PRE-SELF-INJECTIVE RINGS

BY

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In this paper we study commutative rings (with identity) all of whose *proper* homomorphic images are self-injective rings. Such rings will be called *pre-self-injective.*

We were led to the study of such rings by one of the proofs of the structure theorem for finite abelian groups, in which the key lemma is: "Every element x of largest possible order in a finite abelian group G generates a subgroup which is a direct summand of G." If we let n be the order of x, then $nG=0$ (this is easiest to see when n is a power of a prime) so that G is a module over the integers modulo n. Since the subgroup generated by x is also isomorphic to the integers modulo n, the lemma can be restated: The ring of integers modulo n is self-injective when $n \neq 0$.

Our first main result (Theorem 3.5) is that a commutative ring R is pre-self-injective if and only if it is one of the following: (1) A domain (necessarily Prüfer) in which, for every maximal ideal M, R_M is an almost maximal, rank 1 valuation domain; and in which each proper ideal is contained in only finitely many maximal ideals. (2) The direct sum of a finite number of maximal, rank 0 valuation rings. (3) An almost maximal rank 0 valuation ring. (4) A local ring whose maximal ideal has composition length 2 and squares to 0. (By a *local ring* we mean a commutative ring with identity which has exactly one maximal ideal. See §1 for definitions of valuation rings.) If R is noetherian, (1) becomes the class of Dedekind domains, while (2) and (3) become the class of principal ideal rings with DCC (see [6, p. 149]).

Our second main result (Theorem 4.5) is that a domain R with quotient field Q is pre-self-injective if and only if R has Krull dimension 1 and all R-homomorphic images of Q are R-injective. This last condition (Q/H always being R-injective) was studied by E. Matlis who showed [7, Theorems 5 and 4] that every R_M must be an almost maximal valuation domain, and that the converse holds if R is a valuation domain. We show that the general converse is false (Example 4.6).

Finally we show (Theorem 5.1) that over a pre-self-injective domain, every finitely generated module is the direct sum of ideals and cyclic torsion modules.

1. **Preliminaries on valuation rings.**

By a *valuation ring* we will mean a commutative ring with identity such that, for every pair of elements, one divides the other.

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(1.1) *In a valuation ring R the set of ideals is totally ordered by inclusion. If R is a domain the same is true of the set of all R-submodules of its quotient field.*

This follows from the observation that an ideal J is contained in an ideal K if and only if every principal ideal contained in J is also contained in K.

A valuation ring has *rank* 0 if it has only one prime ideal, namely its maximal ideal. Note that a rank 0 valuation ring is a domain only if it is a field. Since the set of nilpotent elements of a commutative ring is the intersection of all of the minimal prime ideals, [8, Chapter I, Corollary 2.6, p. 5], we get:

(1.2) *The maximal ideal of a rank* 0 *valuation ring is nil.*

For a subset J of a ring R we define $\operatorname{ann} J$ (the *annihilator* of J) to be the ideal of all x in R such that $xJ=0$. Clearly $\operatorname{ann}\operatorname{ann} J \supseteq J$ and $\operatorname{ann}\operatorname{ann}\operatorname{ann} J = \operatorname{ann} J$. Hence if J is an ideal of the form $\operatorname{ann} A$, $\operatorname{ann}\operatorname{ann} J = J$. Such an ideal is called an *annihilator* ideal.

PROPOSITION 1.3. *Let R be a valuation ring with maximal ideal M in which* (1) *below holds* (*for example, any rank* 0 *valuation ring*). *Then so do* (2) *and* (3).

(1) $\operatorname{ann}\operatorname{ann}(Rb) = Rb$ *for each b in R.*

(2) *If J is an ideal but not an annihilator ideal, then there is a smallest principal ideal Rb containing J. $Rb = \operatorname{ann}\operatorname{ann} J$ and $Mb = J$.*

(3) *If $J \subset K$* (*ideals*) *then* $\operatorname{ann}\operatorname{ann} J \subseteq K$.

Proof. Suppose that (1) is false, and let $y \in \operatorname{ann}\operatorname{ann}(Rb) - Rb$. Then, by (1.1), $Ry \supset Rb$, say $b = my$ with $m \in M$. Since $y \in \operatorname{ann}\operatorname{ann}(Rb) = \operatorname{ann}\operatorname{ann}(Rmy)$ we get

$$\operatorname{ann}(Ry) \supseteq \operatorname{ann}\operatorname{ann}\operatorname{ann}(Rmy) = \operatorname{ann}(Rmy)$$

so if $rmy = 0$ then $ry = 0$. Hence $\operatorname{ann}(m) \cap Ry = 0$. Now since R is a valuation ring and $Ry \neq 0$, thus $\operatorname{ann}(m) = 0$.

