IDENTITIES INVOLVING THE COEFFICIENTS OF A CLASS OF DIRICHLET SERIES. II

BY

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1. **Introduction.** Let $\phi(s)=\sum_{n=1}^{\infty} a(n)\lambda_n^{-s}$ and $\psi(s)$ be Dirichlet series satisfying a functional equation of the form

$$\Gamma^m(s)\phi(s) = \Gamma^m(r-s)\psi(r-s), \tag{1.1}$$

where m is a positive integer and r is real. Several authors (e.g. see [2], [5], and [6] and the references there given) have derived identities for $\sum_{\lambda_n \leqq x} a(n)(x-\lambda_n)^q$, where q is real. On the contrary, only in a few special cases have identities been given for $\sum_{\lambda_n \leqq x} a(n) \log^q (x/\lambda_n)$. In this paper we derive identities for the aforementioned sum when q is a nonnegative integer and ϕ is a Dirichlet series belonging to the same class as that studied in [2]. We conclude with several examples involving well-known arithmetical functions.

Throughout the sequel we let $s=\sigma+it$ with σ and t both real. If c is real, we denote the integral $\int_{c-i\infty}^{c+i\infty}$ by $\int_{(c)}$. The summation sign $\sum$ appearing with no indices will always mean $\sum_{n=1}^{\infty}$. $J_\nu(x)$ always denotes the usual Bessel function of order ν. c, c_n and c_n', $0 \leqq n < \infty$, always denote constants, not necessarily the same with each occurrence. Also, q always denotes a nonnegative integer.

2. **Definitions and preliminary results.** The following definitions and lemmas will be important in the sequel.

DEFINITION 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers tending to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Consider the functions ϕ and ψ representable as Dirichlet series

$$\phi(s) = \sum a(n)\lambda_n^{-s}, \qquad \psi(s) = \sum b(n)\mu_n^{-s}$$

with finite abscissas of absolute convergence σ_a and σ_a^*, respectively. We say that ϕ and ψ satisfy functional equation (1.1) if there exists in the s-plane a domain D which is the exterior of a bounded closed set S such that in D a holomorphic function $\chi(s)$ exists with these properties:

(i) $\chi(s)=\Gamma^m(s)\phi(s)$, $\sigma>\sigma_a$; $\chi(s)=\Gamma^m(r-s)\psi(r-s)$, $\sigma<r-\sigma_a^*$.

(ii) If $\gamma=\sigma_a^*+p-1/4m$, where the positive integer p and η are chosen so that

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$\gamma > \max(0, \sigma_a, \sigma_a^*)$ and S lies outside $R = \{s : r-\gamma < \sigma < \gamma, |t| \geqq \eta\}$, but in $r-\gamma < \sigma < \gamma$, then for some constant $\theta < 1$,

(2.1) $$\chi(s) = O(\exp[\exp(\theta\pi|s|/(2\gamma - r))]),$$

uniformly in R as $|s| \to \infty$.

LEMMA 2.1. *We have*

$$|\Gamma(s)| \sim (2\pi)^{1/2}|t|^{\sigma-1/2}\exp(-\tfrac{1}{2}\pi|t|),$$

for fixed σ, *as* $|t|$ *tends to* ∞.

LEMMA 2.2. *For* $c > 0$ *and* $q \geqq 0$,

$$\begin{aligned} \frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s^{q+1}}\,ds &= 0, & 0 \leqq x < 1, \\ &= \log^q(x)/q!, & x \geqq 1, \end{aligned}$$

except when $x = 1$ *and* $q = 0$ *in which case the result is* $1/2$.

Lemma 2.2 is a consequence of [4, Lemma 3.62, p. 78].

DEFINITION 2. Let $\phi(s) = \sum a(n)\lambda_n^{-s}$. For $q \geqq 0$ and $x > 0$, let

$$S(x; q) = \frac{1}{q!}\sum_{\lambda_n \leqq x}{}' a(n)\log^q(x/\lambda_n),$$

where the $'$ indicates that if $x = \lambda_n$ and $q = 0$, $a(n)$ is to be multiplied by $1/2$.

