1. Introduction. In this note a theory of Chebyshev approximation is obtained for approximating families with a property which is a generalization of convexity, the betweeness property. This theory is of interest for several reasons. Most of the approximating families for which a tractable theory exists characterize best approximations by the extrema of their error curve. The betweeness property is the weakest easily verifiable condition giving such a characterization of best approximations. The development of the theory sheds considerable light on the well-known linear theory [2], [5] and rational theory [1], [2], [3]. A necessary and sufficient condition for the uniqueness of best approximations is obtained; it is the most general known necessary and sufficient condition for any theory.

Let \( X \) be a compact space and for a function \( g \) define \( \| g \| = \sup \{ |g(x)| : x \in X \} \). Let \( \mathcal{F} \) be a family of real continuous functions with elements \( F, G, H, \ldots \). The Chebyshev problem is: given a continuous function \( f \), to find an element \( G^* \) of \( \mathcal{F} \) to minimize \( e(G) = \| E(G, \cdot) \| \) where \( E(G, x) = f(x) - G(x) \). Such an element \( G^* \) is called a best approximation in \( \mathcal{F} \) to \( f \) on \( X \). It will be assumed throughout the discussion that \( f \) is fixed, and mention of \( f \) is suppressed in the notation \( e(G) \) and \( E(G, \cdot) \).

The author wishes to thank Professor E. Barbeau for his careful criticism of the hypotheses and proofs of this note.

2. The betweeness property.

Definition. A family \( \mathcal{F} \) of real continuous functions is said to have the betweeness property if for any two elements \( G_0 \) and \( G_1 \), there exists a \( \lambda \)-set \( \{ H_\lambda \} \) of elements of \( \mathcal{F} \) such that \( H_0 = G_0, H_1 = G_1 \), and for all \( x \in X \), \( H_\lambda(x) \) is either a strictly monotonic continuous function of \( \lambda \) or a constant, \( 0 \leq \lambda \leq 1 \). (It should be noted that \( H_\lambda(x) \) can be monotone in different senses for different \( x \).)

Lemma 1. Let \( \{ G_k \} \) be a sequence of continuous functions on a compact space \( X \) such that \( \{ G_k \} \) converges pointwise to a continuous function \( G_0 \) and for any \( x \in X \), \( G_k(x) \) is a monotonic sequence, then \( \{ G_k \} \) converges uniformly to \( G_0 \).

Proof. The sequence \( |G_k(x) - G_0(x)| \) is a decreasing sequence of continuous functions, which converges to the continuous limit \( 0 \). By Dini's theorem, the convergence is uniform. From this lemma it can be seen that if \( \{ H_\lambda \} \) is a \( \lambda \)-set for \( G_0 \) and \( G_1 \) then the sequence \( \{ H_{1/k} \} \) converges uniformly to \( G_0 \).
Any linear family $\mathcal{L}$ of continuous functions (and any convex subset of $\mathcal{L}$) has the betweeness property, for a $\lambda$-set is given by $H_\lambda = \lambda G_1 + (1 - \lambda)G_0$.

More generally let $\mathcal{P}$ and $\mathcal{Q}$ be linear families and $\mathcal{F}$ a convex set of pairs $(p, q)$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$ such that $(p, q) \in \mathcal{F}$ implies $(\alpha p, \alpha q) \in \mathcal{F}$ for $\alpha > 0$. A function is an $\mathcal{F}$-admissible rational function if it is of the form $p/q$, $(p, q) \in \mathcal{F}$, $q > 0$. The set $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$-admissible rational functions has the betweeness property, for the $\lambda$-set corresponding to $p_0/q_0$ and $p_1/q_1$ is

$$H_\lambda = \frac{\lambda p_1 + (1 - \lambda)p_0}{\lambda q_1 + (1 - \lambda)q_0},$$

$$dH_\lambda/d\lambda = \frac{(p_1 q_0 - p_0 q_1)/((\lambda q_1 + (1 - \lambda)q_0)^2}$$

being of constant sign for a given point $x$ and vanishing identically at any point $x$ at which $p_0/q_0 - p_1/q_1$ vanishes. In the case where $\mathcal{F}$ consists of all pairs we obtain the family

$$\mathcal{R} = \{p/q : p \in \mathcal{P}, q \in \mathcal{Q}, q > 0\}$$

of admissible rational functions.

