THE POINCARÉ-BENDIXSON THEOREM FOR THE KLEIN BOTTLE(1)

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In 1923 Kneser showed that a continuous flow on the Klein bottle without fixed points has a periodic orbit [1]. The purpose of this paper is to prove a stronger version of this theorem. It states that the Klein bottle cannot support a continuous flow with recurrent points which are not periodic.

1. **Introduction.** Let $\boldsymbol{C}$ be the complex plane. For every pair of integers n and m we define a homeomorphism $T_{n,m}$ of $\boldsymbol{C}$ onto itself by $T_{n,m}(z)=z+n+im$. Letting

$$\mathfrak{F}_0 = \{T_{n,m} : n \text{ and } m \text{ are integers}\},$$

we see that $\mathfrak{F}_0$ is a group of homeomorphisms under the usual composition of maps. We define $\mathfrak{F}$ to be the group of homeomorphisms generated by $T_{0,1}$ and K where $K(z)=\bar{z}+1/2$. Then $\boldsymbol{C}/\mathfrak{F}_0$ and $\boldsymbol{C}/\mathfrak{F}$ are the torus $\mathscr{T}$ and the Klein bottle $\mathscr{K}$ respectively. In addition, $(\boldsymbol{C}, p)$ and $(\boldsymbol{C}, p')$ are the universal covering spaces of $\mathscr{T}$ and $\mathscr{K}$ where p and p' are the canonical maps. Because $\mathfrak{F}_0 \subset \mathfrak{F}$ we get a natural map $p_2 : \mathscr{T} \to \mathscr{K}$ such that $(\mathscr{T}, p_2)$ is the two-fold regular covering space of $\mathscr{K}$ and $p'=p_2 \circ p$.

Let $(\tilde{X}, p)$ be a covering space of X and let $(X, \boldsymbol{R}, \pi)$ be a continuous flow. Then there exists a unique flow $(\tilde{X}, \boldsymbol{R}, \tilde{\pi})$ such that p is a homomorphism of $(\tilde{X}, \boldsymbol{R}, \tilde{\pi})$ onto $(X, \boldsymbol{R}, \pi)$ [2]. Moreover, $\tilde{x} \in \tilde{X}$ is a fixed point of $\tilde{\pi}$ if and only if $p(\tilde{x})$ is a fixed point of π and the covering transformations are automorphisms of $(\tilde{X}, \boldsymbol{R}, \tilde{\pi})$.

Let $(X, \boldsymbol{R}, \pi)$ be a continuous flow on a two-manifold and let $x \in X$. A local cross section of π at x is a subset S of X containing x which is homeomorphic to a nondegenerate closed interval and for which there exists an $\varepsilon > 0$ such that the map $(s, t) \to \pi(s, t)$ is a homeomorphism of $S \times [-\varepsilon, \varepsilon]$ onto the closure of an open neighborhood of x. We call ε the length of the local cross section. If x is an interior point of X which is not a fixed point of π, then there exists a local cross section of π at x [4].

Let S be a simple curve and $a, b \in S$. We denote the open segment of S between a and b by $(a, b)_S$. For $\tau > 0$ and $a \in X$, $(a, a\tau)=\{\pi(a, t) : t \in (0, \tau)\}$ and $[a, a\tau]=\{\pi(a, t) : t \in [0, \tau]\}$, where π is a continuous flow on X. Let $\operatorname{ls}(z, z')$ and $\operatorname{l}(z, z')$

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denote the line segment and the line respectively, determined by z and z' of $\mathbf{C}$. A line L in $\mathbf{C}$ is said to be rational if there exists $T \in \mathfrak{F}_0$ such that $T(L)=L$ and T is not the identity map.

2. **Weil's theorem.** At the Moscow Topology Conference in 1935 A. Weil [3] announced the theorem stated below. The writer does not know of a published proof.

