L^p ESTIMATES FOR INTEGRAL TRANSFORMS

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1. **Introduction.** The following situation is often encountered in real analysis: One is given two σ-finite measure spaces (X, Σ, μ) and (Y, Σ', ν) and a "kernel" $K(x, y)$, a jointly measurable function on $X \times Y$, and one considers the transform $Tf(x) = \int K(x, y) f(y)\, dy$ (for simplicity we write dy and dx for $d\nu(y)$ and $d\mu(x)$) and tries to prove $\left(\int |Tf(x)|^q\, dx\right)^{1/q} \leqq M_{p,q}\left(\int |f(x)|^p\, dx\right)^{1/p}$ for all $f \in L^p$, for certain values of p and q. We call this a (p, q) estimate for K.

We single out three classes of such results. First there are what we may call "elementary" results which give conditions on K which only depend on its absolute value and which hold for a set of p's which is closed. Among these results are the Young inequality for convolutions,

$$\|f * g\|_q \leqq \|f\|_p \|g\|_r, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \quad 1 \leqq p \leqq r'$$

and the following theorem of Hardy-Littlewood-Pólya [10]:

THEOREM A. *Let $X = Y = (0, \infty)$ with Lebesgue measure and suppose K is homogeneous of degree -1, i.e. $K(\alpha x, \alpha y) = \alpha^{-1} K(x, y)$ for all $\alpha > 0$. Then $\|Tf\|_p \leqq M \|f\|_p$ with $M = \int_0^\infty |K(1, y)| y^{-1/p}\, dy$ provided this integral is finite. If K is positive this condition is also necessary.*

The methods generally used to prove "elementary" theorems are Minkowski's inequality, Hölder's inequality, and the Riesz interpolation theorem.

The second class of results also give conditions on K which depend only on its absolute value, but the conclusion is no longer valid for a closed set of p's. An example of such a result is the Hardy-Littlewood-Sobolev fractional integration theorem [21]:

THEOREM B. *Let $X = Y = R^n$, Euclidean n-space with Lebesgue measure, and $K(x, y) = |x - y|^{-n/r}$, $1 < r < \infty$. Then $\|Tf\|_q \leqq M_p \|f\|_p$ where $1/q = 1/p + 1/r - 1$, $1 < p < r'$. In fact, the same conclusion holds if $K(x, y) = \varphi(x - y)$ where $\varphi \in L^{r,\infty}$, i.e. measure $\{x : |\varphi(x)| > \alpha\} \leqq M/\alpha^r$.*

These results, which we will call "positive kernel" theorems, are generally proved using a splitting of functions according to the values they assume, and the Marcinkiewicz interpolation theorem.

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Finally we have the deepest class of results in which the kernel is assumed to have some cancellation property without which the conclusion is evidently false. These are the singular integral theorems of Calderón-Zygmund and their generalizations. The proofs involve splitting functions according to the geometry of their domain.

The purpose of this paper is to extend the above-mentioned theorems. In §2 we extend Theorem A to n-dimensions. In §3 we give a theorem in pure functional analysis (no algebraic or geometric properties of the measure spaces are assumed) which contains Theorem B and also allows the conclusion $\|Tf\|_p \leqq M\|f\|_p$ in some cases. In §4 we give a new way of obtaining variable kernel singular integrals.

Next we apply these results to the problem of convolutions in weighted L^p norms. Let $X=Y$ be a locally compact abelian group with Haar measure (usually R^n with Lebesgue measure), and let $W(x)$, $V(y)$ be nonnegative functions which are finite and nonzero almost everywhere. We wish to find sufficient conditions on h to have $\left(\int |h*f(y)|^q V(y)^q\,dy\right)^{1/q} \leqq \left(\int |f(x)|^p W(x)^p\,dx\right)^{1/p}$. This is easily seen to reduce to a (p, q) estimate for the kernel $K(x, y)=W(x)^{-1}h(x-y)V(y)$.

In §5 we establish an "elementary" result analogous to the Young inequality for convolutions. In §6 we discuss the fractional integration kernel $h(x)=|x|^{-n/r}$ for the weights $|x|^\alpha$. This problem has already been solved by Stein and Weiss [18]. The proof is long and "positive kernel" in flavor and omits the case $p=1$. Our proof is short and "elementary", except for one use of Theorem B in the case $1/q=1/p+1/r-1$. In §7 we consider singular integrals for the weights $|x|^\alpha$. Stein [16] has considered the case of kernels which are bounded on the unit sphere. We discuss what happens when this restriction is removed.

Finally in §8 we discuss the question: to what extent is Theorem B best possible?

Similar problems have been considered by Krée et al. (see [2], [15] and the bibliography there), apparently using different techniques.

We wish to thank Professor Stein for some helpful conversations.

We will use without special reference some standard theorems to be found in [8] and [21].

2. "Elementary" results.

LEMMA 1. *Suppose $T_r f=\int |K(x, y)|^r f(y)\,dy$ satisfies a $(q/r, q/r)$ estimate for some q, r, $1\leqq r\leqq q\leqq\infty$, $r<\infty$. Then $Tf=\int K(x, y)f(y)\,dy$ satisfies a (p, q) estimate for $1/q=1/p+1/r-1$.*

Proof. Suppose first $q<\infty$ and $\|T_r f\|_{q/r}\leqq M_0\|f\|_{q/r}$. Note $p/q+p/r'=1$, hence

$$\left|\int K(x, y)f(y)\,dy\right| \leqq \int |K(x, y)|\,|f(y)|^{p/q}|f(y)|^{p/r'}\,dy$$

$$\leqq \left(\int |K(x, y)|^r|f(y)|^{pr/q}\,dy\right)^{1/r}\left(\int |f(y)|^p\,dy\right)^{1/r'}$$

by Hölder's inequality. Now $|f(y)|^{pr/q} \in L^{q/r}$ hence

$$\int \left(\int |K(x, y)|^r |f(y)|^{pr/q}\, dy \right)^{q/r} dx \leqq M_0^{q/r} \left(\int |f(y)|^p\, dy \right).$$

Thus

$$\int |Tf(x)|^q\, dx \leqq M_0^{q/r} \left(\int |f(y)|^p\, dy \right)^{1+q/r'} \quad \text{and} \quad \|Tf\|_q \leqq M_0^{1/r} \|f\|_p.$$

If $q=\infty$ then $p=r'$ and thus

$$\sup_x \left| \int K(x, y) f(y)\, dy \right| \leqq \sup_x \left(\int |K(x, y)|^r\, dy \right)^{1/r} \|f\|_p$$
$$= \|f\|_p \sup (T_r 1(x))^{1/r} \leqq M_0^{1/r} \|f\|_p,$$

where 1 is the function identically one.

LEMMA 2. *Suppose* $M_0 = \left(\int \left(\int |K(x, y)|^{p'}\, dy \right)^{q/p'} dx \right)^{1/q} < \infty$. *Then* $\|Tf\|_q \leqq M_0 \|f\|_p$. *Similarly if* $M_1 = \left(\int \left(\int |K(x, y)^q\, dx \right)^{p'/q} dy \right)^{1/p} < \infty$ *then* $\|Tf\|_q \leqq M_1 \|f\|_p$. *These results hold for* $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ *if we replace the appropriate integral by an essential supremum.*

Proof. Note that $M_0 \leqq M_1$ if $q \geqq p'$ and $M_1 \leqq M_0$ if $p' \geqq q$, by applying Minkowski's inequality for the q/p' or p'/q norm.

To prove the first half of the lemma we apply Hölder's inequality

$$|Tf(x)| = \left| \int K(x, y) f(y)\, dx \right| \leqq \left(\int |K(x, y)|^{p'}\, dy \right)^{1/p'} \|f\|_p.$$

To prove the second half of the lemma we apply Minkowski's inequality

$$\left\| \int K(x, y) f(y)\, dy \right\|_q \leqq \int \left(\int |K(x, y)|^q\, dx \right)^{1/q} |f(y)|\, dy$$

and then Hölder's inequality.