If R is a rank 0 valuation ring then all the nonunits are nilpotent, which contradicts the above. Hence in a rank 0 valuation ring (1) is true.

To obtain (2), take any $b \in \operatorname{ann}\operatorname{ann}(J) - J$. Then $J \subset Rb \subseteq \operatorname{ann}\operatorname{ann} J$. Taking double annihilators and using (1) gives $\operatorname{ann}\operatorname{ann} J \subseteq Rb \subseteq \operatorname{ann}\operatorname{ann} J$. We have shown that the second annihilator of J equals Rb for every $b \in \operatorname{ann}\operatorname{ann} J - J$, as claimed. Since there are no ideals between M and R, the same is true of Mb and Rb. Therefore $Mb = J$.

(3) follows immediately from (1) and (2).

Finally, we define a valuation ring to be *maximal* if every family of pairwise solvable congruences of the form $x \equiv x_\alpha \pmod{J_\alpha}$ (each $x_\alpha \in R$, each J_α an ideal of R) has a simultaneous solution x. We get the definition of *almost maximal* if we only require a simultaneous solution when $\bigcap_\alpha J_\alpha \neq 0$; equivalently: a valuation ring is *almost maximal* if all of its proper homomorphic images are maximal. For the origin of the term "maximal" see [13, Chapter 2, §§3 and 4].

2. The local case.

LEMMA 2.1. *Let R be a local, self-injective ring with maximal ideal M. Then* ann M *is contained in every nonzero ideal of R.*

Proof. Assume x is a nonzero element of ann M. Then $Mx=0$ and $Rx\neq 0$ show that ann $x=M$. For any nonzero y in R, ann $y\subseteq M=$ann x so an R-homomorphism $\varphi: Ry\to Rx$ can be defined by $ry\to rx$. Since R is self-injective φ is multiplication by some s in R. Hence $x=\varphi(y)=ys\in Ry$. This shows that ann M is contained in every principal ideal ($\neq 0$), hence in every ideal ($\neq 0$) of R.

THEOREM 2.2. *For a local ring R with maximal ideal M, suppose that R/Mx is self-injective whenever $Mx\neq 0$ and R/Rx is self-injective whenever $Rx\neq 0$. Then either R is a valuation ring or else $M^2=0$ and M has composition length* 2.

Proof. Suppose that R is not a valuation ring. Then R has a pair of elements x and y neither of which is a multiple of the other. We first show: *for any such x and y, $Mx=0=My$.* If $Mx\neq 0$ then in the self-injective local ring $\bar{R}=R/Mx$ the annihilator of the maximal ideal $\bar{M}$ contains the nonzero element $\bar{x}$. Moreover, $\bar{y}\neq 0$ since y is not a multiple of x. Hence Lemma 2.1 shows $\bar{x}\in$ ann $\bar{M}\subseteq\bar{R}\bar{y}$, say $x=ry+mx$ ($r\in R, m\in M$), i.e. $(1-m)x=ry$. But since R is local, $1-m$ is invertible, and hence we get the contradiction that x is a multiple of y. Therefore $Mx=0$. Similarly $My=0$.

Next we show that $M^2=0$. If not, $Mu\neq 0$ for some u in M. This implies, by the preceding paragraph (with u in place of y) that either u is a multiple of x or vice versa. If u were a multiple of x we would have $Mu\subseteq Mx=0$. Hence $x=bu$ for some b. Similarly $y=cu$ for some c. Since $0\neq bu\in bM$ and $0\neq cu\in cM$ the preceding paragraph shows that one of b and c must be a multiple of the other; and hence that one of x and y must be a multiple of the other, contrary to our choice of x and y. Thus $M^2=0$.

Since $M^2=0$, M is a vector space over the field R/M of dimension at least 2 (x and y are linearly independent). Thus the R-module M has composition length at least 2. To see that equality holds, take a nonzero m in M which is not a multiple of x. We finally use the hypothesis that $\bar{R}=R/Rx$ is self-injective. Since $\bar{M}^2=\bar{0}$, Lemma 2.1 shows $\bar{m}\in$ ann $\bar{M}\subseteq\bar{R}\bar{y}$, so $m=dy+ex$ ($d, e\in R$). This completes the proof of the theorem.

THEOREM 2.3. *A valuation ring R is self-injective if and only if*

(1) ann ann $(Rb)=Rb$ *for each $b\in R$, and*

(2) *R is maximal.*

Recall that by Proposition 1.3 rank 0 valuation rings satisfy (1).

Proof. Assume R is self-injective. To show (1) repeat the first paragraph of the proof of Proposition 1.3 to get a nonunit, m, whose annihilator is 0. Then multiplication by $1/m$ is a well-defined R homomorphism: $Rm\to R$ which cannot be extended to all of R, contradicting self-injectivity of R.