If $\phi$ is a continuous strictly monotonic function from the real line into the real line and $\mathcal{I}$ has the betweeness property, then the set of elements of the form $\phi(G)$, $G \in \mathcal{I}$ has the betweeness property, for if $\{H_\lambda\}$ is a $\lambda$-set for $G_0$ and $G_1$, $\{\phi(H_\lambda)\}$ is a $\lambda$-set for $\phi(G_0)$ and $\phi(G_1)$.

After the theory of this paper had been obtained it was noticed that Meinardus and Schwedt had used a condition [4, p. 304] quite close to the betweeness property, but developed a different type of theory.

3. Characterization of best approximation. The points at which $E(G, \cdot)$ attains its norm $e(G)$ will be denoted by $M(G)$. By compactness of $X$ and continuity of $E(G, \cdot)$, $M(G)$ is nonempty and closed.

Theorem 1. Let $\mathcal{I}$ have the betweeness property. An element $G_0$ of $\mathcal{I}$ is a best approximation if and only if there exists no element $G_1 \in \mathcal{I}$ such that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$.

Proof. The condition is obviously sufficient for $G_0$ to be a best approximation (we do not need the betweeness property). We now prove necessity. Let us suppose that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$ then by continuity of $E(G_1, \cdot)$ there exists an open cover $U$ of $M(G_0)$ on which this inequality holds. Let $V = X \setminus U$, then if $V$ is empty it is immediate that $G_0$ is not best. We therefore suppose that $V$ is nonempty. Let $H_\lambda$ be a $\lambda$-set corresponding to $G_0$ and $G_1$, $H_0 = G_0$, $H_1 = G_1$. On the set $U$ we have $E(H_\lambda, x)$ on the open interval between $E(G_0, x)$ and $E(G_1, x)$ for $0 < \lambda < 1$, hence

$$E(H_\lambda, x) < e(G_0), \quad 0 < \lambda < 1, x \in U.$$ 

Let $\eta = e(G_0) - \sup \{|E(G_0, x)| : x \in V\}$. As $V$ is compact and $E(G_0, \cdot)$ is continuous, $E(G_0, \cdot)$ attains its supremum on $V$ and this supremum cannot be $e(G_0)$, as
$M(G) \cap V$ is empty, hence $\eta > 0$. The sequence $\{H_n\}$ converges uniformly to $G_0$. Choose $\delta > 0$ such that $\|G_0 - H_\delta\| < \eta$. It follows that for $x \in V$, 
\[
|E(H_\delta, x)| = |f(x) - H_\delta(x)| \leq |f(x) - G_0(x)| + |G_0(x) - H_\delta(x)| < e(G_0) - \eta + \eta = e(G_0).
\]

Combining this inequality and the previous one for $x \in U$, we have 
\[
|E(H_\delta, x)| < e(G_0), \quad x \in X = U \cup V.
\]

and $G_0$ is not best, proving necessity. The theorem is proven.

Let us suppose that $E(G_0, x) \cdot (G_1(x) - G_0(x)) > 0$ for all $x \in M(G_0)$ and $\{H_\lambda\}$ is a $\lambda$-set for $G_0$ and $G_1$. For $\lambda$ sufficiently small, $|E(H_\lambda, x)| < e(G_0)$ for all $x \in M(G_0)$. We then apply Theorem 1 to get 

**Corollary.** Let $\mathcal{G}$ have the betweenness property. An element $G_0$ of $\mathcal{G}$ is a best approximation if and only if there exists no element $G_1 \in \mathcal{G}$ such that $E(G_0, x) \cdot (G_1(x) - G_0(x)) > 0$ for all $x \in M(G_0)$.

4. An error determining set on which best approximations agree. Let $\mathcal{G}^*$ be the set of best approximations to $f$ and $N = \bigcap M(G)$, $G \in \mathcal{G}^*$. We will show in this section that if $\mathcal{G}^*$ is nonempty then $N$ is nonempty, best approximations must agree on $N$ and that $N$ is an error determining set, that is, there exists no approximant $F$ such that $|E(F, x)| < \inf \{e(G) : G \in \mathcal{G}\}$ for $x \in N$. In the cases of approximation by linear or rational families of finite dimension, it can easily be shown that if $\mathcal{G}^*$ is nonempty, there exists an element $F \in \mathcal{G}$ such that $M(F) = N$; in the linear case any element of the convex interior of $\mathcal{G}^*$ is such an $F$. This is not true in general, for let $X = [0, 1]$ and $\mathcal{G}$ be the set of monotonic continuous functions $G$ with $G$ zero in a neighborhood of the point zero. In the approximation of $f=1$ any element $G$ of $\mathcal{G}$ such that $\|1 - G\| = 1$ is a best approximation and $N = \{0\}$, but there is no element $G$ such that $M(G) = N$.