The two halves of the lemma are dual and each may be deduced (for $1 < p$, $q < \infty$) by applying the other to the adjoining operator

$$\int \overline{K(y, x)} \varphi(x)\, dx.$$

We come now to the full generalization of Theorem A to n-dimensions.

THEOREM 1. *Let X and Y be open cones in R^n (here a cone is just a subset closed under dilations). Let $r = |x|$, $x' = x/r$, $s = |y|$, $y' = y/s$, $X' = X \cap S^{n-1}$, and $Y' = Y \cap S^{n-1}$, where S^{n-1} is the unit sphere in R^n. Let dx be Lebesgue measure on R^n restricted to X, and dx' be Lebesgue measure on S^{n-1} restricted to X', and similarly for dy and dy'. As usual we consider $Tf(x) = \int K(x, y) f(y)\, dy$. Suppose K is homogeneous of degree $-n$; i.e., $K(\lambda x, \lambda y) = \lambda^{-n} K(x, y)$ for all $\lambda > 0$. Let $1 \leqq p \leqq \infty$.*

Denote by K_s' the kernel $K(x', sy')$ on $X' \times Y'$. If J' is any kernel on $X' \times Y'$ denote by $\|J'\|_{p,p}$ the norm of the associated operator $\int J'(x', y')\varphi(y')\,dy'$ from $L^p(Y') \to L^p(X')$. Then a sufficient condition for $\|Tf\|_p \leqq M\|f\|_p$ to hold is that

$$\int_0^\infty \|K_s'\|_{p,p} s^{n/p'-1}\,ds \leqq M. \tag{1}$$

If K is positive then the condition

$$\left\| \int_0^\infty K_s'(x', y') s^{n/p'-1}\,ds \right\|_{p,p} \leqq M \tag{2}$$

is necessary.

Proof. We have $Tf(rx') = \int K(rx', ry) r^n f(ry)\,dy = \int K(x', y) f(ry)\,dy$ by the homogeneity of K. Now

$$\int \left| \int K(x', sy') f(rsy')\,dy' \right|^p dx' \leqq \|K_s\|_{p,p}^p \int |f(rsy')|^p\,dy'$$

hence

$$\begin{aligned} \iint \left| \int K(x', sy') f(rsy')\,dy' \right|^p r^{n-1}\,dr\,dx' &\leqq \|K_s\|_{p,p}^p \iint |f(rsy')|^p\,dy' r^{n-1}\,dr \\ &= \|K_s\|_{p,p}^p \|f\|_p^p s^{-n}. \end{aligned}$$

Thus

$$\begin{aligned} \|Tf(x)\|_p &= \left(\int \left| \int K(x', sy') f(rsy')\,dy' s^{n-1}\,ds \right|^p dx \right)^{1/p} \\ &\leqq \int \left(\int \left| \int K(x', sy') f(rsy')\,dy' \right|^p dx \right)^{1/p} s^{n-1}\,ds \\ &\leqq \int \|K_s\|_{p,p} s^{-n/p} \|f\|_p s^{n-1}\,ds, \end{aligned}$$

by Minkowski's inequality. This proves the first assertion. For the second assertion we consider the function $f(y) = \chi(s) s^{-n/p+\varepsilon} \varphi(y')$, where χ is the characteristic function of $(0, a)$, $\varepsilon > 0$, and φ is an arbitrary positive function in $L^p(Y')$. We compute $\|f\|_p = (\varepsilon p)^{-1/p} a^\varepsilon \|\varphi\|_p$. Now

$$\begin{aligned} \|Tf\|_p^p &\geqq \int_{|x| \leqq 1} |Tf(x)|^p\,dx \\ &= \int_0^1 \int_{X'} \left| \int_0^{a/r} \int_{Y'} K(x', sy') s^{-n/p+\varepsilon+n-1} \varphi(y')\,dy'\,ds \right|^p r^{\varepsilon p-1}\,dx'\,dr \\ &\geqq \int_0^1 \int_{X'} \left| \int_0^a \int_{Y'} K(x', sy') s^{n/p'-1+\varepsilon} \varphi(y')\,dy'\,ds \right|^p r^{\varepsilon p-1}\,dx'\,dr \\ &= (1/\varepsilon p) \int_{X'} \left| \int_{Y'} \left[\int_0^a K(x', sy') s^{n/p'-1+\varepsilon}\,ds \right] \varphi(y')\,dy' \right|^p dx'. \end{aligned}$$

Suppose we had $\|Tf\|_p \leqq M\|f\|_p$. Then we would have

$$\int_{X'} \left| \int_{Y'} \left[\int_0^a K(x', sy') s^{n/p'-1+\varepsilon}\,ds \right] \varphi(y')\,dy' \right|^p dx' \leqq a^{\varepsilon p} M^p \|\varphi\|_p^p.$$

We first let $\varepsilon \to 0$ and apply the monotone convergence theorem for $s \leqq 1$ and the dominated convergence theorem for $s > 1$ to obtain the above expression with $\varepsilon = 0$. Note that the right side is then independent of a, so we may let $a \to \infty$ and apply the monotone convergence theorem to obtain

$$\int_{X'} \left| \int_{Y'} \left[\int_0^\infty K_s(x', y') s^{n/p'-1}\, ds \right] \varphi(y')\, dy' \right|^p dx' \leqq M^p \|\varphi\|_p^p.$$

Taking the supremum over φ with $\|\varphi\|_p \leqq 1$ we obtain the desired result.

The proof must be modified in a routine manner for the case $p = \infty$.

We may combine Lemma 1 with the theorem to obtain

COROLLARY 1. *Let X and Y be as above, and suppose K is homogeneous of degree $-n/r$, for some r, $1 \leqq r < \infty$. Suppose $\int \| |K_s|^r \|_{q/r, q/r} s^{nr/p'-1}\, ds = M^r < \infty$. Then $\|Tf\|_q \leqq M \|f\|_p$.*

For the applications it is desirable to have some specific consequences of Theorem 1. In what follows we take $X = Y = R^n$. It is convenient to introduce the group of rotations $SO(n)$. If we pick a "north pole" $x_0 \in S^{n-1}$ then every function f on R^n can be lifted to a function $\tilde{f}$ on $(0, \infty) \times SO(n)$ by the rule $\tilde{f}(r, R) = f(rR(x_0))$. In this way we get all functions on $(0, \infty) \times SO(n)$ which have the invariance property $\tilde{f}(r, RR_0) = \tilde{f}(r, R)$ for every $R_0 \in SO(n)$ which satisfies $R_0(x_0) = x_0$. If dR is the Haar measure on $SO(n)$ we have

$$\int_{R^n} f(x)\, dx = \int_0^\infty \int_{SO(n)} \tilde{f}(r, R) r^{n-1}\, dR\, dr.$$

If we now define a kernel $\tilde{K}_y$ on $SO(n) \times SO(n)$ by $\tilde{K}_y(R, S) = K(R(x_0), S(y'))$ we have $\|K_s'\|_{p,p} = \|\tilde{K}_y\|_{p,p}$, where this last expression is the norm of

$$\int_{SO(n)} \tilde{K}_y(R, S) \varphi(S)\, dS$$

as an operator on $L^p(SO(n))$. Thus (1) becomes $\int_{R^n} \|\tilde{K}_y\|_{p,p} |y|^{-n/p}\, dy \leqq M$ and (2) becomes $\|\int_{R^n} \tilde{K}_y(R, S) |y|^{-n/p}\|_{p,p} \leqq M$. In the case of spherical convolution these two coincide:

COROLLARY 2. *Let $K(x, y)$ be a kernel on $R^n \times R^n$ which is homogeneous of degree $-n$ and such that $K(x, y)$ depends only on $|x|$, $|y|$, and $|x-y|$. Then*

$$\int_{R^n} |K(x_0, y)|\, |y|^{-n/p}\, dy \leqq M < \infty \tag{3}$$

is sufficient for $\|Tf\|_p \leqq M \|f\|_p$. If K is positive then (3) *is also necessary.*

