

# OPERATOR-STABLE PROBABILITY DISTRIBUTIONS ON VECTOR GROUPS

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**1. Introduction.** An operator-stable probability distribution in a group  $G$  is a limit law arising, roughly speaking, from affine modification of the partial sums of a sequence of independent identically distributed  $G$ -valued random variables. This paper is concerned with a more palpable description of an operator-stable distribution in the case where  $G$  is a vector group; i.e. the topological group underlying a  $d$ -dimensional real vector space.

The first step is to reduce the problem to that of finding those probability measures  $\lambda$ , all of whose convolution powers are of the same type; i.e. for each integer  $k \geq 1$ , there is an automorphism  $B_k$  of  $G$  such that  $\lambda^k$  is a translation of the measure  $\lambda \circ B_k^{-1}$ . It is then shown that  $\lambda$  is operator-stable if and only if for each  $t > 0$ ,  $\lambda^t$  is a translation of the measure  $\lambda \circ [\exp(\log t \cdot B)]^{-1}$ , for an automorphism  $B$  of  $G$  characterized by conditions on the spectrum of  $B$  regarded as a linear operator on a vector space.

**2. Notation and definitions.** Throughout this paper, we denote by  $V$  a  $d$ -dimensional vector group. In several proofs, we shall use the same symbol to denote the same group with the additional structure of a vector space, or even an inner product space. Denote by  $\mathcal{P} = \mathcal{P}(V)$  the set of probability measures on  $V$ . With the topology of weak convergence, and multiplication defined by convolution,  $\mathcal{P}$  becomes a topological semigroup. We denote convolution of two measures  $\lambda, \mu$  by  $\lambda * \mu$ , and the  $n$ th convolution power of  $\lambda$  by  $\lambda^n$ .

By  $V^\wedge$ , we mean the character group of  $V$ , "identified" with the dual vector space of the vector space  $V$ . In the sequel,  $x$  will denote the generic element of  $V$  and  $y$  the generic element of  $V^\wedge$ . Let  $(x, y)$  denote the bilinear pairing of  $V$  and  $V^\wedge$  brought about by  $y$  acting as a linear functional on  $x$ . As a group character,  $y$  acts on  $x$  according to  $\exp i(x, y)$ . The characteristic function of a measure  $\lambda \in \mathcal{P}(V)$  is defined by

$$\varphi(y) = \lambda^\wedge(y) = \int_V \exp i(x, y) \lambda(dy).$$

Given  $\lambda \in \mathcal{P}(V)$ , we define  $\lambda^- \in \mathcal{P}(V)$  by  $\lambda^-(E) = \lambda(-E)$ . The mapping  $\lambda \rightarrow \lambda^-$  is an involutive automorphism of  $\mathcal{P}$ . It is easy to check that  $\lambda^{-\wedge} \equiv \lambda^\wedge^-$ , the last

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bar denoting the complex conjugate. The measure  $\lambda$  is called symmetric if and only if  $\lambda = \lambda^-$ . For any  $\lambda \in \mathcal{P}$ , the measure  ${}^\circ\lambda = \lambda * \lambda^-$  is symmetric, and is called the symmetrization of  $\lambda$ .

Let  $H(\lambda) = \{y \in V^\wedge \mid \lambda^\wedge(y) = 1\}$ . It is easy to see that  $H(\lambda)$  is a closed subgroup of  $V^\wedge$ . Note that  $\{y \in V^\wedge \mid |\lambda^\wedge(y)| = 1\} = H({}^\circ\lambda)$  is a closed subgroup of  $V^\wedge$  containing  $H(\lambda)$ .

Denote by  $S(\lambda)$  the support of the measure  $\lambda \in \mathcal{P}(V)$ , viz., the smallest closed set  $F$  in  $V$  with  $\lambda(F) = 1$ . The relation  $S(\lambda * \mu) = \text{Cl}(S(\lambda) + S(\mu))$  is well known. We call a measure  $\lambda \in \mathcal{P}$  full if and only if  $S(\lambda)$  is not contained in any  $(d-1)$ -dimensional hyperplane of  $V$ . If  $\lambda$  is not full, it is called deficient.

**PROPOSITION 1.**  $\lambda \in \mathcal{P}(V)$  is a full measure if and only if  $H({}^\circ\lambda)$  does not contain a 1-dimensional subgroup of  $V^\wedge$ .

**Proof.** Firstly,  $\lambda$  is full if and only if  ${}^\circ\lambda$  is full. If  $S({}^\circ\lambda) \subset X$ , a  $(d-1)$ -dimensional subspace of  $V$ , then

$$\begin{aligned} {}^\circ\lambda^\wedge(y) &= \int_X \exp i(x, y) {}^\circ\lambda(dx) \\ &= 1 \quad \text{if } y \in X^\perp = \{y \in V^\wedge \mid (x, y) = 0, \forall x \in X\}, \end{aligned}$$

and  $X^\perp$  is a 1-dimensional subspace of  $V^\wedge$ . Conversely, if  $H({}^\circ\lambda)$  contains a 1-dimensional subspace  $Y$  of  $V^\wedge$ , then  $\int_V \exp i(x, y) {}^\circ\lambda(dx) = 1$  for all  $y \in Y$  implies  $\exp i(x, y) = 1$ ,  ${}^\circ\lambda$ -almost everywhere, for all  $y \in Y$ , which implies

$${}^\circ\lambda\{x \mid \exp i(x, y) = 1\} = 1 \quad \text{for all } y \in Y$$

so that

$$S({}^\circ\lambda) \subset \{x \mid (x, y) = 0 \pmod{2\pi} \text{ for all } y \in Y\} = \{x \mid (x, y) = 0 \text{ for all } y \in Y\} = Y^\perp,$$

a  $(d-1)$ -dimensional subspace of  $V$ . ■

We mention that the set  $\mathcal{F}$  of full measures in  $\mathcal{P}(V)$  is an open subsemigroup of  $\mathcal{P}$ .

Let us denote by  $\text{End } V$  the ring of continuous endomorphisms of the group  $V$ , identified with the ring of linear transformations of the vector space  $V$ . Let  $\text{Aut } V$  denote the group of continuous automorphisms of  $V$ , identified with the group  $\text{Gl}(V)$  of nonsingular linear transformations of the vector space  $V$ . For any  $A \in \text{End } V$  and  $\lambda \in \mathcal{P}(V)$ , let  $A\lambda$  denote the measure  $A\lambda(F) = \lambda(A^{-1}(F))$ ,  $F$  a Borel subset of  $V$ . If  $A \in \text{Aut } V$  and  $a \in V$ , we call the mapping  $\lambda \rightarrow A\lambda * \delta(a)$  an affine transformation,  $\delta(a)$  denoting the point mass at  $a$ . If instead  $A \in \text{End } V \sim \text{Aut } V$ , the mapping will be called singular affine. The set of affine transformations  $(A, a)$  is denoted by  $\text{Aff } V$ . If  $x$  is a  $V$ -valued random variable having distribution  $\mu$ , then clearly,  $Ax + a$  has distribution  $A\lambda * \delta(a)$ .