THEOREM 2.1. *Let* $\alpha\colon [0, \infty) \to \mathcal{T}$ *be a curve on the torus with no double points. Let* $\tilde{\alpha}\colon [0, \infty) \to \mathbf{C}$ *be any lift of* α *to* $\mathbf{C}$. *If* $|\tilde{\alpha}(t)| \to \infty$ *as* $t \to \infty$, *then* $\lim_{t\to\infty} \tilde{\alpha}(t)/|\tilde{\alpha}(t)|$ *exists.*

LEMMA 2.2. *Let* $\alpha\colon [a, b] \to \mathcal{T}$ *be a simple curve and let* $\tilde{\alpha}\colon [a, b] \to \mathbf{C}$ *be a lift of* α. *Suppose* $J = \mathrm{ls}\,(\tilde{\alpha}(a), \tilde{\alpha}(b)) \cup \tilde{\alpha}([a, b])$ *is a simple closed curve and* $\mathrm{l}\,(\tilde{\alpha}(a), \tilde{\alpha}(b)) \cap \tilde{\alpha}[(a, b)] = \varnothing$. *Then* $T[\tilde{\alpha}(a)]$ *and* $T[\tilde{\alpha}(b)]$ *are not in the interior of* J *for all* T *in* $\mathfrak{F}_0$.

Proof. If the conclusion is false, there exists a T in $\mathfrak{F}_0$ which maps $\tilde{\alpha}(a)$ or $\tilde{\alpha}(b)$ into the interior of J. From the hypothesis it follows that $J \cap T(J) = \varnothing$. Therefore, $T(J \cup \mathrm{Int}\, J) \subset \mathrm{Int}\, J$, which contradicts the Brouwer fixed point theorem.

LEMMA 2.3. *Let* $\alpha\colon [0, \infty) \to \mathcal{T}$ *be a simple curve and let* $\tilde{\alpha}\colon [0, \infty) \to \mathbf{C}$ *be a lift of* α. *Let L be a rational line. If* $|\tilde{\alpha}(t)| \to \infty$ *as* $t \to \infty$, *then* $\tilde{\alpha}(t)$ *does not meet every line parallel to L.*

Proof. Let $V = \{z : |z - \tilde{\alpha}(0)| = 3\}$, let $A = \{s : t > s \Rightarrow |\tilde{\alpha}(t) - \tilde{\alpha}(0)| > 3\}$, and let $t_0 = \text{g.l.b. } A$. Clearly $\tilde{\alpha}(t_0) \in V$. Let R_t be the ray starting at $\tilde{\alpha}(0)$ and containing $\tilde{\alpha}(t)$.

We selected a circle of radius 3 so that any half disk would contain a fundamental region. We use this fact and Lemma 2.2 to show that $R_{t_1} \cup R_{t_2}$ is not a straight line when $t_2 > t_1 > t_0$. Assume it is. In addition, we can assume without any loss of generality that $\tilde{\alpha}[(t_1, t_2)] \cap \{R_{t_1} \cup R_{t_2}\} = \varnothing$. Now Lemma 2.2 applies. But $t_2 > t_1 > t_0$ implies that there is a half disk of radius 3 and hence a fundamental region in the interior of $\mathrm{ls}\,(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2)) \cup \tilde{\alpha}([t_1, t_2])$. The contradiction is obvious.

It follows that there exists a line L' through $\tilde{\alpha}(0)$ and a half plane $\mathscr{H}$ determined by L' such that $\tilde{\alpha}(t) \in \mathscr{H} \cup L'$ when $t > t_0$. In fact, we have $\{\tilde{\alpha}(t) : t > t_0\} \subset \mathscr{H} \cup R'$ where R' is one of the half rays in L' starting at $\tilde{\alpha}(0)$. We now define τ such that $\tilde{\alpha}(\tau) \in R'$ and $\{\tilde{\alpha}(t) : t > \tau\} \subset \mathscr{H} \cup R'$. If there exists t_1 such that $t > t_1$ implies $\tilde{\alpha}(t) \in \mathscr{H}$, then let $\tau = \max\{t : \tilde{\alpha}(t) \in L'\}$. When $\tau \geqq t_0$, it is clear that $\tilde{\alpha}(\tau) \in R'$. If $\tau < t_0$, then either ray can be selected as R' and so we can assume that $\tilde{\alpha}(\tau) \in R'$. For the other case, let $\tau = \min\{t : \tilde{\alpha}(t) \in R' \text{ and } t > t_0\}$.