Proof. That $K(x, y)$ depends only on $|x|$, $|y|$ and $|x-y|$ is equivalent to the fact that $\tilde{K}_y(RT, ST) = \tilde{K}_y(R, S)$. Thus $\int_{SO(n)} \tilde{K}_y(R, S) \varphi(S)\, dS = \int_{SO(n)} \tilde{K}_y(I, SR^{-1}) \varphi(S)\, dS = \int_{SO(n)} \tilde{K}_y(I, S) \varphi(SR)\, dS$, a convolution on $SO(n)$. Thus

$$\|\tilde{K}_y\|_{p,p} \leqq \int_{SO(n)} |\tilde{K}_y(I, S)|\, dS = \int_{SO(n)} |K(X_0, S(y))|\, dS$$

whence we deduce (3) is sufficient. If K is positive then $\int \tilde{K}_y(R, S)|y|^{-n/p}\,dy$ is a positive convolution kernel hence its operator norm is exactly

$$\iint \tilde{K}_y(T,S)|\ y|^{-n/p}\,dy\,dS = \int K(x_0, y)|y|^{-n/p}\,dy.$$

We may use the proof of Corollary 2 to obtain a characterization of all positive, bounded, linear operators on $L^p(R^n)$ that commute with dilation and rotation:

COROLLARY 2′. *Every bounded linear operator $U: L^p(R^n) \to L^p(R^n)$ which is positive and commutes with rotations and dilations has the form $Uf(x) = \int_{R^n} f(y)\,d\mu_x(y)$, where μ_x is a family of positive measures depending measurably on $x \neq 0$ and satisfying*

(i) *$\mu_{\lambda x}(\lambda^{-1}E) = \mu_x(E)$ for every $\lambda > 0$, and every measurable set E.*

(ii) *$\mu_{Rx}(R^{-1}E) = \mu_x(E)$ for every $R \in SO(n)$ and every measurable set E.*

(iii) *$\int_{R^n} |y|^{-n/p}\,d\mu_{x_0}(y) = M < \infty$ for some $x_0 \in S^{n-1}$. The norm of the operator is then exactly M.*

Proof. A slight modification of the proof of Corollary 2 shows that every such μ_x defines an operator with the desired properties. For the converse we must regularize U, applying Corollary 2 and extract a weak* limit. This is a routine but tedious argument, so we omit the details.

Finally, we may combine Lemma 2 with the theorem to obtain

COROLLARY 3. *Let K be a kernel on $R^n \times R^n$ which is homogeneous of degree $-n$ and such that either*

$$\int_{R^n} |y|^{-n/p}\left(\int_{SO(n)} \left[\int_{SO(n)} |K(R(x_0), S(y))|^{p'}\,dS\right]^{p/p'} dR\right)^{1/p} \leqq M < \infty, \tag{4}$$

or

$$\int_{R^n} |y|^{-n/p}\left(\int_{SO(n)} \left[\int_{SO(n)} |K(R(x_0), S(y))|^{p}\,dR\right]^{p'/p} dS\right)^{1/p'} \leqq M < \infty. \tag{5}$$

Then $\|Tf\|_p \leqq M\|f\|_p$.

3. **A "positive kernel" result.** Lemma 2 gives sufficient conditions for an operator to satisfy a (p, q) estimate in terms of a mixed-L^p (in the language of [3]) estimate on the kernel. If we have two such estimates holding (usually one involving M_0 and one involving M_1) then we may use the Riesz interpolation theorem to obtain new results. This is in fact given as an exercise in Dunford-Schwartz [8, Volume I, p. 530]. In the next theorem we give a stronger version of this result in which the mixed-L^p estimates on the kernel are replaced by what we may call mixed-weak-type-L^p estimates. The conclusions are the same for interior (p, q) points, but not for the endpoints.

THEOREM 2. *Suppose there exist exponents r_0, r_1, s_0, s_1 and constants M_0, M_1 with $1<r_0$, $r_1<\infty$, $1<s_0\leqq\infty$ and $0<s_1\leqq\infty$, $r_1<s_0$ such that:*

$$\begin{aligned} m\{x : |K(x,y)| \geqq t\} &= \lambda_0(y,t) \leqq (\varphi_0(y)/t)^{r_0}, \\ m\{y : \varphi_0(y) \geqq u\} &\leqq (M_0/u)^{s_0} \qquad [\varphi_0(y) \leqq M_0 \text{ if } s_0 = \infty], \\ m\{y : |K(x,y)| \geqq t\} &= \lambda_1(x,t) \leqq (\varphi_1(x)/t)^{r_1}, \\ m\{x : \varphi_1(x) \geqq u\} &\leqq (M_1/u)^{s_1} \qquad [\varphi_1(x) \leqq M_1 \text{ if } s_1 = \infty]. \end{aligned}$$

Then $\|Tf\|_q \leqq M\|f\|_p$ for all (p,q) of the form $1/p=(1-\tau)/s_0'+\tau/r_1'$, $1/q=(1-\tau)/r_0+\tau/s_1$ with $0<\tau<1$ and such that $p\leqq q$. The constant M depends only on M_0, M_1, s_0, s_1, r_0, r_1, τ.

Proof. We shall prove the corresponding weak-type estimate $m\{x : |Tf(x)| \geqq t\} \leqq (M\|f\|_p/t)^q$ for all such (p,q) without the restriction $p\leqq q$. The result then follows from the Marcinkiewicz interpolation theorem.

Let α, β and γ be functions of t to be determined later. Assume for simplicity that f and K are nonnegative, and $\|f\|_p=1$. Let $f=f_0+f_1$ where

$$\begin{aligned} f_0(x) &= f(x), \quad \text{if } f(x) \leqq \beta, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Let $K=K_0+K_1$ where

$$\begin{aligned} K_0(x,y) &= K(x,y), \quad \text{if } K(x,y) \leqq \alpha, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Let $T_1f(x) = \int K_1(x,y)f_0(y)\,dy$, $T_2f(x) = \int K_0(x,y)f(y)\,dy$ and $T_3f(x) = \int K_1(x,y)f_1(y)\,dy$ so that $T=T_1+T_2+T_3$. It suffices to prove the weak type estimate for T_1, T_2 and T_3 separately. We use throughout the following identity: if $\lambda_g(u) = m\{x : |g(x)| \geqq u\}$ then $\int |g(x)|^p\,dx = \int_0^\infty p\lambda_g(u)u^{p-1}\,du$.

Now

$$\begin{aligned} T_1f(x) &\leqq \beta \int K_1(x,y)\,dy = \beta \int_\alpha^\infty \lambda_1(x,t)\,dt \\ &\leqq \beta\varphi_1(x)^{r_1} \int_\alpha^\infty t^{-r_1}\,dt = \frac{1}{1-r_1}\beta\varphi_1(x)^{r_1}\alpha^{1-r_1} \end{aligned}$$

since $r_1>1$. Thus we have, for $s_1<\infty$

$$\begin{aligned} m\{x : T_1f(x) \geqq t\} &\leqq m\{x : \varphi_1(x) \geqq t^{1/r_1}\beta^{-1/r_1}\alpha^{1-1/r_1}(1-r_1)^{-1/r_1}\} \\ &\leqq (M_1 t^{-1/r_1}\beta^{1/r_1}\alpha^{1/r_1-1}(1-r_1)^{1/r_1})^{s_1}. \end{aligned}$$

