnegative integer. Then $\mathscr{S}_{[\rho]}$ is coherent and $E^\rho(\mathscr{S}, \mathscr{T})$ is a subvariety of dimension $\leqq \rho$ in X.*

Proof. First we prove the coherence of $\mathscr{S}_{[\rho]}$. Coherence is a local property. Take $x \in X$. By Theorem 2 there exists a subvariety A of dimension $\leqq \rho$ in an open neighborhood U of x in X such that $U \cap E^\rho(\mathscr{S}, \mathscr{T}) \subset A$.

Since $\dim A \leqq \rho$, $\mathscr{S}_{[\rho]}|U = (\mathscr{S}|U)[A]$. Hence $\mathscr{S}_{[\rho]}|U$ is coherent by Theorem 1. $\mathscr{S}_{[\rho]}$ is coherent.

$E^\rho(\mathscr{S}, \mathscr{T}) = E((\mathscr{S} : \mathscr{S}_{[\rho]}), \mathcal{O})$ is a subvariety, because $\mathscr{S} : \mathscr{S}_{[\rho]}$ is coherent. Since $E^\rho(\mathscr{S}, \mathscr{T})$ is locally contained in a subvariety of dimension $\leqq \rho$, $E^\rho(\mathscr{S}, \mathscr{T})$ is a subvariety of dimension $\leqq \rho$ in X. Q.E.D.

COROLLARY. $\mathcal{S}_{[\rho]} = \mathcal{S}[E^\rho(\mathcal{S}, \mathcal{T})]$.

II. **Noether-Lasker decomposition of subsheaves.** Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space. Let $E^\rho(\mathcal{S}, \mathcal{T}) = \bigcup_{i \in I(\rho)} Y_i^\rho$ be the decomposition into irreducible branches. Then we call each nonempty Y_i^ρ, $\rho \geqq 0$, $i \in I(\rho)$, an *associated subvariety* of $\mathcal{S}$ in $\mathcal{T}$ and denote the set of all associated subvarieties of $\mathcal{S}$ in $\mathcal{T}$ by $\mathcal{X}(\mathcal{S}, \mathcal{T})$. From the definition we see readily that $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is locally finite. $\mathcal{S}$ is called a *primary subsheaf* of $\mathcal{T}$ if $\mathcal{S}$ has only one associated subvariety.

The following lemma is a well-known algebraic fact [5, Appendix, Chapter IV]:

LEMMA 2. *Suppose R is a Noetherian ring and N is an R-submodule of a finitely generated R-module M. A prime ideal P in R is an associated prime ideal in the Noether-Lasker decomposition of N as a submodule of M if and only if $P = \sqrt{(N:f)}$ for some $f \in M$.*

THEOREM 4. *Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $x \in X$. Let $\{X_i^\rho \mid \rho \geqq 0, i \in J(\rho)\}$ be the set of all associated subvarieties of $\mathcal{S}$ passing through x, where* $\dim X_i^\rho = \rho$, $i \in J(\rho)$, *and suppose* $(\mathrm{Id}\ X_i^\rho)_x = \bigcap_{j \in K(\rho, i)} P_{ij}^\rho$ *is the decomposition into prime ideals. Then* $\{P_{ij}^\rho \mid \rho \geqq 0, i \in J(\rho), j \in K(\rho, i)\} = P(\mathcal{S}_x, \mathcal{T}_x)$.

Proof. Suppose $P \in P(\mathcal{S}_x, \mathcal{T}_x)$ and $\dim P = \rho$. Then $P = \sqrt{(\mathcal{S}_x : f)}$ for some $f \in \mathcal{T}_x$ by Lemma 2. P defines a subvariety V of dimension ρ in an open neighborhood D of x in X. We can suppose after a shrinking of D that there exists $g \in \Gamma(D, \mathcal{T})$ such that $g_x = f$ and $\mathrm{Id}\ V = \sqrt{((\mathcal{S} \mid D) : g)}$. This implies that

$$\{y \mid y \in D, g_y \in \mathcal{S}_y\} = D - V.$$

Hence $V \subset E^\rho(\mathcal{S}, \mathcal{T})$. Since $\dim V = \rho$ and $\dim E^\rho(\mathcal{S}, \mathcal{T}) \leqq \rho$, $P = P_{ij}^\rho$ for some $i \in J(\rho)$ and some $j \in K(\rho, i)$.

