QUANTITATIVE POLYNOMIAL APPROXIMATION ON CERTAIN PLANAR SETS

BY

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I. Let \( Q \) be a compact space in \( E^k \). The massivity, \( m_n(Q) \), is a sequence defined as follows: Let \( X_n \) be a set of \( n + 1 \) elements of \( Q \); then

\[
m_n(Q) = \max_{X_n \subset Q} \min_{x, y \in X_n, i \neq j} |x_i - x_j|.
\]

Note that \( m_1 \) is what is normally defined to be the diameter of the space. Also note that \( m_n \) decreases to zero monotonely as \( n \to \infty \). E.g., if \( Q \) is a set of measure \( m > 0 \), \( m_n(Q) \) is asymptotic to \( m/n \); if \( Q \) is a \( k \)-dimensional cube, \( m_n(Q) \) is asymptotic to \( c/n^{1/k} \) [6, p. 21].

\( Q_1 \) will represent the interval \( 0 \leq x \leq 1 \), and \( Q_2 \) the square \( 0 \leq x, y \leq 1 \). Given a function \( f(x) \) defined on \( Q \) and a \( \delta > 0 \), the modulus of continuity of the function \( f(x) \), \( \omega_f(\delta) \), is defined as follows:

\[
\omega_f(\delta) = \sup_{|x - y| \leq \delta} |f(x) - f(y)|.
\]

For \( f(x) \) in \( L^2(Q_1) \), we continue \( f(x) \) to have period 1, and we define

\[
\omega_f(\delta) = \sup_{|t| \leq \delta} \left( \int_0^1 |f(x+t) - f(x)|^2 \, dx \right)^{1/2},
\]

while for \( f(x, y) \) in \( L^2(Q_2) \) continued to have period 1 in \( x \) and in \( y \)

\[
\omega_f(\delta) = \sup_{(t, s) \leq \delta} \left( \int_0^1 \int_0^1 |f(x+t, y+s) - f(x, y)|^2 \, dx \, dy \right)^{1/2}.
\]

\( \omega(\delta) \) is a nonnegative, nondecreasing continuous function of \( \delta \). \( f(x) \) is continuous iff \( \omega_f(0^+) = 0 \). Also, \( \omega(\delta) \) is sublinear, and hence \( \omega_f(\delta) \geq c\delta \) for some constant \( c \). If \( \omega_f(\delta) \leq \delta \) for all \( \delta > 0 \), \( f(x) \) is said to be a contraction on \( Q \).

Let \( C(Q) \) be the space of all real valued continuous functions on \( Q \) and let \( P \) be a finite dimensional subspace. Let \( K \) be the class of contractions on \( Q \). We now introduce the degree of approximation

\[
\rho_P = \sup_{f \in K} \inf_{p \in P} \|f - p\|.
\]
We have the following general lower bound theorem for the degree of approximation: If the dimension of $P$ is $n$, then $\rho_P \geq m_n(Q)/2$ [4]. If for some $Q$ there exists $c$ such that $\rho \leq cm_n(Q)$ for all $n$, where $P$ is the class of $n$th degree polynomials, we can then say that $P$ efficiently approximates all contractions on $Q$. If for some family of compact sets $Q$, there exists $c$ such that $\rho_P(Q) \leq cm_n(Q)$ for all $Q$ in $Q$, and for all $n$, then $P$ can be said to approximate contractions efficiently on the family $Q$.

The classic upper bound theorem is that of Dunham Jackson which says, essentially, that polynomials approximate efficiently on $Q_1$ [3, p. 36]. More recently, further results in this direction have been obtained. Yu. A. Brudnji [2] and D. J. Newman simultaneously discovered that for $P_n$, the class of $n$th degree polynomials on linear sets $Q$ of positive measure, the corresponding degree of approximation, $\rho_n$, for $n$ sufficiently large, satisfies the inequality $\rho_n \leq cm/n$, $c$ independent of $Q$. Newman and H. S. Shapiro have shown that polynomials approximate efficiently on the family of cubes, spheres and balls of all dimensions [6]. On the other hand, polynomials are not efficient on the family of all rectangles in the plane (and, hence, the two-dimensional analog of the result for linear sets of positive measure cannot hold). In fact, there is even a (highly pathological) linear set (of measure zero) on which polynomials are not efficient.

An $n$th degree polynomial in more than one variable will refer to a polynomial in which the maximum degree in any single variable is less than $n$. Unless otherwise specified, $P$ and the corresponding $\rho$ will refer to $n$th degree polynomials. In this case, they will sometimes be denoted, respectively, $P_n$ and $\rho_n$. (Note that $P_n$ may have dimension greater than $n$.)

The term normal curve will be used for a compact continuous curve of finite length contained in $Q_2$. In this article we seek estimates for $\rho_P(Q)$ for $Q$ a normal curve.