Next

$$\begin{aligned} T_2f(x) &\leqq \left(\int K_0(x,y)^{p'}\,dy\right)^{1/p'} \|f\|_p \leqq \left(\int_0^\alpha p'\left(\frac{\varphi_1(x)}{t}\right)^{r_1} t^{p'-1}\,dt\right)^{1/p'} \\ &= \left(\frac{p'}{p'-r_1}\right)^{1/p'} \varphi_1(x)^{r_1/p'}\alpha^{1-r_1/p'} \end{aligned}$$

since $r_1 < p'$. Thus we have, for $s_1 < \infty$

$$m\{x : T_2 f(x) \geqq t\} \leqq m\left\{x : \varphi_1(x) \geqq t^{p'/r_1}\alpha^{1-p'/r_1}\left(\frac{p'}{p'-r_1}\right)^{-1/r_1}\right\}$$
$$\leqq \left(M_1 t^{-p'/r_1}\alpha^{p'/r_1-1}\left(\frac{p'}{p'-r_1}\right)^{1/r_1}\right)^{s_1}.$$

Finally, to estimate $T_3 f$ choose $w \geqq 1$ so that $r_0 p'/s_0 < w < r_0$. This is possible since $r_0 > 1$ and $p' < s_0$. Then, by Minkowski's inequality

$$\|T_3 f\|_w \leqq \int\left(\int K_1(x, y)^w\, dx\right)^{1/w} f_1(y)\, dy$$
$$\leqq \int\left(\int_\alpha^\infty \left(\frac{\varphi_0(y)}{t}\right)^{r_0} w t^{w-1}\, dt\right)^{1/w} f_1(y)\, dy$$
$$= \int \varphi_0(y)^{r_0/w}\left(\frac{w}{w-r_0}\right)^{1/w}\alpha^{1-r_0/w} f_1(y)\, dy.$$

We introduce a new parameter γ, depending on t, and split up this last integral according as $\varphi_0(y) \leqq \gamma$ or $\varphi_0(y) > \gamma$. We have

$$\int_{\varphi_0(y) > \gamma} \varphi_0(y)^{r_0/w} f_1(y)\, dy \leqq \|f_1\|_p\left(\int_\gamma^\infty \left(\frac{M_0}{t}\right)^{s_0} t^{r_0 p'/w-1}\frac{r_0 p'}{w}\, dt\right)^{1/p'}$$
$$\leqq M_0^{s_0/p'}\left(\frac{r_0 p'}{w}\right)^{1/p'}\left(\frac{r_0 p'}{w} - s_0\right)^{-1/p'}\gamma^{r_0/w - s_0/p'}$$

since $r_0 p'/w < s_0$. Thus we can estimate

$$\|T_3 f\|_w \leqq \left(\frac{w}{w-r_0}\right)^{1/w}\cdot\alpha^{1-r_0/w}\left[\gamma^{r_0/w}\int f_1(y)\, dy\right.$$
$$\left. + M_0^{s_0/p'}\left(\frac{r_0 p'}{w}\right)^{1/p'}\left(\frac{r_0 p'}{w} - s_0\right)^{-1/p'}\gamma^{r_0/w - s_0/p'}\right]$$
$$\leqq C_1\alpha^{1-r_0/w}\beta^{1-p}\gamma^{r_0/w} + C_2\alpha^{1-r_0/w}\gamma^{r_0/w-s_0/p'}M_0^{s_0/p'}$$

where

$$C_1 = \left(\frac{w}{w-r_0}\right)^{1/w}\frac{1}{1-p} \quad\text{and}\quad C_2 = \left(\frac{w}{w-r_0}\right)^{1/w}\left(\frac{r_0 p'}{w}\right)^{1/p'}\left(\frac{r_0 p'}{w} - s_0\right)^{1/p'}.$$

Finally we have

$$m\{x : T_3 f(x) \geqq t\} \leqq \left(\frac{\|T_3 f\|_w}{t}\right)^w$$
$$\leqq t^{-w}[C_1\alpha^{1-r_0/w}\beta^{1-p}\gamma^{r_0/w} + C_2\alpha^{1-r_0/w}\gamma^{r_0/w-s_0/p'}M_0^{s_0/p'}]^w.$$

Now we must choose $\alpha_1\beta$ and γ as functions of t so that all three estimates become a constant times t^{-q}. If we set $\alpha^{p'(s_1/r_1)-s_1} = t^{-q+p'(s_1/r_1)}$ and $\beta = (\alpha/t)^{(p'-1)}$ then we have

$$m\{x : T_1 f(x) \geqq t\} \leqq M_1^{s_1}(1-r_1)^{s_1/r_1}t^{-q}$$

and

$$m\{x : T_2 f(x) \geqq t\} \leqq M_1^{s_1}\left(\frac{p'}{p'-r_1}\right)^{s_1/r_1}t^{-q}.$$

Finally set $\gamma=\beta^{(p'/s_0)(p-1)}$. We then have

$$\begin{aligned} m\{x : T_3f(x) \geqq t\} &\leqq (C_1+M_0^{s_0/p'}C_2)^w t^{-w}\alpha^{w-r_0}\beta^{(1-p)w+(r_0p'/s_0)(p-1)} \\ &= (C_1+M_0^{s/p'}C_2)^w\alpha^{r_0p'/s_0-r_0}t^{-r_0p'/s_0}. \end{aligned}$$

Thus p and q must satisfy the identity

$$\left(-q+p'\frac{s_1}{r_1}\right)\left(\frac{r_0}{s_0}p'-r_0\right) = \left(-q+p'\frac{r_0}{s_0}\right)\left(\frac{s_1}{r_1}p'-s_1\right),$$

which follows from $1/p'=(1-\tau)/s_0+\tau/r_1$, $1/q=(1-\tau)/r_0+\tau/s_1$.

It remains to indicate how the proof is modified in case $s_0=\infty$ or $s_1=\infty$. If $s_1=\infty$, we have

$$T_1f(x) \leqq \frac{1}{1-r_1}\beta M_1^{r_1}\alpha^{1-r_1}$$

and

$$T_2f(x) \leqq \left(\frac{p'}{p'-r_1}\right)^{1/p'} M_1^{r_1/p'}\alpha^{1-r_1/p'}.$$

We now choose

$$\frac{1}{1-r_1}\beta M_1^{r_1}\alpha^{1-r_1} = \left(\frac{p'}{p'-r_1}\right)^{1/p'} M_1^{r_1/p'}\alpha^{1-r_1/p'} = \tfrac{1}{2}t$$

so that $m\{x : T_1f(x)\geqq t\}=m\{x : T_2f(x)\geqq t\}=0$. We then handle T_3f by a suitable choice of γ. If $s_0=\infty$ then

$$\|T_3f\|_w \leqq M_0^{r_0/w}\left(\frac{w}{w-r_0}\right)^{1/w}\alpha^{1-r_0/w}\|f_1\|_1 = C_1M_0^{r_0/w}\alpha^{1-r_0/w}\beta^{1-p}.$$

Thus

$$m\{x : T_3f(x) \geqq t\} \leqq t^{-w}C_1^wM_0^{r_0}\alpha^{w-r_0}\beta^{(1-p)w} = C'\alpha^{-r_0}$$

and the proof proceeds as before. Note, however, in this case we have the weak-type estimate holding for $p=1$, $q=r_0$.

REMARK 1. Theorem B is contained in the above theorem if we let $r_0=r_1=r$, $s_0=s_1=\infty$. Of course the proof given above was modelled on one of the proofs of Theorem B, so we do not have a new proof of this result.

REMARK 2. Theorem 2 is incomplete in the following respect: if we consider for fixed p and q the set of all kernels which satisfy a (p, q) estimate as a consequence of Theorem 2, this set is not closed under addition. In other words, to apply Theorem 2 we may first have to split the kernel into a sum of several others, and then apply the theorem to each part with different values of r_0, r_1, s_0, s_1. This suggests that there may be a more general result that contains Theorem 2.

REMARK 3. The case $s_0=s_1=\infty$ has been announced by Krée [15].

4. **Variable kernel singular integral operators.** In this section we change our notation to conform to that of singular integrals. We will consider operators $Sf(x)=\int_{R^n} H(x, y)f(x-y)\,dy$ on functions defined on R^n, or more generally

operators $S_\varepsilon f(x)=\int_{|y|\geqq\varepsilon} H(x, y)f(x-y)\,dy$ such that $S_\varepsilon f\to Sf$ in some sense although Sf does not exist as an absolutely convergent integral.

