

OPERATOR-STABLE PROBABILITY DISTRIBUTIONS ON VECTOR GROUPS

BY
MICHAEL SHARPE⁽¹⁾

1. Introduction. An operator-stable probability distribution in a group G is a limit law arising, roughly speaking, from affine modification of the partial sums of a sequence of independent identically distributed G -valued random variables. This paper is concerned with a more palpable description of an operator-stable distribution in the case where G is a vector group; i.e. the topological group underlying a d -dimensional real vector space.

The first step is to reduce the problem to that of finding those probability measures λ , all of whose convolution powers are of the same type; i.e. for each integer $k \geq 1$, there is an automorphism B_k of G such that λ^k is a translation of the measure $\lambda \circ B_k^{-1}$. It is then shown that λ is operator-stable if and only if for each $t > 0$, λ^t is a translation of the measure $\lambda \circ [\exp(\log t \cdot B)]^{-1}$, for an automorphism B of G characterized by conditions on the spectrum of B regarded as a linear operator on a vector space.

2. Notation and definitions. Throughout this paper, we denote by V a d -dimensional vector group. In several proofs, we shall use the same symbol to denote the same group with the additional structure of a vector space, or even an inner product space. Denote by $\mathcal{P} = \mathcal{P}(V)$ the set of probability measures on V . With the topology of weak convergence, and multiplication defined by convolution, \mathcal{P} becomes a topological semigroup. We denote convolution of two measures λ, μ by $\lambda * \mu$, and the n th convolution power of λ by λ^n .

By V^\wedge , we mean the character group of V , "identified" with the dual vector space of the vector space V . In the sequel, x will denote the generic element of V and y the generic element of V^\wedge . Let (x, y) denote the bilinear pairing of V and V^\wedge brought about by y acting as a linear functional on x . As a group character, y acts on x according to $\exp i(x, y)$. The characteristic function of a measure $\lambda \in \mathcal{P}(V)$ is defined by

$$\varphi(y) = \lambda^\wedge(y) = \int_V \exp i(x, y) \lambda(dy).$$

Given $\lambda \in \mathcal{P}(V)$, we define $\lambda^- \in \mathcal{P}(V)$ by $\lambda^-(E) = \lambda(-E)$. The mapping $\lambda \rightarrow \lambda^-$ is an involutive automorphism of \mathcal{P} . It is easy to check that $\lambda^{-\wedge} \equiv \lambda^\wedge^-$, the last

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bar denoting the complex conjugate. The measure λ is called symmetric if and only if $\lambda = \lambda^-$. For any $\lambda \in \mathcal{P}$, the measure ${}^\circ\lambda = \lambda * \lambda^-$ is symmetric, and is called the symmetrization of λ .

Let $H(\lambda) = \{y \in V^\wedge \mid \lambda^\wedge(y) = 1\}$. It is easy to see that $H(\lambda)$ is a closed subgroup of V^\wedge . Note that $\{y \in V^\wedge \mid |\lambda^\wedge(y)| = 1\} = H({}^\circ\lambda)$ is a closed subgroup of V^\wedge containing $H(\lambda)$.

Denote by $S(\lambda)$ the support of the measure $\lambda \in \mathcal{P}(V)$, viz., the smallest closed set F in V with $\lambda(F) = 1$. The relation $S(\lambda * \mu) = \text{Cl}(S(\lambda) + S(\mu))$ is well known. We call a measure $\lambda \in \mathcal{P}$ full if and only if $S(\lambda)$ is not contained in any $(d-1)$ -dimensional hyperplane of V . If λ is not full, it is called deficient.

PROPOSITION 1. $\lambda \in \mathcal{P}(V)$ is a full measure if and only if $H({}^\circ\lambda)$ does not contain a 1-dimensional subgroup of V^\wedge .

Proof. Firstly, λ is full if and only if ${}^\circ\lambda$ is full. If $S({}^\circ\lambda) \subset X$, a $(d-1)$ -dimensional subspace of V , then

$$\begin{aligned} {}^\circ\lambda^\wedge(y) &= \int_X \exp i(x, y) {}^\circ\lambda(dx) \\ &= 1 \quad \text{if } y \in X^\perp = \{y \in V^\wedge \mid (x, y) = 0, \forall x \in X\}, \end{aligned}$$

and X^\perp is a 1-dimensional subspace of V^\wedge . Conversely, if $H({}^\circ\lambda)$ contains a 1-dimensional subspace Y of V^\wedge , then $\int_V \exp i(x, y) {}^\circ\lambda(dx) = 1$ for all $y \in Y$ implies $\exp i(x, y) = 1$, ${}^\circ\lambda$ -almost everywhere, for all $y \in Y$, which implies

$${}^\circ\lambda\{x \mid \exp i(x, y) = 1\} = 1 \quad \text{for all } y \in Y$$

so that

$$S({}^\circ\lambda) \subset \{x \mid (x, y) = 0 \pmod{2\pi} \text{ for all } y \in Y\} = \{x \mid (x, y) = 0 \text{ for all } y \in Y\} = Y^\perp,$$

a $(d-1)$ -dimensional subspace of V . ■

We mention that the set \mathcal{F} of full measures in $\mathcal{P}(V)$ is an open subsemigroup of \mathcal{P} .

Let us denote by $\text{End } V$ the ring of continuous endomorphisms of the group V , identified with the ring of linear transformations of the vector space V . Let $\text{Aut } V$ denote the group of continuous automorphisms of V , identified with the group $\text{Gl}(V)$ of nonsingular linear transformations of the vector space V . For any $A \in \text{End } V$ and $\lambda \in \mathcal{P}(V)$, let $A\lambda$ denote the measure $A\lambda(F) = \lambda(A^{-1}(F))$, F a Borel subset of V . If $A \in \text{Aut } V$ and $a \in V$, we call the mapping $\lambda \rightarrow A\lambda * \delta(a)$ an affine transformation, $\delta(a)$ denoting the point mass at a . If instead $A \in \text{End } V \sim \text{Aut } V$, the mapping will be called singular affine. The set of affine transformations (A, a) is denoted by $\text{Aff } V$. If x is a V -valued random variable having distribution μ , then clearly, $Ax + a$ has distribution $A\lambda * \delta(a)$.