Let L be a rational line through $\tilde{\alpha}(\tau)$ such that $\tilde{\alpha}(t)$ meets every line parallel to L. Clearly $L \neq L'$. Since L is rational, there exists $T \in \mathfrak{F}_0$ such that $T(\tilde{\alpha}(\tau)) \in \mathscr{H}$ and $T(L) = L$. Denote $T \circ \tilde{\alpha}$ by $\tilde{\alpha}_1$. Let

$$r = |\tilde{\alpha}(\tau) - \tilde{\alpha}(0)| + |\tilde{\alpha}(\tau) - \tilde{\alpha}_1(\tau)| \quad \text{and} \quad D = \{z : |z - \tilde{\alpha}(\tau)| < r\} \cap \mathscr{H}.$$

The set $L' \cap \{z : |z - \tilde{\alpha}(\tau)| = r\}$ consists of two points, q_1 and q_2, and we can assume

$q_1 \notin R'$ and $q_2 \in R'$. Select $\tau_1 > \tau$ such that $t > \tau_1$ implies $\tilde{\alpha}(t) \notin D$. In the half plane determined by L and q_1 we select a line L_1 parallel to L such that $L \cap \tilde{\alpha}([\tau, \tau_1]) = \varnothing$ and $L_1 \cap \bar{D} = \varnothing$. Let $y_1 = L_1 \cap L'$. Let $\tau_2 = \min\{t : \tilde{\alpha}(t) \in L_1\}$. Now select L_2 parallel to L such that $L_2 \cap \tilde{\alpha}([\tau, \tau_2]) = \varnothing$, $L_2 \cap \bar{D} = \varnothing$, and $L_2 \cap L' = y_2 \in R'$. Let $\tau_3 = \min\{t : \tilde{\alpha}(t) \in L_2\}$. There exists $\tau_4 \in [\tau_2, \tau_3]$ such that $\tilde{\alpha}(\tau_4) \in L_1$ and such that $t \in [\tau_2, \tau_3]$ and $\tilde{\alpha}(t) \in L_1$ together imply $|\tilde{\alpha}(\tau_4) - y_1| \leqq |\tilde{\alpha}(t) - y_1|$. Finally, let

$$\tau_5 = \min\{t : t \in [\tau_4, \tau_3] \text{ and } \tilde{\alpha}(t) \in L_2 \cup \operatorname{ls}(q_2, y_2)\}.$$

Set

$$J = \operatorname{ls}(y_1, \tilde{\alpha}(\tau_4)) \cup \tilde{\alpha}([\tau_4, \tau_5]) \cup \operatorname{ls}(\tilde{\alpha}(\tau_5), y_2) \cup \operatorname{ls}(y_2, y_1)$$

or

$$J = \operatorname{ls}(y_1, \tilde{\alpha}(\tau_4)) \cup \tilde{\alpha}([\tau_4, \tau_5]) \cup \operatorname{ls}(\tilde{\alpha}(\tau_5), y_1)$$

according as $\tilde{\alpha}(\tau_5)$ is in L_2 or L'. In either case J is a simple closed curve with D and hence $\tilde{\alpha}_1(\tau)$ in its interior.