Fix $\rho \geqq 0$ and $i \in J(\rho)$. By definition X_i^ρ is an irreducible branch of $E^\sigma(\mathcal{S}, \mathcal{T})$ for some $\sigma \geqq \rho$. Let U be a Stein open neighborhood of x in X such that $U \cap X_i^\rho = \bigcup_{j \in K(\rho, i)} X_{ij}^\rho$ is the decomposition into irreducible branches and $P_{ij}^\rho = (\mathrm{Id}\ X_{ij}^\rho)_x$, $j \in K(\rho, i)$. Fix $j \in K(\rho, i)$. Let Z^1 be the union of irreducible branches of $E^\sigma(\mathcal{S}, \mathcal{T}) \cap U$ other than X_{ij}^ρ and let

$$Z = Z^1 \cup (E^{\rho-1}(\mathcal{S}, \mathcal{T}) \cap U).$$

Take $y \in X_{ij}^\rho - Z$.

$$(\mathcal{S} \mid U)[X_{ij}^\rho]_y = \mathcal{S}_{[\sigma]y} \neq \mathcal{S}_y.$$

Since $(\mathcal{S} \mid U)[X_{ij}^\rho]$ is generated by global sections [1, Satz 4, §2], there exists $t \in \Gamma(U, (\mathcal{S} \mid U)[X_{ij}^\rho])$ such that $t_y \notin \mathcal{S}_y$.

Let $Y = E(\mathcal{S} \mid U, (\mathcal{S} \mid U) + (\mathcal{O} \mid U)t)$. Since X_{ij}^ρ is irreducible, if $Y \neq X_{ij}^\rho$, then $\dim Y < \rho$ and $t_y \in \mathcal{S}_{[\rho-1]y} = \mathcal{S}_y$ (contradiction). Hence $Y = X_{ij}^\rho$. $P_{ij}^\rho = (\mathrm{Id}\ X_{ij}^\rho)_x = (\mathrm{Id}\ Y)_x = \sqrt{(\mathcal{S}_x : t_x)}$. By Lemma 2, $P_{ij}^\rho \in P(\mathcal{S}_x, \mathcal{T}_x)$. Q.E.D.

This theorem gives us a characterization of associated subvarieties and tells us that the subvariety-germs defined by associated prime ideals in the Noether-Lasker decomposition of the stalks of $\mathscr{S}$ can be pieced together to form global subvarieties.

COROLLARY 1. *If* $Y \in \mathscr{X}(\mathscr{S}, \mathscr{T})$, *then* $E(\mathscr{S}, \mathscr{S}[Y]) = Y$.

Proof. Obviously $E(\mathscr{S}, \mathscr{S}[Y]) \subset Y$. Suppose $x \in Y$ and $P \in P((\mathrm{Id}\ Y)_x, \mathcal{O}_x)$. Then by Theorem 4, $P \in P(\mathscr{S}_x, \mathscr{T}_x)$. By Lemma 2, $P = \sqrt{(\mathscr{S}_x : s)}$ for some $s \in \mathscr{T}_x$. $s \in \mathscr{S}[Y]_x - \mathscr{S}_x$. Q.E.D.

COROLLARY 2. *Suppose* $\mathscr{R}$ *is a coherent analytic subsheaf of* $\mathscr{T}$ *and* $\mathscr{S} \subset \mathscr{R} \subset \mathscr{T}$ *and* $\mathscr{X}(\mathscr{S}, \mathscr{T}) = \{X^i \mid i \in I\}$. *Then* $\mathscr{X}(\mathscr{S}, \mathscr{R}) \subset \mathscr{X}(\mathscr{S}, \mathscr{T})$. *Hence there is a subset* J *of* I *such that* $E(\mathscr{S}, \mathscr{R}) = \bigcup \{X^i \mid i \in J\}$.

Proof. Suppose $Y \in \mathscr{X}(\mathscr{S}, \mathscr{R})$. Take $y \in Y$ and $P \in P((\mathrm{Id}\ Y)_y, \mathcal{O}_y)$. By Theorem 4, $P \in P(\mathscr{S}_y, \mathscr{R}_y)$. By Lemma 2, $P = \sqrt{(\mathscr{S}_y : s)}$ for some $s \in \mathscr{R}_y$. Since $s \in \mathscr{T}_y$, by Lemma 2, $P \in P(\mathscr{S}_y, \mathscr{T}_y)$.