We now show R is maximal. Let a family of congruences

(3) $$x \equiv x_\alpha \pmod{J_\alpha} \qquad (\alpha \in A)$$

be given which are pairwise solvable. This is equivalent to saying that for all α and β, $x_\alpha - x_\beta$ belongs to the larger of J_α and J_β. In particular, if there is a smallest J_α, then that x_α is a simultaneous solution of (3). Consequently we can suppose that there is no smallest J_α.

Let $A_\alpha = \operatorname{ann} J_\alpha$, and consider the multiplication maps: $A_\alpha \to R$ by x_α. Pairwise solvability shows that if $J_\alpha \subseteq J_\beta$, and hence $A_\alpha \supseteq A_\beta$, the multiplication maps: $A_\alpha \to R$ (by x_α) and $A_\beta \to R$ (by x_β) coincide on their common domain A_β. We can therefore take the union of these maps to get an R-homomorphism

$$f\colon A = \bigcup A_\alpha \to R.$$

By the hypothesis that R is self-injective, f is a multiplication by some element x of R. We show x solves the congruences (3).

Let α be given. Then for a_α in A_α we get $(x - x_\alpha)a_\alpha = xa_\alpha - x_\alpha a_\alpha = f(a_\alpha) - f(a_\alpha) = 0$. Thus

(4) $$x - x_\alpha \in \operatorname{ann} A_\alpha = \operatorname{ann} \operatorname{ann} J_\alpha \qquad (\text{all } \alpha).$$

If the right-hand side equals J_α we are done. Otherwise we can use the hypothesis that there is no smallest J_α to find a $J_\beta \subset J_\alpha$. Then $x - x_\alpha = (x - x_\beta) + (x_\beta - x_\alpha)$ with $x - x_\beta \in \operatorname{ann} \operatorname{ann} J_\beta \subseteq J_\alpha$ by (4) and Proposition 1.3 and $x_\beta - x_\alpha \in J_\alpha$ by pairwise solvability. Thus $x - x_\alpha \in J_\alpha$ as desired.

Now suppose that R satisfies (1) and (2) and let an R-homomorphism $f\colon J \to R$ be given with J a left ideal of R. We show that f is a multiplication map. Take j in J and note that $0 = f((\operatorname{ann} j)j) = (\operatorname{ann} j)f(j)$ shows that

$$f(j) \in \operatorname{ann} \operatorname{ann} j = Rj.$$

Thus $f(j) = jx_j$ for some x_j in R. To put these multiplication maps together, consider the system of congruences

(5) $$x \equiv x_j \pmod{\operatorname{ann} Rj} \qquad (j \in J).$$

These congruences are pairwise solvable: If $\operatorname{ann} Rj \subseteq \operatorname{ann} Rk$ and hence $Rj \supseteq Rk$ ($= \operatorname{ann} \operatorname{ann} Rk$) we can write $k = rj$ so that $k(x_j - x_k) = rjx_j - kx_k = f(rj) - f(k) = 0$ so $x_j - x_k \in \operatorname{ann} Rk$. Therefore, by maximality we have an x in R which simultaneously solves all the congruences (5). Then, for j in J, $f(j) = jx_j = jx$ since $j(x - x_j) = 0$.

LEMMA 2.4. *Every nonzero prime ideal of a pre-self-injective ring is maximal.*

Proof. We are trying to prove that every self-injective integral domain is a field. But this follows from the fact that over an integral domain every injective module is divisible [2, Chapter 7, Proposition 12].

Finally, we can determine all pre-self-injective valuation rings:

COROLLARY 2.5. *For a valuation ring R the following are equivalent*:

(i) *R is pre-self-injective.*

(ii) *R/Rx is self-injective whenever $x \neq 0$; and R has rank 0 or is a domain of rank 1.*

(iii) *R is an almost maximal rank 0 valuation ring or an almost maximal rank 1 valuation domain.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2.4, while (iii) $\Rightarrow$ (i) follows from Theorem 2.3.

To get (ii) $\Rightarrow$ (iii) note that every R/Rx is self-injective (when $x \neq 0$) hence maximal by Theorem 2.3. Since every nonzero ideal of R contains a nonzero principal ideal, it follows that R is almost maximal.

EXAMPLES 2.6. It is easy to see that every discrete rank 1 valuation domain is almost maximal and that these are the only noetherian examples of almost maximal valuation domains. Examples of maximal nonnoetherian valuation domains, in the form of "long" power series are given by Kaplansky [3, §4, p. 314] or [6, Remark (i), p. 151]; and additional examples are given by a theorem of Krull which states that every valuation domain is contained in a maximal one having the same value group and residue class field as the original domain [5, Theorem 24, p. 191]. Taking proper homomorphic images of rank 1 almost maximal valuation domains we get examples of rank 0 maximal valuation rings; and all the examples we know of rank 0 almost maximal valuation rings have this form.

