CHEBYSHEV APPROXIMATION BY FAMILIES WITH
THE BETWEENESS PROPERTY

BY

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1. Introduction. In this note a theory of Chebyshev approximation is obtained for approximating families with a property which is a generalization of convexity, the betweeness property. This theory is of interest for several reasons. Most of the approximating families for which a tractable theory exists characterize best approximations by the extrema of their error curve. The betweeness property is the weakest easily verifiable condition giving such a characterization of best approximations. The development of the theory sheds considerable light on the well-known linear theory [2], [5] and rational theory [1], [2], [3]. A necessary and sufficient condition for the uniqueness of best approximations is obtained; it is the most general known necessary and sufficient condition for any theory.

Let $X$ be a compact space and for a function $g$ define $\|g\| = \sup \{|g(x)| : x \in X\}$. Let $\mathcal{F}$ be a family of real continuous functions with elements $F, G, H, \ldots$. The Chebyshev problem is: given a continuous function $f$, to find an element $G^*$ of $\mathcal{F}$ to minimize $e(G) = \|E(G, \cdot)\|$ where $E(G, x) = f(x) - G(x)$. Such an element $G^*$ is called a best approximation in $\mathcal{F}$ to $f$ on $X$. It will be assumed throughout the discussion that $f$ is fixed, and mention of $f$ is suppressed in the notation $e(G)$ and $E(G, \cdot)$.

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2. The betweeness property.

Definition. A family $\mathcal{F}$ of real continuous functions is said to have the betweeness property if for any two elements $G_0$ and $G_1$, there exists a $\lambda$-set $\{H_\lambda\}$ of elements of $\mathcal{F}$ such that $H_0 = G_0$, $H_1 = G_1$, and for all $x \in X$, $H_\lambda(x)$ is either a strictly monotonic continuous function of $\lambda$ or a constant, $0 \leq \lambda \leq 1$. (It should be noted that $H_\lambda(x)$ can be monotone in different senses for different $x$.)

Lemma 1. Let $\{G_k\}$ be a sequence of continuous functions on a compact space $X$ such that $\{G_k\}$ converges pointwise to a continuous function $G_0$ and for any $x \in X$, $G_k(x)$ is a monotonic sequence, then $\{G_k\}$ converges uniformly to $G_0$.

Proof. The sequence $|G_k(x) - G_0(x)|$ is a decreasing sequence of continuous functions, which converges to the continuous limit 0. By Dini's theorem, the convergence is uniform. From this lemma it can be seen that if $\{H_\lambda\}$ is a $\lambda$-set for $G_0$ and $G_1$ then the sequence $\{H_{1/k}\}$ converges uniformly to $G_0$.

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Any linear family $\mathcal{L}$ of continuous functions (and any convex subset of $\mathcal{L}$) has the betweenness property, for a $\lambda$-set is given by $H_\lambda = \lambda G_1 + (1 - \lambda)G_0$.

More generally let $\mathcal{P}$ and $\mathcal{Q}$ be linear families and $\mathcal{F}$ a convex set of pairs $(p, q)$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$ such that $(p, q) \in \mathcal{F}$ implies $(ap, aq) \in \mathcal{F}$ for $a > 0$. A function is an $\mathcal{F}$-admissible rational function if it is of the form $p/q$, $(p, q) \in \mathcal{F}, q > 0$. The set $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$-admissible rational functions has the betweenness property, for the $\lambda$-set corresponding to $p_0/q_0$ and $p_1/q_1$ is

$$H_\lambda = \frac{(\lambda p_1 + (1 - \lambda)p_0)(\lambda q_1 + (1 - \lambda)q_0)}{(\lambda q_1 + (1 - \lambda)q_0)^2},$$

being of constant sign for a given point $x$ and vanishing identically at any point $x$ at which $p_0/q_0 - p_1/q_1$ vanishes. In the case where $\mathcal{F}$ consists of all pairs we obtain the family

$$\mathcal{R} = \{p/q : p \in \mathcal{P}, q \in \mathcal{Q}, q > 0\}$$

of admissible rational functions.

If $\phi$ is a continuous strictly monotonic function from the real line into the real line and $\mathcal{G}$ has the betweenness property, then the set of elements of the form $\phi(G)$, $G \in \mathcal{G}$ has the betweenness property, for if $\{H_\lambda\}$ is a $\lambda$-set for $G_0$ and $G_1$, $\{\phi(H_\lambda)\}$ is a $\lambda$-set for $\phi(G_0)$ and $\phi(G_1)$.