By Theorem 4, $P \in P((\mathrm{Id}\ X^i)_y, \mathcal{O}_y)$ for some $i \in I$ such that $y \in X^i$. Since the two irreducible subvarieties X^i and Y have a branch-germ in common at y, $X^i = Y$. Hence $\mathscr{X}(\mathscr{S}, \mathscr{R}) \subset \mathscr{X}(\mathscr{S}, \mathscr{T})$. The existence of J follows from

$$E(\mathscr{S}, \mathscr{R}) = \bigcup \{Y \mid Y \in \mathscr{X}(\mathscr{S}, \mathscr{R})\}. \qquad \text{Q.E.D.}$$

THEOREM 5. *Suppose* $\mathscr{S}$ *is a coherent analytic subsheaf of a coherent analytic sheaf* $\mathscr{T}$ *on a complex space* $(X, \mathcal{O})$ *and* A *is a subvariety of* X. *Suppose* $\mathscr{X}(\mathscr{S}, \mathscr{T}) = \{X^i \mid i \in I\}$, $x \in A$, *and* $I' = \{i \in I \mid x \in X^i\}$. *Suppose*

$$P((\mathrm{Id}\ X^i)_x, \mathcal{O}_x) = \{P_{ij} \mid j \in J_i\}, \qquad i \in I'.$$

Let $\mathscr{S}_x = \bigcap \{Q_{ij} \mid i \in I', j \in J_i\}$ *be a Noether-Lasker decomposition of* $\mathscr{S}_x$, *where the radical of* Q_{ij} *is* P_{ij}, $i \in I', j \in J_i$, *and let* $K = \{i \mid i \in I',\ X^i \not\subset A\}$. *Then*

$$(\mathscr{S}[A])_x = \bigcap \{Q_{ij} \mid i \in K, j \in J_i\}.$$

Proof. Let $\mathscr{A} = \mathrm{Id}\ A$. By Theorem 1

$$\mathscr{S}[A]_x = \bigcup_{n=1}^{\infty} (\mathscr{S}_x : \mathscr{A}_x^n) = \bigcup_{n=1}^{\infty} \bigcap \{Q_{ij} : \mathscr{A}_x^n \mid i \in I', j \in J_i\}$$

$$= \bigcap \left\{ \bigcup_{n=1}^{\infty} (Q_{ij} : \mathscr{A}_x^n) \mid i \in I', j \in J_i \right\}.$$

For $i \in I' - K$, $P_{ij} \supset \mathscr{A}_x$ and hence $Q_{ij} : \mathscr{A}_x^n = \mathscr{T}_x$ for n sufficiently large. For $i \in K$, $P_{ij} \not\supset \mathscr{A}_x$ and hence $Q_{ij} : \mathscr{A}_x^n = Q_{ij}$ for every n. Therefore

$$\mathscr{S}[A]_x = \bigcap \{Q_{ij} \mid i \in K, j \in J_i\}. \qquad \text{Q.E.D.}$$

COROLLARY. $\mathscr{X}(\mathscr{S}[A], \mathscr{T}) = \{X^i \mid X^i \not\subset A\}$ *and*

$$E(\mathscr{S}[A], \mathscr{T}) = \bigcup \{Y \mid Y \in \mathscr{X}(\mathscr{S}, \mathscr{T}),\ Y \not\subset A\}.$$

Proof. The first assertion follows from Theorems 4 and 5 and the second assertion follows from the first. Q.E.D.

LEMMA 3. *Suppose $\mathscr{S} \subset \mathscr{R}$ are coherent analytic subsheaves of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$. Suppose $\mathscr{I}$ is the ideal-sheaf of $E(\mathscr{S}, \mathscr{R})$ and $x \in E(\mathscr{S}, \mathscr{R})$. Then there is a natural number k such that $((\mathscr{I}^k\mathscr{T} + \mathscr{S}) \cap \mathscr{R})_x = \mathscr{S}_x$.*

Proof. $E(0, \mathscr{R}/\mathscr{S}) = E(\mathscr{S}, \mathscr{R})$. By Hilbert Nullstellensatz there is a natural number l such that $\mathscr{I}^l_x \subset (0 : \mathscr{R}/\mathscr{S})_x$. By the Lemma of Artin-Rees [5, Theorem 4′; §2, Chapter VIII] there exists a natural number k such that $(\mathscr{I}^k(\mathscr{T}/\mathscr{S}) \cap (\mathscr{R}/\mathscr{S}))_x \subset (\mathscr{I}^l(\mathscr{R}/\mathscr{S}))_x$. Hence $((\mathscr{I}^k\mathscr{T} + \mathscr{S}) \cap \mathscr{R})_x = \mathscr{S}_x$. Q.E.D.