II. Lemma 1. If $Q$ is a normal curve, there exist $c_1, c_2 > 0$ such that $c_1/n \leq m_n(Q) \leq c_2/n$.

Proof. The projection of $Q$ on the $x$-axis (or the $y$-axis) must be a line segment $[a, b]$. Consider the points $\{a+k(b-a)/n, k=0, 1, \ldots, n\}$. Invert back to $n+1$ corresponding points of $Q$ to get the set $X_n$. The $x$-coordinate (or the $y$-coordinate) of the minimum distance between any two points of $X_n$ is $(b-a)/n$, giving $m_n(Q) \geq (b-a)/n$. Also, $m_n(Q) \leq 2L/n$, where $L$ is the length of $Q$. For, otherwise, there would be $n+1$ nonintersecting discs of radius $L/n$, with their centers being elements of $Q$. The portion of $Q$ contained within these discs would have length $\geq 2L$. This contradiction establishes Lemma 1.

Lemma 2. On $Q_2$, there exist $c_1, c_2 > 0$ such that $c_1/n < \rho_n < c_2/n$.

This lemma is a special case of the more general theorem of Newman and Shapiro [6, Theorem 4, p. 212].
Lemma 3. Let \( f(x) \) be a function defined on \( S \subseteq Q, \) \( Q \) a compact set in \( E^n, \) where \( f(x) \) has modulus of continuity \( \omega(\delta). \) Then there is a function \( f^*(x) \) defined on \( Q \) with the following properties:

(a) \( f^*(x) = f(x), x \in S, \)

(b) \(|f^*(x) - f^*(y)| \leq \omega(d(x, y)), \) where \( d(x, y) \) is the distance from \( x \) to \( y. \) (In other words, a function can be extended without changing its modulus of continuity.)

Proof. It is sufficient to show that \( f \) can be properly extended to one point \( \bar{x} \) not in \( S, \) for the result would then follow by transfinite induction.

For all \( x, y \) in \( S, \)

\[ f(x) - f(y) \leq \omega(d(x, y)) \]

\[ \leq \omega(d(x, \bar{x}) + d(y, \bar{x})), \] by the triangle inequality and by the fact that \( \omega \) is nondecreasing,

\[ \leq \omega(d(x, \bar{x}))(d(y, \bar{x})), \] by sublinearity of \( \omega. \)

Hence, for any \( x \) we can find \( \alpha \) so that

\[ f(x) - \omega(d(x, \bar{x})) \leq \alpha \leq f(x) + \omega(d(x, \bar{x})). \]

Defining \( f^*(\bar{x}) \) to be \( \alpha, \) we have

\[ |f(x) - f(\bar{x})| \leq \omega(d(x, \bar{x})). \] Q.E.D.

Corollary. Let \( f \) be a contraction on \( S \subseteq Q, Q \subseteq E^n. \) Then \( f \) can be extended as a contraction on \( Q. \)

Lemma 4. The binomials \( x^iy^j, 0 \leq i, j \leq n, \) generate on the curve \( \sum_{k=0}^{N} \sum_{m=0}^{N} a_{km}x^ky^m = 0 \) a vector space of dimension less than \( 3N^2. \)

Proof. Let \( S \) be the space generated by the \( x^iy^j, 0 \leq i, j \leq n \) on the entire plane. Consider the space generated by the \( x^iy^j, 0 \leq i, j \leq n \) on the restriction to the given algebraic curve. A basis of the binomials for this space spans, on the plane, a subspace \( S_0 \) of \( S. \) Another subspace \( S_1 \) of \( S \) is generated by those binomials spanning \( S \) which are not included in the basis for \( S_0. \) Note that the dimension of \( S \) is \( (n+1)^2. \) We must show that \( \dim S_0 < 3N^2. \) For \( n < N, \) the result is trivial. Now,

\[ x^iy^j \sum_{k=0}^{N} a_{km}x^ky^m, \quad 0 \leq i, j \leq n - N, \]

are linearly independent elements of \( S_1. \) Hence \( \dim S_1 \geq (n-N)^2. \) Thus

\[ \dim S_0 = \dim S - \dim S_1 \leq (n+1)^2 - (n-N)^2 < 3nN, \quad n \geq N, \]

and the lemma is proved.

Lemma 5. If \( Q \) is a normal curve, then there exist \( c_1, c_2 > 0 \) such that \( c_1/n^2 < \rho_n < c_2/n \) for all \( n. \)
Proof. Since the dimension of the space spanned by \( n \)th degree polynomials on \( Q \) is \( \leq n^2 \), we have \( \rho_n \geq m_n/2 \geq c_1/n^2 \), by the general lower bound theorem and Lemma 1. To prove the right hand inequality, we can consider any contraction on \( Q \) to be extended, by the corollary to Lemma 3, as a contraction to \( Q_2 \). Now \( \rho_n(Q) \leq \rho_n(Q_2) \leq c_2/n^2 \), by Lemma 2, completing the proof.

**Theorem 1.** If \( Q \) is the restriction of an algebraic curve

\[
\sum_{t \leq 0 \text{ or } f \leq 0} a_i x^i y^j = 0
\]

to \( Q_2 \), then there exist \( c_1, c_2 > 0 \) such that \( c_1/n \leq \rho_n(Q) \leq c_2/n \).

Proof. From Lemma 4, the general lower bound theorem and Lemma 1, \( \rho_n \geq 1/6Nn \), while \( \rho_n \leq c/n \) by Lemma 5.