After the theory of this paper had been obtained it was noticed that Meinardus and Schwedt had used a condition [4, p. 304] quite close to the betweeness property, but developed a different type of theory.

3. Characterization of best approximation. The points at which $E(G, \cdot)$ attains its norm $e(G)$ will be denoted by $M(G)$. By compactness of $X$ and continuity of $E(G, \cdot)$, $M(G)$ is nonempty and closed.

**Theorem 1.** Let $\mathcal{G}$ have the betweenness property. An element $G_0$ of $\mathcal{G}$ is a best approximation if and only if there exists no element $G_1 \in \mathcal{G}$ such that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$.

**Proof.** The condition is obviously sufficient for $G_0$ to be a best approximation (we do not need the betweenness property). We now prove necessity. Let us suppose that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$ then by continuity of $E(G_1, \cdot)$ there exists an open cover $U$ of $M(G_0)$ on which this inequality holds. Let $V = X \sim U$, then if $V$ is empty it is immediate that $G_0$ is not best. We therefore suppose that $V$ is non-empty. Let $H_\lambda$ be a $\lambda$-set corresponding to $G_0$ and $G_1$, $H_0 = G_0, H_1 = G_1$. On the set $U$ we have $E(H_\lambda, x)$ on the open interval between $E(G_0, x)$ and $E(G_1, x)$ for $0 < \lambda < 1$, hence

$$E(H_\lambda, x) < e(G_0), \quad 0 < \lambda < 1, x \in U.$$  

Let $\eta = e(G_0) - \sup \{|E(G_0, x)| : x \in V\}$. As $V$ is compact and $E(G_0, \cdot)$ is continuous, $E(G_0, \cdot)$ attains its supremum on $V$ and this supremum cannot be $e(G_0)$, as
M(G) ∩ V is empty, hence η > 0. The sequence \{H_{1/k}\} converges uniformly to G₀.

Choose δ > 0 such that \|G₀ - H₀\| < η. It follows that for x ∈ V,

\[|E(H_δ, x)| = |f(x) - H_δ(x)| ≤ |f(x) - G₀(x)| + |G₀(x) - H_δ(x)|< e(G₀) - η + η = e(G₀).

Combining this inequality and the previous one for x ∈ U, we have

\[|E(H_δ, x)| < e(G₀), \quad x ∈ X = U ∪ V.

and G₀ is not best, proving necessity. The theorem is proven.

Let us suppose that \(E(G₀, x) \cdot (G_1(x) - G₀(x)) > 0\) for all x ∈ M(G₀) and \{H_{λ}\} is a λ-set for G₀ and G₁. For λ sufficiently small, \(|E(H_λ, x)| < e(G₀)|\) for all x ∈ M(G₀). We then apply Theorem 1 to get

**Corollary.** Let \(\mathcal{G}\) have the betweenness property. An element G₀ of \(\mathcal{G}\) is a best approximation if and only if there exists no element G₁ ∈ \(\mathcal{G}\) such that \(E(G₀, x) \cdot (G₁(x) - G₀(x)) > 0\) for all x ∈ M(G₀).

4. An error determining set on which best approximations agree. Let \(\mathcal{G}^*\) be the set of best approximations to f and \(N = \bigcap M(G), G ∈ \mathcal{G}^*\). We will show in this section that if \(\mathcal{G}^*\) is nonempty then N is nonempty; best approximations must agree on N and that N is an error determining set, that is, there exists no approximant F such that \(|E(F, x)| < \inf \{e(G) : G ∈ \mathcal{G}\}\) for x ∈ N. In the cases of approximation by linear or rational families of finite dimension, it can easily be shown that if \(\mathcal{G}^*\) is nonempty, there exists an element F ∈ \(\mathcal{G}\) such that \(M(F) = N\); in the linear case any element of the convex interior of \(\mathcal{G}^*\) is such an F. This is not true in general, for let \(X = [0, 1]\) and \(\mathcal{G}\) be the set of monotonic continuous functions G with G zero in a neighborhood of the point zero. In the approximation of f = 1 any element G of \(\mathcal{G}\) such that \(||1 - G|| = 1\) is a best approximation and \(N = \{0\}\), but there is no element G such that \(M(G) = N\).