LEMMA 4. *Suppose $\mathscr{S} \subset \mathscr{R}$ are coherent analytic subsheaves of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$. Then there exists a coherent analytic subsheaf $\mathscr{Q}$ of $\mathscr{T}$ such that $E(\mathscr{Q}, \mathscr{T}) = E(\mathscr{S}, \mathscr{R})$ and $\mathscr{Q} \cap \mathscr{R} = \mathscr{S}$.*

Proof. Suppose $\mathscr{X}(\mathscr{S}, \mathscr{R}) = \{X^i \mid i \in I\}$. Let $\mathscr{I}_i$ be the ideal-sheaf of X^i. Take $x^i \in X^i$. By Lemma 3 and Corollary 1 to Theorem 4, there exists a natural number $k(i)$ such that

$$((\mathscr{I}_i^{k(i)}\mathscr{T} + \mathscr{S}) \cap \mathscr{S}[X^i]_{\mathscr{R}})_{x^i} = \mathscr{S}_{x^i}, \qquad i \in I. \tag{2}$$

Since $\mathscr{X}(\mathscr{S}, \mathscr{R})$ is locally finite, $\mathscr{Q} = \bigcap_{i \in I} (\mathscr{I}_i^{k(i)}\mathscr{T} + \mathscr{S})$ is a coherent analytic subsheaf of $\mathscr{T}$. Obviously $E(\mathscr{Q}, \mathscr{T}) \subset E(\mathscr{S}, \mathscr{R})$. We are going to prove that $\mathscr{Q} \cap \mathscr{R} = \mathscr{S}$.

From Corollaries 1 and 2 to Theorem 4 and (2) we conclude that

$$E(\mathscr{S}, (\mathscr{I}_i^{k(i)}\mathscr{T} + \mathscr{S}) \cap \mathscr{S}[X^i]_{\mathscr{R}}) \subset \bigcup \{X^j \mid j \in I,\ X^j \subsetneqq X^i\}, \qquad i \in I. \tag{3}$$

Obviously $\mathscr{S} \subset \mathscr{Q} \cap \mathscr{R}$. Let $Y = E(\mathscr{S}, \mathscr{Q} \cap \mathscr{R})$. By Corollary 2 to Theorem 4 there exists a subset of J of I such that $Y = \bigcup \{X^i \mid i \in J\}$. Suppose $Y \neq \varnothing$. Then take $y \in Y$. Take a relatively compact open neighborhood of U of y in X. Let $F = \{i \mid i \in J,\ X^i \cap U \neq \varnothing\}$. F is a finite set. Take $i \in F$ such that

$$\dim X^i = \max \{\dim X^j \mid j \in F\}.$$

Take an open neighborhood W of a point z of X^i in U such that

$$W \cap (\bigcup \{X^j \mid j \in I,\ X^j \not\supset X^i\}) = \varnothing.$$

$Y \cap W = X^i \cap W$. $(\mathscr{Q} \cap \mathscr{R}) | W \subset \mathscr{S}[X^i]_{\mathscr{R}} | W$. By (3) $(\mathscr{I}_i^{k(i)}\mathscr{T} + \mathscr{S}) \cap \mathscr{S}[X^i]_{\mathscr{R}} | W = \mathscr{S} | W$. Hence

$$\mathscr{S} | W \subset \mathscr{Q} \cap \mathscr{R} | W \subset (\mathscr{I}_i^{k(i)}\mathscr{T} + \mathscr{S}) \cap \mathscr{S}[X^i]_{\mathscr{R}} | W = \mathscr{S} | W.$$

$z \notin Y$ (contradiction). Hence $Y = \varnothing$ and $\mathscr{S} = \mathscr{Q} \cap \mathscr{R}$.

Suppose $E(\mathscr{Q}, \mathscr{T}) \neq E(\mathscr{S}, \mathscr{R})$. Take $x \in E(\mathscr{S}, \mathscr{R}) - E(\mathscr{Q}, \mathscr{T})$. Then $\mathscr{Q}_x = \mathscr{T}_x$. $\mathscr{S}_x = \mathscr{Q}_x \cap \mathscr{R}_x = \mathscr{R}_x$, contradicting that $x \in E(\mathscr{S}, \mathscr{R})$. Hence $E(\mathscr{Q}, \mathscr{T}) = E(\mathscr{S}, \mathscr{R})$. Q.E.D.

LEMMA 5. *Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $\mathcal{X}(\mathcal{S}, \mathcal{T})=\{X^i \mid i \in I\}$. Let $J=\{i \mid i \in I,\ X^i$ is maximal in $\mathcal{X}(\mathcal{S}, \mathcal{T})\}$. Then there exist a coherent analytic subsheaf $\mathcal{R}$ of $\mathcal{T}$ and primary subsheaves $\mathcal{Q}_i$ of $\mathcal{T}$, $i \in J$, such that* (i) $E(\mathcal{R}, \mathcal{T})=\bigcup_{i \in K} X^i$, *where* $K=I-J$, (ii) $E(\mathcal{Q}_i, \mathcal{T})=X^i$, $i \in J$, *and* (iii) $(\bigcap_{i \in J} \mathcal{Q}_i) \cap \mathcal{R}=\mathcal{S}$.