It is not difficult to verify that for any bounded continuous function  $f$  on  $V$ ,

$$\int_V f d(A\lambda) = \int_V f \circ A d\lambda,$$

and that  $S(A\lambda) = AS(\lambda)$ ,  $(AB)\lambda = A(B\lambda)$ ,  $A(\lambda * \mu) = A\lambda * A\mu$  and  $(A\lambda)^\wedge(y) = \lambda^\wedge(A^*y)$ , where  $A^*$  denotes the adjoint map of  $V^\wedge \rightarrow V^\wedge$  induced by the bilinear pairing  $(\cdot, \cdot)$ ; i.e.  $(Ax, y) \equiv (x, A^*y)$ . We give  $\text{End } V$  the compact-open topology as a set of functions from  $V$  to  $V$ . This topology is equivalent to the norm topology of  $\text{End } V$  as a space of linear operators on  $V$ , when  $V$  is provided with a vector norm. It is easy to check that the mapping  $\langle A, \lambda \rangle \rightarrow A\lambda$  of  $\text{End } V \times \mathcal{P}(V) \rightarrow \mathcal{P}(V)$  is jointly continuous. We define an equivalence relation in  $\mathcal{P}(V)$  by writing  $\lambda \sim \mu$  if and only if there is an affine transformation  $(A, a)$  such that  $\lambda = A\mu * \delta(a)$ . The measures  $\lambda, \mu$  are then said to be of the same type, and an equivalence class of measures is called a type. Clearly, random variables  $x$  and  $Ax + a$  will have distributions of the same type.

**3. Statement of the problem.** In terms of random variables, the problem we study is enunciated as follows: suppose that  $\{x_n\}$  is a sequence of  $V$ -valued random variables with common distribution  $\mu$ , and assume that the terms of the sequence are mutually independent; assume, further, that  $(A_n, a_n)$  is a sequence of affine transformations of  $V$  such that the distribution of  $A_n(x_1 + \dots + x_n) + a_n$  converges to a measure  $\lambda \in \mathcal{P}(V)$ ; what can be said about the limit measure  $\lambda$ ?

Converting this to a problem involving only measures, we ask which measures  $\lambda$  can arise as limits of sequences  $A_n\mu^n * \delta(a_n)$ .

Our aim is to characterize those  $\lambda$  which are full and are such limits—denote by  $\mathcal{S}$  the class of such measures. For reasons which do not become clear until the problem is posed in a different form,  $\mathcal{S}$  is called the class of operator-stable measures on  $V$ . It is clear that  $\mathcal{S}$  is invariant under affine transformations.

We refer the reader to Feller [1] for an account of the class  $\mathcal{S}$  when  $V$  reduces to the real line. It is possible in this case to give an explicit formula for the characteristic function of a stable distribution. In fact (Feller [1, p. 542]) any such characteristic function  $\lambda^\wedge$  must have the form  $\lambda^\wedge(y) = \exp(a\psi_{\alpha, \delta}(y) + iby)$ , where  $0 < \alpha \leq 2$ ,  $-1 \leq \delta \leq 1$ , and

$$\begin{aligned} \psi_{\alpha, \delta}(y) &= -|y|^\alpha [1 - i \operatorname{sgn} y \delta \tan \pi\alpha/2], & \alpha \neq 1, \\ &= -|y| [1 + i \operatorname{sgn} y \delta \log |y|], & \alpha = 1. \end{aligned}$$

In particular,  $\lambda^{\wedge t}(y) = \lambda^\wedge(t^{1/\alpha}y) \exp ib(t)y$ ,  $t > 0$  where  $b(t)$  is real. It is easily seen that if  $\alpha \neq 1$ , there exists  $b$  such that  $\mu = \lambda * \delta(b)$  satisfies  $\mu^{\wedge t}(y) = \lambda^\wedge(t^{1/\alpha}y)$  but if  $\alpha = 1$ , it is not generally possible to make such a centering, and  $b(t)$  has the form  $b(t) = ct \log t$ . These results are obtained from the fact that the Khintchine-Lévy measure  $M$  (see next paragraph) of a stable distribution must be given by  $M\{[x, \infty)\} = cp x^{-\alpha}$ ,  $M\{(-\infty, -x]\} = cq x^{-\alpha}$  where  $c \geq 0$ ,  $p + q = 1$ ,  $p \geq 0$  and  $q \geq 0$ .

All that has been done so far in the multi-dimensional case is to find limits of distributions of sequences  $A_n(x_1 + \dots + x_n) + a_n$  where  $A_n$  is a multiple of the identity operator. By the same techniques as in the one-dimensional case, one finds the Khintchine-Lévy representing measures as mixtures of one-dimensional K-L measures for stable distributions concentrated in rays starting at the origin.

Our results will provide analogues of these facts, except that an explicit representation for a general operator-stable characteristic function does not seem possible.

**4. The Khintchine-Lévy formula.** A stable distribution is infinitely divisible and a major tool in the analysis of  $\mathcal{S}$  will be the multi-dimensional form of the Khintchine-Lévy representation. The form we use is a slight modification of the original Lévy [4] version, and we state it as

**PROPOSITION 2 (KHINTCHINE-LÉVY).** *To an infinitely divisible measure  $\lambda \in \mathcal{P}(V)$ , there corresponds a triple  $(c, \phi, M)$  consisting of an element  $c \in V$ , a nonnegative quadratic form  $\phi$  on  $V$  and a nonnegative Radon measure  $M$  on the locally compact space  $V \sim \{0\}$  satisfying*

- (i)  *$M$  is finite off every neighborhood of 0, and*
- (ii)  *$\int_{K \sim \{0\}} \|x\|^2 M(dx) < \infty$  for every compact subset  $K$  of  $V$ ,  $\|\cdot\|$  being any vector norm on  $V$ , such that*

$$\lambda^\wedge(y) = \exp \left\{ i(c, y) - \phi(y) + \int_{V \sim \{0\}} [\exp i(x, y) - 1 - i(\tau(x), y)] M(dx) \right\}.$$

Here,  $\tau: V \rightarrow V$  is any continuous function satisfying

- (a)  $\tau(x) = x + O(\|x\|^2)$  ( $x \rightarrow 0$ ), and
- (b)  $\tau$  is bounded.

A change in the function  $\tau$  produces only a change in the term  $c$ .

For uniqueness, we know that if, for  $j=1, 2$ ,

$$\psi_j(y) = i(c_j, y) - \phi_j(y) + \int_{V \sim \{0\}} [\exp i(x, y) - 1 - i(\tau_j(x), y)] M_j(dx)$$

and  $\psi_1(y) = \psi_2(y)$  for all  $y \in V^\wedge$ , then  $\phi_1 = \phi_2$  and  $M_1 = M_2$ . If, further,  $\tau_1 = \tau_2$ , then  $c_1 = c_2$ .

We describe this representation by saying that  $\lambda$  has representing triple  $(c, \phi, M)$ . Any nonnegative Radon measure  $M$  on  $V \sim \{0\}$  satisfying (i) and (ii) will be called a K-L measure. Note that the element  $c$  determines merely a translation of  $\lambda$ , and  $\phi$  determines the Gaussian component.

The next proposition is pure calculation, and sets out the manner in which the representing triple changes with an affine transformation of  $\lambda$ .