**Lemma 3.** Let \(\mathcal{G}\) have the betweenness property and \(\mathcal{G}^*\) be nonempty. Given a finite number G₁, \ldots, Gₙ of elements of \(\mathcal{G}^*\) there exists an element G₀ of \(\mathcal{G}^*\) such that \(\bigcap_{k=1}^{n} M(G_k) \supset M(G₀)\).

**Proof.** Let G₁ and G₂ be any two best approximations and \(G₁\) be any element of the λ-set corresponding to G₁ and G₂, 0 < λ < 1, then for all x ∈ X, \(G₁(x)\) lies between G₁(x) and G₂(x),

\[|E(G₁, x)| ≤ \sup \{|E(G₁, x)|, |E(G₂, x)|\}

with equality only if G₁(x) = G₂(x). It follows that \(G₁\) is a best approximation and \(M(G₁) \subset M(G₁) ∩ M(G₂)\). Similarly, there exists \(Gₖ\) such that \(Gₖ\) is in the λ-set corresponding to \(Gₖ₋₁\) and \(Gₖ₊₁\), 0 < λ < 1, and \(M(Gₖ) \subset \bigcap_{k=1}^{n} M(Gₖ)\), k = 2, \ldots, n − 1. The required approximant in \(\mathcal{G}^*\) is \(G₋₁\) and the lemma is proven.

**Corollary.** Let G₀, G₁ ∈ \(\mathcal{G}^*\), then the λ-set \(\{Hₖ\}\) for G₀ and G₁ is contained in \(\mathcal{G}^*\).
Lemma 4. Let $\mathcal{S}$ have the betweeness property. If $\mathcal{S}^*$ is nonempty, $N$ is nonempty.

Proof. If $N$, an intersection of a nonempty family of closed sets, were empty, it could be expressed as a finite intersection of these sets.

$$N = \bigcap_{k=1}^{n} M(G_k), \quad G_k \in \mathcal{S}^*.$$  

By the previous lemma there exists $G_0 \in \mathcal{S}^*$ such that $\bigcap_{k=1}^{n} M(G_k) \supset M(G_0)$. It follows from the definition of $N$ that $N = M(G_0)$. But $M(G_0)$ is nonempty and so we have a contradiction proving the lemma.

Lemma 5. Let the family $\mathcal{S}$ of real continuous functions have the betweeness property. Let $G_0, G_1 \in \mathcal{S}^*$, then $G_0(x) = G_1(x)$ for all $x \in N$.

Proof. Let $G_0, G_1 \in \mathcal{S}^*$ be given and select a $\lambda$-set $\{H_\lambda\}$ corresponding to $G_0$ and $G_1$, $0 < \lambda < 1$. If $G_0(x) \neq G_1(x)$ for some $x$, then

$$|E(H_\lambda, x)| < \max \{|E(G_0, x)|, |E(G_1, x)|\}$$

for $0 < \lambda < 1$ and since $\{H_\lambda\} \in \mathcal{S}^*$, $x \notin N$.

Lemma 6. Let $\mathcal{S}$ have the betweeness property. If $\mathcal{S}^*$ is nonempty there exists no approximant $G$ such that $|E(G, x)| < \inf \{e(G) : G \in \mathcal{S}\}$ for all $x \in N$.

Proof. Suppose such a $G$ exists, then by continuity of $E(G, \cdot)$, the inequality

$$|E(G, x)| < \inf \{e(G) : G \in \mathcal{S}\} = \Delta(f, \mathcal{S})$$

holds on an open cover $U$ of $N$. Let $V = X \sim U$, then $V$ is nonempty (for otherwise $e(G) < \Delta(f, \mathcal{S})$, which is impossible).

Since

$$\{\bigcap (V \cap M(F)) : F \in \mathcal{S}^*\} = N = \emptyset$$

is an intersection of closed sets in a compact space, there exists a finite set $G_1, \ldots, G_n$ of elements of $\mathcal{S}^*$ such that $\bigcap_{k=1}^{n} (V \cap M(G_k)) = \emptyset$. Applying Lemma 3, there exists $G_0 \in \mathcal{S}^*$ such that $M(G_0) \subset \bigcap_{k=1}^{n} M(G_k) \subset U$. Now let $\{H_{\lambda}\}$ be a $\lambda$-set corresponding to $G_0$ and $G$, $H_0 = G_0$, $H_1 = G$. Since $E(H_{\lambda}, x)$ is between $E(G_0, x)$ and $E(G, x)$ for $0 < \lambda < 1$ and $x \in U$,