For \( Q \) a normal curve, the trivial estimates of \( c_1/n \) and \( c_2/n^2 \) as upper and lower bounds for \( \rho_n \) have been established. If the dimension of \( n \)th degree polynomials on \( Q \) is equal to \( O(n) \), then \( c_1/n < \rho_n < c_2/n \) (as in the case where \( Q \) is an algebraic curve). If the dimension of \( n \)th degree polynomials on \( Q \) is of greater order than \( n \), say of order \( n^2 \), to what extent can the trivial estimates for \( \rho_n \) be improved? Theorem 2 and Theorem 5 provide a partial answer to this question. It is toward the establishment of these theorems that we conclude this section with several additional lemmata. In Theorem 1 \( \rho_n \) can be taken in the \( L^2 \) norm or the sup norm, where what is meant by \( \|/\|_{L^2} \) on the normal curve \( y=g(x) \) is \( (\int_0^1 |f(x, g(x))|^2 \, dx)^{1/2} \).

In all following results \( \rho_P \) will be considered in \( L^2 \) only.

We denote by \( C_P \) the set of all \( \varphi(x) \) with \( L^2[0, 1] \) norm one which are in the orthogonal complement of \( P \).

**Lemma 6.**

\[
\rho_P = \sup_{\varphi \in C_P} \left\| \int_0^x \varphi(t) \, dt \right\|_{L^2}.
\]

For proof, cf. [5, Lemma 2, p. 942].

We recall that \( F(x) \) is said to be in the Paley-Wiener class for the upper half plane, \( PW \), if

(a) \( F \) is analytic in the upper half plane, and

\[
\int_{-\infty}^{\infty} |F(x + iy)|^2 \, dx < M \quad \text{for all } y > 0,
\]

or, equivalently,

(b) \( F(z) = \int_0^\infty e^{ix} \varphi(x) \, dx, \quad \varphi(x) \in L^2[0, \infty] \).

For proof of the equivalence of (a) and (b), and for the general Paley-Wiener theory, the reader is referred to [7, pp. 1–13].
Lemma 7. Let $P = \{x^\lambda, \lambda \in \Lambda\}$, where $0 \in \Lambda$, $\lambda \geq 0$. Then
\[
\rho^2(\mathcal{Q}_1) = \sup_{F \in \mathcal{P} \mathcal{W}} \int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{x^2 + 1/4 + \sum_{\lambda \in \Lambda} \frac{x^2 + (\lambda - 1/2)^2}{x^2 + (\lambda + 3/2)^2}} \, dx \]
\[
\int_{-\infty}^{\infty} |F(x)|^2 \, dx
\]
For proof, cf. [5, Lemma 3, p. 943].

It is convenient to adopt the following notation:
\[
M(\Lambda) = \max_x \frac{1}{x^2 + 1/4 + \sum_{\lambda \in \Lambda} \frac{x^2 + (\lambda - 1/2)^2}{x^2 + (\lambda + 3/2)^2}}
\]

Lemma 8. For $P$ defined as in Lemma 7, $\rho^2(\mathcal{Q}_1) \leq M(\Lambda)$.

Since, by Parseval's Identity,
\[
\int_{-\infty}^{\infty} |F(x+i)|^2 \, dx \leq \int_{-\infty}^{\infty} |F(x)|^2 \, dx, \quad \text{for all } F \in \mathcal{P} \mathcal{W},
\]
Lemma 8 is a corollary to Lemma 7.

Lemma 9. If $|\sum_{k=0}^{n} a_k x^k| \leq 1$ whenever $0 \leq x \leq 1$, then
\[
|a_k| \leq 2^{2k} \frac{n(n+k-1)!}{(n-k)!(2k)!}
\]

Furthermore, these are the best possible bounds for $|a_k|$.

To derive this bound, one demonstrates that $\cos 2n(\arccos x^{1/2})$, which is equal to
\[
\sum (-1)^{k} 2^{2k} \frac{n(n+k-1)!}{(n-k)!(2k)!} x^k,
\]
is maximal for each coefficient. For proof, see [1, p. 30]. Note that the above upper bounds yield the estimate $|a_k| \leq 3^{2^n}$ for all $k$.

Lemma 10. If $|\sum_{k=0}^{n} a_k x^{k/n}| \leq 1$, $0 < \delta \leq x \leq 1$, then
\[
|a_k| \leq \left( \frac{3}{1-\delta} \right)^{2^n}, \quad k = 0, 1, \ldots, n^2.
\]

Proof. Assume $|\sum_{k=0}^{n} a_k w^k| \leq 1$, $0 < \delta \leq w \leq 1$. Let $y = (w - \delta)/(1 - \delta)$. Then
\[
\left| \sum_{k=0}^{n^2} a_k ((1-\delta)y + \delta)^k \right| = \left| \sum_{k=0}^{n^2} b_k y^k \right| \leq 1, \quad 0 \leq y \leq 1.
\]

Then, by Lemma 9, $|b_k| \leq 3^{2^n}$. This, in turn, gives
\[
|a_k| < \left( \frac{3}{1-\delta} \right)^{2^n}.
\]

Letting $w = x^{1/n}$, the result follows.
Lemma 11. Let $p, q, n$ be integers, $p < q \leq n$, with $p, q$ relatively prime. The inequality $kq < pm_1 + qm_2 < (k+1)q$, $m_1, m_2$, $k$ positive integers, has at least $k$ solutions in $m_1, m_2$ whenever $k \leq q-1$. For $q-1 < k \leq n$, the inequality has $q-1$ solutions.