It is not difficult to verify that for any bounded continuous function f on V ,

$$\int_V f d(A\lambda) = \int_V f \circ A d\lambda,$$

and that $S(A\lambda) = AS(\lambda)$, $(AB)\lambda = A(B\lambda)$, $A(\lambda * \mu) = A\lambda * A\mu$ and $(A\lambda)^\wedge(y) = \lambda^\wedge(A^*y)$, where A^* denotes the adjoint map of $V^\wedge \rightarrow V^\wedge$ induced by the bilinear pairing (\cdot, \cdot) ; i.e. $(Ax, y) \equiv (x, A^*y)$. We give $\text{End } V$ the compact-open topology as a set of functions from V to V . This topology is equivalent to the norm topology of $\text{End } V$ as a space of linear operators on V , when V is provided with a vector norm. It is easy to check that the mapping $\langle A, \lambda \rangle \rightarrow A\lambda$ of $\text{End } V \times \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ is jointly continuous. We define an equivalence relation in $\mathcal{P}(V)$ by writing $\lambda \sim \mu$ if and only if there is an affine transformation (A, a) such that $\lambda = A\mu * \delta(a)$. The measures λ, μ are then said to be of the same type, and an equivalence class of measures is called a type. Clearly, random variables x and $Ax + a$ will have distributions of the same type.

3. Statement of the problem. In terms of random variables, the problem we study is enunciated as follows: suppose that $\{x_n\}$ is a sequence of V -valued random variables with common distribution μ , and assume that the terms of the sequence are mutually independent; assume, further, that (A_n, a_n) is a sequence of affine transformations of V such that the distribution of $A_n(x_1 + \dots + x_n) + a_n$ converges to a measure $\lambda \in \mathcal{P}(V)$; what can be said about the limit measure λ ?

Converting this to a problem involving only measures, we ask which measures λ can arise as limits of sequences $A_n\mu^n * \delta(a_n)$.

Our aim is to characterize those λ which are full and are such limits—denote by \mathcal{S} the class of such measures. For reasons which do not become clear until the problem is posed in a different form, \mathcal{S} is called the class of operator-stable measures on V . It is clear that \mathcal{S} is invariant under affine transformations.

We refer the reader to Feller [1] for an account of the class \mathcal{S} when V reduces to the real line. It is possible in this case to give an explicit formula for the characteristic function of a stable distribution. In fact (Feller [1, p. 542]) any such characteristic function λ^\wedge must have the form $\lambda^\wedge(y) = \exp(a\psi_{\alpha, \delta}(y) + iby)$, where $0 < \alpha \leq 2$, $-1 \leq \delta \leq 1$, and

$$\begin{aligned} \psi_{\alpha, \delta}(y) &= -|y|^\alpha [1 - i \operatorname{sgn} y \delta \tan \pi\alpha/2], & \alpha \neq 1, \\ &= -|y| [1 + i \operatorname{sgn} y \delta \log |y|], & \alpha = 1. \end{aligned}$$

In particular, $\lambda^{\wedge t}(y) = \lambda^\wedge(t^{1/\alpha}y) \exp ib(t)y$, $t > 0$ where $b(t)$ is real. It is easily seen that if $\alpha \neq 1$, there exists b such that $\mu = \lambda * \delta(b)$ satisfies $\mu^{\wedge t}(y) = \lambda^\wedge(t^{1/\alpha}y)$ but if $\alpha = 1$, it is not generally possible to make such a centering, and $b(t)$ has the form $b(t) = ct \log t$. These results are obtained from the fact that the Khintchine-Lévy measure M (see next paragraph) of a stable distribution must be given by $M\{[x, \infty)\} = cpx^{-\alpha}$, $M\{(-\infty, -x]\} = cqx^{-\alpha}$ where $c \geq 0$, $p + q = 1$, $p \geq 0$ and $q \geq 0$.

All that has been done so far in the multi-dimensional case is to find limits of distributions of sequences $A_n(x_1 + \dots + x_n) + a_n$ where A_n is a multiple of the identity operator. By the same techniques as in the one-dimensional case, one finds the Khintchine-Lévy representing measures as mixtures of one-dimensional K-L measures for stable distributions concentrated in rays starting at the origin.

Our results will provide analogues of these facts, except that an explicit representation for a general operator-stable characteristic function does not seem possible.

4. The Khintchine-Lévy formula. A stable distribution is infinitely divisible and a major tool in the analysis of \mathcal{S} will be the multi-dimensional form of the Khintchine-Lévy representation. The form we use is a slight modification of the original Lévy [4] version, and we state it as

PROPOSITION 2 (KHINTCHINE-LÉVY). *To an infinitely divisible measure $\lambda \in \mathcal{P}(V)$, there corresponds a triple (c, ϕ, M) consisting of an element $c \in V$, a nonnegative quadratic form ϕ on V and a nonnegative Radon measure M on the locally compact space $V \sim \{0\}$ satisfying*

- (i) *M is finite off every neighborhood of 0, and*
- (ii) *$\int_{K \sim \{0\}} \|x\|^2 M(dx) < \infty$ for every compact subset K of V , $\|\cdot\|$ being any vector norm on V , such that*

$$\lambda^\wedge(y) = \exp \left\{ i(c, y) - \phi(y) + \int_{V \sim \{0\}} [\exp i(x, y) - 1 - i(\tau(x), y)] M(dx) \right\}.$$

Here, $\tau: V \rightarrow V$ is any continuous function satisfying

- (a) $\tau(x) = x + O(\|x\|^2)$ ($x \rightarrow 0$), and
- (b) τ is bounded.

A change in the function τ produces only a change in the term c .

For uniqueness, we know that if, for $j=1, 2$,

$$\psi_j(y) = i(c_j, y) - \phi_j(y) + \int_{V \sim \{0\}} [\exp i(x, y) - 1 - i(\tau_j(x), y)] M_j(dx)$$

and $\psi_1(y) = \psi_2(y)$ for all $y \in V^\wedge$, then $\phi_1 = \phi_2$ and $M_1 = M_2$. If, further, $\tau_1 = \tau_2$, then $c_1 = c_2$.

We describe this representation by saying that λ has representing triple (c, ϕ, M) . Any nonnegative Radon measure M on $V \sim \{0\}$ satisfying (i) and (ii) will be called a K-L measure. Note that the element c determines merely a translation of λ , and ϕ determines the Gaussian component.

The next proposition is pure calculation, and sets out the manner in which the representing triple changes with an affine transformation of λ .