$$E(H_\lambda, x) < e(G_0), \quad 0 < \lambda < 1, x \in U.$$  

Now let $\eta = e(G_0) - \sup \{|E(G_0, x)| : x \in V\}$. As the sequence $\{H_{1/k}\}$ converges uniformly to $G_0$, there exists $\delta > 0$ such that $\|G_0 - H_\delta\| < \eta$. For $x \in V$ we have

$$|E(H_\delta, x)| = |f(x) - H_\delta(x)|$$

$$\leq |f(x) - G_0(x)| + |G_0(x) - H_\delta(x)| < e(G_0) - \eta + \eta = e(G_0).$$

Combining this inequality for $x \in V$ with the earlier one for $x \in U$, we have $E(H_\delta, x) < e(G_0), x \in X$, and so

$$e(H_\delta) < \inf \{e(G) : G \in \mathcal{S}\}.$$  

This is a contradiction and the lemma is proven.
5. Uniqueness results. Lemmas 5 and 6 are very powerful results. Using them we can obtain many uniqueness results. We give below the most general uniqueness result, a generalization of Haar’s classical result concerning necessary and sufficient conditions for best linear approximations to be unique. After this result was obtained it was noted that it includes a uniqueness result of Singer [6] for approximation by arbitrary linear subspaces of $C(X)$.

**Definition.** A family $\mathcal{G}$ of real continuous functions is said to have zero-sign compatibility if for any two distinct elements $G$ and $H$, any closed subset $Z$ of the zeros of $G - H$, and any continuous function $s$ which takes the values $+1$ or $-1$ on $Z$, there exists $F \in \mathcal{G}$ such that

\[ \text{sgn} \ (F(x) - G(x)) = s(x), \quad x \in Z. \]

Without loss of generality we can assume $\|s\| = 1$.

**Theorem 2.** Let $\mathcal{G}$ have the betweeness property. A necessary and sufficient condition that for every continuous function a best approximation is unique is that $\mathcal{G}$ have zero-sign compatibility.

**Proof.** Suppose that for two distinct elements $G$ and $H$, a closed subset $Z$ of the zeros of $G - H$, and a continuous function $s$, $\|s\| = 1$, which takes the values $+1$ or $-1$ on $Z$, there exists no element $F$ for which (*) holds.

Define:

\[ f(x) = G(x) + s(x)\|G - H\| - |G(x) - H(x)|, \]

then

\[ E(G, \cdot) = s(x)\|G - H\| - |G(x) - H(x)|. \]

For $x \in Z$ we have $E(G, x) = s(x)\|G - H\|$, hence $Z \subseteq M(G)$. If a better approximant $F$ existed it would satisfy

\[ \text{sgn} \ (F(x) - G(x)) = s(x), \quad x \in Z, \]

which is impossible by hypothesis. Hence $G$ is a best approximation to $f$ and since

\[ |f(x) - H(x)| \leq |f(x) - G(x)| + |G(x) - H(x)| \]
\[ \leq \|G - H\| - |G(x) - H(x)| + |G(x) - H(x)| = \|G - H\|, \]

$H$ is also a best approximation to $f$, proving necessity.

**Remark.** The proof of necessity assumes nothing about $\mathcal{G}$ and therefore shows that zero-sign compatibility is necessary for uniqueness, $\mathcal{G}$ any approximating family.

Suppose now that $\mathcal{G}$ has zero-sign compatibility and $G, G_1$ are distinct best approximations. Therefore $G(x) = G_1(x)$ for $x \in N$ by Lemma 5. Let the function $s$ be $E(G, \cdot)/\|E(G, \cdot)\|$ then by zero-sign compatibility there exists an element $F$ such that $\text{sgn} \ (F(x) - G(x)) = \text{sgn} \ (E(G, x))$ for $x \in N$. Let $(H_\lambda)$ be a $\lambda$-set for $G$ and $F$, $H_0 = G$, $H_1 = F$. The sequence $(H_{\lambda})$ converges uniformly to $G$ so for some $\delta > 0$, $E(H_{\lambda}, x)$ will be between $E(G, x)$ and $-E(G, x)$ for all $x \in N$, hence $|E(H_{\lambda}, x)| < |E(G, x)| = e(G)$ for all $x \in N$. This contradicts Lemma 6 so sufficiency is proven.
We now consider approximation on a compact subset \( Y \) of \( X \). If the betweenness property holds on \( X \), it holds on \( Y \).