Proof. The solutions in $m_1, m_2$ of $kq < pm_1 + qm_2 < (k+1)q$ are the same as the solutions in $m_1, m_2$ of $k < m_1 p/q + m_2 < k+1$. $m_1 p/q$ is a nonintegral rational which is less than $k$ for any $m_1$ in $\{1, 2, \ldots, k\}$. Thus, letting $m_2 = 1 + \left\lceil k - m_1 p/q \right\rceil$, there is exactly one solution for $m_1 = 1, 2, \ldots, k$, hence at least $k$ solutions.

Similarly, for $k \geq q$, $m_1 p/q + 1 + \left\lceil k - m_1 p/q \right\rceil$ are solutions for $m_1 = 1, 2, \ldots, q-1$, and the lemma is proved.

III. Our main theorems are Theorem 2 and Theorem 5.

Theorem 2. If $Q$ is the set $(x, e^x)$, $0 \leq x \leq 1$, then there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{n^{1/2}} \leq \max_{f \in P_n} \min_{p \in P_n} \|f(x, e^x) - p(x, e^x)\|_{L^2} \leq \frac{c_2}{n^{3/2}},$$

i.e., $c_1/n^{3/2} \leq \rho_n \leq c_2/n^{3/2}$ in the $L^2$ norm.

Let $\alpha$ be an irrational number. $\alpha$ will be called of degree $f(n)$ if there exist $p, n$ such that $|\alpha - p/n| < 1/f(n)$ for infinitely many $n$.

Theorem 5. Let $\alpha$, $0 < \varepsilon \leq \alpha \leq 1$ be an irrational of degree $n^4(3/(1-\delta))^{2n^2}$, $0 < \delta < 1$. Let $Q$ be the set of points $(x, e^x)$, $\delta \leq x \leq 1$. Then there exists a subsequence $\rho_n$ of $\rho_n$ and constants $c_1, c_2 > 0$ such that, in the $L^2$ norm, $c_1/n^{3/2} \leq \rho_n \leq c_2/n^{3/2}$.

Proof of Theorem 2. We are approximating with linear combinations of $x^k e^{mx}$, $k, m \leq n$. By Lemma 6,

$$\rho_n = \sup_{p \in P_n} \left\| \int_0^x \varphi(t) \, dt \right\|,$$

where $\int_0^1 \varphi(x)x^k e^{mx} \, dx = 0$, $k, m \leq n$,

or, letting $te = e^x$, where

$$\int_{1/e}^1 \varphi(t)(\log t)^k t^{m-1} \, dt = 0, \quad k, m \leq n. \quad (1)$$

Let

$$F(z) = \int_{1/e}^1 \varphi(t)t^{-(iz+1/2)} \, dt.$$

Letting $t = e^{-u}$, we note that $F \in PW$. Setting $k = 0$ in (1), we get $F(m+1/2)i = 0$, $m \leq n-1$. Setting $k = 1$ in (1) and integrating by parts, we get $F'((m+1/2)i) = 0$, $m \leq n-1$. For general $k$, integration by parts $k$ times yields

$$F^{(k)}(m+1/2)i = 0, \quad m \leq n-1. \quad (2)$$

(In each case the boundary terms drop out by the orthogonality of $\varphi$ to $x^k e^{mx}$.)


Since $F(x) = \int_{-\infty}^{\infty} e^{iu} e^{-u^{2}/2} \varphi(e^{-u}) \, du$, Parseval’s Identity yields

(3) \[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x)|^2 \, dx = \int_{-\infty}^{\infty} e^{-u} |\varphi(e^{-u})|^2 \, du = \int_{0}^{1} |\varphi(t)|^2 \, dt = 1. \]

Integrating by parts,

\[ F(z) = t^{-(iz + 1/2)} \int_{0}^{1} \varphi(u) \, du + (iz + 1/2) \int_{1/e}^{t} t^{-(iz + 3/2)} \int_{0}^{t} \varphi(u) \, du \, dt. \]

But $\varphi$ is orthogonal to 1, hence

\[ F(z) = (iz + 1/2) \int_{1/e}^{1} t^{-(iz + 3/2)} \int_{0}^{t} \varphi(u) \, du \, dt. \]

Thus

\[ \frac{F(x+i)}{ix + 1/2} = \int_{1/e}^{1} t^{-(ix + 1/2)} \int_{0}^{t} \varphi(u) \, du \, dt \]

\[ = \int_{-\infty}^{\infty} e^{i\pi x - u^2/2} \int_{0}^{\infty} \varphi(u) \, du \, dv. \]

Hence, by Parseval’s Identity,

(4) \[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{x^2 + 1/4} \, dx = \int_{-\infty}^{\infty} e^{-v} \left( \int_{0}^{\infty} \varphi(u) \, du \right)^2 \, dv = \int_{0}^{1} \left( \int_{0}^{t} \varphi(u) \, du \right)^2 \, dt. \]