The first results of Calderón-Zygmund [5] dealt with kernels H which are independent of x, homogeneous of degree $-n$, and have mean value zero on the sphere. For such kernels, under additional mild hypotheses, $\|S_\varepsilon f\|_p\leqq A_p\|f\|_p$ and $S_\varepsilon f\to Sf$ in L^p, for $1<p<\infty$. Later [6], [7] they consider kernels $H(x, y)$ which, for each x, are homogeneous of degree $-n$ in y and have mean value zero on the sphere; similar results are proven for these kernels.

Here we take up the study of variable kernel operators from a new point of view. Our results partly overlap with those of [6], [7], but neither contains the other.

The idea is to consider the auxiliary vector-valued operator

$$S^*f(x) = \int H(t, y)f(x-y)\,dy,$$

which sends scalar valued functions on R^n into functions of x taking values in some space B of functions of t. We then use the vector-valued singular integral theory to obtain an estimate $\int \|S^*f(x)\|_B^p\,dx\leqq A_p^p\int |f(x)|^p\,dx$. If we choose B so that all functions in B are bounded and continuous, with $|g(t)|\leqq M\|g\|_B$, we may set $t=x$ and obtain $\|Sf\|_p\leqq MA_p\|f\|_p$. For technical reasons it is convenient to have B a Hilbert space. This suggests taking B to be the Sobolev space $H^m(R^n)$ for $m>n/2$. We shall take m an integer to simplify the statement of our results. Thus in the sequel m is the smallest integer greater than $n/2$.

LEMMA 3. *Suppose that for almost every x, $H(x, y)\in L^2$. Denote by $\hat{H}(x, \eta)$ the Fourier transform in the y variable (hence $\hat{H}(x, \eta)$, for almost every x, is in L^2). Suppose*

(1) $\int |(\partial/\partial x)^\alpha \hat{H}(x, \eta)|^2\,dx\leqq M^2<\infty$ *for all $\eta\in R^n$ and all α with $|\alpha|\leqq m$. Then $\|Sf\|_2\leqq C_m M\|f\|_2$ for all $f\in L^2$.*

Proof. Note that for almost every t, $S^*f(x)=\int H(t, y)f(x-y)\,dy$ is well defined as an absolutely convergent integral for every x. Now by the Plancherel theorem for Hilbert-space-valued functions

$$\begin{aligned}\int \|S^*f(x)\|_B^2\,dx &= \int \|(S^*f)\hat{}(\xi)\|_B^2\,d\xi = \int \|\hat{H}(t, \xi)\hat{f}(\xi)\|_B^2\,d\xi\\ &= \int \sum_{|\alpha|\leqq m}\int \left|\left(\frac{\partial}{\partial t}\right)^\alpha \hat{H}(t, \xi)\right|^2 dt\,|\hat{f}(\xi)|^2\,d\xi\\ &\leqq C_m' M^2\int |\hat{f}(\xi)|^2\,d\xi = C_m' M^2\|f\|_2^2.\end{aligned}$$

But by the Sobolev inequality (see [1])

$$|Sf(x)| = \left|\int H(x, y)f(x-y)\,dy\right| \leqq C_m''\left\|\int H(t, y)f(x-y)\,dy\right\|_B.$$

Combining the two inequalities we obtain the desired result.

THEOREM 3. *Suppose that in addition to the hypotheses of Lemma* 3, *H also satisfies*

(2) $\sup_y \int_{|x|\geqq c|y|} \left(\int |(\partial/\partial t)^\alpha H(t, x+y)-(\partial/\partial t)^\alpha H(t, x)|^2\, dt\right)^{1/2} dx \leqq M<\infty$ *for all* α, $|\alpha|\leqq m$, *and some fixed c. Then* $\|Sf\|_p \leqq A_p M \|f\|_p$ *for* $1<p<\infty$.

Proof. As before it suffices to show $\left(\int \|S^*f(x)\|_B^p\, dx\right)^{1/p} \leqq A_p' M \|f\|_p$. But by the vector-valued singular integral theory [3] this is a consequence of the L^2 estimate and

(3) $\sup_y \int_{|x|\geqq c|y|} \|H(\cdot, x+y)-H(\cdot, x)\|_B\, dx \leqq M$ which is just a restatement of (2).

COROLLARY 1. *Suppose* $H(x, y)=\Omega(x, y')/|y|^n$ *where* Ω *is a kernel on* $R^n\times S^{n-1}$ *which for each fixed x is integrable and has mean value zero on* S^{n-1}. *Suppose* $\Omega_\alpha=(\partial/\partial x)^\alpha\Omega$, $|\alpha|\leqq m$, *exists in the distribution sense as a bounded function once continuously differentiable in the y variable, such that*

(4) $\int \sup_{y'} |(\partial/\partial y_j)\Omega_\alpha(x, y')|^2\, dx \leqq M^2<\infty$ *for* $j=1, \ldots, n$. *Let* $S_\varepsilon f(x)=\int_{|y|>\varepsilon} H(x, y)f(x-y)\, dy$. *Then*

(a) $\|S_\varepsilon f\|_p \leqq A_p\|f\|_p$, $1<p<\infty$, *where* A_p *depends only on p, M and the bounds for* Ω_α.

(b) *There exists* $Sf\in L^p$ *such that* $S_\varepsilon f\to Sf$ *in* L^p.

Proof. Let $\varphi(x, y)=1$ if $|y|\geqq 1$, 0 otherwise. Let $H_1(x, y)=\varphi(x, y)H(x, y)$.

We will show that H_1 satisfies (1) and (2). The corollary then follows from the standard singular integrals arguments (see [5], [17]).

Now

$$\left(\frac{\partial}{\partial x}\right)^\alpha \hat{H}_1(x, \eta) = \left(\left(\frac{\partial}{\partial x}\right)^\alpha H_1\right)^\wedge (x, \eta)$$

is the Fourier transform of $\varphi(x, y)\Omega_\alpha(x, y')/|y|^n$. Now Calderón-Zygmund [6] show that this is a bounded function provided Ω_α is in some L^p class for $p>1$. Thus, in particular $|(\partial/\partial x)^\alpha \hat{H}_1(x, \eta)|\leqq C\left(\int_{S^{n-1}} |\Omega_\alpha(x, y')|^2\, dy'\right)^{1/2}$ hence

$$\int \left|\left(\frac{\partial}{\partial x}\right)^\alpha \hat{H}_1(x, \eta)\right|^2 dx \leqq C^2 \iint |\Omega_\alpha(x, y')|^2\, dy'\, dx. \tag{5}$$

Now since $\Omega_\alpha(x, y')$ is continuous in y' and has mean value zero on the sphere, it must vanish at some point, hence

$$\int_{S^{n-1}} |\Omega_\alpha(x, y')|^2\, dy' \leqq C' \sum_{j=1}^n \int \left|\frac{\partial}{\partial y_j}\Omega_\alpha(x, y')\right|^2 dy' \tag{6}$$

(see [1, p. 73]). Combining (4), (5), and (6) we obtain (1).