**PROPOSITION 3.** *If  $\lambda$  is infinitely divisible in  $\mathcal{P}(V)$  and has representing triple  $(c, \phi, M)$ , and if  $A \in \text{Aut } V$  and  $a \in V$ , then the representing triple of  $A\lambda * \delta(a)$  is  $(c', \phi \circ A^*, AM)$  for some  $c' \in V$ .*

**Proof.** In the calculation, it is only necessary to note that if  $\tau$  satisfies (a) and (b) of Proposition 2, then so does the function  $A \circ \tau \circ A^{-1}$ . ■

Finally, if  $\lambda$  is infinitely divisible with representing triple  $(c, \phi, M)$  and  $t > 0$ , we let  $\lambda^t$  denote the  $t$ th power of  $\lambda$ ; viz., the infinitely divisible measure with representing triple  $(tc, t\phi, tM)$ . The semigroup  $\{\lambda^t \mid t > 0\}$  is then weakly continuous.

**5. Reduction of the problem.** The following lemma is used repeatedly in the sequel. It is a generalization of a lemma which is well known in the one-dimensional set-up (see e.g. Feller [1, p. 246, Lemma 1]).

**PROPOSITION 4 (THE COMPACTNESS LEMMA).** *Suppose that for  $n=1, 2, \dots$ ,  $\lambda_n \in \mathcal{P}(V)$ ,  $(A_n, a_n)$  is an affine transformation, and assume that*

- (i)  $\lambda_n \rightarrow \lambda \in \mathcal{P}(V)$ ,
- (ii)  $A_n \lambda_n * \delta(a_n) \rightarrow \mu \in \mathcal{P}(V)$ .

*Then, if  $\lambda$  and  $\mu$  are full in  $V$ , the set  $\{A_n \mid n=1, 2, \dots\}$  is precompact in  $\text{Aut } V$ ,  $\{a_n \mid n=1, 2, \dots\}$  is precompact in  $V$ , and if  $A$  and  $a$  are limit points in these respective sets, then  $A\lambda * \delta(a) = \mu$ .*

In other words, if for  $n=1, 2, \dots$ ,  $\lambda_n \sim \mu_n$ , if  $\lambda_n \rightarrow \lambda$ ,  $\mu_n \rightarrow \mu$  and  $\lambda$  and  $\mu$  are full, then  $\lambda \sim \mu$ .

**Proof.** Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $V$ , and let  $\|x\|^2 = \langle x, x \rangle$ . If it is possible to prove that

- (A)  $\{a_n \mid n=1, 2, \dots\}$  is bounded in  $V$ , and
- (B)  $\{\|A_n\| \mid n=1, 2, \dots\}$  is bounded in  $\mathbb{R}$ ,

then any sequence  $\{n_k\}$  of positive integers will have a subsequence  $\{n'_k\}$  such that  $a_{n'_k}$  converges to an element  $a$  of  $V$ , and  $A_{n'_k}$  converges to an element  $E$  of  $\text{End } V$ . By joint continuity,

$$A_{n'_k} \lambda_{n'_k} * \delta(a_{n'_k}) \rightarrow E\lambda * \delta(a) \quad \text{as } k \rightarrow \infty.$$

But the left side converges to  $\mu$ , so  $\mu = E\lambda * \delta(a)$ . Since  $S(\mu) = ES(\lambda) + a$  is not contained in a  $(d-1)$ -dimensional hyperplane,  $E$  must be invertible, so that all limit points of  $\{A_n \mid n=1, 2, \dots\}$  are in  $\text{Aut } V$ , showing  $\{A_n \mid n=1, 2, \dots\}$  to be precompact in  $\text{Aut } V$ . It suffices, therefore, to prove (A) and (B).

We start with (B). Firstly, the conditions (i) and (ii) imply that  ${}^\circ\lambda_n \rightarrow {}^\circ\lambda$  and  $A_n({}^\circ\lambda_n) \rightarrow {}^\circ\mu$ , where  ${}^\circ\lambda$  and  ${}^\circ\mu$  are full. In proving (B), we may therefore assume that  $a_n = 0$  for all  $n$ . With this assumption in force temporarily, we shall assume, for purposes of obtaining a contradiction, that  $\{\|A_n\| \mid n=1, 2, \dots\}$  is not bounded.

Let us choose a fixed orthonormal basis for the inner product space  $(V, \langle \cdot, \cdot \rangle)$ , and think of the operators  $A_n$  as matrices, relative to this basis. Now,  $A_n$  can be factored into polar form,  $A_n = U_n P_n$ , where  $P_n$  is positive self-adjoint and  $U_n$  is orthogonal. The fact that  $P_n$  is diagonalizable by orthogonal matrices implies that  $A_n = V_n D_n W_n$  where  $D_n$  is diagonal, and  $V_n$  and  $W_n$  are orthogonal. Passing to a subsequence, if necessary, it may be assumed that one entry of  $D_n$  tends to infinity as  $n \rightarrow \infty$ . Modifying  $V_n$  if necessary, it may be assumed that it is the first entry,  $\alpha_n^{-1}$ , which does so. Then  $\alpha_n \rightarrow 0$ . Since the orthogonal group is compact, it may be assumed that  $W_n$  converges to an orthogonal matrix  $W$ . Set  $\nu_n = W_n \lambda_n$ . We have  $A_n W_n^{-1}(\nu_n) = A_n \lambda_n \rightarrow \mu$  and  $\nu_n \rightarrow W\lambda = \nu$ , also full. Replacing  $\lambda_n$  and  $A_n$  by  $\nu_n$  and  $A_n W_n^{-1}$ , we see that it may be assumed that  $A_n$  can be factored into the form

$V_n D_n, V_n$  orthogonal and  $D_n$  diagonal. Let  $\varepsilon > 0$  be given: let  $\Delta_r = \{x \mid \|x\| \leq r\}$  and let  $r$  be chosen so large that

- (iii)  $\lambda_n(\Delta) > 1 - \varepsilon/2$  for all  $n$ ,
  - (iv)  $A_n \lambda_n(\Delta) > 1 - \varepsilon/2$  for all  $n$ ,
  - (v)  $\Delta$  is a continuity set for  $\lambda$  and  $\mu$  (i.e.  $\lambda(\text{bdry } \Delta) = \mu(\text{bdry } \Delta) = 0$ ).
- Now,  $A_n^{-1}(\Delta) = D_n^{-1} V_n^{-1}(\Delta) = D_n^{-1}(\Delta)$ , so by (iii) and (iv),

$$\lambda_n(D_n^{-1}(\Delta) \cap \Delta) > 1 - \varepsilon \quad \text{for all } n,$$

and

$$\lambda_n(D_n^{-1}(\Delta)) \rightarrow \mu(\Delta) > 1 - \varepsilon.$$

Notice that  $x \in D_n^{-1}(\Delta) \cap \Delta$  implies  $\|x\| \leq r$  and  $\|D_n x\| \leq r$  so that  $|x_j| \leq r$  for  $j = 2, \dots, d$  and  $|x_1| \leq r \alpha_n$ . Let  $L_n$  be the rectangle

$$\{x \mid |x_1| \leq \alpha_n r, |x_j| \leq r \text{ for } j = 2, \dots, d\}.$$

Then  $D_n^{-1}(\Delta) \cap \Delta \subset L_n$ . Define  $f_K(x) = \max [(1 - K|x_1|), 0]$ ,  $K > 0$ . We have

$$\begin{aligned} \int_V f_K(x) \lambda_n(dx) &\geq \int_{L_n} f_K(x) \lambda_n(dx) \\ &\geq \min \{f_K(x) \mid x \in L_n\} \cdot \lambda_n(L_n) \\ &\geq \lambda_n(D_n^{-1}(\Delta) \cap \Delta) \cdot (1 - K \alpha_n r) \\ &> (1 - \varepsilon)(1 - K \alpha_n r) \quad \text{for all } n, K > 0. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we find that  $\int_V f_K(x) \lambda(dx) \geq 1 - \varepsilon$  for all  $K > 0$ , hence

$$\lambda\{x \mid |x_1| \geq 1/K\} \geq 1 - \varepsilon \quad \text{for all } K > 0,$$

implying  $\lambda\{x \mid x_1 = 0\} \geq 1 - \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary,  $\lambda\{x \mid x_1 = 0\} = 1$  and  $\lambda$  must be deficient. This contradiction implies the truth of (B).