PROPOSITION 3. *If λ is infinitely divisible in $\mathcal{P}(V)$ and has representing triple (c, ϕ, M) , and if $A \in \text{Aut } V$ and $a \in V$, then the representing triple of $A\lambda * \delta(a)$ is $(c', \phi \circ A^*, AM)$ for some $c' \in V$.*

Proof. In the calculation, it is only necessary to note that if τ satisfies (a) and (b) of Proposition 2, then so does the function $A \circ \tau \circ A^{-1}$. ■

Finally, if λ is infinitely divisible with representing triple (c, ϕ, M) and $t > 0$, we let λ^t denote the t th power of λ ; viz., the infinitely divisible measure with representing triple $(tc, t\phi, tM)$. The semigroup $\{\lambda^t \mid t > 0\}$ is then weakly continuous.

5. Reduction of the problem. The following lemma is used repeatedly in the sequel. It is a generalization of a lemma which is well known in the one-dimensional set-up (see e.g. Feller [1, p. 246, Lemma 1]).

PROPOSITION 4 (THE COMPACTNESS LEMMA). *Suppose that for $n=1, 2, \dots$, $\lambda_n \in \mathcal{P}(V)$, (A_n, a_n) is an affine transformation, and assume that*

- (i) $\lambda_n \rightarrow \lambda \in \mathcal{P}(V)$,
- (ii) $A_n \lambda_n * \delta(a_n) \rightarrow \mu \in \mathcal{P}(V)$.

*Then, if λ and μ are full in V , the set $\{A_n \mid n=1, 2, \dots\}$ is precompact in $\text{Aut } V$, $\{a_n \mid n=1, 2, \dots\}$ is precompact in V , and if A and a are limit points in these respective sets, then $A\lambda * \delta(a) = \mu$.*

In other words, if for $n=1, 2, \dots$, $\lambda_n \sim \mu_n$, if $\lambda_n \rightarrow \lambda$, $\mu_n \rightarrow \mu$ and λ and μ are full, then $\lambda \sim \mu$.

Proof. Let $\langle \cdot, \cdot \rangle$ be an inner product on V , and let $\|x\|^2 = \langle x, x \rangle$. If it is possible to prove that

- (A) $\{a_n \mid n=1, 2, \dots\}$ is bounded in V , and
- (B) $\{\|A_n\| \mid n=1, 2, \dots\}$ is bounded in \mathbb{R} ,

then any sequence $\{n_k\}$ of positive integers will have a subsequence $\{n'_k\}$ such that $a_{n'_k}$ converges to an element a of V , and $A_{n'_k}$ converges to an element E of $\text{End } V$. By joint continuity,

$$A_{n'_k} \lambda_{n'_k} * \delta(a_{n'_k}) \rightarrow E\lambda * \delta(a) \quad \text{as } k \rightarrow \infty.$$

But the left side converges to μ , so $\mu = E\lambda * \delta(a)$. Since $S(\mu) = ES(\lambda) + a$ is not contained in a $(d-1)$ -dimensional hyperplane, E must be invertible, so that all limit points of $\{A_n \mid n=1, 2, \dots\}$ are in $\text{Aut } V$, showing $\{A_n \mid n=1, 2, \dots\}$ to be precompact in $\text{Aut } V$. It suffices, therefore, to prove (A) and (B).

We start with (B). Firstly, the conditions (i) and (ii) imply that ${}^\circ\lambda_n \rightarrow {}^\circ\lambda$ and $A_n({}^\circ\lambda_n) \rightarrow {}^\circ\mu$, where ${}^\circ\lambda$ and ${}^\circ\mu$ are full. In proving (B), we may therefore assume that $a_n = 0$ for all n . With this assumption in force temporarily, we shall assume, for purposes of obtaining a contradiction, that $\{\|A_n\| \mid n=1, 2, \dots\}$ is not bounded.

Let us choose a fixed orthonormal basis for the inner product space $(V, \langle \cdot, \cdot \rangle)$, and think of the operators A_n as matrices, relative to this basis. Now, A_n can be factored into polar form, $A_n = U_n P_n$, where P_n is positive self-adjoint and U_n is orthogonal. The fact that P_n is diagonalizable by orthogonal matrices implies that $A_n = V_n D_n W_n$ where D_n is diagonal, and V_n and W_n are orthogonal. Passing to a subsequence, if necessary, it may be assumed that one entry of D_n tends to infinity as $n \rightarrow \infty$. Modifying V_n if necessary, it may be assumed that it is the first entry, α_n^{-1} , which does so. Then $\alpha_n \rightarrow 0$. Since the orthogonal group is compact, it may be assumed that W_n converges to an orthogonal matrix W . Set $\nu_n = W_n \lambda_n$. We have $A_n W_n^{-1}(\nu_n) = A_n \lambda_n \rightarrow \mu$ and $\nu_n \rightarrow W\lambda = \nu$, also full. Replacing λ_n and A_n by ν_n and $A_n W_n^{-1}$, we see that it may be assumed that A_n can be factored into the form

$V_n D_n, V_n$ orthogonal and D_n diagonal. Let $\varepsilon > 0$ be given: let $\Delta_r = \{x \mid \|x\| \leq r\}$ and let r be chosen so large that

- (iii) $\lambda_n(\Delta) > 1 - \varepsilon/2$ for all n ,
 - (iv) $A_n \lambda_n(\Delta) > 1 - \varepsilon/2$ for all n ,
 - (v) Δ is a continuity set for λ and μ (i.e. $\lambda(\text{bdry } \Delta) = \mu(\text{bdry } \Delta) = 0$).
- Now, $A_n^{-1}(\Delta) = D_n^{-1} V_n^{-1}(\Delta) = D_n^{-1}(\Delta)$, so by (iii) and (iv),

$$\lambda_n(D_n^{-1}(\Delta) \cap \Delta) > 1 - \varepsilon \quad \text{for all } n,$$

and

$$\lambda_n(D_n^{-1}(\Delta)) \rightarrow \mu(\Delta) > 1 - \varepsilon.$$

Notice that $x \in D_n^{-1}(\Delta) \cap \Delta$ implies $\|x\| \leq r$ and $\|D_n x\| \leq r$ so that $|x_j| \leq r$ for $j = 2, \dots, d$ and $|x_1| \leq r \alpha_n$. Let L_n be the rectangle

$$\{x \mid |x_1| \leq \alpha_n r, |x_j| \leq r \text{ for } j = 2, \dots, d\}.$$

Then $D_n^{-1}(\Delta) \cap \Delta \subset L_n$. Define $f_K(x) = \max [(1 - K|x_1|), 0]$, $K > 0$. We have

$$\begin{aligned} \int_V f_K(x) \lambda_n(dx) &\geq \int_{L_n} f_K(x) \lambda_n(dx) \\ &\geq \min \{f_K(x) \mid x \in L_n\} \cdot \lambda_n(L_n) \\ &\geq \lambda_n(D_n^{-1}(\Delta) \cap \Delta) \cdot (1 - K \alpha_n r) \\ &> (1 - \varepsilon)(1 - K \alpha_n r) \quad \text{for all } n, K > 0. \end{aligned}$$