Let $\tau_6 = \min\{t : \tilde{\alpha}_1(t) \in J \text{ and } t > \tau\}$, and let $x_i = L_i \cap T(L')$. Either $\tilde{\alpha}_1(\tau_6) \in \operatorname{ls}(x_1, \tilde{\alpha}(\tau_4))$ or $\tilde{\alpha}_1(\tau_6) \in \operatorname{ls}(x_2, \tilde{\alpha}(\tau_3))$ because α is simple and $\{\tilde{\alpha}_1(t) : t \geqq \tau\} \subset T(\mathscr{H} \cup R') \subset \mathscr{H}$. If $\tilde{\alpha}_1(\tau_3) \in \operatorname{ls}(x_2, \tilde{\alpha}(\tau_3))$, then $|\tilde{\alpha}_1(\tau_3) - \tilde{\alpha}_1(\tau)| < |\tilde{\alpha}(\tau_3) - \tilde{\alpha}(\tau)|$ which is impossible. Therefore $\tilde{\alpha}_1(\tau_3)$ is in the exterior of J, $\tau_6 < \tau_3$, and $\tilde{\alpha}_1(\tau_6) \in \operatorname{ls}(x_1, \tilde{\alpha}(\tau_4))$. Since $\{\tilde{\alpha}(t) : t \in (\tau, \tau_2)\}$ is between L_1 and L_2, $\tau_6 \in [\tau_2, \tau_3)$. We now have the following contradiction:

$$|\tilde{\alpha}_1(\tau_6) - \tilde{\alpha}_1(\tau)| < |\tilde{\alpha}(\tau_4) - \tilde{\alpha}(\tau)| \leqq |\tilde{\alpha}(\tau_6) - \tilde{\alpha}(\tau)|.$$

The proof is completed.

Proof of Theorem 2.1. Set

$$B = \bigcap_{n=1}^{\infty} \operatorname{Cl}\{\tilde{\alpha}(t)/|\tilde{\alpha}(t)| : t \geqq n\}.$$

The set B is a closed connected subset of $U = \{z : |z| = 1\}$. Clearly $\lim_{t\to\infty} \tilde{\alpha}(t)/|\tilde{\alpha}(t)|$ exists if and only if B is a point. We argue by contradiction. Let a and b be two distinct points of B. Let U_1 and U_2 be the two components of $U - \{a, b\}$. Since the arcwise connected set $\{\tilde{\alpha}(t)/|\tilde{\alpha}(t)| : t \geqq n\}$ meets every open interval containing a or b, we can assume that

$$U_1 \subset \{\tilde{\alpha}(t)/|\tilde{\alpha}(t)| : t \geqq n\}$$

for all n. Let R be a ray starting at 0 with rational slope so that $z_0 = R \cap U \in U_1$. Pick z_1 and z_2 in U_1 so that they are separated by z_0. Let L be any line parallel to R, and let R_1 and R_2 be the rays starting at 0 and determined by z_1 and z_2 respectively. There exists $r > 0$ such that the sets $\{z : |z| > r \text{ and } z \in R_i\}$, $i = 1, 2$ are on opposite sides of L. Since $U_1 \subset \{\tilde{\alpha}(t)/|\tilde{\alpha}(t)| : t \geqq n\}$ for all n, $\tilde{\alpha}(t)$ meets the above sets and hence L. But this is contrary to the conclusion of Lemma 2.3, and the proof is completed.

COROLLARY 2.4. *Let α, β: $[0, \infty) \to \mathscr{T}$ be simple curves which do not intersect. Let $\tilde{\alpha}$, $\tilde{\beta}$: $[0, \infty) \to \boldsymbol{C}$ be lifts of α and β respectively. If $|\tilde{\alpha}(t)| \to \infty$ and $|\tilde{\beta}(t)| \to \infty$ as $t \to \infty$, then*

$$\lim_{t\to\infty} \frac{\tilde{\alpha}(t)}{|\tilde{\alpha}(t)|} = \pm \lim_{t\to\infty} \frac{\tilde{\beta}(t)}{|\tilde{\beta}(t)|}.$$