Next consider (2). By the mean value theorem

$$\left|\left(\frac{\partial}{\partial t}\right)^\alpha H(t, x+y)-\left(\frac{\partial}{\partial t}\right)^\alpha H(t, x)\right| \leqq |y| \left|\nabla_x\left(\frac{\partial}{\partial t}\right)^\alpha H(t, x_0)\right|$$

for some x_0 on the segment between x and $x+y$. But

$$|y|\left|\nabla_x\left(\frac{\partial}{\partial t}\right)^\alpha H(t,x_0)\right| = |y|\left|\nabla_x \frac{\Omega_\alpha(t,x_0')}{|x_0|^n}\right|$$
$$\leqq c\sum_{j=1}^n |y| \sup_{z'\in S^{n-1}}\left|\frac{\partial}{\partial z_j}\Omega_\alpha(t,z')\right| |x|^{-n-1}$$

if $|x|\geqq 4|y|$. Thus

$$\left(\int\left|\left(\frac{\partial}{\partial t}\right)^\alpha H(t,x+y)-\left(\frac{\partial}{\partial t}\right)^\alpha H(t,x)\right|^2 dt\right)^{1/2}$$
$$\leqq c'|y|\,|x|^{-n-1}\left(\int \sup_{z'}\left|\frac{\partial}{\partial z_j}\Omega_\alpha(t,z')\right|^2 dt\right)^{1/2}$$
$$\leqq c'M|y|\,|x|^{-n-1}$$

and $\int_{|x|\geqq 4|y|}|y|\,|x|^{-n-1}\,dx=\frac{1}{4}$, proving (2).

REMARK. These results may be carried over to singular integrals with mixed homogeneity [9]. Similar results are given there using the spherical harmonic decomposition introduced in [7]. Our results are different in that they require more smoothness in the first variable but less smoothness in the second variable. More precise techniques are available in the ordinary homogeneous case [6] which give better results than our corollary. These methods, however, seem to be special to this case.

We can also prove variable kernel Marcinkiewicz multiplier type theorems using the same ideas. We give one example using the Hörmander version, although we can evidently obtain similar results using other variants of the Marcinkiewicz theorem [17].

THEOREM 4. *Suppose $\varphi(x,y)$ is a bounded function on $R^n\times R^n$ such that*

$$\int_{r\leqq|y|\leqq 2r}\int_{R^n}|y|^{2|\beta|}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial y}\right)^\beta\varphi(x,y)\right|^2 dx\,dy \leqq M<\infty \tag{7}$$

independent of r, for all α,β with $|\alpha|\leqq m$, $|\beta|\leqq m$. Then the transformation $Uf(x)=\int\varphi(x,y)\hat{f}(y)e^{-2\pi i x\cdot y}\,dy$, originally defined for $f\in\mathscr{S}$ satisfies $\|Uf\|_p\leqq A_pM\|f\|_p$ for $1<p<\infty$, hence can be extended to all L^p with the same inequality.

Proof. As before we consider $U^*f(x)=\int\varphi(t,y)\hat{f}(y)e^{-2\pi i x\cdot y}\,dy$ as a vector-valued multiplier with values in $H^m(R^n)$. The condition (7) is just what is needed to apply the vector-valued version of Hörmander's theorem [3, Theorem 8]. We then set $t=x$ and use the Sobolev inequality to get the desired estimate for Uf.

5. **Convolutions with weighted norms.** We consider nonnegative functions on R^n which are positive almost everywhere. These will be denoted w or v and referred to as weight functions. We denote by $L^p(w)$ the space of functions f satisfying $\int_{R^n}|f(x)|^p w(x)^p\,dx=M^p<\infty$ with the norm $\|f\|_{p,w}=M$. $L^\infty(w)$ is the space of functions f with $fw\in L^\infty$. We will use:

INTERPOLATION THEOREM [19]. *If a linear operator T satisfies $\|Tf\|_{q_i,w_i} \leqq M_i\|f\|_{p_i,v_i}$ for $i=0, 1$, then T satisfies $\|Tf\|_{q,w} \leqq M_0^{1-t}M_1^t\|f\|_{p,v}$ for $1/p=(1-t)/p_0+t/p_1$, $1/q=(1-t)/q_0+t/q_1$, $w=w_0^{1-t}w_1^t$, $v=v_0^{1-t}v_1^t$ and $0 \leqq t \leqq 1$.*

For simplicity we assume that all weight functions are symmetric; i.e., $w(-x)=w(x)$.

PROPOSITION 1. *Let T be a linear operator which commutes with translations and satisfies an estimate $\|Tf\|_{q,w} \leqq M\|f\|_{p,v}$, $1 \leqq p, q < \infty$.* (a) *Then T is convolution with a tempered distribution Φ.* (b) *T also satisfies the estimate $\|Tf\|_{q(t),w(t)} \leqq M\|f\|_{p(t),v(t)}$ for $1/q(t)=(1-t)/q+t/q'$, $1/p(t)=(1-t)/p+t/p'$, $w(t)=w^{1-t}v^{-t}$, $v(t)=v^{1-t}w^{-t}$ and $0 \leqq t \leqq 1$.* (c) *If T is positive then it is convolution with a positive, locally bounded measure μ: $Tf(x)=\int f(x-y)\,d\mu(y)$.* (d) *If $p=q$ and $w=v$ then μ is a bounded measure and T satisfies the estimate $\|Tf\|_{r,w^\alpha} \leqq M\|f\|_{r,w^\alpha}$ for $(1/r, \alpha)$ in the closed parallelogram in R^2 bounded by the points $(1/p, 1)$, $(1/p', -1)$, $(0, 0)$ and $(0, 1)$. In particular $\|Tf\|_{p,w^\alpha} \leqq M\|f\|_{p,w^\alpha}$ for $1 \geqq \alpha \geqq 1-p$.*

Proof. The first assertion is an exercise in distribution theory (see Hörmander [11] for the unweighted case).

Consider the standard pairing $\langle f, g\rangle = \int f(x)g(x)\,dx$. Under this pairing the dual space of $L^p(w)$ is $L^{p'}(w^{-1})$. If T^* denotes the adjoint of T then $\|T^*f\|_{p',v^{-1}} \leqq M\|f\|_{q',w^{-1}}$. But T^* is just convolution with $\Phi(-x)$. Since v and w are symmetric T must satisfy the same estimate as T^*. Assertion (b) now follows by interpolation. Assertion (c) is merely the fact that a positive tempered distribution is a positive, locally bounded measure.

If $p=q$ and $w=v$ then, setting $t=\frac{1}{2}$ in (b), we obtain $\|Tf\|_2 \leqq M\|f\|_2$. It is well known that for positive T this implies $\|\mu\| \leqq M$ (to see this take f to be the characteristic function of a large ball) hence $\|Tf\|_1 \leqq M\|f\|_1$ and $\|Tf\|_\infty \leqq M\|f\|_\infty$. Assertion (d) now follows by interpolation.

PROPOSITION 2. *Let μ be a positive measure such that $\mu * w \leqq Mv$ almost everywhere. Then $\|\mu * f\|_{q,w^{1/p}v^{-1/p'}} \leqq M\|f\|_{p,v^{1/p}w^{-1/p'}}$ for p, $1 \leqq p \leqq \infty$.*

Proof. We prove this for $p=1$ and ∞ and then use interpolation. For $p=\infty$ the result is trivial. For $p=1$ we have

$$\int |\mu * f(x)|w(x)\,dx \leqq \iint |f(x-y)|w(x)\,d\mu(y)\,dx$$
$$= \int |f(x)| \int w(x+y)\,d\mu(y)\,dx \leqq M\int |f(x)|v(x)\,dx$$

using the symmetry of w and v.

There is also a direct proof using Hölder's inequality which we leave as an exercise.

6. **Fractional integration with weighted norms.** In this section we consider the transformation $I_\lambda f(x)=\int_{R^n} f(x-y)|y|^{-\lambda}\,dy$ for λ, $0<\lambda<n$.