There exist self-injective rings (necessarily *not* domains) of rank $\neq 0$: Let M be a group of type Z_{p^∞}. Its ring of endomorphisms is the p-adic integers P. Hence we can make the group direct sum $R = P \oplus M$ into a ring by defining multiplication in M to be zero and defining xm (x in P, m in M) to be $x(m)$. Barbara Osofsky has shown that R is a self-injective valuation ring and that every ideal of R is either a subgroup of M or has the form $Rp^t = Pp^t \oplus M$ [10, Example 1, p. 378]. We observe that every homomorphic image of R except one (namely P) is self-injective: If $0 \subseteq J \subset M$, then since every homomorphic image ($\neq 0$) of M is again isomorphic to M, $R/J \cong R =$ self-injective. If $J \supset M$, then (since R is maximal by our Theorem 2.3) R/J is maximal of rank 0 and hence (2.3 again) self-injective.

An example of a local ring whose maximal ideal has composition length 2 and squares to zero (see Theorem 2.2) is the polynomial ring $F[x, y]$ over a field F with relations $x^2 = xy = y^2 = 0$. These rings are discussed in more detail in [6, Remark (ii), p. 152].

3. The global case.

LEMMA 3.1. *Let S be a multiplicatively closed subset of a ring R, and suppose that every prime ideal of R is maximal. Then the natural map*: $R \to R_S$ *is an epimorphism.*

Proof. First consider the case that R is local with maximal ideal M. Then every element of S is either outside M, hence a unit, or an element of M and hence nilpotent (by the proof of (1.2)). Thus if S consists wholly of elements outside M, $R_S=R$; while if S contains an element of M then $0\in S$ and hence $R_S=0$. In either case $R\to R_S$ is an epimorphism.

For the general case, it is sufficient to show that, for every maximal ideal M of R, the induced map

$$R_M\to(R_S)_M\cong(R_M)_S$$

is an epimorphism; and this reduces the problem to the local case. (For notation see [8, Chapter I, §6]. To show the above reduction apply [8, Theorem 8.9, p. 23] to the image of the natural map.)

PROPOSITION 3.2. *Let S be a multiplicatively closed subset of a pre-self-injective ring R. Then R_S is pre-self-injective. If R is also self-injective, so is R_S.*

Proof. Choose an arbitrary nonzero ideal of R_S and write it in the form J_S where J is a nonzero ideal of R. By hypothesis R/J is self-injective. Since R is pre-self-injective, every nonzero prime ideal of R is maximal (Lemma 2.4), and hence *every* prime ideal of R/J is maximal. Thus the lemma shows that $R_S/J_S\cong(R/J)_S$ is a homomorphic image of R/J and hence a proper homomorphic image of R; hence self-injective. Therefore R_S is pre-self-injective. If R is also self-injective, then we can also do the above proof for $J=0$.

Note that we have *not* shown that if R is a self-injective ring, then so is every R_S.

The proof of the following lemma was suggested by Barbara Osofsky.

LEMMA 3.3. *Every injective, pre-self-injective commutative ring is semilocal* (*i.e. has only a finite number of maximal ideals*).

Proof. First we show that if K is any ideal of a regular (in the sense of von Neumann) commutative ring R for which R/K is self-injective, then R/K is also R-injective. To do this let f be an R-homomorphism: $J\to R/K$ with J an ideal of R. It will suffice to prove that $f(J\cap K)=0$. For then f will induce an R-homomorphism $\bar f\colon \bar J\to\bar R=R/K$ where $\bar J=(J+K)/K$. Since $\bar f$ is also an $\bar R$-homomorphism we can use self-injectivity of $\bar R$ to extend $\bar f$ to a map $\bar g\colon \bar R\to\bar R$. The composition of $\bar g$ with the natural map: $R\to\bar R$ is an extension of f and this shows that $\bar R$ is R-injective.

To see that $f(J\cap K)=0$, suppose not. Since R is regular, every principal ideal is generated by an idempotent element. Thus $J\cap K$ contains an idempotent element e such that $f(e)\neq 0$. But this leads to the contradiction $0\neq f(ee)=ef(e)\subseteq K(R/K)=0$.

Now, to prove the lemma, let R be any injective, pre-self-injective ring. Then the number of maximal ideals of R is the same as that of $R/\mathrm{rad}\,R$ (rad = Jacobson radical); and a theorem of Utumi [14, Lemma 4.1 and Theorem 4.7] states that if R is any left self-injective ring, then $R/\mathrm{rad}\,R$ is a left self-injective regular ring.