The proof of (A) now proceeds as follows: by assumption,  $\{A_n \lambda_n * \delta(a_n)\}$  is precompact, and since (B) is true,  $\{A_n \lambda_n\}$  is precompact; there is therefore a compact subset  $C$  of  $V$  such that  $A_n \lambda_n(C) > 3/4$  and  $(A_n \lambda_n * \delta(a_n))(C) > 3/4$  for all  $n$ , so that  $A_n \lambda_n(C \cap (C - a_n)) > 1/2$  and certainly  $C \cap (C - a_n) \neq \emptyset$ ; we conclude that for all  $n$ ,  $|a_n| \leq \text{diam } C < \infty$ , and (A) is proven. ■

**COROLLARY 1.** *If  $\lambda$  is any full measure in  $\mathcal{P}(V)$  and if we define  $\text{Inv } \lambda = \{(A, a) \in \text{Aff } V \mid A \lambda * \delta(a) = \lambda\}$ , then  $\text{Inv } \lambda$  is a compact subgroup of  $\text{Aff } V$ .*

**COROLLARY 2.** *In the open subsemigroup  $\mathcal{F}$  of full measures in  $\mathcal{P}(V)$ , the relation “ $\sim$ ” induces compact equivalence classes.*

Both of these corollaries follow trivially from the compactness lemma.

**6. A characterization of operator-stable measures.** The class  $\mathcal{S}$  will now be characterized in a manner more amenable to analysis than that of the original definition.

**THEOREM 1.** *A full measure  $\mu \in \mathcal{P}(V)$  is stable (i.e.,  $\mu \in \mathcal{S}$ ) if and only if for each integer  $n \geq 1$ , there is an affine transformation  $(B_n, b_n)$  such that  $\mu^n = B_n \mu * \delta(b_n)$ . In other words,  $\mu$  is in  $\mathcal{S}$  if and only if  $\mu$  is full and all powers of  $\mu$  are of the same type.*

**Proof.** Firstly, if all powers of  $\mu$  are of the same type, then for each  $n$ ,

$$\mu = C_n \mu^n * \delta(c_n), \quad (C_n, c_n) \in \text{Aff } V$$

so  $\mu = \lim C_n \mu^n * \delta(c_n)$  implying  $\mu \in \mathcal{S}$ .

Suppose now that  $\mu$  is full and that

(i)  $\mu = \lim A_n \lambda^n * \delta(a_n)$ ,  $\lambda \in \mathcal{P}(V)$ ,  $(A_n, a_n) \in \text{Aff } V$ .

Since convolution is a jointly continuous operation, we obtain from (i) the equation

$$\mu^m = \lim_n [A_n \lambda^n * \delta(a_n)]^m \quad \text{for } m = 1, 2, \dots$$

and  $A_n$  being an automorphism of  $\mathcal{P}(V)$ , this last equation means

(ii)  $\mu^m = \lim_n A_n \mu^{nm} * \delta(m \cdot a_n)$ .

Taking the arithmetic subsequence  $\{nm \mid n = 1, 2, \dots\}$  of  $\{n \mid n = 1, 2, \dots\}$ , we obtain from (i) that

(iii)  $\mu = \lim_n A_{nm} \mu^{nm} * \delta(a_{nm})$ .

Let  $\mu_n = A_{nm} \mu^{nm} * \delta(a_{nm})$ . By (iii),

(iv)  $\mu_n \rightarrow \mu$ .

Equation (ii) can now be written

(v)  $C_n \mu_n * \delta(c_n) \rightarrow \mu^m$

where  $C_n = A_n A_{nm}^{-1}$  and  $c_n = m \cdot a_n - A_n A_{nm}^{-1} a_{nm}$ . Since  $\mu$  and  $\mu^m$  are full measures, the compactness lemma can be invoked to infer from (iv) and (v) that  $\mu$  and  $\mu^m$  are of the same type, for every positive integer  $n$ . ■

Our next aim is to extend the last theorem to include all real positive powers of  $\mu$ . In so doing, we get more information about the affine transformations which appear.

**THEOREM 2.** *If  $\mu$  is full and operator-stable, there is an automorphism  $B$  of  $V$  such that*

$$\mu^t = \exp \{ \log t \cdot B \} \mu * \delta(b(t)); \quad t > 0,$$

for some  $b(t) \in V$ . The converse is clearly true.

While the proof is not at all difficult, it involves many steps, and for better organization, we proceed with a sequence of simple lemmas. To start with, define  $G_t = \{A \in \text{Aut } V \mid \mu^t = A \mu * \delta(a) \text{ for some } a \in V\}$ . The set  $G_t$  may, a priori, be empty for some  $t > 0$ , but the last theorem shows that  $G_t$  is nonempty whenever  $t$  is a positive integer.

**LEMMA 1.**  $G_t \neq \emptyset$  for any  $t > 0$ .

**Proof.** If  $t > 0$  is rational and equal to  $j/k$ ,  $G_j$  and  $G_k$  are nonempty and  $\mu^j$  and  $\mu^k$  are of the same type. Therefore, there is an affine transformation  $(B, b)$  such that

$\mu^j = B\mu^k * \delta(b)$ . But then,  $(B\mu * \delta(1/k \cdot b))^k = \mu^j$ , and so  $B\mu * \delta(1/k \cdot b)$  is the unique infinitely divisible  $k$ th root of  $\mu^j$ , so  $B\mu * \delta(1/k \cdot b) = \mu^{j/k} = \mu^t$ . Thus,  $G_t \neq \emptyset$  if  $t$  is rational, and  $\mu^t$  is of the same type as  $\mu$  if  $t$  is rational. If  $t$  is any positive real number, let  $t_n$  be rational, and let  $t_n \rightarrow t$ . Then  $\mu^{t_n} \rightarrow \mu^t$  and, if

$$\mu = C_n \mu^{t_n} * \delta(c_n), \quad (C_n, c_n) \in \text{Aff } V,$$

the compactness lemma implies that  $\mu^t$  is of the same type as  $\mu$ , so that  $G_t \neq \emptyset$ . ■

LEMMA 2.  $G_t^{-1} = G_{1/t}$  and  $G_{s \cdot t} = G_s \cdot G_t$  for all  $s, t > 0$ .