Proof. For $i \in J$ let $Y^i=\bigcup \{X^j \mid j \in I \text{ and } j \neq i\}$ and define $\mathcal{Q}_i=\mathcal{S}[Y^i]$.

Then by Corollary to Theorem 5, $\{X^i\}=\mathcal{X}(\mathcal{Q}_i, \mathcal{T})$, $i \in J$. Hence $E(\mathcal{Q}_i, \mathcal{T})=X^i$ and $\mathcal{Q}_i$ is a primary subsheaf of $\mathcal{T}$, $i \in J$. Take $y^i \in X^i - Y^i$, $i \in J$. Then $(\mathcal{Q}_i)_{y^i}=\mathcal{S}_{y^i}$, $i \in J$. Since $\mathcal{Q}_i \supset \mathcal{S}$, $i \in J$, we have

$$\Big(\bigcap_{j \in J} \mathcal{Q}_j\Big)_{y^i} = \mathcal{S}_{y^i}, \qquad i \in J. \tag{4}$$

Since $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is locally finite, $\bigcap_{i \in J} \mathcal{Q}_i$ is a coherent analytic subsheaf of $\mathcal{T}$. By (4) and Corollary 2 to Theorem 4 $E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i) \subset \bigcup_{i \in K} X^i$.

Suppose $x \in \bigcup_{i \in K} X^i - E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i)$. Then $x \in X^j$ for some $j \in K$ and $\mathcal{S}_x=\bigcap_{i \in J} (\mathcal{Q}_i)_x$. Let $L=\{i \mid i \in J,\ x \in X^i\}$. Since

$$P((\mathcal{Q}_i)_x, \mathcal{T}_x) = P((\operatorname{Id} X^i)_x, \mathcal{O}_x), \qquad i \in L,$$

$$P(\mathcal{S}_x, \mathcal{T}_x) \subset \bigcup_{i \in L} P((\operatorname{Id} X^i)_x, \mathcal{O}_x).$$

Since

$$P((\operatorname{Id} X^j)_x, \mathcal{O}_x) \cap \Big(\bigcup_{i \in L} P((\operatorname{Id} X^i)_x, \mathcal{O}_x)\Big) = \varnothing,$$

Theorem 4, which asserts that $P((\operatorname{Id} X^j)_x, \mathcal{O}_x) \subset P(\mathcal{S}_x, \mathcal{T}_x)$, is contradicted. Hence $E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i)=\bigcup_{i \in K} X^i$. By Lemma 4 there exists a coherent analytic subsheaf $\mathcal{R}$ of $\mathcal{T}$ such that $(\bigcap_{i \in J} \mathcal{Q}_i) \cap \mathcal{R}=\mathcal{S}$ and $E(\mathcal{R}, \mathcal{T})=\bigcup_{i \in K} X^i$. Q.E.D.

THEOREM 6 (NOETHER-LASKER DECOMPOSITION OF COHERENT SUBHEAVES). *Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $\mathcal{X}(\mathcal{S}, \mathcal{T})=\{X^i \mid i \in I\}$. Then for every $i \in I$, there exists a primary subsheaf $\mathcal{Q}_i$ of $\mathcal{T}$ such that $E(\mathcal{Q}_i, \mathcal{T})=X^i$ and $\bigcap_{i \in I} \mathcal{Q}_i=\mathcal{S}$.*

Proof. For $Y \in \mathcal{X}(\mathcal{S}, \mathcal{T})$ define the *depth* of Y in $\mathcal{X}(\mathcal{S}, \mathcal{T})$ to be $\sup \{l \mid$ there exist $Y_j \in \mathcal{X}(\mathcal{S}, \mathcal{T})$, $0 \leqq j \leqq l$, such that $Y_0=Y$ and $Y_j \subsetneqq Y_{j+1}$ for $0 \leqq j < l\}$. If $Y_j \in \mathcal{X}(\mathcal{S}, \mathcal{T})$, $0 \leqq j \leqq l$, and $Y_j \subsetneqq Y_{j+1}$, $0 \leqq j < l$, then for $x \in Y_0$, $\dim_x Y_j < \dim_x Y_{j+1}$ (because Y_j is irreducible) and $l \leqq \dim_x X$. So the depth of Y in $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is finite for $Y \in \mathcal{X}(\mathcal{S}, \mathcal{T})$. For $i \in I$ denote the depth of X^i by d_i, and, for any nonnegative integer d, let $I_d=\{i \mid i \in I\ d_i=d\}$, $J_d=\bigcup_{l \leqq d} I_l$, and $K_d=I-J_d$.