Proof. The equation $\lim_{t\to\infty} \tilde{\alpha}(t)/|\tilde{\alpha}(t)| = e^{i\theta'}$ means given $\varepsilon > 0$ there exists $\tau > 0$ such that

$$\tilde{\alpha}(t) \in \{re^{i\theta} : r > 0 \text{ and } \theta \in (\theta' - \varepsilon, \theta' + \varepsilon)\}$$

when $t > \tau$. Using this and the hypothesis it follows that $\tilde{\alpha}(\boldsymbol{R}^+) \cap \tilde{\beta}(\boldsymbol{R}^+) \neq \varnothing$ if the conclusion does not hold.

3. **The torus.** Let α: $[0, 1] \to \mathscr{T}$ be a closed curve which is not null-homotopic, and let $\tilde{\alpha}$: $[0, 1] \to \boldsymbol{C}$ be any lift of α to $\boldsymbol{C}$. There exists $T \in \mathfrak{F}_0$ such that $\tilde{\alpha}(1) = T[\tilde{\alpha}(0)]$ and T is not the identity. The curve $\tilde{\alpha}_u$: $\boldsymbol{R} \to \boldsymbol{C}$ defined by $\tilde{\alpha}_u(t) = T^n[\tilde{\alpha}(t')]$ where $t = n + t'$ and $t' \in [0, 1)$ is called a universal lift of α. If $T' \in \mathfrak{F}_0$, then obviously $T' \circ \tilde{\alpha}_u$ is another universal lift of α.

REMARK 3.1. Let α: $[0, 1] \to \mathscr{T}$ be a closed curve which is not null-homotopic, and let $\tilde{\alpha}_u$ be any universal lift of α.

(a) The image of $\tilde{\alpha}_u$ is between two parallel rational lines.

(b) If $\tilde{\alpha}_u(\boldsymbol{R})$ lies between the lines L_1 and L_2, then $\tilde{\alpha}_u(\boldsymbol{R}) \cap L \neq \varnothing$ whenever L is not parallel to L_1.

(c) The curve α is simple if and only if $\tilde{\alpha}_u$ is simple and for any T in $\mathfrak{F}_0$, $T \circ \tilde{\alpha}_u(\boldsymbol{R}) \cap \tilde{\alpha}_u(\boldsymbol{R})$ is either $\tilde{\alpha}_u(\boldsymbol{R})$ or the empty set.

(d) If α is simple and $T_{n,m}$ is the element of $\mathfrak{F}_0$ used to define $\tilde{\alpha}_u$, then n and m are relatively prime.

Proof. Use the definition of $\tilde{\alpha}_u$ and the properties of the transformations in $\mathfrak{F}_0$.

Let $(\mathscr{T}, \boldsymbol{R}, \pi)$ be a continuous flow on the torus and let $(\boldsymbol{C}, \boldsymbol{R}, \tilde{\pi})$ be its lift to the complex plane.

In the classical Poincaré-Bendixson theory one applies the Jordan curve theorem to curves obtained from local cross sections and orbits. We form the same kind of curves on the torus, but then we take the universal lifts and use the Jordan curve theorem on the sphere.

Let S be a local cross section of π at $w \in \mathscr{T}$. Suppose $t_0 > 0$, $wt_0 \in S$, $w \neq wt_0$, and $[w, wt_0] \cap (w, wt_0)_S = \varnothing$, then $\gamma = [w, wt_0] \cup (w, wt_0)_S$ is a simple closed curve ($wt_0 = \pi(w, t_0)$). When γ is not null-homotopic we call the universal lift $\tilde{\gamma}_u$ of γ a *control curve*. Let $z_0 \in \tilde{\gamma}_u$ such that $p(z_0) \in (w, wt_0)_S$. By 3.1 $\tilde{\gamma}_u$ divides $\boldsymbol{C}$ into two parts. Clearly $z_0(-\varepsilon)$ lies in one and $z_0\varepsilon$ in the other where ε is the length of S. We call them Γ^- and Γ^+ respectively. The definition of Γ^+ does not depend on z_0. If $z \in \tilde{\gamma}_u \cap p^{-1}(w)$, then $z(-\varepsilon) \in \Gamma^-$ and $z(t_0 + \varepsilon) \in \Gamma^+$.