**Proof.** It is easy to see that  $G_t^{-1} \subset G_{1/t}$  and  $G_s \cdot G_t \subset G_{st}$ . Replacing  $t$  by  $1/t$  in the first inclusion and  $s$  and  $t$  by  $1/s$  and  $st$  in the second, the reverse inclusions hold. ■

LEMMA 3.  $G_s \cap G_t = \emptyset$  if  $s \neq t$ .

**Proof.** If  $A \in G_s \cap G_t$ , say  $\mu^s = A\mu * \delta(a)$  and  $\mu^t = A\mu * \delta(a')$ , then  ${}^\circ\mu^s = {}^\circ\mu^t$  implying  $|\mu^\wedge(y)|^{2s} \equiv |\mu^\wedge(y)|^{2t}$  for all  $y$ . If  $s \neq t$ , this implies that  $|\mu^\wedge(y)| = 0$  or  $1$ , so  $|\mu^\wedge(y)| \equiv 1$ ,  $\mu^\wedge$  being continuous. But this would mean that  $\mu$  is degenerate. ■

LEMMA 4.  $G = \bigcup \{G_t \mid t > 0\}$  is a closed subgroup of  $\text{Aut } V$ .

**Proof.** That  $G$  is a subgroup is the content of Lemma 2. To prove that  $G$  is closed in  $\text{Aut } V$ , let us assume that a sequence  $\{A_n\}$  of members of  $G$  converges in  $\text{Aut } V$  to an automorphism  $A$ . We must show  $A \in G$ . Suppose  $A_n \in G_{t_n}$ . If  $\{t_n\}$  contains a subsequence which converges to 0 in  $R$  it may be assumed for the purposes of this argument that  $t_n \rightarrow 0$ . Then  $A_n \mu = \mu^{t_n} * \delta(a_n)$  for some  $a_n \in V$ , so  $A_n({}^\circ\mu) = ({}^\circ\mu)^{t_n} \rightarrow \delta(0)$  as  $n \rightarrow \infty$ . Thus,  $A({}^\circ\mu) = \delta(0)$ , and since  ${}^\circ\mu$  is full,  $A = 0$ , a contradiction which establishes the fact that  $\{t_n\}$  cannot have a subsequence tending to 0. On the other hand, if  $t_n$  has a subsequence tending to  $\infty$ , by setting  $B_n = A_n^{-1} \in G_{1/t_n}$ ,  $B_n \rightarrow A^{-1}$  and  $1/t_n \rightarrow 0$ . The last argument implies that  $A^{-1} = 0$ . We conclude that  $\{t_n\}$  must be bounded away from 0 and  $\infty$ . Let  $t$  be any limit point of  $\{t_n\}$ ; it may be assumed that  $t_n \rightarrow t$ . Then  $A_n \mu \rightarrow A\mu$  but

$$A_n \mu = \mu^{t_n} * \delta(a_n) \quad \text{and} \quad \mu^{t_n} \rightarrow \mu^t.$$

Since  $A\mu$  and  $\mu$  are full, the compactness lemma implies  $\{a_n\}$  is precompact in  $V$ , so if  $a$  is any limit point, we have

$$A\mu = \mu^t * \delta(a), \quad \text{so that } A \in G_t \subset G. \quad \blacksquare$$

LEMMA 5. The mapping  $\eta: G \rightarrow R^+$ , defined by  $\eta(A) = t$  if  $A \in G_t$ , is well defined, and is a continuous open homomorphism of  $G$  onto the positive reals under multiplication. The kernel of  $\eta$  is  $G_1$ , a compact normal subgroup of  $G$ .

**Proof.** The mapping  $\eta$  is well defined by Lemma 3. It is a homomorphism because of Lemma 2. To prove that  $\eta$  is continuous, suppose  $A_n \rightarrow A_0$  in  $G$ , with

$\eta(A_n) = t_n$ , so that  $A_n \in G_{t_n}$ . We have  $\mu^{t_n} = A_n \mu * \delta(a_n)$  for some  $a_n \in V$ , so  $(\circ\mu)^{t_n} = A_n(\circ\mu) \rightarrow A_0(\circ\mu) - (\circ\mu)^{t_0}$ . But this means  $|\mu^\wedge(y)|^{2t_n} \rightarrow |\mu^\wedge(y)|^{2t_0}$ , hence  $t_n \rightarrow t_0$ . The openness of  $\eta$  is now an automatic consequence of its continuity and the  $\sigma$ -compactness of  $G$ —see, e.g. Theorem 5.29 of Hewitt and Ross [2]. ■

**Proof of Theorem 2.** The mapping  $\xi = \log \eta$  is a real additive continuous homomorphism of  $G$  onto  $(R, +)$ . Firstly, we demonstrate the existence of a one-parameter subgroup  $H$  of  $G$  with  $\xi(H) = R$ . Since  $G$  is a closed subgroup of  $\text{Gl}(V)$ , it is a linear Lie group, so the component  $G^\circ$  of the identity in  $G$  is an open normal subgroup of  $G$ . Since, by Lemma 5,  $\eta$  is open,  $\xi$  is open, implying that  $\xi(G^\circ)$  is an open subgroup of  $(R, +)$  and hence must itself be  $R$ . Since  $G^\circ$  is the union of its one-parameter subgroups, one of them,  $H$  say, must map onto  $R$  under the mapping  $\xi$ . Now,  $H$ , being a one-parameter subgroup of  $\text{Gl}(V)$ , must be of the form  $\{e^{sB} \mid -\infty < s < \infty\}$  for some operator  $B$ . The mapping  $s \rightarrow e^{sB}$  is a continuous homomorphism of  $(R, +)$  into  $G$ , so  $s \rightarrow e^{sB} \rightarrow \xi(e^{sB})$  is a continuous homomorphism of  $(R, +)$  onto  $(R, +)$  so that  $\xi(e^{sB}) = Ks$  for some constant  $K \in R$ . Replacing  $B$  by  $K^{-1}B$ , we may assume that  $\xi(e^{sB}) = s$ , giving  $\eta(e^{sB}) = e^s$  or  $\eta(e^{\log t \cdot B}) = t$ . Thus  $\exp \{\log t \cdot B\} \in G_t$  for every  $t > 0$ .

It remains only to show that  $B$  is invertible, if  $\mu$  is full. If  $B$  were noninvertible,  $B^*$  would be noninvertible and there would exist  $y \in V^\wedge$  such that  $B^*y = 0, y \neq 0$ . In this case,  $\exp \{\log t \cdot B^*\}y = y$  for all  $t > 0$ , implying

$$|\mu^\wedge{}^t(sy)| = |\mu^\wedge(\exp \{\log t \cdot B^*\}sy)| = |\mu^\wedge(sy)|.$$

This would mean  $|\mu^\wedge| \equiv 1$  on the subspace generated by  $y$ , a contradiction to the fullness of  $\mu$ , by Proposition 1. ■

REMARK. We shall denote  $\exp \{\log t \cdot B\}$  by the notation  $t^B$ .