Thus we can suppose that R is a regular ring all of whose homomorphic images (including R itself) are self-injective. The first part of this proof then shows that every cyclic R-module is (R-) injective. But a theorem of Barbara Osofsky [11] or [12] asserts that any ring (commutative or not) with this last property must be semisimple with minimum condition; in the commutative case this means that R is the direct sum of a finite number of fields. In particular, R has only a finite number of maximal ideals.

Next we generalize the fact that a commutative ring with DCC is the direct sum of a finite number of local rings.

LEMMA 3.4. *Suppose that a commutative ring R has only a finite number of maximal ideals $M(1), \ldots, M(n)$, and that every prime ideal of R is maximal. Then*

$$R \cong R_{M(1)} \oplus \cdots \oplus R_{M(n)}. \tag{1}$$

Proof. Let $\nu_i\colon R \to R_{M(i)}$ be the natural map $r \to r/1$. Since $\nu_i(r)=0$ for all i only if $r=0$, the map $\nu\colon r \to (\nu_1(r), \ldots, \nu_n(r))$ imbeds R monomorphically in the right side of (1).

Since each ν_i is an epimorphism (Lemma 3.1), we can show that ν is an epimorphism by finding an element x of R such that $x/1$ is invertible in $R_{M(1)}$ and zero in each other $R_{M(i)}$.

By one of the forms of the Chinese Remainder Theorem, there is an element z in R such that $z \equiv 1$ $(M(1))$ and $z \equiv 0$ modulo each other $M(i)$. Since all primes of R are maximal, the maximal ideal of each $R_{M(i)}$ is its smallest prime ideal and hence is nil. Thus the element $z/1$ of $R_{M(i)}$ (for $i \neq 1$) is nilpotent; and since only a finite number of maximal ideals exist, there is a positive integer d such that $(z/1)^d=0$ in $R_{M(2)}, \ldots, R_{M(n)}$. Note that z^d is still congruent to 1 modulo $M(1)$. Hence, in the local ring $R_{M(1)}$, $z^d/1$ is outside the maximal ideal, and hence invertible. Hence $x=z^d$ is the element we want.

MAIN THEOREM 3.5. *A commutative ring* (*with* 1) *is pre-self-injective if and only if it is one of the following.*

(1) *An integral domain* (*necessarily a Prüfer domain*) *in which, for every maximal ideal M, R_M is an almost maximal rank* 1 *valuation domain*; *and every proper ideal is contained in only finitely many maximal ideals.*

(2) *The direct sum of a finite number of maximal rank* 0 *valuation rings.* (*Here R is also self-injective.*)

(3) *An almost maximal rank* 0 *valuation ring.*

(4) *A local ring whose maximal ideal M has composition length* 2 *and satisfies $M^2=0$.* (*Here R is not self-injective.*)

Proof. Let R be pre-self-injective. Note that by 3.3 every nonzero ideal (or element) is contained in only finitely many maximal ideals. We suppose first that R has an infinite number of maximal ideals M and show that R must be a domain. So suppose $xy=0$ with $x \neq 0$. Then x is contained in only finitely many maximal ideals, and $xy=0$ is in every maximal ideal. Since maximal ideals are prime, y is in infinitely many maximal ideals so $y=0$.

Now suppose R is a pre-self-injective domain. We show it is of type (1). For each maximal ideal M, R_M is a pre-self-injective local domain (Lemma 3.2) and hence an almost maximal valuation domain (by Theorem 2.2 and Corollary 2.5). Since each R_M is a valuation domain all of its finitely generated ideals are invertible (in fact, principal) and hence the same is true of R [1, Chapter II, Theorem 4, p. 148], that is, R is a Prüfer domain.

Now consider the case that R is not a domain. Then it can have only a finite number of maximal ideals. Since R is pre-self-injective, every nonzero prime ideal is maximal (Lemma 2.4), and since R is not a domain, 0 is not prime. Hence (Lemma 3.4) R is the direct sum of a finite number of local rings.

If the number of summands is 1, that is R is local, then we know (Theorem 2.2 and Corollary 2.5) that R must be of type (3) or (4). Observe that type (4) rings are not self-injective. For let x and y be independent elements of the maximal ideal M, then $Rx \cong R/M \cong Ry$ but this isomorphism cannot be multiplication by some element of R.

Now if the number of summands is at least 2, then each of them is a self-injective, pre-self-injective ring, and by Theorems 2.2 and 2.3 each summand must be a maximal rank 0 valuation ring.

For the converse note that Corollary 2.5 and Theorem 2.3 show that rings of type (3) are pre-self-injective while those of type (2) are self-injective and pre-self-injective. Rings of type (4) are pre-self-injective since their proper homomorphic images have at most one proper ideal, hence are maximal rank 0 valuation rings. Finally, let R be a domain of type (1). Then every nonzero prime ideal of R is maximal (since this is true of each R_M). Consequently, for each nonzero ideal J of R, the ring R/J has only a finite number of maximal ideals and all of its primes are maximal. Lemma 3.4 then shows R/J to be the direct sum of a finite number of rings of the form $(R/J)_{\overline{M}} \cong R_M/J_M$ (M a maximal ideal of R, $\overline{M}$ its image in R/J), a maximal rank 0 valuation ring and hence self-injective (Theorem 2.3). Therefore R/J is self-injective and the proof of the theorem is complete.