It follows from (2) and the general Paley-Wiener theory that

\[ G(z) = F(z) \prod_{m \geq n-1} \frac{z + i(m + 1/2)^n}{z - i(m + 1/2)^n} \]

is in $PW$. Further, since $|x + i(m + 1/2)| = |x - i(m + 1/2)|$, (3) gives

(5) \[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(x)|^2 \, dx = 1, \]

and (4) becomes

(6) \[ \int_{0}^{t} \left( \int_{0}^{t} \varphi(u) \, du \right)^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|G(x+i)|^2}{x^2 + 1/4} \prod_{m \geq n-1} \left( \frac{x^2 + (m - 1/2)^2}{x^2 + (m + 3/2)^2} \right)^n \, dx. \]

From (5), (6) and Lemma 6 we get

\[ \rho_n^2 = \sup_{G \in PW} \frac{\int_{-\infty}^{\infty} \frac{|G(x+i)|^2}{x^2 + 1/4} \prod_{m \geq n-1} \left( \frac{x^2 + (m - 1/2)^2}{x^2 + (m + 3/2)^2} \right)^n \, dx}{\int_{-\infty}^{\infty} |G(x)|^2 \, dx}. \]

But $\|G(x+i)\|_{2^2} \leq \|G(x)\|_{2^2}$. Hence

\[ \rho_n^2 \leq \max_x \frac{1}{x^2 + 1/4} \prod_{m \geq n-1} \left( \frac{x^2 + (m - 1/2)^2}{x^2 + (m + 3/2)^2} \right)^n \]

\[ = \max_x \frac{(x^2 + 1/4)^{2n-1}}{(x^2 + (n - 1/2)^2)^n(x^2 + (n + 3/2)^2)^n}. \]
We note that the function \( x = x(n) \) at which the max is taken on must go to infinity with \( n \), for, if not, the upper bound thus obtained for \( \rho_n \) would grossly violate the general lower bound theorem. This being the case,

\[
\rho_n < c \max_x \frac{x^{2n-1}}{(x^2 + n^2)^n}.
\]

The max occurs at \( x^2 = 2n^3 - n^2 \), and the maximum value is \( c(2n^3 - n^2)^{-1/2}(1 - 1/2n)^n \), which tends to \( c/(2e)^{1/2}n^{3/2} \), giving the desired upper bound.

Choosing \( G(z) = 1/(z + in^{3/2}) \) we get from (7) that

\[
\int_{-\infty}^{\infty} \frac{x^2}{x^2 + n^3} dx
\]

\[
\rho_n \geq \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{x^2 + n^3} dx
\]

\[
\int_{-\infty}^{\infty} \frac{c^*}{x^2 + n^3} dx = \frac{c^*}{n^3}
\]

Q.E.D.

Our interest will now, temporarily, be turned from linear combinations of \( x^k, k = 0, 1, \ldots, n \) as approximating functions to approximation with linear combinations of \( x^{k/q}, k = 0, 1, 2, \ldots, nq \), on the interval \( Q_1 \). We refer to the latter space and the corresponding degree of approximation as \( P_{nq}^{*} \) and \( \rho_{nq}^{*} \).

Since \( P_n \subseteq P_{nq} \), it follows from Jackson’s theorem that there is a \( c_1 \) such that \( \rho_{nq}^{*} \leq c_1/nq \). By the general lower bound theorem, there is a \( c_2 \) such that \( \rho_{nq}^{*} \geq c_2/n \). The following result, in addition to being of independent interest, will be used in the proof of Theorem 4.

**Theorem 3.** There exist constants \( c_1, c_2 > 0 \) such that \( c_1/nq^{1/2} \leq \rho_{nq}^{*} \leq c_2/nq^{1/2} \).

**Proof.** By Lemma 7,

\[
\rho_{nq}^{*} = \sup_{F \in \mathcal{W}} \frac{\int_{-\infty}^{\infty} \left| F(x+i) \right|^2 dx}{\left( x^2 + 1/4 \right)^{1/4} \prod_{k=0}^{nq} \left( x^2 + (k/q - 1/2)^2 \right) dx},
\]

while, by Lemma 8,

\[
\rho_{nq}^{*} \leq M(\Lambda), \quad \text{where } \Lambda = \{ k/q \mid k = 0, 1, \ldots, nq \}.
\]

But, by cancellation,

\[
M(\Lambda) = \max_x \left( x^2 + 1/4 \right)^{-1/2} \prod_{k=0}^{2q-1} \left( x^2 + (k/q - 1/2)^2 \right) \prod_{j=nq-2q+1}^{\infty} \left( x^2 + (j/q + 3/2)^2 \right)^{-1}.
\]

By (9) and by the general lower bound theorem, \( M \) cannot fall off faster than \( c/n^4 \). The point \( x = x(n, q) \) where the max is taken on must go to infinity with \( n \) or \( q \),
for, if not, (10) would yield $M = O(1/n^2q)$. We now have, from (10), that there exist positive constants $k_1, k_2$ such that

$$k_1 M \leq \frac{1}{x^2} \left( \frac{x^2}{x^2 + n^2} \right)^{2q} \leq k_2 M.$$  

The max in (11) is determined to be at $x^2 = 2qn^2 - n^2$, yielding $M = O(1/n^2q)$. Thus, from (9) we have

$$\rho_{nq}^* = O(1/nq^{1/2}).$$  

Choosing $F(z) = 1/(z + inq^{1/2})$, we have, from (8) and (11),

$$\rho_{nq}^* \geq \frac{nq^{1/2}}{\pi} \int_{nq^{1/2}}^{2nq^{1/2}} c \left( \frac{x^2}{x^2 + n^2} \right)^{2q} dx \int_{-\infty}^{\infty} \frac{dx}{x^2 + n^2q} \geq \frac{c}{nq^{1/2}} \left( 1 + 1/q \right)^{-2q} \int_{-\infty}^{\infty} \frac{dx}{x^2 + n^2q} \geq \frac{c}{nq^{1/2}}$$

which, together with (12), proves the theorem.