Letting $n \rightarrow \infty$, we find that $\int_V f_K(x) \lambda(dx) \geq 1 - \varepsilon$ for all $K > 0$, hence

$$\lambda\{x \mid |x_1| \geq 1/K\} \geq 1 - \varepsilon \quad \text{for all } K > 0,$$

implying $\lambda\{x \mid x_1 = 0\} \geq 1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lambda\{x \mid x_1 = 0\} = 1$ and λ must be deficient. This contradiction implies the truth of (B).

The proof of (A) now proceeds as follows: by assumption, $\{A_n \lambda_n * \delta(a_n)\}$ is precompact, and since (B) is true, $\{A_n \lambda_n\}$ is precompact; there is therefore a compact subset C of V such that $A_n \lambda_n(C) > 3/4$ and $(A_n \lambda_n * \delta(a_n))(C) > 3/4$ for all n , so that $A_n \lambda_n(C \cap (C - a_n)) > 1/2$ and certainly $C \cap (C - a_n) \neq \emptyset$; we conclude that for all n , $|a_n| \leq \text{diam } C < \infty$, and (A) is proven. ■

COROLLARY 1. *If λ is any full measure in $\mathcal{P}(V)$ and if we define $\text{Inv } \lambda = \{(A, a) \in \text{Aff } V \mid A \lambda * \delta(a) = \lambda\}$, then $\text{Inv } \lambda$ is a compact subgroup of $\text{Aff } V$.*

COROLLARY 2. *In the open subsemigroup \mathcal{F} of full measures in $\mathcal{P}(V)$, the relation “ \sim ” induces compact equivalence classes.*

Both of these corollaries follow trivially from the compactness lemma.

6. A characterization of operator-stable measures. The class \mathcal{S} will now be characterized in a manner more amenable to analysis than that of the original definition.

THEOREM 1. *A full measure $\mu \in \mathcal{P}(V)$ is stable (i.e., $\mu \in \mathcal{S}$) if and only if for each integer $n \geq 1$, there is an affine transformation (B_n, b_n) such that $\mu^n = B_n \mu * \delta(b_n)$. In other words, μ is in \mathcal{S} if and only if μ is full and all powers of μ are of the same type.*

Proof. Firstly, if all powers of μ are of the same type, then for each n ,

$$\mu = C_n \mu^n * \delta(c_n), \quad (C_n, c_n) \in \text{Aff } V$$

so $\mu = \lim C_n \mu^n * \delta(c_n)$ implying $\mu \in \mathcal{S}$.

Suppose now that μ is full and that

(i) $\mu = \lim A_n \lambda^n * \delta(a_n)$, $\lambda \in \mathcal{P}(V)$, $(A_n, a_n) \in \text{Aff } V$.

Since convolution is a jointly continuous operation, we obtain from (i) the equation

$$\mu^m = \lim_n [A_n \lambda^n * \delta(a_n)]^m \quad \text{for } m = 1, 2, \dots$$

and A_n being an automorphism of $\mathcal{P}(V)$, this last equation means

(ii) $\mu^m = \lim_n A_n \mu^{nm} * \delta(m \cdot a_n)$.

Taking the arithmetic subsequence $\{nm \mid n = 1, 2, \dots\}$ of $\{n \mid n = 1, 2, \dots\}$, we obtain from (i) that

(iii) $\mu = \lim_n A_{nm} \mu^{nm} * \delta(a_{nm})$.

Let $\mu_n = A_{nm} \lambda^{nm} * \delta(a_{nm})$. By (iii),

(iv) $\mu_n \rightarrow \mu$.

Equation (ii) can now be written

(v) $C_n \mu_n * \delta(c_n) \rightarrow \mu^m$

where $C_n = A_n A_{nm}^{-1}$ and $c_n = m \cdot a_n - A_n A_{nm}^{-1} a_{nm}$. Since μ and μ^m are full measures, the compactness lemma can be invoked to infer from (iv) and (v) that μ and μ^m are of the same type, for every positive integer n . ■

Our next aim is to extend the last theorem to include all real positive powers of μ . In so doing, we get more information about the affine transformations which appear.

THEOREM 2. *If μ is full and operator-stable, there is an automorphism B of V such that*

$$\mu^t = \exp \{ \log t \cdot B \} \mu * \delta(b(t)); \quad t > 0,$$

for some $b(t) \in V$. The converse is clearly true.

While the proof is not at all difficult, it involves many steps, and for better organization, we proceed with a sequence of simple lemmas. To start with, define $G_t = \{A \in \text{Aut } V \mid \mu^t = A \mu * \delta(a) \text{ for some } a \in V\}$. The set G_t may, a priori, be empty for some $t > 0$, but the last theorem shows that G_t is nonempty whenever t is a positive integer.

LEMMA 1. $G_t \neq \emptyset$ for any $t > 0$.

Proof. If $t > 0$ is rational and equal to j/k , G_j and G_k are nonempty and μ^j and μ^k are of the same type. Therefore, there is an affine transformation (B, b) such that

$\mu^j = B\mu^k * \delta(b)$. But then, $(B\mu * \delta(1/k \cdot b))^k = \mu^j$, and so $B\mu * \delta(1/k \cdot b)$ is the unique infinitely divisible k th root of μ^j , so $B\mu * \delta(1/k \cdot b) = \mu^{j/k} = \mu^t$. Thus, $G_t \neq \emptyset$ if t is rational, and μ^t is of the same type as μ if t is rational. If t is any positive real number, let t_n be rational, and let $t_n \rightarrow t$. Then $\mu^{t_n} \rightarrow \mu^t$ and, if

$$\mu = C_n \mu^{t_n} * \delta(c_n), \quad (C_n, c_n) \in \text{Aff } V,$$

the compactness lemma implies that μ^t is of the same type as μ , so that $G_t \neq \emptyset$. ■

LEMMA 2. $G_t^{-1} = G_{1/t}$ and $G_{s \cdot t} = G_s \cdot G_t$ for all $s, t > 0$.