LEMMA 2.3. *For* $c > \max(0, \sigma_a)$,

$$S(x; q) = \frac{1}{2\pi i}\int_{(c)} \frac{\phi(s)x^s}{s^{q+1}}\,ds.$$

This result is an easy consequence of Lemma 2.2.

DEFINITION 3. For $u, v > 0$ and $q \geqq 0$, define

$$s(u, v; q) = \sum_{j=0}^{q} \frac{(-1)^j \log^j u \log^{q-j} v}{j!(q-j)!}.$$

For $q > 0$ we observe

(2.2) $$\partial s(u, v; q)/\partial v = s(u, v; q-1)/v,$$

(2.3) $$\partial s(u, v; q)/\partial u = -s(u, v; q-1)/u,$$

and

(2.4) $$s(v, v; q) = 0.$$

DEFINITION 4. Let k and m be positive integers with $k \leqq m$. If $\min(\mu, \nu)+3/2>0$, define

$$K_\nu(x;\mu;m_k) = \int_0^\infty u_{m-1}^{\nu-\mu-1} J_\mu(u_{m-1})\,du_{m-1}\cdots\int_0^\infty u_k^{\nu-\mu-1} J_\mu(u_k)\,du_k \cdot\int_0^\infty u_{k-1}^{-1} J_\nu(u_{k-1})\,du_{k-1}\cdots\int_0^\infty u_1^{-1} J_\nu(u_1) J_\nu(x/u_1\cdots u_{m-1})\,du_1.$$

For $k=1$, let $K_\nu(x;\mu;m_1)=K_\nu(x;\mu;m)$, as in [2]. For $m=1$, $K_\nu(x;\mu;1)=J_\nu(x)$.

LEMMA 2.4. *For* $0<c<\min \frac{1}{2}(\mu,\nu)+3/4$,

$$x^{-\nu}K_\nu(x;\mu;m_k) = \frac{1}{2\pi i}\int_{(c)} \frac{2^{2ms-(m-k)\mu-k\nu-m+1}\Gamma^m(s)x^{-2s}}{\Gamma^{m-k}(\mu+1-s)\Gamma^k(\nu+1-s)}\,ds.$$

Lemma 2.4 is a consequence of a general theorem on multiple Mellin transforms (see e.g. [14, pp. 53, 60 and 196]).

LEMMA 2.5. *For* $\min \frac{1}{2}(\mu,\nu)+3/4>0$,

$$K_\nu(x;\mu;m_k) \sim x^{[(\nu-\mu)(m-k)-m+1/2]/m}\sum_{n=0}^\infty c_n x^{-j/m}\cos(mx^{1/m}+c_n'),$$

as $x\to\infty$.

Chandrasekharan and Narasimhan [7, Lemma 1] have proven a similar result, and the proof of Lemma 2.5 follows along the same lines as their proof.

LEMMA 2.6. *As* $x\to 0$, $K_\nu(x;\mu;m)=O(x^\nu)$.

This is a consequence of the fact that $J_\nu(x)=O(x^\nu)$ as $x\to 0$.

LEMMA 2.7. *Let q be a positive integer and* $\xi_j>0$, $1\leqq j\leqq q$. *Then,*

$$\int_{\xi_q}^\infty \frac{d\xi_{q-1}}{\xi_{q-1}}\cdots\int_{\xi_2}^\infty \frac{d\xi_1}{\xi_1}\int_{\xi_1}^\infty u^a K_\nu(u;\mu;m)\,du = (-1)^{q-1}\int_{\xi_q}^\infty u^a K_\nu(u;\mu;m)s(u,\xi_q;q-1)\,du,$$

provided $a<[(\mu-\nu)(m-1)+1/2]/m$.