LEMMA 3.2. *Let $\tilde{\gamma}_u$ be a control curve. Then Γ^+ and $\Gamma^+ \cup \tilde{\gamma}_u$ are positively invariant, and Γ^- and $\Gamma^- \cup \tilde{\gamma}_u$ are negatively invariant.*

Proof. Suppose $z \in \Gamma^+$ and $\boldsymbol{O}^+(z) \not\subset \Gamma^+$. Let τ be the smallest positive number such that $z\tau \in \tilde{\gamma}_u$. By our choice of τ, $p(z\tau) \in [w, wt_0)_S$. But this implies $z(\tau-\varepsilon) \in \Gamma^-$, a contradiction. Clearly $\mathrm{Cl}\,(\Gamma^+) = \Gamma^+ \cup \tilde{\gamma}_u$ is also positively invariant.

THEOREM 3.3. *Let $z_0 \in \boldsymbol{C}$. If $p(z_0)$ is {positively recurrent} {negatively recurrent} and not periodic, then $\{\boldsymbol{O}^+(z_0)\}$ $\{\boldsymbol{O}^-(z_0)\}$ does not lie between two parallel rational lines.*

Proof. Suppose $p(z_0)$ is positively recurrent and not periodic. Assume that $\boldsymbol{O}^+(z_0)$ lies between two parallel rational lines L' and L''.

Since $\mathfrak{F}_0$ is a properly discontinuous group of transformations, there exists a finite set of lines, $\{L_1, \ldots, L_q\}$, parallel to L' such that $z' \sim z_0$ and z' between L' and L'' implies $z' \in L_i$ for some i. Since $p(z_0)$ is positively recurrent, there exists at least one i such that given any $\varepsilon > 0$ we can find $z' \in L_i$ satisfying $z_0 \sim z'$ and $\boldsymbol{O}^+(z_0) \cap S(z', \varepsilon) \neq \varnothing$ $(S(z', \varepsilon) = \{z : |z-z'| < \varepsilon\})$. We consider two cases.

For the first case we assume that there exists an i satisfying the above such that $z_0 \notin L_i$. For definiteness, assume that $z_0 \in L_1$ and $i=2$. Let T be a covering transformation mapping L_1 on L_2. We prove by induction that given any positive integer m and any $\varepsilon > 0$, there exists $z' \sim z_0$ such that $z' \in T^m(L_1)$ and $\boldsymbol{O}^+(z_0) \cap S(z', \varepsilon) \neq \varnothing$. It is true for $m=1$. Suppose it is true for m. There exists $z' \in L_2$ and $\tau > 0$ such that $z_1 \sim z_0$ and $|z_0\tau - z_1| < \varepsilon$. We can assume that $T(z_0) = z_1$. Choose $\delta > 0$ such that $S(z_0, \delta)\tau \subset S(z_1, \varepsilon)$. Using the isometric and automorphic properties of the covering transformations, we see that $S(z', \delta)\tau \subset S(T(z'), \varepsilon)$ for all $z' \sim z_0$. By the induction assumption there exists $z_2 \sim z_0$ and $\tau' > 0$ such that $z_2 \in T^m(L_1)$ and $|z_0\tau' - z_2| < \delta$. Therefore, $z_0(\tau+\tau') \in S(T(z_2), \varepsilon)$ and $T(z_2) \in T^{m+1}(L_1)$. Since $d(T^m(L_1), L_1) \to \infty$ as $m \to \infty$, $\boldsymbol{O}^+(z_0)$ can not lie between L' and L''.