Proof. It is easy to see that $G_t^{-1} \subset G_{1/t}$ and $G_s \cdot G_t \subset G_{st}$. Replacing t by $1/t$ in the first inclusion and s and t by $1/s$ and st in the second, the reverse inclusions hold. ■

LEMMA 3. $G_s \cap G_t = \emptyset$ if $s \neq t$.

Proof. If $A \in G_s \cap G_t$, say $\mu^s = A\mu * \delta(a)$ and $\mu^t = A\mu * \delta(a')$, then ${}^\circ\mu^s = {}^\circ\mu^t$ implying $|\mu^\wedge(y)|^{2s} \equiv |\mu^\wedge(y)|^{2t}$ for all y . If $s \neq t$, this implies that $|\mu^\wedge(y)| = 0$ or 1 , so $|\mu^\wedge(y)| \equiv 1$, μ^\wedge being continuous. But this would mean that μ is degenerate. ■

LEMMA 4. $G = \bigcup \{G_t \mid t > 0\}$ is a closed subgroup of $\text{Aut } V$.

Proof. That G is a subgroup is the content of Lemma 2. To prove that G is closed in $\text{Aut } V$, let us assume that a sequence $\{A_n\}$ of members of G converges in $\text{Aut } V$ to an automorphism A . We must show $A \in G$. Suppose $A_n \in G_{t_n}$. If $\{t_n\}$ contains a subsequence which converges to 0 in R it may be assumed for the purposes of this argument that $t_n \rightarrow 0$. Then $A_n \mu = \mu^{t_n} * \delta(a_n)$ for some $a_n \in V$, so $A_n({}^\circ\mu) = ({}^\circ\mu)^{t_n} \rightarrow \delta(0)$ as $n \rightarrow \infty$. Thus, $A({}^\circ\mu) = \delta(0)$, and since ${}^\circ\mu$ is full, $A = 0$, a contradiction which establishes the fact that $\{t_n\}$ cannot have a subsequence tending to 0. On the other hand, if t_n has a subsequence tending to ∞ , by setting $B_n = A_n^{-1} \in G_{1/t_n}$, $B_n \rightarrow A^{-1}$ and $1/t_n \rightarrow 0$. The last argument implies that $A^{-1} = 0$. We conclude that $\{t_n\}$ must be bounded away from 0 and ∞ . Let t be any limit point of $\{t_n\}$; it may be assumed that $t_n \rightarrow t$. Then $A_n \mu \rightarrow A\mu$ but

$$A_n \mu = \mu^{t_n} * \delta(a_n) \quad \text{and} \quad \mu^{t_n} \rightarrow \mu^t.$$

Since $A\mu$ and μ are full, the compactness lemma implies $\{a_n\}$ is precompact in V , so if a is any limit point, we have

$$A\mu = \mu^t * \delta(a), \quad \text{so that } A \in G_t \subset G. \quad \blacksquare$$

LEMMA 5. The mapping $\eta: G \rightarrow R^+$, defined by $\eta(A) = t$ if $A \in G_t$, is well defined, and is a continuous open homomorphism of G onto the positive reals under multiplication. The kernel of η is G_1 , a compact normal subgroup of G .

Proof. The mapping η is well defined by Lemma 3. It is a homomorphism because of Lemma 2. To prove that η is continuous, suppose $A_n \rightarrow A_0$ in G , with

$\eta(A_n) = t_n$, so that $A_n \in G_{t_n}$. We have $\mu^{t_n} = A_n \mu * \delta(a_n)$ for some $a_n \in V$, so $(\circ\mu)^{t_n} = A_n(\circ\mu) \rightarrow A_0(\circ\mu) - (\circ\mu)^{t_0}$. But this means $|\mu^\wedge(y)|^{2t_n} \rightarrow |\mu^\wedge(y)|^{2t_0}$, hence $t_n \rightarrow t_0$. The openness of η is now an automatic consequence of its continuity and the σ -compactness of G —see, e.g. Theorem 5.29 of Hewitt and Ross [2]. ■

Proof of Theorem 2. The mapping $\xi = \log \eta$ is a real additive continuous homomorphism of G onto $(R, +)$. Firstly, we demonstrate the existence of a one-parameter subgroup H of G with $\xi(H) = R$. Since G is a closed subgroup of $\text{Gl}(V)$, it is a linear Lie group, so the component G° of the identity in G is an open normal subgroup of G . Since, by Lemma 5, η is open, ξ is open, implying that $\xi(G^\circ)$ is an open subgroup of $(R, +)$ and hence must itself be R . Since G° is the union of its one-parameter subgroups, one of them, H say, must map onto R under the mapping ξ . Now, H , being a one-parameter subgroup of $\text{Gl}(V)$, must be of the form $\{e^{sB} \mid -\infty < s < \infty\}$ for some operator B . The mapping $s \rightarrow e^{sB}$ is a continuous homomorphism of $(R, +)$ into G , so $s \rightarrow e^{sB} \rightarrow \xi(e^{sB})$ is a continuous homomorphism of $(R, +)$ onto $(R, +)$ so that $\xi(e^{sB}) = Ks$ for some constant $K \in R$. Replacing B by $K^{-1}B$, we may assume that $\xi(e^{sB}) = s$, giving $\eta(e^{sB}) = e^s$ or $\eta(e^{\log t \cdot B}) = t$. Thus $\exp \{\log t \cdot B\} \in G_t$ for every $t > 0$.

It remains only to show that B is invertible, if μ is full. If B were noninvertible, B^* would be noninvertible and there would exist $y \in V^\wedge$ such that $B^*y = 0, y \neq 0$. In this case, $\exp \{\log t \cdot B^*\}y = y$ for all $t > 0$, implying

$$|\mu^\wedge{}^t(sy)| = |\mu^\wedge(\exp \{\log t \cdot B^*\}sy)| = |\mu^\wedge(sy)|.$$

This would mean $|\mu^\wedge| \equiv 1$ on the subspace generated by y , a contradiction to the fullness of μ , by Proposition 1. ■

REMARK. We shall denote $\exp \{\log t \cdot B\}$ by the notation t^B .