We are going to prove by induction on d the following:

(5) For every $d \geqq 0$ there exist primary subsheaves $\mathcal{Q}_i$ of $\mathcal{T}$ for $i \in I_d$ and a coherent analytic subsheaf $\mathcal{R}_d$ of $\mathcal{T}$ such that (i) $E(\mathcal{Q}_i, \mathcal{T})=X^i, i \in I_d$, (ii) $E(\mathcal{R}_d, \mathcal{T})=\bigcup_{i \in K_d} X^i$, and (iii) $(\bigcap_{i \in J_d} \mathcal{Q}_i) \cap \mathcal{R}_d=\mathcal{S}$.

$J_0=\{X^i \mid X^i$ is maximal in $\mathscr{X}(\mathscr{S},\mathscr{T})\}$. By Lemma 5, (5) is true for $d=0$. Suppose (5) is proved for $0\leqq d\leqq e$. Since $E(\mathscr{R}_e,\mathscr{T})=\bigcup_{i\in K_e} X^i$ and

$$E(\mathscr{R}_e,\mathscr{T}) = \bigcup\{Y \mid Y\in\mathscr{X}(\mathscr{R}_e,\mathscr{T})\},$$

$\{X^i \mid i\in I_{e+1}\}=\{Y\in\mathscr{X}(\mathscr{R}_e,\mathscr{T}) \mid Y$ is maximal in $\mathscr{X}(\mathscr{R}_e,\mathscr{T})\}$. By Lemma 5 there exist primary subsheaves $\mathscr{Q}_i$ of $\mathscr{T}$, $i\in I_{e+1}$, and a coherent analytic subsheaf $\mathscr{B}$ of $\mathscr{T}$ such that (i) $E(\mathscr{Q}_i,\mathscr{T})=X^i$, $i\in I_{e+1}$, (ii) $E(\mathscr{B},\mathscr{T})$ is thin in $\bigcup_{i\in I_{e+1}} X^i$, and (iii) $(\bigcap_{i\in I_{e+1}}\mathscr{Q}_i)\cap\mathscr{B}=\mathscr{R}_e$. Hence $(\bigcap_{i\in J_{e+1}}\mathscr{Q}_i)\cap\mathscr{B}=\mathscr{S}$. Let

$$Z = \{Y \mid Y\in\mathscr{X}(\mathscr{B},\mathscr{T}),\ Y\not\subset \textstyle\bigcup_{i\in K_{e+1}} X^i\}.$$

By Corollary to Theorem 5, $E(\mathscr{B}[Z]_{\mathscr{T}},\mathscr{T})\subset\bigcup_{i\in K_{e+1}} X^i$. Let

$$V = E\Big(\mathscr{S}, \big(\bigcap_{i\in J_{e+1}}\mathscr{Q}_i\big)\cap\mathscr{B}[Z]_{\mathscr{T}}\Big).$$

Then $V\subset Z$. Since Z is thin in $\bigcup_{i\in I_{e+1}} X^i$, by Corollary 2 to Theorem 4, $V\subset\bigcup_{i\in K_{e+1}} X^i$.

By Lemma 4 there exists a coherent analytic subsheaf $\mathscr{H}$ of $\mathscr{T}$ such that $E(\mathscr{H},\mathscr{T})=V$ and $(\bigcap_{i\in J_{e+1}}\mathscr{Q}_i)\cap\mathscr{B}[Z]_{\mathscr{T}}\cap\mathscr{H}=\mathscr{S}$. Let $\mathscr{R}_{e+1}=\mathscr{B}[Z]_{\mathscr{T}}\cap\mathscr{H}$. Then $(\bigcap_{i\in J_{e+1}}\mathscr{Q}_i)\cap\mathscr{R}_{e+1}=\mathscr{S}$ and $E(\mathscr{R}_{e+1},\mathscr{T})\subset\bigcup_{i\in K_{e+1}} X^i$. Suppose $E(\mathscr{R}_{e+1},\mathscr{T})\neq\bigcup_{i\in K_{e+1}} X^i$. Take $x\in\bigcup_{i\in K_{e+1}} X^i-E(\mathscr{R}_{e+1},\mathscr{T})$. $x\in X^j$ for some $j\in K_{e+1}$ and $\mathscr{S}_x=\bigcap_{i\in J_{e+1}}(\mathscr{Q}_i)_x$. Let $L=\{i \mid i\in J_{e+1}, x\in X^i\}$. Since $P((\mathscr{Q}_i)_x,\mathscr{T}_x)=P((\mathrm{Id}\ X^i)_x,\mathscr{O}_x)$, $i\in L$, $P(\mathscr{S}_x,\mathscr{T}_x)\subset\bigcup_{i\in L}P((\mathrm{Id}\ X^i)_x,\mathscr{O}_x)$. However,

$$P((\mathrm{Id}\ X^j)_x,\mathscr{O}_x)\cap\Big(\bigcup_{i\in L}P((\mathrm{Id}\ X^i)_x,\mathscr{O}_x)\Big) = \varnothing,$$

contradicting Theorem 4, which asserts $P((\mathrm{Id}\ X^j)_x,\mathscr{O}_x)\subset P(\mathscr{S}_x,\mathscr{T}_x)$. Hence $E(\mathscr{R}_{e+1},\mathscr{T})=\bigcup_{i\in K_{e+1}} X^i$. The induction process is complete and (5) is proved.