**Lemma 3.** Let $\mathcal{G}$ have the betweenness property and $\mathcal{G}^*$ be nonempty. Given a finite number $G_1, \ldots, G_n$ of elements of $\mathcal{G}^*$ there exists an element $G_0$ of $\mathcal{G}^*$ such that $\bigcap_{k=1}^n M(G_k) \supset M(G_0)$.

**Proof.** Let $G_1$ and $G_2$ be any two best approximations and $G_1$ be any element of the $\lambda$-set corresponding to $G_1$ and $G_2$, $0 < \lambda < 1$, then for all $x \in X$, $G_1(x)$ lies between $G_1(x)$ and $G_2(x)$, 
\[
|E(G_1, x)| \leq \sup \{|E(G_1, x)|, |E(G_2, x)|\}
\]

with equality only if $G_1(x) = G_2(x)$. It follows that $G_1$ is a best approximation and $M(G_1) \subseteq M(G_1) \cap M(G_2)$. Similarly, there exists $G_k \in \mathcal{G}^*$ such that $G_k$ is in the $\lambda$-set corresponding to $G_k$ and $G_k+1$, $0 < \lambda < 1$, and $M(G_k) \subseteq \bigcap_{k=1}^{n+1} M(G_k)$, $k = 2, \ldots, n - 1$. The required approximant in $\mathcal{G}^*$ is $G_{n-1}$ and the lemma is proven.

**Corollary.** Let $G_0, G_1 \in \mathcal{G}^*$, then the $\lambda$-set $\{H_\lambda\}$ for $G_0$ and $G_1$ is contained in $\mathcal{G}^*$.
Lemma 4. Let $\mathcal{G}$ have the betweenness property. If $\mathcal{G}^*$ is nonempty, $N$ is nonempty.

Proof. If $N$, an intersection of a nonempty family of closed sets, were empty, it could be expressed as a finite intersection of these sets.

$$N = \bigcap_{k=1}^{n} M(G_k), \quad G_k \in \mathcal{G}^*. $$

By the previous lemma there exists $G_0 \in \mathcal{G}^*$ such that $\bigcap_{k=1}^{n} M(G_k) \supseteq M(G_0)$. It follows from the definition of $N$ that $N = M(G_0)$. But $M(G_0)$ is nonempty and so we have a contradiction proving the lemma.

Lemma 5. Let the family $\mathcal{G}$ of real continuous functions have the betweenness property. Let $G_0, G_1 \in \mathcal{G}^*$, then $G_0(x) = G_1(x)$ for all $x \in N$.

Proof. Let $G_0, G_1 \in \mathcal{G}^*$ be given and select a $\lambda$-set $\{H_\lambda\}$ corresponding to $G_0$ and $G_1$, $0 < \lambda < 1$. If $G_0(x) \neq G_1(x)$ for some $x$, then

$$|E(H_\lambda, x)| < \max \{|E(G_0, x)|, |E(G_1, x)|\}$$

for $0 < \lambda < 1$ and since $\{H_\lambda\} \in \mathcal{G}^*, x \notin N$.

Lemma 6. Let $\mathcal{G}$ have the betweenness property. If $\mathcal{G}^*$ is nonempty there exists no approximant $G$ such that $|E(G, x)| < \inf \{e(G) : G \in \mathcal{G}\}$ for all $x \in N$.

Proof. Suppose such a $G$ exists, then by continuity of $E(G, \cdot)$, the inequality

$$|E(G, x)| < \inf \{e(G) : G \in \mathcal{G}\} = \Delta(f, \mathcal{G})$$

holds on an open cover $U$ of $N$. Let $V = X \sim U$, then $V$ is nonempty (for otherwise $e(G) < \Delta(f, \mathcal{G})$, which is impossible).