**7. The class  $\mathcal{B}$ .** The class of operators,  $B$ , which can occur in a representation  $\mu^t = t^B \mu * \delta(b(t))$  for some full measure  $\mu$  will be denoted by  $\mathcal{B}$ . So far, we know only that  $\mathcal{B}$  consists solely of nonsingular operators. It is easy to check, now, that if  $B \in \mathcal{B}$  and  $A$  is any automorphism,  $ABA^{-1} \in \mathcal{B}$ . In other words,  $\mathcal{B}$  is closed under similarity transformations, and  $\mathcal{B}$  will be describable through spectral properties. In fact,

**THEOREM 3.** *Necessary and sufficient conditions for an operator  $B$  to be in the class  $\mathcal{B}$  are*

- (i) *The spectrum of  $B$  is in the half-plane  $\text{Re } z \geq 1/2$ , and*
- (ii) *The eigenvalues lying on the line  $\text{Re } z = 1/2$  are simple—i.e. the elementary divisors of  $B$  associated with these eigenvalues are of first degree.*

Once again, the rather complicated proof obliges us to break it down to more organizable parts. Firstly, we record some facts about the representing triple of a stable distribution.

**PROPOSITION 5.** *If  $\lambda \in S$  has representing triple  $(c, \phi, M)$  and  $\lambda^t = t^B \lambda * \delta(b(t))$ ,  $B \in \mathcal{B}$ , then*

- (a)  $\phi(t^{B^*}y) = t\phi(y)$  for  $y \in V^\wedge$  and  $t > 0$ ,
- (b)  $t^B M = tM$  for  $t > 0$ .

**Proof.** A direct application of Theorem 2 on Proposition 3. ■  
 The heart of the proof of Theorem 3 lies in

**LEMMA 6.** *A measure  $M$  concentrated on an orbit  $\{t^B x_0 \mid t > 0\}$  and satisfying  $t^B M = tM$  is a K-L measure if and only if every eigenvalue of  $B$  in the cyclic subspace generated by  $x_0$  has real part greater than  $1/2$ .*

**Proof.** Assume firstly that  $S(M) \subset \{t^B x_0 \mid t > 0\}$  and  $t^B M = tM$ . Let  $X = [x_0, Bx_0, B^2x_0, \dots] = [x_0]$ , the cyclic subspace generated by  $x_0$ . Since  $B$  is non-singular,  $BX = X$ , and since our interest is only in the behavior of  $B$  on  $X$ , we assume  $B = B|X$ .  $B$  is then a cyclic operator, and by structure theory for such operators (see, e.g. Jacobson [3, p. 73]), to each elementary divisor of  $B$ , there is a subspace  $X_j$ , such that

- (i)  $BX_j = X_j$ , and
- (ii)  $X = X_1 \oplus \dots \oplus X_k$ .

Also, the minimum polynomial  $q_j^{m_j}$  of  $B|X_j$  is a power of a polynomial which is irreducible over the real field.

Now, set  $F(s) = M\{t^B x_0 \mid t > s\}$ . The condition  $t^B M \equiv tM$  implies that  $F(st) = 1/tF(s)$ . Thus,  $F(t) = K/t$  for some constant  $K > 0$ . The measure  $M$  is a K-L measure if and only if  $\int \|x\|^2 M(dx) < \infty$  in a neighborhood of 0. This is the case if and only if

$$\int_0^1 \|t^B x_0\|^2 (-dF/dt) dt < \infty,$$

or

$$(*) \int_0^1 \|t^B x_0\|^2 t^{-2} dt < \infty.$$

Here,  $\|\cdot\|$  is any vector norm on  $X$ . Any other vector norm would suffice, for all such norms are equivalent on  $X$ . We shall specify a norm on  $X$  which facilitates computation. Let  $\|\cdot\|_j$  be a vector norm on  $X_j$ , to be chosen later, and let  $\|\sum x_j\| = \sum \|x_j\|_j$ . Then  $\|\cdot\|$  is a norm on  $X$  with the property that  $\|\sum x_j\| \geq \|x_r\|_r$  for each  $r$ ,  $1 \leq r \leq k$ . Suppose  $x_0 = \sum_{j=1}^k x_j$ ,  $x_j \in X_j$ . Then  $x_j \neq 0$  for each  $j$ , otherwise  $x_0$  fails to be cyclic in  $X$ . We have then  $t^B x_0 = \sum_{j=1}^k t^B x_j = \sum_{j=1}^k t^{B_j} x_j$ , where  $B_j = B|X_j$ , and  $\|t^B x_0\| \geq \|t^{B_r} x_r\|_r$ , for each  $r$ . For the rest of the proof, let  $r$  be arbitrary but fixed,  $1 \leq r \leq k$ . We now choose  $\|\cdot\|_r$  in  $X_r$  as follows: extend  $X_r$  to its complexification  $X_r^c$ , and let  $B_r^c$  be the extension of  $B_r$  to  $X_r^c$  in the usual way. We shall define a norm  $\|\cdot\|_r^c$  for  $X_r^c$  and let  $\|\cdot\|_r$  be the restriction of  $\|\cdot\|_r^c$  to  $X_r$ . For notational convenience, let  $J(a_1, \dots, a_k)$  denote the  $k \times k$  matrix having all entries equal to zero below the diagonal,  $a_1$  on the principal diagonal,  $a_2$  on the super-diagonal,

... ,  $a_k$  in  $(1, k)$  position. To choose  $\|\cdot\|_r^c$ , let us firstly choose a (complex) basis  $\{\xi_1, \dots, \xi_p\}$  for  $X_r^c$  so that the matrix of  $B_1^c$  with respect to  $\{\xi_1, \dots, \xi_p\}$  is  $J(\alpha, 1, 0, \dots, 0)$  if  $q_j$  is linear, and

$$\begin{pmatrix} J(\alpha, 1, 0, \dots, 0) & 0 \\ 0 & J(\bar{\alpha}, 1, 0, \dots, 0) \end{pmatrix}$$

if  $q_j$  is quadratic. The complex numbers  $\alpha, \bar{\alpha}$  are the eigenvalues of  $B_j$ , hence of  $B$ . Choose  $\|\cdot\|_r^c$  by making  $\|\sum \alpha_j \xi_j\|_r^c = \sum |\alpha_j|$ . Then, we have  $\|t^{B_r X_r}\|_r = \|t^{B_r X_r}\|_r^c$ , and since the (complex) matrix of  $t^{B_r}$  can be seen by an easy computation to be

$$J(t^\alpha, t^\alpha \log t, \dots, 1/(p-1)! t^\alpha (\log t)^{p-1})$$

if  $q_j$  is linear, or the obvious extension if  $q_j$  is quadratic, ( $\|t^{B_r X_r}\|_r^c$ )<sup>2</sup> is seen to be a linear combination, with coefficients depending on  $x_r$ , of terms of the form  $t^\alpha (\log t)^m t^{\bar{\alpha}} (\log t)^n$ ,  $m \leq p-1, n \leq p-1$ , and these terms are each  $t^{2 \operatorname{Re} \alpha} (\log t)^q$ , for some  $q \leq 2p-2$ . Then

$$\begin{aligned} \int_0^1 \|t^{B_r X_r}\|^2 t^{-2} dt &\geq \int_0^1 \text{const } t^{2 \operatorname{Re} \alpha} t^{-2} dt \\ &\geq \text{const} \int_0^1 t^{2 \operatorname{Re} \alpha - 2} dt. \end{aligned}$$

The first term is finite, by (\*), so  $2 \operatorname{Re} \alpha - 2 > -1$ , implying  $\operatorname{Re} \alpha > 1/2$ .