THEOREM 5 (cf. [18]). *I_λ is a bounded transformation from $L^p(|x|^\alpha)$ to $L^q(|x|^{-\beta})$ provided $\alpha+\beta\geqq 0$, $\alpha+\beta+\lambda=n/p'+n/q$, $-n/p<\alpha<n/p'$, $-n/q'<\beta<n/q$, and $1\leqq p\leqq q\leqq\infty$, except if $\alpha+\beta=0$ we require $1<p\leqq q<\infty$.*

Proof. Only the case $\alpha+\beta=0$ is nonelementary. If $\alpha+\beta>0$ it suffices to prove the special case $p=q$. For, by Lemma 1, it suffices to show that the kernel $|\,|y|^{-\alpha}|x-y|^{-\lambda}|x|^{-\beta}\,|^{n/(\alpha+\beta+\lambda)}$ satisfies a $((\alpha+\beta+\lambda)q/n, (\alpha+\beta+\lambda)q/n)$ estimate, i.e. $I_{\lambda n/(\alpha+\beta+\lambda)}$ is bounded from $L^{(\alpha+\beta+\lambda)q/n}(|x|^{\alpha n/(\alpha+\beta+\lambda)})$ to $L^{(\alpha+\beta+\lambda)q/n}(|x|^{-\beta n/(\alpha+\beta+\lambda)})$. Note the conditions on the parameters are again satisfied, with $\alpha+\beta>0$ required to keep $\lambda n/(\alpha+\beta+\lambda)<n$.

Now if $p=q$ the kernel $|y|^{-\alpha}|x-y|^{-\lambda}|x|^{-\beta}$ is homogeneous of degree $-n$ since $\alpha+\beta+\lambda=n$. Thus Corollary 2 of Theorem 1 applies and gives an exact bound of $\int |x_0-y|^{-\lambda}|y|^{-\alpha-n/p}\,dy$ for any $x_0\in R^n$ with $|x_0|=1$. This integral is finite provided $\lambda<n$, $\alpha<n/p'$ and $n/p'-\lambda<\alpha$, and these conditions are equivalent to those given in the statement of the theorem.

It is also possible to prove the case $p=q$ from Proposition 2 of §5 and interpolation using the "semigroup property" of I_α, namely $I_\alpha\circ I_\beta=I_\gamma$ provided $\alpha+\beta-\gamma=n$ and $0<\alpha,\beta,\gamma<n$. We leave the details as an exercise.

If $\alpha+\beta=0$ then $\lambda=n/p'+n/q$ and we have, by the usual fractional integration theorem (Theorem B) that $|x-y|^{-\lambda}$ satisfies a (p,q) estimate. Thus it suffices to show $|y|^\beta|x-y|^{-\lambda}|x|^{-\beta}-|x-y|^{-\lambda}=|x-y|^{-\lambda}(1-|y|^\beta/|x|^\beta)$ satisfies a (p,q) estimate. This result is again elementary. By Corollary 2 of Theorem 1 and Lemma 1 it suffices to show

$$\int |x_0-y|^{-n}(|1-|y|^\beta|)^{n/\lambda}|y|^{-n^2/\lambda q}\,dy = M < \infty.$$

The singularities of the integrand near $y=0$ and $y=\infty$ are integrable provided $n/q-\lambda<\beta<n/q$. Near x_0 we have $|1-|y|^\beta|\leqq c_\beta|x_0-y|$ hence the integrand is $O(|x_0-y|^{-n+n/\lambda})$ and is integrable for all β.

7. **Singular integrals in weighted norms.** Here we consider singular integral operators

$$Sf(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} f(x-y)\,\frac{\Omega(y)}{|y|^n}\,dy$$

where Ω is homogeneous of degree zero, integrable on the sphere with mean value zero, and such that the even part $\Omega(y)+\Omega(-y)$ is in $L\log L^+(S^{n-1})$. The fundamental Calderón-Zygmund inequality [6] $\|Sf\|_p\leqq A_p\|f\|_p$ for $1<p<\infty$ was extended by Stein [16] to $\|Sf\|_{p,|x|^\alpha}\leqq A_{p,\alpha}\|f\|_{p,|x|^\alpha}$ for $-n/p<\alpha<n/p'$ under the additional hypothesis that Ω be bounded. His proof uses the Calderón-Zygmund inequality and a lemma which is a consequence of our Corollary 2 of Theorem 1.

For the range $-1/p<\alpha<1/p'$, Krée [14] shows that no additional assumptions on Ω are necessary. His method is to use the corresponding fact for the Hilbert transform and follows quite closely the Calderón-Zygmund argument in [6].

Because the assumption $-1/p<\alpha<1/p'$ is necessary for the Hilbert transform result, this suggests that outside $-1/p<\alpha<1/p'$ it will be necessary to impose additional hypotheses on Ω. In fact this is the case, as the following example shows:

In R^2, let ψ be the characteristic function of the infinite rectangle

$$Q = \{x_1 \geqq 1,\ -1 \leqq x_2 \leqq 1\}.$$

The functions we consider are $f_\beta(x)=x_1^\beta\psi(x)$. Now Ω is a function on the circle $x_1^2+x_2^2=1$, which we parametrize in the usual way $x_1=\sin\theta$, $x_2=\cos\theta$. Let Ω be odd and nonnegative on the left half-circle $x_1\leqq 0$.

We compute $\|f_\beta\|_{p,|x|^\alpha}<\infty$ if and only if $\beta<-\alpha-1/p$. Suppose $|x|<\frac{1}{2}$. Then for $\varepsilon<\frac{1}{2}$ we have

$$S_\varepsilon f(x) = Sf(x) = \int_0^\infty \int_{\pi/2}^{3\pi/2} f(x-(r,\theta))\Omega(\theta)\,dr/r.$$

Note that the integrand is positive. For fixed θ sufficiently near π, the ray $x-(r,\theta)$ intersects the rectangle Q on a segment whose projection on the x_1-axis extends at least between $1/(2|\theta-\pi|)$ and $3/(4|\theta-\pi|)$, along which the function $f_\beta(x)=x_1^{-\beta}$ $\geqq\frac{1}{2}r^{-\beta}$. Thus the integral above is at least

$$\tfrac{1}{2}\int_{\pi-\varepsilon}^{\pi+\varepsilon}\Omega(\theta)\int_{1/(2|\theta-\pi|)}^{3/(4|\theta-\pi|)} r^{\beta-1}\,dr\,d\theta = c\int_{\pi-\varepsilon}^{\pi+\varepsilon}\Omega(\theta)|\theta-\pi|^{-\beta}\,d\theta.$$

If $\beta>0$ this integral is not finite for all $\Omega\in L^q$ unless $q>1/(1-\beta)$, i.e. unless $1/q\leqq 1+\alpha+1/p$. Thus for this to hold for all α, $-1/p\geqq\alpha>-2/p$ requires $1/q\leqq 1/p'$.

By the usual duality argument we obtain the condition $1/q\leqq 1-\alpha+1/p'$, which for α in the range $1/p'<\alpha<2/p'$ implies $1/q\leqq 1/p$. Thus we see that, in 2 dimensions, Stein's result cannot be strengthened beyond replacing $\Omega\in L^\infty$ by $\Omega\in L^q$ for $q=\max\{p,p'\}$. However, as the following result shows, this condition is roughly sufficient.

THEOREM 6. *Suppose $q>2(n-1)$ and $\Omega\in L^q(S^{n-1})$ with mean value zero. Then*

$$Sf(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} f(x-y)\frac{\Omega(y')}{|y|^n}\,dy$$

is a bounded operator on $L^p(|x|^\alpha)$ for $-n/p<\alpha<n/p'$ and $q'\leqq p\leqq q$.

Proof. Since we already have a (p,p) estimate for the kernel $\Omega(x-y)/|x-y|^n$ it suffices to establish a (p,p) estimate for $\Omega(x-y)|x-y|^{-n}(1-|x|^\alpha/|y|^\alpha)$. We do this for $p=q$ and $p=q'$, and the result follows by interpolation.