7. The class \mathcal{B} . The class of operators, B , which can occur in a representation $\mu^t = t^B \mu * \delta(b(t))$ for some full measure μ will be denoted by \mathcal{B} . So far, we know only that \mathcal{B} consists solely of nonsingular operators. It is easy to check, now, that if $B \in \mathcal{B}$ and A is any automorphism, $ABA^{-1} \in \mathcal{B}$. In other words, \mathcal{B} is closed under similarity transformations, and \mathcal{B} will be describable through spectral properties. In fact,

THEOREM 3. *Necessary and sufficient conditions for an operator B to be in the class \mathcal{B} are*

- (i) *The spectrum of B is in the half-plane $\text{Re } z \geq 1/2$, and*
- (ii) *The eigenvalues lying on the line $\text{Re } z = 1/2$ are simple—i.e. the elementary divisors of B associated with these eigenvalues are of first degree.*

Once again, the rather complicated proof obliges us to break it down to more organizable parts. Firstly, we record some facts about the representing triple of a stable distribution.

PROPOSITION 5. *If $\lambda \in S$ has representing triple (c, ϕ, M) and $\lambda^t = t^B \lambda * \delta(b(t))$, $B \in \mathcal{B}$, then*

- (a) $\phi(t^{B^*}y) = t\phi(y)$ for $y \in V^\wedge$ and $t > 0$,
- (b) $t^B M = tM$ for $t > 0$.

Proof. A direct application of Theorem 2 on Proposition 3. ■
 The heart of the proof of Theorem 3 lies in

LEMMA 6. *A measure M concentrated on an orbit $\{t^B x_0 \mid t > 0\}$ and satisfying $t^B M = tM$ is a K-L measure if and only if every eigenvalue of B in the cyclic subspace generated by x_0 has real part greater than $1/2$.*

Proof. Assume firstly that $S(M) \subset \{t^B x_0 \mid t > 0\}$ and $t^B M = tM$. Let $X = [x_0, Bx_0, B^2x_0, \dots] = [x_0]$, the cyclic subspace generated by x_0 . Since B is non-singular, $BX = X$, and since our interest is only in the behavior of B on X , we assume $B = B|X$. B is then a cyclic operator, and by structure theory for such operators (see, e.g. Jacobson [3, p. 73]), to each elementary divisor of B , there is a subspace X_j , such that

- (i) $BX_j = X_j$, and
- (ii) $X = X_1 \oplus \dots \oplus X_k$.

Also, the minimum polynomial $q_j^{m_j}$ of $B|X_j$ is a power of a polynomial which is irreducible over the real field.

Now, set $F(s) = M\{t^B x_0 \mid t > s\}$. The condition $t^B M \equiv tM$ implies that $F(st) = 1/tF(s)$. Thus, $F(t) = K/t$ for some constant $K > 0$. The measure M is a K-L measure if and only if $\int \|x\|^2 M(dx) < \infty$ in a neighborhood of 0. This is the case if and only if

$$\int_0^1 \|t^B x_0\|^2 (-dF/dt) dt < \infty,$$

or

$$(*) \int_0^1 \|t^B x_0\|^2 t^{-2} dt < \infty.$$

Here, $\|\cdot\|$ is any vector norm on X . Any other vector norm would suffice, for all such norms are equivalent on X . We shall specify a norm on X which facilitates computation. Let $\|\cdot\|_j$ be a vector norm on X_j , to be chosen later, and let $\|\sum x_j\| = \sum \|x_j\|_j$. Then $\|\cdot\|$ is a norm on X with the property that $\|\sum x_j\| \geq \|x_r\|_r$ for each r , $1 \leq r \leq k$. Suppose $x_0 = \sum_{j=1}^k x_j$, $x_j \in X_j$. Then $x_j \neq 0$ for each j , otherwise x_0 fails to be cyclic in X . We have then $t^B x_0 = \sum_{j=1}^k t^B x_j = \sum_{j=1}^k t^{B_j} x_j$, where $B_j = B|X_j$, and $\|t^B x_0\| \geq \|t^{B_r} x_r\|_r$, for each r . For the rest of the proof, let r be arbitrary but fixed, $1 \leq r \leq k$. We now choose $\|\cdot\|_r$ in X_r as follows: extend X_r to its complexification X_r^c , and let B_r^c be the extension of B_r to X_r^c in the usual way. We shall define a norm $\|\cdot\|_r^c$ for X_r^c and let $\|\cdot\|_r$ be the restriction of $\|\cdot\|_r^c$ to X_r . For notational convenience, let $J(a_1, \dots, a_k)$ denote the $k \times k$ matrix having all entries equal to zero below the diagonal, a_1 on the principal diagonal, a_2 on the super-diagonal,

..., a_k in $(1, k)$ position. To choose $\|\cdot\|_r^c$, let us firstly choose a (complex) basis $\{\xi_1, \dots, \xi_p\}$ for X_r^c so that the matrix of B_1^c with respect to $\{\xi_1, \dots, \xi_p\}$ is $J(\alpha, 1, 0, \dots, 0)$ if q_j is linear, and

$$\begin{pmatrix} J(\alpha, 1, 0, \dots, 0) & 0 \\ 0 & J(\bar{\alpha}, 1, 0, \dots, 0) \end{pmatrix}$$

if q_j is quadratic. The complex numbers $\alpha, \bar{\alpha}$ are the eigenvalues of B_j , hence of B . Choose $\|\cdot\|_r^c$ by making $\|\sum \alpha_j \xi_j\|_r^c = \sum |\alpha_j|$. Then, we have $\|t^{B_r X_r}\|_r = \|t^{B_r X_r}\|_r^c$, and since the (complex) matrix of t^{B_r} can be seen by an easy computation to be

$$J(t^\alpha, t^\alpha \log t, \dots, 1/(p-1)! t^\alpha (\log t)^{p-1})$$

if q_j is linear, or the obvious extension if q_j is quadratic, ($\|t^{B_r X_r}\|_r^c$)² is seen to be a linear combination, with coefficients depending on x_r , of terms of the form $t^\alpha (\log t)^m t^{\bar{\alpha}} (\log t)^n$, $m \leq p-1, n \leq p-1$, and these terms are each $t^{2 \operatorname{Re} \alpha} (\log t)^q$, for some $q \leq 2p-2$. Then

$$\begin{aligned} \int_0^1 \|t^{B_r X_r}\|^2 t^{-2} dt &\geq \int_0^1 \operatorname{const} t^{2 \operatorname{Re} \alpha} t^{-2} dt \\ &\geq \operatorname{const} \int_0^1 t^{2 \operatorname{Re} \alpha - 2} dt. \end{aligned}$$

The first term is finite, by (*), so $2 \operatorname{Re} \alpha - 2 > -1$, implying $\operatorname{Re} \alpha > 1/2$.