Let $Q$ be the curve $y = x^{p/q}$, $p, q$ relatively prime, $\delta \leq x \leq 1$. Consider the family of all such $Q$. Since these curves are algebraic, Theorem 1 implies $c_1/n \leq \rho_n \leq c_2/n$. However, for each $Q$ in the family, the constants, $c_1, c_2$, although independent of $n$, might well depend on which particular curve from the family is under consideration, i.e., $c_1$ and/or $c_2$ may be nontrivial functions of $p$ and/or $q$. We shall now concern ourselves with more precise estimates for $\rho_n$—estimates that will yield the order of magnitude as a function of which $Q$ from the family is under consideration. The following result in this direction will be crucial to the proof of Theorem 5.

**Theorem 4.** Let $Q$ be the set $(x, x^{p/q})$, $\delta \leq x \leq 1$, $p, q$ relatively prime. Then there exist $c_1, c_2 > 0$ such that, in the $L^2$ norm, $c_1/nq^{1/2} \leq \rho_n \leq c_2/nq^{1/2}$ for $n, p, q$ such that $n \geq q$.

**Proof.** We note that for $y = x^{p/q}$, approximating with $n$th degree polynomials in $x$ and $y$ is the same as approximating with $x^i + jy^j$, $0 \leq i, j \leq n$.

Let $f^*(x) = f(x, x^{p/q})$ be defined for $\delta \leq x \leq 1$. Now

$$\|f(x, x^{p/q}) - p_n(x, x^{p/q})\| = \|f^*(x) - \sum a_{ij}x^i + jy^j\|.$$

Also, the slope of $f^*(x)$,

$$\frac{\|f^*(y) - f^*(x)\|}{d(x, y)} = \frac{d((x, x^{p/q}), (y, y^{p/q}))}{d((x, x^{p/q}), (y, y^{p/q}))} \leq (1 + m^2)^{1/2},$$

where $m$ is the maximum slope for $y = x^{p/q}$ on $[\delta, 1]$. Since $m \leq 1/\delta$, we have
\[ \| f^*(y) - f^*(x) \| \leq M |y - x|, \quad M = (1 + 1/\delta^2)^{1/2}. \] Thus \( f^*(x)/M \) is in \( K(\delta, 1) \). Hence the assertion of our theorem is equivalent to the existence of \( c_1, c_2 \) such that

\[ \tag{13} \frac{c_1}{nq^{1/2}} \leq \max_{f \in K(\delta, 1)} \min_{(a_i)} \left\| f(x) - \sum_{i=0}^{n} a_i x_i^{i+p/q} \right\| \leq \frac{c_2}{nq^{1/2}}. \]

First, the left inequality in (13) will be established. Define

\[ \rho^*_{nq}(a, b) = \max_{f \in K(a, b)} \min_{(a_i)} \left\| f(x) - \sum_{i=0}^{nq} a_i x_i^{i/q} \right\|. \]

**Lemma 12.** \( \rho^*_{2nq}(\delta, 1) \geq (1 - \delta) \rho^*_{2nq}(0, 1) \).

**Proof.** With respect to \( \rho^*_{2nq}(0, 1) \), consider the coefficients \( \{b_k\} \) for the function, \( g \), that maximizes. Then

\[ \left\| g(x) - \sum_{i=0}^{nq} b_k x_i^{i/q} \right\| = \rho^*_{2nq}(0, 1). \]

Let \( f(x) = (1 - \delta) g((x - \delta)/(1 - \delta)) \). \( f(x) \) is in \( K(\delta, 1) \). Now

\[ \sum_{i=0}^{nq} a_k x_i^{i/q} = \sum (1 - \delta) b_k ((x - \delta)/(1 - \delta))^{k/q} \]

is the best approximating polynomial to \( f(x) \) on \([\delta, 1]\). For, suppose the contrary, i.e.,

\[ \left\| \sum c_k x_i^{i/q} - f(x) \right\| < \left\| \sum (1 - \delta) b_k ((x - \delta)/(1 - \delta))^{k/q} - f(x) \right\| \]

on \([\delta, 1]\). Then, on \([0, 1]\),

\[ \left\| \sum c_k ((1 - \delta)x + \delta)^i - f((1 - \delta)x + \delta) \right\| \leq \left\| \sum (1 - \delta) b_k x_i^{i/q} - f((1 - \delta)x + \delta) \right\| \leq \left\| \sum (1 - \delta) b_k x_i^{i/q} - g(x) \right\| \]

contrary to the assumption that the \( b_k \) minimized for \( g \). Hence,

\[ \rho^*_{2nq}(\delta, 1) \geq \left\| \sum (1 - \delta) b_k ((x - \delta)/(1 - \delta))^{k/q} - f(x) \right\| \quad \text{on } [\delta, 1] \]

\[ = (1 - \delta) \left\| \sum b_k ((x - \delta)/(1 - \delta))^{k/q} - g((x - \delta)/(1 - \delta)) \right\| \quad \text{on } [\delta, 1] \]

\[ = (1 - \delta) \left\| \sum b_k x_i^{i/q} - g(x) \right\| \quad \text{on } [0, 1] \]

\[ = (1 - \delta) \rho^*_{2nq}(0, 1), \]

and Lemma 12 is established. Now,

\[ \max_{f \in K(\delta, 1)} \min_{(a_i)} \left\| f(x) - \sum a_i x_i^{i+p/q} \right\| \geq \rho^*_{2nq}(\delta, 1) \geq (1 - \delta) \rho^*_{2nq}(0, 1) \geq \frac{c}{nq^{1/2}}, \]

the last inequality following from Theorem 3.