**Lemma 7.** Let \( X \) be a compact normal space and \( Y \) a compact subset of \( X \). If \( \mathcal{G} \) has zero-sign compatibility on \( X \), \( \mathcal{G} \) has zero-sign compatibility on \( Y \).

**Proof.** Let \((G, H)\) be a pair of distinct elements of \( \mathcal{G} \). Let \( Z \) be a closed subset of \( Y \cap \{x : G(x) - H(x) = 0\} \) then \( Z \) is a closed subset in \( X \) of the zeros of \( G - H \). Let \( s' \) be a continuous mapping of \( Y \) into \([-1, 1]\) taking values +1 or −1 on \( Z \). Since \( X \) is a normal space, there exists by the Tietze extension theorem \( s \in C(X), \|s\| = 1, s(x) = s'(x) \) for \( x \in Y \). Let \( \mathcal{G} \) have zero-sign compatibility on \( X \); then there exists \( F \in \mathcal{G} \) such that

\[
\text{sgn} (F(x) - G(x)) = s(x) = s'(x), \quad x \in Z.
\]

From the lemma and Theorem 2 we obtain

**Corollary.** Let \( X \) be a compact normal space. Let \( \mathcal{G} \) have the betweeness property and best approximations on \( X \) to any continuous function be unique, then best approximations on any compact subset of \( X \) are unique to any continuous function.

We now consider approximation by an open subset \( \mathcal{G}' \) of \( \mathcal{G} \). If \( \mathcal{G} \) has the betweenness property, it follows that the function \( F \) in the definition of zero-sign compatibility can be chosen arbitrarily close to the function \( G \) of that definition. If \( G \in \mathcal{G}' \) it follows that \( F \) can be chosen such that \( F \in \mathcal{G}' \). It follows that if \( \mathcal{G} \) has zero-sign compatibility, so does \( \mathcal{G}' \). We obtain:

**Corollary.** Let both \( \mathcal{G} \) and \( \mathcal{G}' \), an open subset of \( \mathcal{G} \), have the betweeness property. If every continuous function has at most one best approximation from \( \mathcal{G} \), every continuous function has at most one best approximation from \( \mathcal{G}' \).

Less general but simpler uniqueness results can be developed in terms of the sign changing property and property \( Z \).

**Definition.** \( \mathcal{G} \) has the sign changing property of degree \( n \) at \( G \) if for any \( n \) distinct points \( \{x_1, \ldots, x_n\} \) and \( n \) real numbers \( w_1, \ldots, w_n \) which are either +1 or −1, there exists an approximant \( F \) such that

\[
\text{sgn} (F(x_k) - G(x_k)) = w_k, \quad k = 1, \ldots, n.
\]

We need not specify the closeness of \( F \) to \( G \) in the above definition since if such an \( F \) exists, there exists with the betweeness property such an \( F \) arbitrarily close to \( G \).

**Definition.** \( \mathcal{G} \) has property \( Z \) of degree \( n \) at \( G \) if \( G - F \) having \( n \) zeros implies \( F = G \).

Let \( \mathcal{G} \) have the betweenness property. The \( F \) in the definition of the sign changing property can be chosen such that for given \( \epsilon > 0, \|F - G\| < \epsilon \). Let \( G \in \mathcal{G}^* \). If \( \mathcal{G} \) has the sign changing property of degree \( n \) at \( G \) then \( G \) either coincides with the function \( f \) being approximated or \( N \) has at least \( n + 1 \) points, for if it had less we could find \( F \).
such that $|E(F, x)| < e(G)$ for $x \in N(X)$, which contradicts Lemma 6. If $\mathcal{G}$ has property $Z$ of degree $n$ at $G$ then by Lemma 5 best approximations must be identical if $N$ has $n$ or more points. We therefore have:

**Theorem 3.** Let $\mathcal{G}$ have the betweeness property and $G \in \mathcal{G}^*$. If $\mathcal{G}$ has property $Z$ of degree $n + 1$ at $G$ and the sign changing property of degree $n$ at $G$ then $G$ is a unique best approximation.

**Bibliography**


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