To obtain the converse, note that $\operatorname{Re} \alpha > 1/2$ implies $\int_0^1 t^{2 \operatorname{Re} \alpha - 2} dt < \infty$ and an integration by parts shows that all the terms $t^\alpha (\log t)^m t^{\bar{\alpha}} (\log t)^n$, as above, have finite integrals at 0. Taking the same norms as above, we find

$$\int_0^1 \|t^{B_r X_r}\|_r^2 t^{-2} dt < \infty,$$

so

$$\int_0^1 \|t^{B_r X}\|^2 t^{-2} dt \leq \operatorname{const} \Sigma_r \int_0^1 \|t^{B_r X_r}\|^2 t^{-2} dt < \infty. \quad \blacksquare$$

We now proceed with the

Proof of Theorem 3. Let $\Lambda(B) = \{M \mid M \text{ a K-L measure on } V \sim \{0\} \text{ and } t^B M = tM\}$. Note that if $M \in \Lambda(B)$, $S(M)$ is invariant under t^B for all $t > 0$, so that $S(M)$ is a union of orbits of t^B . The M may be infinite measures, but they are essentially finite in the sense that $W(dx) = \|x\|^2 / (1 + \|x\|^2) M(dx)$ is a finite measure in V . To apply the theory of finite measures, let $\mathfrak{M}(V)$ be the real linear space of finite measures in V , with the topology of weak convergence. Let

$$\Omega(B) = \{W \in \mathfrak{M}(V) \mid W(dx) = \|x\|^2 / (1 + \|x\|^2) M(dx), M \in \Lambda(B)\}.$$

$\Omega(B)$ is easily seen to be a convex cone in \mathfrak{M} , and $\Omega_1(B) = \{W \in \Omega \mid W(V) \leq 1\}$ is a compact convex subset of \mathfrak{M} . Each W in Ω has the property that $S(W)$ is a union of orbits of t^B . Thus, it is easily seen that the extreme points of $\Omega_1(B)$ are the

measures concentrated along a single orbit $\{t^B x_0 \mid t > 0\}$. Thus, the set of convex combinations of such measures is dense in Ω_1 , and this shows that in $\Lambda(B)$, the linear combinations of M 's which are concentrated in a single orbit are dense in $\Lambda(B)$.

Let X now be decomposed into a direct sum of subspaces X_j ($1 \leq j \leq r$) such that $BX_j = X_j$, and such that the minimum polynomial of $B|X_j$ is a power of a real-irreducible polynomial. Assume that the eigenvalues in X_1, \dots, X_k lie in $\text{Re } z > 1/2$ and those in X_{k+1}, \dots, X_r lie in $\text{Re } z \leq 1/2$. Let $X_0 = X_{k+1} + \dots + X_r$. Now, if $x \in X$, and $x = \sum_0^k x_j$, $x_j \in X_j$ then, by Lemma 6, the orbit $\{t^B x \mid t > 0\}$ supports a nonzero K-L measure $M \in \Lambda(B)$ if and only if $x_0 = 0$. Hence, since linear combinations of such M are dense in $\Lambda(B)$, M is concentrated in $X_1 + \dots + X_k$, for all $M \in \Lambda(B)$.

To find how B behaves in X_0 , examine the adjoint $B^* : V^\wedge \rightarrow V^\wedge$. Let, in V^\wedge ,

$$Y_j = (X_0 \oplus \dots \oplus X_{j-1} \oplus X_{j+1} \oplus \dots \oplus X_k)^\perp, \quad 0 \leq j \leq k.$$

Y_j is then the dual of X_j , and $B^*|Y_j = (B|X_j)^*$. It is a consequence of the Khintchine-Lévy formula that

$$\log \circ \mu^\wedge(y) = -2\phi(y) + \int (\cos(x, y) - 1)M(dx)$$

and if $y \in Y_0$, since M is concentrated in $X_1 \oplus \dots \oplus X_k$,

$$\log \circ \mu^\wedge(y) = -2\phi(y).$$

Since μ is assumed full, $\phi(y) \neq 0$ if $y \in Y_0$, $y \neq 0$. Thus, the quadratic form $\phi|Y_0$ is nondegenerate, and there is a basis $\{y_1, \dots, y_q\}$ in Y_0 such that

$$\phi\left(\sum_1^q a_j y_j\right) = \sum_1^q a_j^2.$$

With respect to this basis $\{y_1, \dots, y_q\}$, a bilinear form $\langle \cdot, \cdot \rangle$ is defined, and adjoints C' of operators C are defined by

$$\langle Cy_1, y_2 \rangle = \langle y_1, C'y_2 \rangle.$$

Then, the condition 5(a) reads that for all $y \in Y_0$ and $t > 0$,

$$\phi(t^{B^*}y) = t\phi(y).$$

Since $B^*Y_0 = Y_0$, it will not hurt to assume in this section that $B^* = B^*|Y_0$. Then, for all $y \in Y_0$,

$$\langle t^{B^*}y, t^{B^*}y \rangle = t\langle y, y \rangle.$$

Thus,

$$\langle t^{B^*}t^{B^*}y, y \rangle = \langle ty, y \rangle$$

or

$$t^{B^*}t^{B^*} \equiv tI \equiv t^{1/2}I \cdot t^{1/2}I.$$

Since $t^{1/2}I \equiv t^{1/2}$, this can be written as

$$t^{(B^*-I/2)}t^{(B^*-I/2)'} \equiv I.$$

Setting $C = B^* - I/2$, we find that $t^{C+C'} \equiv I$ for all $t > 0$, so that $C + C' = 0$. Thus, in this basis, C is a skew operator. Hence $B^* = I/2 + C$ is a normal operator, and all its elementary divisors are linear. This proves the necessity of the conditions in the theorem. To prove the sufficiency of the conditions, let B satisfy (i) and (ii). As in the proof of the necessity, let $X = X_1 \oplus \dots \oplus X_r$, X_j having the same meaning as before. If $x_j \in X_j$, $x_j \neq 0$, ($1 \leq j \leq k$), then, by Lemma 6, the orbit $\{t^B x_j \mid t > 0\}$ supports a nonzero K-L measure $M_j \in \Lambda(B)$. Since the orbit $\{t^B x_j \mid t > 0\}$ generates X_j , the stable measure λ_j with triple $(0, 0, M_j)$ is supported, and is full, in X_j . On $X_0 = X_{k+1} \oplus \dots \oplus X_r$, we construct a Gaussian measure μ satisfying $\mu^t = \exp \{ \log t \cdot B \mid X_0 \} \mu$. We can imagine that $X = X_0$, hence that B may be put into a canonical form $\text{diag} \{ J_1, \dots, J_q \}$ where J_j is either a 1×1 matrix with element $1/2$, or a 2×2 matrix