We claim that $\mathscr{S}=\bigcap_{i\in I}\mathscr{Q}_i$. Obviously $\mathscr{S}\subset\bigcap_{i\in I}\mathscr{Q}_i$. Take $x\in X$.

$$F = \{i \mid i\in I,\ x\in X^i\}$$

is a finite set. Take $d\geqq\max\{d_i \mid i\in F\}$. Then $x\notin\bigcup_{i\in K_d} X^i$. $(\mathscr{R}_d)_x=\mathscr{T}_x$.

$$\mathscr{S}_x = \Big(\bigcap_{i\in J_d}(\mathscr{Q}_i)_x\cap(\mathscr{R}_d)\Big)_x = \bigcap_{i\in J_d}(\mathscr{Q}_i)_x \supset \bigcap_{i\in I}(\mathscr{Q}_i)_x. \qquad \text{Q.E.D.}$$

REMARK. The decomposition $\mathscr{S}=\bigcap_{i\in I}\mathscr{Q}_i$ is irredundant, i.e. $\mathscr{S}\neq\bigcap_{i\in I-\{j\}}\mathscr{Q}_i$ for any $j\in I$; for otherwise by Theorem 4 we have

$$\mathscr{X}(\mathscr{S},\mathscr{T}) \subset \bigcup_{i\in I-\{j\}}\mathscr{X}(\mathscr{Q}_i,\mathscr{T}) = \{X^i \mid i\in I-\{j\}\}.$$

In general, $\mathscr{Q}_i$, $i\in I$, is not uniquely determined. For example, when $(X,\mathscr{O})$ is $\boldsymbol{C}^2$ with coordinate-functions z_1 and z_2, then $(\mathscr{O}z_1)\cap(\mathscr{O}z_1^2+\mathscr{O}z_2)=(\mathscr{O}z_1)\cap(\mathscr{O}z_1+\mathscr{O}z_2)^2$ are two different irredundant Noether-Lasker decompositions.

However, corresponding to the uniqueness of isolated ideal components in the usual Noether-Lasker decomposition in rings, we have the following:

A subset L of $\mathscr{X}(\mathscr{S}, \mathscr{T})$ is called *an isolated system of associated subvarieties* if $Y_1 \subset Y_2$, $Y_i \in \mathscr{X}(\mathscr{S}, \mathscr{T})$, $i=1, 2$, and $Y_1 \in L$ imply $Y_2 \in L$. If L is an isolated system of associated subvarieties, then $\bigcap \{\mathscr{Q}_i \mid X^i \in L\}$ is unique, because it is equal to $\mathscr{S}[\bigcup \{X^i \mid X^i \notin L\}]$ by Corollary to Theorem 5.

THEOREM 7. *Suppose $\mathscr{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathscr{T}$ on a complex space $(X, \mathscr{O})$. If $\mathscr{S}$ is primary, then $\Gamma(X, \mathscr{S})$ is a primary $\Gamma(X, \mathscr{O})$-submodule of $\Gamma(X, \mathscr{T})$. The converse is true if $(X, \mathscr{O})$ is Stein and $\Gamma(X, \mathscr{S}) \neq \Gamma(X, \mathscr{T})$.*

Proof. (i) Suppose $\mathscr{S}$ is primary. Let $\{Y\} = \mathscr{X}(\mathscr{S}, \mathscr{T})$. We are going to prove that $\Gamma(X, \mathscr{S})$ is a primary $\Gamma(X, \mathscr{O})$-submodule of $\Gamma(X, \mathscr{T})$ with $\Gamma(X, \text{Id } Y)$ as its radical.

Take $f \in \Gamma(X, \text{Id } Y)$. Fix $y \in Y$. Since $E(\mathscr{S}:\mathscr{T}, \mathscr{O}) = Y$, by Hilbert Nullstellensatz $f_y^k \mathscr{T}_y \subset \mathscr{S}_y$ for some natural number k. $\mathscr{X}(\mathscr{S}, f^k\mathscr{T} + \mathscr{S}) \subset \{Y\}$ by Corollary 2 to Theorem 4. $y \notin E(\mathscr{S}, f^k\mathscr{T} + \mathscr{S})$ implies $E(\mathscr{S}, f^k\mathscr{T} + \mathscr{S}) = \varnothing$. $f^k\Gamma(X, \mathscr{T}) \subset \Gamma(X, \mathscr{S})$.