Since

$$\bigcap \{V \cap M(F) : F \in \mathcal{G}^*\} = V \cap N = \emptyset$$

is an intersection of closed sets in a compact space, there exists a finite set $G_1, \ldots, G_n$ of elements of $\mathcal{G}^*$ such that $\bigcap_{k=1}^{n} (V \cap M(G_k)) = \emptyset$. Applying Lemma 3, there exists $G_0 \in \mathcal{G}^*$ such that $M(G_0) \subset \bigcap_{k=1}^{n} M(G_k) \subset U$. Now let $\{H_\lambda\}$ be a $\lambda$-set corresponding to $G_0$ and $G_1$. $H_0 = G_0, H_1 = G$. Since $E(H_\lambda, x)$ is between $E(G_0, x)$ and $E(G, x)$ for $0 < \lambda < 1$ and $x \in U$,

$$E(H_\lambda, x) < e(G_0), \quad 0 < \lambda < 1, x \in U.$$ 

Now let $\eta = e(G_0) - \sup \{|E(G_0, x) : x \in V\}$. As the sequence $\{H_{1/n}\}$ converges uniformly to $G_0$, there exists $\delta > 0$ such that $\|G_0 - H_\delta\| < \eta$. For $x \in V$ we have

$$|E(H_\delta, x)| = |f(x) - H_\delta(x)| \leq |f(x) - G_0(x)| + |G_0(x) - H_\delta(x)| < e(G_0) - \eta + \eta = e(G_0).$$

Combining this inequality for $x \in V$ with the earlier one for $x \in U$, we have $E(H_\delta, x) < e(G_0), x \in X$, and so

$$e(H_\delta) < \inf \{e(G) : G \in \mathcal{G}\}.$$

This is a contradiction and the lemma is proven.
5. Uniqueness results. Lemmas 5 and 6 are very powerful results. Using them we can obtain many uniqueness results. We give below the most general uniqueness result, a generalization of Haar's classical result concerning necessary and sufficient conditions for best linear approximations to be unique. After this result was obtained it was noted that it includes a uniqueness result of Singer [6] for approximation by arbitrary linear subspaces of $C(X)$.

**Definition.** A family $\mathcal{G}$ of real continuous functions is said to have zero-sign compatibility if for any two distinct elements $G$ and $H$, any closed subset $Z$ of the zeros of $G - H$, and any continuous function $s$ which takes the values $+1$ or $-1$ on $Z$, there exists $F \in \mathcal{G}$ such that

\[
\text{sgn} \ (F(x) - G(x)) = s(x), \quad x \in Z.
\]

Without loss of generality we can assume $\|s\| = 1$.

**Theorem 2.** Let $\mathcal{G}$ have the betweeness property. A necessary and sufficient condition that for every continuous function a best approximation is unique is that $\mathcal{G}$ have zero-sign compatibility.

**Proof.** Suppose that for two distinct elements $G$ and $H$, a closed subset $Z$ of the zeros of $G - H$, and a continuous function $s$, $\|s\| = 1$, which takes the values $+1$ or $-1$ on $Z$, there exists no element $F$ for which (*) holds.

Define:

\[
f(x) = G(x) + s(x)\left[\|G - H\| - |G(x) - H(x)|\right],
\]

then

\[
E(G, \cdot) = s(x)\left[\|G - H\| - |G(x) - H(x)|\right].
\]

For $x \in Z$ we have $E(G, x) = s(x)\|G - H\|$, hence $Z \subset M(G)$. If a better approximant $F$ existed it would satisfy

\[
\text{sgn} \ (F(x) - G(x)) = s(x), \quad x \in Z,
\]

which is impossible by hypothesis. Hence $G$ is a best approximation to $f$ and since

\[
|f(x) - H(x)| \leq |f(x) - G(x)| + |G(x) - H(x)|
\]

\[
\leq \|G - H\| - |G(x) - H(x)| + |G(x) - H(x)| = \|G - H\|,
\]

$H$ is also a best approximation to $f$, proving necessity.

**Remark.** The proof of necessity assumes nothing about $\mathcal{G}$ and therefore shows that zero-sign compatibility is necessary for uniqueness, $\mathcal{G}$ any approximating family.

Suppose now that $\mathcal{G}$ has zero-sign compatibility and $G, G_1$ are distinct best approximations. Therefore $G(x) = G_1(x)$ for $x \in N$ by Lemma 5. Let the function $s$ be $E(G, \cdot)/\|E(G, \cdot)\|$ then by zero-sign compatibility there exists an element $F$ such that $\text{sgn} \ (F(x) - G(x)) = \text{sgn} \ (E(G, x))$ for $x \in N$. Let $\{H_\lambda\}$ be a $\lambda$-set for $G$ and $F$, $H_0 = G$, $H_1 = F$. The sequence $\{H_{\lambda, x}\}$ converges uniformly to $G$ so for some $\delta > 0$, $E(H_{\lambda, x}, x)$ will be between $E(G, x)$ and $-E(G, x)$ for all $x \in N$, hence $|E(H_{\delta}, x)| < |E(G, x)| = \varepsilon(G)$ for all $x \in N$. This contradicts Lemma 6 so sufficiency is proven.
We now consider approximation on a compact subset $Y$ of $X$. If the betweeness property holds on $X$, it holds on $Y$.