REMARK. For an example showing that the two conditions appearing in (1) of the theorem are independent, see Example 4.6.

COROLLARY 3.6. *Every nonnoetherian pre-self-injective ring has cardinality at least* $2^{\aleph_0}$.

Proof. First let R be a maximal rank 0 valuation ring which is not noetherian. Then R does not satisfy the DCC on ideals, so R has an infinite decreasing sequence of ideals $A_1 \supset A_2 \supset \cdots$.

For each i let $a_i \in A_i - A_{i+1}$. Then the system of congruences

$$(1) \qquad x \equiv a_1 + a_2 + \cdots + a_{n-1} \pmod{A_n} \qquad (n = 2, 3, 4, \ldots)$$

is pairwise solvable and hence has a solution in R which we may conveniently think of as an "infinite sum" $a_1 + a_2 + \cdots$. Note that if one or more of the a_i are replaced by 0, (1) remains solvable. Thus R contains $2^{\aleph_0}$ "infinite sums" of the form $c_1+c_2+\cdots$ where each c_i is either a_i or 0.

To complete the proof of the corollary it will be sufficient to show that every nonnoetherian pre-self-injective ring R can be mapped onto a nonnoetherian, rank 0 maximal valuation ring. Referring to our main theorem, we see that this is trivial when R is not a domain. But since every nonnoetherian domain has a proper homomorphic image which is again nonnoetherian, and since no proper homomorphic image of a pre-self-injective domain is again a domain (nonzero primes of a pre-self-injective domain are maximal by Lemma 2.4), the problem for domains is reduced to that for nondomains.

Combining the above corollary with (1) of the main theorem we get:

COROLLARY 3.7. *A domain of algebraic numbers is pre-self-injective if and only if it is a Dedekind domain.*

4. **Pre-injectivity of the quotient field.** In this section R will be an integral domain with quotient field Q. The R-module Q is always injective, since it is torsion-free and divisible [2, Chapter VII, Proposition 1.3, p. 128]. Our object here will be to relate R-injectivity of the proper R-homomorphic images of Q to pre-self-injectivity of R.

PROPOSITION 4.1. *Let R be a domain with quotient field Q and let I be an ideal of R. Assume Q/I is R-injective and*

$$(*) \qquad \text{if } q \in Q \quad \text{and} \quad qI \subseteq I \quad \text{then} \quad q \in R.$$

Then R/I is self-injective.

Proof. Let J be an ideal of R with $I \subseteq J \subseteq R$, and let $f: J/I \to R/I$ be an R/I-homomorphism. Now f is also an R-homomorphism: $J/I \to Q/I$ and since Q/I is R-injective, there is some q in Q such that $f(j+I)=jq+I$ for all j in J. However for every element $i \in I$, $f(i+I)=iq+I=f(0+I)=0+I$ showing $qI \subseteq I$, and by $(*)$ therefore $q \in R$.

An extension of f to an R/I-map: $R/I \to R/I$ is now given by $(r+I) \to (r+I)(q+I)$. Thus R/I is self-injective.

When is condition $(*)$ satisfied? For valuation domains the answer is:

LEMMA 4.2. *Let R be a valuation domain with quotient field Q. Then R has rank 1 if and only if for every nonzero ideal I of R:*

$$(*) \qquad \text{if} \quad q \in Q \quad \text{and} \quad qI \subseteq I \quad \text{then} \quad q \in R.$$

Proof. Suppose that R has rank 1 and $(*)$ is false. That is, $qI \subseteq I$ for some ideal $I \neq 0$ and some $q \in Q-R$. Since R is a valuation domain, $r=1/q$ belongs to R (in fact to the maximal ideal M of R since $q=1/r \notin R$) and $I=rI$. Now let K be any ideal properly contained in I. Then in the rank 0 valuation ring $\bar{R}=R/K$ we get

$\bar{I}=\bar{r}\bar{I}=\bar{r}^2\bar{I}=\bar{r}^3\bar{I}=\cdots$. Since the maximal ideal $\bar{M}$ of $\bar{R}$ is nil (1.2) we see that $\bar{I}=\bar{0}$, that is, $I\subseteq K$, a contradiction. Hence (*) holds.