Suppose $g \in \Gamma(X, \mathscr{O}) - \Gamma(X, \text{Id } Y)$ and $s \in \Gamma(X, \mathscr{T})$ such that $gs \in \Gamma(X, \mathscr{S})$. For some $y \in Y$ g does not vanish at y. Then $s_y \in \mathscr{S}_y$. $\mathscr{X}(\mathscr{S}, \mathscr{O}s + \mathscr{S}) \subset \{Y\}$ by Corollary 2 to Theorem 4. $y \notin E(\mathscr{S}, \mathscr{O}s + \mathscr{S})$ implies $E(\mathscr{S}, \mathscr{O}s + \mathscr{T}) = \varnothing$. $s \in \Gamma(X, \mathscr{T})$.

(ii) Suppose $\Gamma(X, \mathscr{S})$ is a primary $\Gamma(X, \mathscr{O})$-submodule of $\Gamma(X, \mathscr{T})$. Suppose $\Gamma(X, \mathscr{S}) \neq \Gamma(X, \mathscr{T})$ and $(X, \mathscr{O})$ is Stein. Let $P \subset \Gamma(X, \mathscr{O})$ be the radical of $\Gamma(X, \mathscr{S})$. P defines a subvariety Y in X. Clearly $E(\mathscr{S}, \mathscr{J}) \subset Y$ and $E(\mathscr{S}, \mathscr{J}) \neq \varnothing$.

We claim $P = \Gamma(X, \text{Id } Y)$. Take $f \in \Gamma(X, \text{Id } Y)$. Fix $y \in Y$. By Hilbert Nullstellensatz $f_y^k \in (\sum_{i=1}^l \mathscr{O}g_i)_y$ for some natural number k and some $g_1, \ldots, g_l \in P$. $y \notin E((\sum_{i=1}^l \mathscr{O}g_i):f, \mathscr{O})$. By Cartan Theorem A there exists $h \in \Gamma(X, (\sum_{i=1}^l \mathscr{O}g_i):f)$ such that h does not vanish at y. $hf \in \Gamma(X, \sum_{i=1}^l \mathscr{O}g_i)$ and $h \notin P$. Let $\varphi: \mathscr{O}^l \to \sum_{i=1}^l \mathscr{O}g_i$ be the sheaf-epimorphism defined by $\varphi(t_1, \ldots, t_l) = \sum_{i=1}^l t_i(g_i)_x$ for $x \in X$ and $(t_1, \ldots, t_l) \in \mathscr{O}_x^p$. Since $H^1(X, \text{Ker } \varphi) = 0$ by Cartan Theorem B, $hf = \sum_{i=1}^l c_i g_i$ for some $c_1, \ldots, c_l \in \Gamma(X, \mathscr{O})$. $hf \in P$. $h \notin P$ implies $f \in P$. Hence $P = \Gamma(X, \text{Id } Y)$.

Y is irreducible, for otherwise $Y = Y_1 \cup Y_2$ for some subvarieties $Y_1 \neq Y$ and $Y_2 \neq Y$ and $P = \Gamma(X, \text{Id } Y) = \Gamma(X, \text{Id } Y_1) \cap \Gamma(X, \text{Id } Y_2)$ with $\Gamma(X, \text{Id } Y_i) \neq \Gamma(X, \text{Id } Y)$, $i = 1, 2$.

Suppose $Z \in \mathscr{X}(\mathscr{S}, \mathscr{T})$. $Z \subset Y$. Suppose $Y \neq Z$. Then take $y \in Y - Z$. By Corollary 1 to Theorem 4, $E(\mathscr{S}:\mathscr{S}[Z], \mathscr{O}) = Z$. By Cartan Theorem A there exists $f \in \Gamma(X, \mathscr{S}:\mathscr{S}[Z])$ such that f does not vanish at y and there exists $s \in \Gamma(X, \mathscr{S}[Z]) - \Gamma(X, \mathscr{S})$. Hence $f \notin P$, $s \notin \Gamma(X, \mathscr{S})$, and $fs \in \Gamma(X, \mathscr{S})$. Contradiction. $\mathscr{X}(\mathscr{S}, \mathscr{T}) = \{Y\}$. $\mathscr{S}$ is primary. Q.E.D.

REMARKS. (i) Theorem 7 justifies the term *primary subsheaf.*

(ii) Theorem ′ of [3] follows from Theorems 6 and 7.

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