**Lemma 7.** Let $X$ be a compact normal space and $Y$ a compact subset of $X$. If $\mathcal{G}$ has zero-sign compatibility on $X$, $\mathcal{G}$ has zero-sign compatibility on $Y$.

**Proof.** Let $(G, H)$ be a pair of distinct elements of $\mathcal{G}$. Let $Z$ be a closed subset of $Y \cap \{ x : G(x) - H(x) = 0 \}$ then $Z$ is a closed subset in $X$ of the zeros of $G - H$. Let $s'$ be a continuous mapping of $Y$ into $[-1, 1]$ taking values $+1$ or $-1$ on $Z$. Since $X$ is a normal space, there exists by the Tietze extension theorem $s \in C(X)$, $\|s\| = 1$, $s(x) = s'(x)$ for $x \in Y$. Let $\mathcal{G}$ have zero-sign compatibility on $X$; then there exists $F \in \mathcal{G}$ such that

$$\text{sgn} (F(x) - G(x)) = s(x) = s'(x), \quad x \in Z.$$ 

From the lemma and Theorem 2 we obtain

**Corollary.** Let $X$ be a compact normal space. Let $\mathcal{G}$ have the betweeness property and best approximations on $X$ to any continuous function be unique, then best approximations on any compact subset of $X$ are unique to any continuous function.

We now consider approximation by an open subset $\mathcal{G}'$ of $\mathcal{G}$. If $\mathcal{G}$ has the betweeness property, it follows that the function $F$ in the definition of zero-sign compatibility can be chosen arbitrarily close to the function $G$ of that definition. If $G \in \mathcal{G}'$ it follows that $F$ can be chosen such that $F \in \mathcal{G}'$. It follows that if $\mathcal{G}$ has zero-sign compatibility, so does $\mathcal{G}'$. We obtain:

**Corollary.** Let both $\mathcal{G}$ and $\mathcal{G}'$, an open subset of $\mathcal{G}$, have the betweeness property. If every continuous function has at most one best approximation from $\mathcal{G}$, every continuous function has at most one best approximation from $\mathcal{G}'$.

Less general but simpler uniqueness results can be developed in terms of the sign changing property and property $Z$.

**Definition.** $\mathcal{G}$ has the sign changing property of degree $n$ at $G$ if for any $n$ distinct points $\{x_1, \ldots, x_n\}$ and $n$ real numbers $w_1, \ldots, w_n$ which are either $+1$ or $-1$, there exists an approximant $F$ such that

$$\text{sgn} (F(x_k) - G(x_k)) = w_k, \quad k = 1, \ldots, n.$$ 

We need not specify the closeness of $F$ to $G$ in the above definition since if such an $F$ exists, there exists with the betweeness property such an $F$ arbitrarily close to $G$.

**Definition.** $\mathcal{G}$ has property $Z$ of degree $n$ at $G$ if $G - F$ having $n$ zeros implies $F = G$.

Let $\mathcal{G}$ have the betweeness property. The $F$ in the definition of the sign changing property can be chosen such that for given $\epsilon > 0$, $\|F - G\| < \epsilon$. Let $G \in \mathcal{G}^*$. If $\mathcal{G}$ has the sign changing property of degree $n$ at $G$ then $G$ either coincides with the function $f$ being approximated or $N$ has at least $n + 1$ points, for if it had less we could find $F$
such that $|E(F, x)| < e(G)$ for $x \in N(X)$, which contradicts Lemma 6. If $\mathcal{G}$ has property $Z$ of degree $n$ at $G$ then by Lemma 5 best approximations must be identical if $N$ has $n$ or more points. We therefore have:

**THEOREM 3.** Let $\mathcal{G}$ have the betweeness property and $G \in \mathcal{G}^*$. If $\mathcal{G}$ has property $Z$ of degree $n+1$ at $G$ and the sign changing property of degree $n$ at $G$ then $G$ is a unique best approximation.

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