Conversely, suppose that R has a prime ideal P such that $P\subset M$, M again the maximal ideal of R. Take $m\in M-P$. Then $P\subseteq Pm$ ($P\subset Rm$ since the ideals of R are totally ordered by inclusion; writing an element of P in the form xm, the fact that P is prime then implies that $x\in P$), and the opposite inclusion is trivial. Thus $P=Pm$; or, equivalently, $(1/m)P=P$ with $1/m\notin R$ since $m\in M$. Thus (*) is violated.

A globalization argument, to be given presently, will now complete the proof that if Q is pre-injective and R has Krull dimension 1, then R is pre-self-injective. We therefore turn to the converse.

Proposition 4.3. *Let R be a domain with quotient field Q and I be an ideal. If R/Ic is self-injective for all $c\neq 0$ in R then Q/I is R-injective.*

Proof. Let J be an ideal of R and $f\colon J\to Q/I$ be an R-homomorphism. Let $c\neq 0$ be in the kernel of f ($\operatorname{Ker} f\neq 0$ since Q/I is a torsion module). Let ν be the natural map $J\to J/Ic$ and φ be the isomorphism $Q/I\to Q/Ic$ given by $q+I\to qc+Ic$.

$$\begin{array}{ccc} J & \xrightarrow{f} & Q/I \\ \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\varphi} \\ J/Ic & \overset{f'}{\dashrightarrow} & Q/Ic \end{array}$$

Since $\ker\nu\subseteq\ker f$, we can define an R-homomorphism $f'\colon J/Ic\to Q/Ic$ by $f'=\varphi f\nu^{-1}$. We show $\operatorname{Im} f'\subseteq R/Ic$. Let $j+Ic\in J/Ic$ and assume $f(j)=r/s+I$. Then $f(cj)=0+I=rc/s+I$ showing $f'(j+Ic)=rc/s+Ic\in I/Ic\subseteq R/Ic$. Hence f' is an R/Ic-homomorphism: $J/Ic\to R/Ic$. Now R/Ic is self-injective so f' may be extended to an R/Ic homomorphism: $R/Ic\to R/Ic$ which we will again call f'.

The desired extension of f to a map $R\to Q/I$ is now given by $\varphi^{-1}f\nu$ (see the diagram below). Thus Q/I is R-injective.

$$\begin{array}{ccc} R & \dashrightarrow & Q/I \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R/Ic & \xrightarrow{f'} & Q/Ic \end{array} \qquad (=\text{ multiplication by } c)$$

We now have to obtain R-injectivity of Q/I where I is an arbitrary R-submodule of Q, not merely an ideal of R. To do this we prove a "primary decomposition theorem" for torsion modules over a pre-self-injective domain.

Proposition 4.4. *Let R be an integral domain in which proper prime ideals are maximal and in which each nonzero ideal is contained in only finitely many maximal ideals. Then for every torsion R-module T, $T\cong\bigoplus T_M$ with M ranging over the maximal ideals.*

Proof. (This is part of Corollary 8.6 of *Cotorsion modules*, E. Matlis, Mem. Amer. Math. Soc. No. 49 (1964). The present proof is included for the convenience of the reader.) First we note that if A is an ideal and if M and N are maximal ideals of R,

(1) $R_M \otimes (R/A)=0$ except for a finite number of M.

(2) $R_N \otimes R_M \otimes (R/A)=0$ whenever $M\neq N$.

To obtain (1) note that the left-hand side is isomorphic to R_M/A_M which is zero whenever A is not contained in M. To obtain (2), note that by Lemma 3.4, R/A is the direct sum of a finite number of local rings (which are also R-modules). Since tensor products commute with direct sums, we can suppose that A is contained in only one maximal ideal of R. Thus either $R_M \otimes (R/A)=0$ or $R_N \otimes (R/A)=0$.

For each t in T, (1) shows that $R_M \otimes (Rt)=0$ except for finitely many M. Thus we can use the canonical maps $t \to 1_M \otimes t$ to define a map $\nu: T \to \bigoplus R_M \otimes T$. To see that ν is an isomorphism, localize at a maximal ideal N and use (2), getting $1_N \otimes \nu: R_N \otimes T \to R_N \otimes R_N \otimes T$ which is an isomorphism. Hence so is ν.

We now obtain the main result of this section.

THEOREM 4.5. *For an integral domain R with quotient field Q the following are equivalent*

(1) *R is pre-self-injective.*

(2) *Every R-homomorphic image of Q is R-injective and R has Krull dimension* 1.

Proof. (2) $\Rightarrow$ (1). Since Q is pre-injective, R_M is an almost maximal valuation domain for every maximal ideal M (by Matlis's Theorem [7, Theorem 5, p. 61]). Let $I\neq 0$ be an ideal of R. We have to verify condition (*) of Proposition 4.1, so let $qI\subseteq I$. Since R has Krull dimension 1, the same is true of each valuation ring R_M. Thus by Lemma 4.2 (applied to $qI_M \subseteq I_M$), $q\in R_M$ for each M, and hence $q \in R$, establishing (1).