Let $p=q$. We use the first part of Corollary 3 of Theorem 1. Define r by $1/p'=1/r+1/p$. Then

$$\left(\int\left|\Omega(R(x_0)-S(y))|R(x_0)-S(y)|^{-n}\left(1-\frac{1}{|y|^\alpha}\right)\right|^{p'}dS\right)^{1/p'}$$
$$\leqq\left(\int|\Omega(R(x_0)-S(y))|^p\,dS\right)^{1/p}\left(\int\left||R(x_0)-S(y)|^{-n}\left(1-\frac{1}{|y|^\alpha}\right)\right|^r dS\right)^{1/r}.$$

Note that the second factor is independent of R, and that

$$\left(\int\int |\Omega(R(x_0)-S(y))|^p\, dS\, dR\right)^{1/p} = \|\Omega\|_p.$$

Thus

$$\left(\int\left(\int |\Omega(R(x_0)-S(y))|R(x_0)-S(y)|^{-n}(1-|y|^{-\alpha})|^{p'}\, dS\right)^{p/p'} dR\right)^{1/p}$$
$$\leqq \|\Omega\|_p\left(\int |\,|x_0-S(y)|^{-n}(1-|y|^{-\alpha})|^r\, dS\right)^{1/r}.$$

We estimate this integral for $|y|<\frac{1}{2}$, $\frac{1}{2}\leqq|y|\leqq 2$ and $2<|y|$. In the first domain $|x_0-S(y)|^{-n}\leqq 2^n$ so the integral is $\leqq 2^n(1-|y|^{-\alpha})$. In the third domain $|x_0-S(y)|^{-n}\leqq 2^n|y|^{-n}$ so the integral is $\leqq 2^n(1-|y|^{-\alpha})|y|^{-n}$. In the middle domain we compute in polar coordinates

$$\int |x_0-S(y)|^{-nr}\, dS = \int_0^\pi |1+|y|^2-2|y|\cos\theta|^{-nr/2}|\sin\theta|^{n-2}\, d\theta$$
$$\leqq \int_0^\infty |(1-|y|)^2+\theta^2|^{-nr/2}\theta^{n-2}\, d\theta = c|1-|y|\,|^{-nr+n-1}.$$

Also $|1-|y|^{-\alpha}|\leqq c|1-|y|\,|$ so the integral we are estimating is $\leqq c|1-|y|\,|^{(n-1)/r'}$.

Now to apply the criterion of Corollary 3 we must show that this integral is in $L^1(|y|^{-n/p})$. For $|y|<\frac{1}{2}$ we require $\alpha<n/p'$. For $|y|>2$ we require $\alpha>-n/p$. For $\frac{1}{2}<|y|<2$ we require $1/r'<1/(n-1)$, or in other words $q>2(n-1)$.

The case $p=q'$ is treated almost identically using the second criterion of Corollary 3 of Theorem 1. We omit the details.

REMARK. It is possible to interpolate between this result and Krée's result. We consider Sf as a bilinear operator depending on f and Ω and we use multilinear interpolation (this is best done within the framework of [4]). The fact that we require the mean value of Ω to be zero is a minor difficulty which may be overcome by [20, Theorem 5.4], for example. We thus obtain inequalities in p, q and α which express the fact that the point $(1/p, 1/q, \alpha)$ in R^3 lies in the interior of the convex hull of the points $(1/p,1,1/p')$, $(1/p,1,-1/p)$, $(1/p,1/r,n/p')$ and $(1/p,1/r,-n/p)$ where $1/r=\min\{1/p, 1/p'\}$ for $r>2(n-1)$ and $1<p<\infty$.

It seems likely that the condition $r>2(n-1)$ is not really necessary, but we have not been able to prove this.

8. **Is the fractional integration theorem "best possible"?** We begin with a weak converse to the fractional integration theorem. Recall that two functions f and g are called equimeasurable if their distribution functions $\lambda_f(t)=m\{x : |f(x)|>t\}$ and $\lambda_g(t)=m\{y : |g(y)|>t\}$ are equal almost everywhere.

THEOREM 7. *Suppose that f is a function on R^n such that $\|g*h\|_q\leqq A_g\|h\|_p$ for fixed p, q with $1<p<q<\infty$, and every g equimeasurable with f. Then $f\in L^{r,\infty}(R^n)$ where $1/r=1/q+1/p'$.*

Proof. Let $\psi_s(x)$ denote the characteristic function of the ball $\{|x| \leqq s\}$, and let c be the generic symbol for a constant that depends only on n, so that different occurrences of c may stand for different constants.

Take g to be the nonnegative radial function nonincreasing in $|x|$, which is equimeasurable with f. Then the set $\{x : g(x) > t\}$ is the ball of radius $c\lambda_f(t)^{1/n}$ about the origin. Thus $g(x) \geqq t\psi_s(x)$ for $s = c\lambda_f(t)^{1/n}$. Since $\psi_s * \psi_s(x) \geqq cs^n\psi_{s/2}(x)$ we have $g * \psi_s \geqq cts^n\psi_{s/2}$. But by hypothesis $\|g * \psi_s\|_q \leqq A_g\|\psi_s\|_p$, which implies $ts^n s^{n/q} \leqq cA_g s^{n/p}$ which simplifies to $\lambda_f(t) \leqq (cA_g/t)^r$.

Thus there can be no condition on the distribution function λ_f alone that guarantees $\|f * h\|_q \leqq M\|h\|_p$ and which is weaker than that of Theorem B. In this sense Theorem B is best possible.

We may ask the same question for any locally compact abelian group. For the torus groups T^n and the lattice groups Z^n it is clear that Theorem 7 remains true with almost the identical proof. In general we do not know the answer.

We may go a step further and ask if there are conditions on the absolute value $|f(x)|$ which imply $\|f * h\|_q \leqq M\|h\|_p$. We reformulate this as follows:

PROBLEM. Characterize all positive operators from L^p to L^q which commute with translations. Does the answer depend only on $1/p - 1/q$?

For $p = q$ the answer is known: convolutions with bounded, positive measures. For $p < q$ we do not know the answer. The following examples show, however, that Theorem B is far from being the answer.

EXAMPLE 1. On the circle T^1 there exist positive singular measures μ_α with Fourier-Steiltjes series $\sum c_n e^{inx}$ with $c_n = O(|n|^{-\alpha})$, for every α, $0 < \alpha < \frac{1}{2}$, [12]. By the Paley theorem on Fourier multipliers [21, II, p, 127] we must have $\|\mu * f\|_q \leqq M\|f\|_p$ for $1 < p \leqq 2 \leqq q < \infty$ and $1/r = 1/p' + 1/q = \alpha$.

EXAMPLE 2. On R^2 the function $f(x_1, x_2) = |x_1 x_2|^{-1/r}$ has an everywhere infinite distribution function, yet $\|f * g\|_q \leqq M_p\|g\|_p$ for $1/p' + 1/q = 1/r$, $1 < p$, $q < \infty$. This follows from the following "Fubini-type" theorem:

PROPOSITION 2. *Let $G = G_1 \oplus G_2$, G, G_1, G_2 locally compact abelian groups. Suppose $f(x, y)$ is a function on G such that, for almost every $y \in G_2$, $f(x, y) \in L^{r,\infty}(G_1)$, and $\|f(\cdot, y)\|_{L^{r,\infty}(G_1)} \in L^{r,\infty}(G_2)$, with norm M. Then $\|f * g\|_q \leqq cM\|g\|_p$ for $1 < p$, $q < \infty$, $1/r = 1/p' + 1/q$.*

Proof. By Theorem 2 (in fact Theorem B for G_1)

$$\left(\int_{G_1}\left|\int_{G_1} f(x', y')g(x - x', y - y')\,dx'\right|^q dx\right)^{1/q} \leqq c\|f(\cdot, y')\|_{L^{r,\infty}(G_1)}\|g(\cdot, y - y')\|_{L^p(G_1)}.$$

Thus by Minkowski's inequality we have

$$\|f * g(\cdot, y)\|_{L^q(G_1)} \leqq c\int_{G_2}\|f(\cdot, y')\|_{L^{r,\infty}(G_1)}\|g(\cdot, y - y')\|_{L^p(G_1)}\,dy'$$

which is a convolution on G_2. To this convolution we again apply fractional integration to obtain

$$\|f * g\|_q \leqq c\|\,\|f(\cdot, y)\|_{L^{r,\infty}(G_1)}\|_{L^{r,\infty}(G_2)}\|g\|_p.$$

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