We now prove the upper bound in (13). Let \( f \) be extended, as a contraction, to \([0, 1]\). Lemma 8 now reduces the proof of the desired inequality to showing that

\[ M(\Lambda) < c/n^2q, \quad \text{where } \Lambda = \{i+jp/q, 0 \leq i, j \leq n\}. \]
Let $x = x(n, q)$ be the point $x$ at which the max in $M(\Lambda)$ is taken on. As in the proof of Theorem 2, $x(n, q)$, for fixed $q$, must tend to infinity with $n$. Observe that \((x^2 + (\lambda - 1/2)^2)/(x^2 + (\lambda + 3/2)^2)\) is an increasing function of $\lambda$ for $\lambda > 4(1 + x^2)^{1/2} - 2$, and decreasing for $\lambda$ less than this quantity. Let $q$ be fixed. We consider two possible cases:

**Case 1.** $1/x(n, q) = O(1/n)$.

Then there is some $c$ such that \((x^2 + (\lambda - 1/2)^2)/(x^2 + (\lambda + 3/2)^2)\) is decreasing for all $\lambda < cn$. Let $\Lambda_0$ be $\{\lambda \in \Lambda \mid \lambda < cn\}$. $M(\Lambda_0) \leq M(\Lambda)$. The facts that

\[
\frac{(x^2 + (\lambda - 1/2)^2)/(x^2 + (\lambda + 3/2)^2)}{\text{is decreasing for } \lambda < cn}
\]

that this quotient is a decreasing function of $\lambda$, and that, by Lemma 11, there are at least $\min(k, q-1)$ members of $\Lambda_0$ between $k$ and $k+1$ for $k + 1 < cn$ imply that $M(\Lambda_1) > M(\Lambda_0)$, where

\[
\Lambda_1 = \{i+j/q, 0 \leq i \leq \lfloor cn \rfloor, 0 \leq j \leq \min(i, q)\}.
\]

That is,

\[
\begin{align*}
1, & \quad 1 + 1/q, \\
2, & \quad 2 + 1/q, \quad 2 + 2/q, \\
3, & \quad 3 + 1/q, \quad 3 + 2/q, \quad 3 + 3/q, \\
4, & \quad 4 + 1/q, \quad 4 + 2/q, \quad 4 + 3/q, \\
& \vdots \\
q-1, & \quad q-1 + 1/q, \quad q-1 + 2/q, \quad \ldots, \quad q-1 + (q-1)/q, \\
q, & \quad q + 1/q, \quad q + 2/q, \quad \ldots, \quad q + (q-1)/q, \\
& \vdots \\
\lfloor cn-1 \rfloor, & \quad \lfloor cn-1 \rfloor + 1/q, \quad \lfloor cn-1 \rfloor + 2/q, \quad \ldots, \quad \lfloor cn-1 \rfloor + (q-1)/q.
\end{align*}
\]

By the cancellation of the terms in the product (all but the first two and last two entries in each column cancel) we have

\[
M(\Lambda_1) = \max_x \frac{1}{x^2 + \frac{1}{4}} \prod_{k=0}^{q} \frac{x^2 + (k + k/q - 1/2)^2}{x^2 + ([cn] + k/q + 1/2)^2} \frac{x^2 + (k + k/q + 1/2)^2}{x^2 + ([cn] + k/q + 3/2)^2}
\]

which is of the order of

\[
\max_x \frac{1}{x^2} \left( \prod \left(1 - \frac{c^2n^2 - k^2}{x^2 + c^2n^2} \right) \right) < \max_x \frac{1}{x^2} \exp \left(-2 \sum_{k=0}^{q} \frac{c^2n^2 - k^2}{x^2 + c^2n^2} \right),
\]

or

\[
\max_x \frac{1}{x^2} \exp \left(- \frac{2c^2n^2 - q^2/3}{x^2 + n^2} \right).
\]