(1) $\Rightarrow$ (2). That R has Krull dimension 1 is a restatement of Lemma 2.4. Also, each nonzero ideal of R is contained in only a finite number of maximal ideals (3.5). Let H be an R-submodule of Q. We may suppose $H\neq 0$ since ${}_RQ$ is injective. Hence Q/H is a torsion module, so by our primary decomposition theorem (4.4) $Q/H\cong \bigoplus (Q/H)_M \cong \bigoplus Q/H_M$. Thus to show Q/H injective it is sufficient to show (i) For every R-homomorphism $f: J\to \bigoplus(Q/H)_M$ where J is an ideal of R, $f(J)$ has only finitely many nonzero coordinates; and (ii) each Q/H_M is R-injective.

To establish (i) let $f(J)\cong J/A$. Then $(J/A)_M \cong J_M/A_M$. Since J/A is a torsion module, A is nonzero and hence is contained in only finitely many maximal ideals. Thus $J_M/A_M=R_M/R_M=0$ for all but a finite number of M.

To establish (ii), choose an M. We may suppose $H_M\neq Q$ (otherwise there is nothing to prove). Then $1/q \notin H_M$ for some $q\in Q$. Since R_M is a valuation domain (3.5), the R_M-submodules of Q are totally ordered by inclusion (1.1). Thus $qH_M \subset R_M$. The ideal qH_M of R_M has the form I_M for an ideal I of R, namely $I=qH_M \cap R$.

Finally, observe that

$$Q/H_M \cong Q/qH_M = Q/I_M$$

which, by the primary decomposition theorem, is a direct summand of Q/I; and Q/I is injective by Proposition 4.3. This completes the proof.

EXAMPLE 4.6. Let Q be the subfield of the complex numbers generated by the rationals and pth roots of 1 for all primes p, and R the ring of all algebraic integers in Q. Nakano has shown that every R_M is a discrete rank 1 (hence almost maximal) valuation domain and that every prime number is contained in an infinite number of maximal ideals of R [9]. It is easy to see that the proper prime ideals of R are maximal.

It follows from Theorem 3.5 that R is not pre-self-injective. Thus the two conditions in (1) of Theorem 3.5 are independent. (To see the other half of the independence, take any valuation domain which is not almost maximal.) Since Theorem 4.5 shows Q is not pre-injective, the question raised by Matlis [7, Remark 2, p. 61], "If every R_M is an almost maximal valuation domain, is Q pre-injective?" has a negative answer.

5. **Finitely generated modules.**

We close with the observation:

THEOREM 5.1. *Over a pre-self-injective domain R every finitely generated module is the direct sum of ideals of R and cyclic R-modules.*

Proof. Let M be the given finitely generated module. If M is a torsion module, then since our main theorem shows that R is h-local and Prüfer, the result about M is a special case of [E. Matlis, *Decomposable modules*, Trans. Amer. Math. Soc. **125** (1966), 147–179; Theorem 5.7]. For the general case, the fact that R is a Prüfer domain shows that $M \cong M/T \oplus T$ (T the torsion submodule of M) and M/T is isomorphic to a direct sum of ideals of R [4, Theorem 1].

The theorem is spoiled somewhat by the fact that the only examples we know of pre-self-injective domains are Dedekind domains and rank 1 almost maximal valuation domains, and in both of these cases the theorem is already known [4, Theorem 1, Remarks, p. 332, and Theorem 14]. Therefore we have not explored other similar analogues from abelian group theory.

REFERENCES

1. N. Bourbaki, *Algèbre commutative*, Chapters I and II, Hermann, Paris, 1961.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

3. I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. **19** (1942), 303–321.

4. ———, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. **72** (1952), 327–340.

5. W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. **167** (1932), 160–196.

6. L. S. Levy, *Commutative rings whose homomorphic images are self-injective*, Pacific J. Math. **18** (1966), 149–153.

7. E. Matlis, *Injective modules over Prüfer rings*, Nagoya Math. J. **15** (1959), 57–69.

8. M. Nagata, *Local rings*, Interscience, New York, 1962.

9. N. Nakano, *Idealtheorie in einem speziellen unendlichen algebraischen Zahlkörper*, J. Sci. Hiroshima Univ. Ser. A-I Math. **16** (1953), 425–439.

10. B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387.

11. ———, *Noncyclic injective modules*, Proc. Amer. Math. Soc. (to appear).

12. ———, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.

13. O. F. G. Schilling, *Valuation theory*, Math. Surveys, No. 4, Amer. Math. Soc., Providence, R. I., 1950.

14. Y. Utumi, *On continuous rings and self-injective rings*, Trans. Amer. Math. Soc. **118** (1965), 158–173.

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