If the first case does not hold, assuming $z_0 \in L_1$, there exists $\varepsilon_1 > 0$ such that $z' \sim z_0$ and $\boldsymbol{O}^+(z_0) \cap S(z', \varepsilon_1) \neq \varnothing$ together imply that $z' \in L_1$. We can assume that $S(z_0, \varepsilon_1)$ is a canonical neighborhood of z_0. Clearly z_0 is not recurrent. Thus there exists a local cross section $\tilde{S}$ of $\tilde{\pi}$ at z_0 such that $\boldsymbol{O}(z_0) \cap \tilde{S} = z_0$ and $\tilde{S} \subset S(z_0, \varepsilon_1)$. It follows that $S = p(\tilde{S})$ is a local cross section of π at w. Since w is positively recurrent and not periodic we can find $t_2 > t_1 > 0$ satisfying

(i) $wt_i \in S$,

(ii) $[w, wt_1] \cap (w, wt_1)_S = \varnothing$, and

(iii) $wt_2 \in (w, wt_1)_S$.

Set $\gamma = [w, wt_1] \cup (w, wt_1)_S$. From (ii) we see that γ is simple. If γ is null-homotopic, then $z_0 t_1 \in \tilde{S}$ which contradicts our choice of $\tilde{S}$. Therefore, we have a control curve $\tilde{\gamma}_u$ through z_0 which is between two lines parallel to L_1 because $\tilde{S} \subset S(z_0, \varepsilon_1)$. Clearly $z_0 t_2 \in \Gamma^+$ and $z_0 t_2 \in T(\tilde{\gamma}_u)$ for some $T \in \mathfrak{F}_0$. Because $|z_0 t_2 - z'| < \varepsilon'$, where $z' \sim z_0$, $T(L_1) = L_1$ and $T(\tilde{\gamma}_u) = \tilde{\gamma}_u$. Consequently, $z_0 t_2 \in \Gamma^+ \cap \tilde{\gamma}_u = \varnothing$. Thus the second case cannot occur and the proof is completed.

THEOREM 3.4. *Let $z_0 \in \boldsymbol{C}$ and let A be any compact set in $\boldsymbol{C}$. Suppose that $p(z_0) \in \mathscr{T}$*

is {positively recurrent} {negatively recurrent} and not periodic. Then there exists a neighborhood V of z_0 and $\tau > 0$ such that $Vt \cap A = \varnothing$ when $\{t > \tau\}$ $\{-t > \tau\}$.

Proof. Consider the first reading. We can find a control curve $\tilde{\gamma}_u$ and a deck transformation T such that $A \cup z_0 \subset \Gamma^- \cap T(\Gamma^+)$. By 3.1, 3.2, and 3.3 there exists $\tau > 0$ such that $z_0\tau \in \Gamma^+$ and hence a neighborhood V of z_0 such that $V\tau \subset \Gamma^+$. Now apply Lemma 3.2.

COROLLARY 3.5. *Let $z_0 \in \mathbf{C}$. If $p(z_0)$ is {positively recurent} {negatively recurrent} and not periodic, then $|z_0 t| \to \infty$ as $\{t \to \infty\}$ $\{t \to -\infty\}$ and*

$$\left\{\lim_{t\to\infty} \frac{z_0 t}{|z_0 t|}\right\} \qquad \left\{\lim_{t\to-\infty} \frac{z_0 t}{|z_0 t|}\right\}$$

exists.

Proof. Use 3.4 and 2.1.

THEOREM 3.6. *Let $z_0, z_1 \in \mathbf{C}$ such that $p(z_0)$ and $p(z_1)$ are {positively recurrent} {negatively recurrent} and not periodic. Then*

$$\left\{\lim_{t\to\infty} \frac{z_0 t}{|z_0 t|} = \lim_{t\to\infty} \frac{z_1 t}{|z_1 t|}\right\} \qquad \left\{\lim_{t\to-\infty} \frac{z_0 t}{|z_0 t|} = \lim_{t\to-\infty} \frac{z_1 t}{|z_1 t|}\right\}.$$

Proof. Use 2.4 and the translates of a control curve.