To obtain the converse, note that  $\operatorname{Re} \alpha > 1/2$  implies  $\int_0^1 t^{2 \operatorname{Re} \alpha - 2} dt < \infty$  and an integration by parts shows that all the terms  $t^\alpha (\log t)^m t^{\bar{\alpha}} (\log t)^n$ , as above, have finite integrals at 0. Taking the same norms as above, we find

$$\int_0^1 \|t^{B_r X_r}\|_r^2 t^{-2} dt < \infty,$$

so

$$\int_0^1 \|t^{B_r X_r}\|^2 t^{-2} dt \leq \text{const } \Sigma_r \int_0^1 \|t^{B_r X_r}\|_r^2 t^{-2} dt < \infty. \quad \blacksquare$$

We now proceed with the

**Proof of Theorem 3.** Let  $\Lambda(B) = \{M \mid M \text{ a K-L measure on } V \sim \{0\} \text{ and } t^B M = tM\}$ . Note that if  $M \in \Lambda(B)$ ,  $S(M)$  is invariant under  $t^B$  for all  $t > 0$ , so that  $S(M)$  is a union of orbits of  $t^B$ . The  $M$  may be infinite measures, but they are essentially finite in the sense that  $W(dx) = \|x\|^2 / (1 + \|x\|^2) M(dx)$  is a finite measure in  $V$ . To apply the theory of finite measures, let  $\mathfrak{M}(V)$  be the real linear space of finite measures in  $V$ , with the topology of weak convergence. Let

$$\Omega(B) = \{W \in \mathfrak{M}(V) \mid W(dx) = \|x\|^2 / (1 + \|x\|^2) M(dx), M \in \Lambda(B)\}.$$

$\Omega(B)$  is easily seen to be a convex cone in  $\mathfrak{M}$ , and  $\Omega_1(B) = \{W \in \Omega \mid W(V) \leq 1\}$  is a compact convex subset of  $\mathfrak{M}$ . Each  $W$  in  $\Omega$  has the property that  $S(W)$  is a union of orbits of  $t^B$ . Thus, it is easily seen that the extreme points of  $\Omega_1(B)$  are the

measures concentrated along a single orbit  $\{t^B x_0 \mid t > 0\}$ . Thus, the set of convex combinations of such measures is dense in  $\Omega_1$ , and this shows that in  $\Lambda(B)$ , the linear combinations of  $M$ 's which are concentrated in a single orbit are dense in  $\Lambda(B)$ .

Let  $X$  now be decomposed into a direct sum of subspaces  $X_j$  ( $1 \leq j \leq r$ ) such that  $BX_j = X_j$ , and such that the minimum polynomial of  $B|X_j$  is a power of a real-irreducible polynomial. Assume that the eigenvalues in  $X_1, \dots, X_k$  lie in  $\text{Re } z > 1/2$  and those in  $X_{k+1}, \dots, X_r$  lie in  $\text{Re } z \leq 1/2$ . Let  $X_0 = X_{k+1} + \dots + X_r$ . Now, if  $x \in X$ , and  $x = \sum_0^k x_j$ ,  $x_j \in X_j$  then, by Lemma 6, the orbit  $\{t^B x \mid t > 0\}$  supports a nonzero K-L measure  $M \in \Lambda(B)$  if and only if  $x_0 = 0$ . Hence, since linear combinations of such  $M$  are dense in  $\Lambda(B)$ ,  $M$  is concentrated in  $X_1 + \dots + X_k$ , for all  $M \in \Lambda(B)$ .

To find how  $B$  behaves in  $X_0$ , examine the adjoint  $B^* : V^\wedge \rightarrow V^\wedge$ . Let, in  $V^\wedge$ ,

$$Y_j = (X_0 \oplus \dots \oplus X_{j-1} \oplus X_{j+1} \oplus \dots \oplus X_k)^\perp, \quad 0 \leq j \leq k.$$

$Y_j$  is then the dual of  $X_j$ , and  $B^*|Y_j = (B|X_j)^*$ . It is a consequence of the Khintchine-Lévy formula that

$$\log \circ \mu^\wedge(y) = -2\phi(y) + \int (\cos(x, y) - 1)M(dx)$$

and if  $y \in Y_0$ , since  $M$  is concentrated in  $X_1 \oplus \dots \oplus X_k$ ,

$$\log \circ \mu^\wedge(y) = -2\phi(y).$$

Since  $\mu$  is assumed full,  $\phi(y) \neq 0$  if  $y \in Y_0$ ,  $y \neq 0$ . Thus, the quadratic form  $\phi|Y_0$  is nondegenerate, and there is a basis  $\{y_1, \dots, y_q\}$  in  $Y_0$  such that

$$\phi\left(\sum_1^q a_j y_j\right) = \sum_1^q a_j^2.$$

With respect to this basis  $\{y_1, \dots, y_q\}$ , a bilinear form  $\langle \cdot, \cdot \rangle$  is defined, and adjoints  $C'$  of operators  $C$  are defined by

$$\langle Cy_1, y_2 \rangle = \langle y_1, C'y_2 \rangle.$$

Then, the condition 5(a) reads that for all  $y \in Y_0$  and  $t > 0$ ,

$$\phi(t^{B^*}y) = t\phi(y).$$

Since  $B^*Y_0 = Y_0$ , it will not hurt to assume in this section that  $B^* = B^*|Y_0$ . Then, for all  $y \in Y_0$ ,

$$\langle t^{B^*}y, t^{B^*}y \rangle = t\langle y, y \rangle.$$

Thus,

$$\langle t^{B^*}t^{B^*}y, y \rangle = \langle ty, y \rangle$$

or

$$t^{B^*}t^{B^*} \equiv tI \equiv t^{1/2}I \cdot t^{1/2}I.$$

Since  $t^{1/2}I \equiv t^{1/2}$ , this can be written as

$$t^{(B^*-I/2)}t^{(B^*-I/2)'} \equiv I.$$

Setting  $C = B^* - I/2$ , we find that  $t^{C+C'} \equiv I$  for all  $t > 0$ , so that  $C + C' = 0$ . Thus, in this basis,  $C$  is a skew operator. Hence  $B^* = I/2 + C$  is a normal operator, and all its elementary divisors are linear. This proves the necessity of the conditions in the theorem. To prove the sufficiency of the conditions, let  $B$  satisfy (i) and (ii). As in the proof of the necessity, let  $X = X_1 \oplus \dots \oplus X_r$ ,  $X_j$  having the same meaning as before. If  $x_j \in X_j$ ,  $x_j \neq 0$ , ( $1 \leq j \leq k$ ), then, by Lemma 6, the orbit  $\{t^B x_j \mid t > 0\}$  supports a nonzero K-L measure  $M_j \in \Lambda(B)$ . Since the orbit  $\{t^B x_j \mid t > 0\}$  generates  $X_j$ , the stable measure  $\lambda_j$  with triple  $(0, 0, M_j)$  is supported, and is full, in  $X_j$ . On  $X_0 = X_{k+1} \oplus \dots \oplus X_r$ , we construct a Gaussian measure  $\mu$  satisfying  $\mu^t = \exp \{ \log t \cdot B \mid X_0 \} \mu$ . We can imagine that  $X = X_0$ , hence that  $B$  may be put into a canonical form  $\text{diag} \{ J_1, \dots, J_q \}$  where  $J_j$  is either a  $1 \times 1$  matrix with element  $1/2$ , or a  $2 \times 2$  matrix