$$\begin{pmatrix} 1/2 & \beta \\ -\beta & 1/2 \end{pmatrix}.$$

With the dual basis $\{ \eta_1, \dots, \eta_p \}$ in Y_0 , B^* has canonical form $\text{diag} \{ J_1^*, \dots, J_q^* \}$. We have to construct a quadratic form ϕ on Y_0 such that $\phi(t^{B^*} y) \equiv t\phi(y)$. Consider $\phi(\sum a_i \eta_i) = \sum a_i^2$. If $[\eta_1] = X_j$, $t^{J_j} = t^{1/2} I$ and $\phi(t^{J_j} a \eta_i) = t\phi(a \eta_i)$. If $[\eta_i, \eta_{i+1}] = X_j$, we have

$$t^{J_j} = t^{1/2} \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}$$

and, once again, $\phi(t^{J_j}(a \eta_i + b \eta_{i+1})) = t\phi(a \eta_i + b \eta_{i+1})$. Thus, $e^{-\phi(y)}$ is a Gaussian characteristic function, and its distribution is full and concentrated in X_0 . The measure $\nu = \lambda_1 * \dots * \lambda_k * \mu$ is now full and stable, and $\nu^t = t^B \nu$. ■

As by-products of the proof of Theorem 3, we can assert the following:

THEOREM 4. Any full operator-stable measure λ on V can be decomposed into a product $\lambda = \lambda_1 * \lambda_2$ of measures λ_i concentrated in subspaces V_i , $V = V_1 \oplus V_2$, where λ_1 is a full Gaussian measure in V_1 and λ_2 is a full operator-stable measure on V_2 having no Gaussian component.

THEOREM 5. Any K-L measure M for a full operator-stable measure λ on V can be represented as a mixture of K-L measures M_θ where M_θ is a K-L measure concentrated in an orbit, θ , of t^B and satisfies $t^B M_\theta = t M_\theta$. The measure M_θ is characterized by the condition that $s M_\theta \{ t^B x_0 \mid t > s \}$ is constant for all s , when x_0 is a generator of the orbit θ .

If λ is full and operator-stable, and satisfies $\lambda^t = t^B \lambda * \delta(b(t))$ for some $b(t)$, we shall call B an exponent for λ . A measure λ may possess more than one exponent. For example, if V is given an inner product with respect to which λ is a rotation-invariant Gaussian measure, then $B = I/2$ is an exponent for λ , as is any operator of the form $I/2 + C$ where C is skew.

The operators in the class \mathcal{B} can be interpreted as exponents of the normalizing factors which give rise to operator-stable laws. That is, if x_1, x_2, \dots are independent

full operator-stable random variables having the same distribution, then each has the same distribution as some translate of $n^{-B}(x_1 + \dots + x_n)$, for $n = 1, 2, \dots$

8. Centering. We turn, finally, to an examination of the term $b(t)$ in the formula

$$(8.1) \quad \lambda^t = t^B \lambda * \delta(b(t)) \quad \text{for all } t > 0,$$

satisfied by a full operator-stable measure λ with exponent B . Taking sth powers on both sides of (8.1), we find that

$$\lambda^{st} = t^B \lambda^s * \delta(sb(t)) = t^{Bs} \lambda * \delta(t^B b(s) + sb(t)).$$

But $\lambda^{st} = (st)^B \lambda * \delta(b(st))$, from (8.1), so that the vector-valued function $b(\cdot)$ must satisfy the functional equation

$$(8.2) \quad b(st) = t^B b(s) + sb(t) \quad \text{for all } s > 0, t > 0.$$

One consequence, obtained by setting $s = t = 1$, is that $b(1) = 0$.

THEOREM 6. *If 1 is not in the spectrum of B , the general solution of (8.2) is*

$$(8.3) \quad b(t) = tx_0 - t^B x_0, \quad t > 0,$$

for some $x_0 \in V$, and in this case, when the full operator-stable measure λ satisfying (8.1) is centered at x_0 , the measure $\mu = \lambda * \delta(-x_0)$ satisfies $\mu^t = t^B \mu$.

REMARK. By analogy with the one-dimensional case, an operator-stable measure μ satisfying $\mu^t = t^B \mu$ could be called strictly stable. Our theorem shows that when 1 is not in the spectrum of B , any full operator-stable measure can be centered so as to become strictly stable.

Proof. Once we have proven (8.3), the second assertion of the theorem follows by a standard calculation. As a first step in proving (8.3), note that if $b_1(\cdot)$ and $b_2(\cdot)$ are solutions of (8.2), then $b_1(\cdot) - b_2(\cdot)$ is a solution of (8.2). Note also that if $b(\cdot)$ is a solution of (8.2) such that $b(t_0) = 0$ for some $t_0 \neq 1$, then

$$b(st_0) = t_0 b(s) = t_0^B b(s) \quad \text{for all } s > 0.$$

Since 1 is not an eigenvalue of B , t_0 is not an eigenvalue of t_0^B , so we must have $b(s) \equiv 0$. These observations imply that any two solutions of (8.2) which agree at even one point must in fact be identical.

Now, the operator $(tI - t^B)$ is invertible, for $t \neq 1$, hence we can always solve an equation

$$(tI - t^B)x_0 = x_1 \quad \text{for } x_0 \in V, \text{ for all } x_1 \in V.$$

Hence, the solutions $b(t) = tx_0 - t^B x_0$ constitute all possible solutions of (8.2). ■

In the event that B has 1 as an eigenvalue, it may not be possible to center λ as in the last theorem. This is in analogy with the one-dimensional case of the so-called asymmetric Cauchy distribution. An example is given by taking $B = I$ so that (8.2) becomes

$$b(st) = sb(t) + tb(s),$$

which is satisfied by the function $b(t) = t \log tx_0$, for all $x_0 \in V$. We can clearly not absorb such a factor to center λ .

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UNIVERSITY OF CALIFORNIA AT SAN DIEGO,
LA JOLLA, CALIFORNIA
YALE UNIVERSITY,
NEW HAVEN, CONNECTICUT