4. **The Klein bottle.** We are now ready to prove the main theorem.

THEOREM 5.1. *Let $(\mathscr{K}, \mathbf{R}, \pi)$ be a continuous flow on the Klein bottle. Then every positively or negatively recurrent orbit is periodic.*

Proof. Let $(\mathscr{T}, \mathbf{R}, \hat{\pi})$ and $(\mathbf{C}, \mathbf{R}, \tilde{\pi})$ be the lifts of π to $\mathscr{T}$ and $\mathbf{C}$. Assume $w \in \mathscr{K}$ is positively recurrent and not periodic. Let $p_2^{-1}(w) = \{\hat{w}_1, \hat{w}_2\}$. Clearly $\hat{w}_1$ and $\hat{w}_2$ are not periodic. There exists a deck transformation $\hat{K}: \mathscr{T} \to \mathscr{T}$ which permutes $\hat{w}_1$ and $\hat{w}_2$. In addition, $\hat{K}$ is induced by K and is an automorphism of $\hat{\pi}$. Since w is positively recurrent, either $\hat{w}_1$ or $\hat{w}_2$ is in $\omega(\hat{w}_1) = \omega$-limit points of $\hat{w}_1$. If $\hat{w}_2 \in \omega(w_1)$, then by applying $\hat{K}$ we see that $\hat{w}_1 \in \omega(\hat{w}_2) \subset \omega(\hat{w}_1)$. Therefore, $\hat{w}_1$ and $\hat{w}_2$ are positively recurrent and not periodic.

Let $z_1 \in p^{-1}(\hat{w}_1)$. Then $p[K(z_1)] = \hat{w}_2$. Using 3.5 we conclude that $\lim_{t\to\infty} (z_1 t)/|z_1 t|$ and $\lim_{t\to\infty} K(z_1 t)/|K(z_1 t)|$ exist. Clearly their respective limits are complex conjugates. Therefore, by 3.6 $\lim_{t\to\infty} (z_1 t)/|z_1 t| = \pm 1$. We will consider only the $+1$ case.

Since $\mathbf{O}^+(z_1)$ can not lie between two parallel rational lines, $\operatorname{Im}(z_1 t)$ is unbounded for positive t. The next step is to show that $\operatorname{Im}(z_1 t)$ has neither upper nor lower bound for $t > 0$. There is no loss in generality in assuming that $\operatorname{Re}(z_1 t) > \operatorname{Re}(z_1)$ if $t > 0$. The half plane $\{z : \operatorname{Re}(z) > \operatorname{Re}(z_1)\}$ is divided into two parts by $\mathbf{O}^+(z_1)$. Clearly $K(z_1)$ lies in one and $T_{0,n}[K(z_1)]$ in the other for a suitable n. Observe that $\operatorname{Im} K(z_1 t) = -\operatorname{Im}(z_1 t)$ and $\operatorname{Im} T_{0,n}[K(z_1 t)] = -\operatorname{Im}(z_1 t) + n$. It follows $\operatorname{Im}(z_1 t)$ has neither an upper nor a lower bound. Therefore, $O^+(z_1)$ meets every line parallel to the x-axis which contradicts Lemma 2.3, because $t \to \hat{w}_1 t$ is a simple curve on the torus.

COROLLARY 5.2 (KNESER). *Let* $(\mathscr{X}, \boldsymbol{R}, \pi)$ *be a continuous flow on the Klein bottle without fixed points. Then there exists a periodic orbit.*

Proof. Let M be a minimal set under π, and let $w \in M$. We know that w is almost periodic. Therefore, by Theorem 5.1 w is periodic or fixed, but π has no fixed points.

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