$$\begin{pmatrix} 1/2 & \beta \\ -\beta & 1/2 \end{pmatrix}.$$

With the dual basis  $\{\eta_1, \dots, \eta_p\}$  in  $Y_0$ ,  $B^*$  has canonical form  $\text{diag} \{ J_1^*, \dots, J_q^* \}$ . We have to construct a quadratic form  $\phi$  on  $Y_0$  such that  $\phi(t^{B^*} y) \equiv t\phi(y)$ . Consider  $\phi(\sum a_i \eta_i) = \sum a_i^2$ . If  $[\eta_1] = X_j$ ,  $t^{J_j} = t^{1/2} I$  and  $\phi(t^{J_j} a \eta_i) = t\phi(a \eta_i)$ . If  $[\eta_i, \eta_{i+1}] = X_j$ , we have

$$t^{J_j} = t^{1/2} \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}$$

and, once again,  $\phi(t^{J_j}(a\eta_i + b\eta_{i+1})) = t\phi(a\eta_i + b\eta_{i+1})$ . Thus,  $e^{-\phi(y)}$  is a Gaussian characteristic function, and its distribution is full and concentrated in  $X_0$ . The measure  $\nu = \lambda_1 * \dots * \lambda_k * \mu$  is now full and stable, and  $\nu^t = t^B \nu$ . ■

As by-products of the proof of Theorem 3, we can assert the following:

**THEOREM 4.** *Any full operator-stable measure  $\lambda$  on  $V$  can be decomposed into a product  $\lambda = \lambda_1 * \lambda_2$  of measures  $\lambda_i$  concentrated in subspaces  $V_i$ ,  $V = V_1 \oplus V_2$ , where  $\lambda_1$  is a full Gaussian measure in  $V_1$  and  $\lambda_2$  is a full operator-stable measure on  $V_2$  having no Gaussian component.*

**THEOREM 5.** *Any K-L measure  $M$  for a full operator-stable measure  $\lambda$  on  $V$  can be represented as a mixture of K-L measures  $M_\theta$  where  $M_\theta$  is a K-L measure concentrated in an orbit,  $\theta$ , of  $t^B$  and satisfies  $t^B M_\theta = t M_\theta$ . The measure  $M_\theta$  is characterized by the condition that  $sM_\theta\{t^B x_0 \mid t > s\}$  is constant for all  $s$ , when  $x_0$  is a generator of the orbit  $\theta$ .*

If  $\lambda$  is full and operator-stable, and satisfies  $\lambda^t = t^B \lambda * \delta(b(t))$  for some  $b(t)$ , we shall call  $B$  an exponent for  $\lambda$ . A measure  $\lambda$  may possess more than one exponent. For example, if  $V$  is given an inner product with respect to which  $\lambda$  is a rotation-invariant Gaussian measure, then  $B = I/2$  is an exponent for  $\lambda$ , as is any operator of the form  $I/2 + C$  where  $C$  is skew.

The operators in the class  $\mathcal{B}$  can be interpreted as exponents of the normalizing factors which give rise to operator-stable laws. That is, if  $x_1, x_2, \dots$  are independent

full operator-stable random variables having the same distribution, then each has the same distribution as some translate of  $n^{-B}(x_1 + \dots + x_n)$ , for  $n = 1, 2, \dots$

**8. Centering.** We turn, finally, to an examination of the term  $b(t)$  in the formula

$$(8.1) \quad \lambda^t = t^B \lambda * \delta(b(t)) \quad \text{for all } t > 0,$$

satisfied by a full operator-stable measure  $\lambda$  with exponent  $B$ . Taking sth powers on both sides of (8.1), we find that

$$\lambda^{st} = t^B \lambda^s * \delta(sb(t)) = t^B s^B \lambda * \delta(t^B b(s) + sb(t)).$$

But  $\lambda^{st} = (st)^B \lambda * \delta(b(st))$ , from (8.1), so that the vector-valued function  $b(\cdot)$  must satisfy the functional equation

$$(8.2) \quad b(st) = t^B b(s) + sb(t) \quad \text{for all } s > 0, t > 0.$$

One consequence, obtained by setting  $s = t = 1$ , is that  $b(1) = 0$ .

**THEOREM 6.** *If 1 is not in the spectrum of  $B$ , the general solution of (8.2) is*

$$(8.3) \quad b(t) = tx_0 - t^B x_0, \quad t > 0,$$

for some  $x_0 \in V$ , and in this case, when the full operator-stable measure  $\lambda$  satisfying (8.1) is centered at  $x_0$ , the measure  $\mu = \lambda * \delta(-x_0)$  satisfies  $\mu^t = t^B \mu$ .

**REMARK.** By analogy with the one-dimensional case, an operator-stable measure  $\mu$  satisfying  $\mu^t = t^B \mu$  could be called strictly stable. Our theorem shows that when 1 is not in the spectrum of  $B$ , any full operator-stable measure can be centered so as to become strictly stable.

**Proof.** Once we have proven (8.3), the second assertion of the theorem follows by a standard calculation. As a first step in proving (8.3), note that if  $b_1(\cdot)$  and  $b_2(\cdot)$  are solutions of (8.2), then  $b_1(\cdot) - b_2(\cdot)$  is a solution of (8.2). Note also that if  $b(\cdot)$  is a solution of (8.2) such that  $b(t_0) = 0$  for some  $t_0 \neq 1$ , then

$$b(st_0) = t_0 b(s) = t_0^B b(s) \quad \text{for all } s > 0.$$

Since 1 is not an eigenvalue of  $B$ ,  $t_0$  is not an eigenvalue of  $t_0^B$ , so we must have  $b(s) \equiv 0$ . These observations imply that any two solutions of (8.2) which agree at even one point must in fact be identical.

Now, the operator  $(tI - t^B)$  is invertible, for  $t \neq 1$ , hence we can always solve an equation

$$(tI - t^B)x_0 = x_1 \quad \text{for } x_0 \in V, \text{ for all } x_1 \in V.$$

Hence, the solutions  $b(t) = tx_0 - t^B x_0$  constitute all possible solutions of (8.2). ■

In the event that  $B$  has 1 as an eigenvalue, it may not be possible to center  $\lambda$  as in the last theorem. This is in analogy with the one-dimensional case of the so-called asymmetric Cauchy distribution. An example is given by taking  $B = I$  so that (8.2) becomes

$$b(st) = sb(t) + tb(s),$$

which is satisfied by the function  $b(t) = t \log tx_0$ , for all  $x_0 \in V$ . We can clearly not absorb such a factor to center  $\lambda$ .

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#### REFERENCES

1. W. Feller, *An introduction to probability theory and its applications*, Vol. II, Wiley, New York, 1966.
2. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. I, Springer-Verlag, Berlin, 1963.
3. N. Jacobson, *Lectures in abstract algebra*, Vol. II, Van Nostrand, Princeton, N. J., 1953.
4. P. Lévy, *Théorie de l'addition des variable aléatoires*, 2nd ed., Gauthier-Villars, Paris, 1954.

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