But the maximum (away from the origin) is taken at $x^2 = n^2(2cq^2 - 1) - q^2/3$, where we get $M(\Lambda_1) = O(1/qn^2)$, and the estimate is established for Case 1.
Case II. We now suppose that $1/x(n,q)$ is not $=O(1/n)$. Then there is a subsequence of \( \{n\} \) for which $x(n,q) = O(n)$. Let \( \Lambda_2 = \{ \lambda \in \Lambda \mid \lambda > [4x] \} \). \( M(\Lambda_2) > M(\Lambda) \). Now, that $\frac{(x^2 + (\lambda - 1/2)^2)(x^2 + (\lambda + 3/2)^2)}{(x^2 + (\lambda + 1/2)^2)(x^2 + (\lambda - 1/2)^2)}$ is an increasing function of $\lambda$ bounded by 1 implies, in view of Lemma 11, that $M(\Lambda_3) > M(\Lambda_2)$, where
\[
\Lambda_3 = \{ i - j/q \mid [4x] \leq i \leq n, 0 \leq j \leq \min (i, q) \}.
\]
That is,
\[
\Lambda_3 = \{ [4x] - \min ([4x], q - 1)/q, \ldots, [4x] - 2/q, [4x] - 1/q, [4x],
[4x] + 1 - \min ([4x] + 1, q - 1)/q, \ldots, [4x] + 1 - 2/q, [4x] + 1 - 1/q, [4x] + 1,
[4x] + 2 - \min ([4x] + 2, q - 1)/q, \ldots, [4x] + 2 - 2/q, [4x] + 2 - 1/q, [4x] + 2,
\vdots
n - (q - 1)/q, \ldots, n - 2/q, n - 1/q, n \}.
\]
By cancellation of terms in the product, we have
\[
M(\Lambda_3) \leq \max_x \frac{1}{x^2 + \frac{1}{4}} \prod_{k=0}^{\min([4x],q)} \frac{x^2 + ([4x] - k/q - 1/2)^2}{x^2 + ([4x] - k/q - 3/2)^2}.
\]
Thus, for the subsequence of \( n \) for which $x = O(n)$, we have
\[
M(\Lambda_3) = O(cx^2/n^2)^\alpha, \quad \alpha = \min ([4x], q).
\]
Hence, for these values of \( n \), $M(\Lambda) < M(\Lambda_3) < 1/n^5$, which is impossible by the lower bound theorem. Thus Case II cannot occur and the theorem is established.

We are now in a position where we can give a proof of Theorem 5: Consider all \( n \) for which there exist \( p \) such that $|\alpha - p/n| < 1/n^2(3/(1 - \delta))^2n^2$. This is a subsequence \( n_1 \) of \( n \), and defines our subsequence \( p_{n_1} \) of \( p_n \). For convenience of notation, the subscripts will be dropped, i.e., \( n_1 \) will be referred to as \( n \). It will be shown that
\[
\varepsilon c_1/2n^{3/2} \leq p_n \leq 2c_2/en^{3/2}.
\]
(Here \( c_1 \) and \( c_2 \) are the same constants as those in Theorem 4.)

Extend $f(x, y)$, now defined as a contraction on $y = x^a$, to be a contraction on the square $0 \leq x, y \leq 1$. That this can be done follows, again, from the corollary to Lemma 3. Consider the curve $y = x^{p/n}$. Then
\[
|x^a - x^{p/n}| < 1/en^4(3/(1 - \delta))^2n^2.
\]
By Theorem 4, there exists $p_n(x, y)$ such that
\[
\|p_n(x, x^{p/n}) - f(x, x^{p/n})\| \leq c_2/n^{3/2}.
\]
Now,
\[
\|p_n(x, x^a) - f(x, x^a)\| \leq \|p_n(x, x^a) - p_n(x, x^{p/n})\| + \|p_n(x, x^{p/n}) - f(x, x^{p/n})\| + \|f(x, x^{p/n}) - f(x, x^a)\|.
\]
Since $f$ is a contraction, it follows from (15) that
\[(18) \| f(x, x^{p/n}) - f(x, x^p) \| \leq \frac{1}{en^4(3/(1 - \delta))^{2n^2}}.\]

Now,
\[
\| p_n(x, x^p) - p_n(x, x^{p/n}) \| \leq \max_{\delta \leq x \leq 1} | p_n(x, x^p) - p_n(x, x^{p/n}) | \\
\leq \max_{\delta \leq x \leq 1} \sum_{k=1}^{n} \left( \frac{d^k}{dy^k} p_n(x, y) \right)_{y = x^{p/n}} | x^p - x^{p/n} |^k,
\]
by Taylor's Theorem. But this is
\[
< \frac{1}{en^4(3/(1 - \delta))^{2n^2}} \sum_{k=1}^{n} \left( \frac{d^k}{dy^k} p_n(x, y) \right)_{y = x^{p/n}}
\]
by (15). But, by Lemma 10, the coefficients of $p_n(x, x^{p/n})$ are bounded by $(3/(1 - \delta))^{2n^2}$. Hence, we have
\[(19) \| p_n(x, x^p) - p_n(x, x^{p/n}) \| < \frac{1}{en^4}.\]

Substitution of (16), (18), and (19) in (17) yields the upper bound asserted in (14).

To establish the lower bound, we note that
\[
\| p_n(x, x^p) - f(x, x^p) \| + \| p_n(x, x^p) - p_n(x, x^{p/n}) \| + \| f(x, x^{p/n}) - f(x, x^p) \| \\
\geq \| p_n(x, x^{p/n}) - f(x, x^{p/n}) \|,
\]
and hence that by (18), (19) and Theorem 4,
\[
\| p_n(x, x^p) - f(x, x^p) \| \geq \frac{c_1}{n^{3/2}} - \frac{1}{n^4} - \frac{1}{n^6} \geq \frac{c_1}{2n^{3/2}}. \text{ Q.E.D.}
\]

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