Proof. We use induction on q. If $q=1$, the result is trivial. Assume that for $q\geqq 2$,

$$\int_{\xi_{q-1}}^\infty \frac{d\xi_{q-2}}{\xi_{q-2}}\cdots\int_{\xi_2}^\infty \frac{d\xi_1}{\xi_1}\int_{\xi_1}^\infty u^a K_\nu(u;\mu;m)\,du = (-1)^{q-2}\int_{\xi_{q-1}}^\infty u^a K_\nu(u;\mu;m)s(u,\xi_{q-1};q-2)\,du.$$

Multiply both sides by $1/\xi_{q-1}$ and integrate over (ξ_q, ∞) to obtain

$$\int_{\xi_q}^{\infty} \frac{d\xi_{q-1}}{\xi_{q-1}} \cdots \int_{\xi_2}^{\infty} \frac{d\xi_1}{\xi_1} \int_{\xi_1}^{\infty} u^a K_\nu(u; \mu; m)\, du$$

$$= (-1)^{q-2} \int_{\xi_q}^{\infty} \frac{d\xi_{q-1}}{\xi_{q-1}} \int_{\xi_{q-1}}^{\infty} u^a K_\nu(u; \mu; m) s(u, \xi_{q-1}; q-2)\, du$$

$$= (-1)^{q-2} \int_{\xi_q}^{\infty} u^a K_\nu(u; \mu; m)\, du \int_{\xi_q}^{u} \frac{\partial s}{\partial \xi_{q-1}} (u, \xi_{q-1}; q-1)\, d\xi_{q-1}$$

$$= (-1)^{q-1} \int_{\xi_q}^{\infty} u^a K_\nu(u; \mu; m) s(u, \xi_q; q-1)\, du,$$

upon the use of (2.2) and (2.4). The interchange in order of integration is valid by a theorem in [8, p. 349].

LEMMA 2.8. *Let q be a positive integer and $\xi_j > 0$, $1 \leq j \leq q$. Then,*

$$\int_0^{\xi_q} \frac{d\xi_{q-1}}{\xi_{q-1}} \cdots \int_0^{\xi_2} \frac{d\xi_1}{\xi_1} \int_0^{\xi_1} u^a K_\nu(u; \mu; m)\, du = \int_0^{\xi_q} u^a K_\nu(u; \mu; m) s(u, \xi_q; q-1)\, du,$$

provided $a+\nu+1>0$.

The proof is analogous to that of Lemma 2.7.

LEMMA 2.9. *We have*

$$d[x^\nu K_\nu(x; \mu; m)]/dx = x^\nu K_{\nu-1}(x; \mu; m)$$

and

$$d[x^\nu K_\nu(x; \nu-1; m_k)]/dx = x^{\nu-1} K_\nu(x; \nu-1; m_{k-1}).$$

These formulas are easily derived from Lemma 2.4.

LEMMA 2.10. *Let q be a nonnegative integer and suppose $b = ma + (\nu-\mu)(m-1) - \frac{1}{2} < 0$. Then, as $\xi \to \infty$,*

$$\int_{\xi}^{\infty} u^a K_\nu(u; \mu; m) \log^q u\, du$$

$$\sim \xi^{b/m} \left(\sum_{n=0}^{\infty} \xi^{-n/m} \sum_{k=0}^{\min(n,q)} \log^{q-k} \xi (c_k \exp(im\xi^{1/m}) + c_k' \exp(-im\xi^{1/m})) \right).$$

Proof. In view of Lemma 2.5 it is sufficient to examine

$$\int_{\xi}^{\infty} u^{(b-m+1)/m} \log^q u \exp(imu^{1/m})\, du$$

$$= c \int_{m\xi^{1/m}}^{\infty} u^b \log^q u\, e^{iu}\, du$$

$$= \exp(im\xi^{1/m}) \sum_{j=0}^{n-1} \xi^{(b-j)/m} \sum_{k=0}^{\min(j,q)} c_k \log^{q-k} \xi + c \int_{m\xi^{1/m}}^{\infty} u^{b-n} \sum_{j=0}^{q} c_j' \log^{q-j} u\, e^{iu}\, du,$$

upon $n \geq q$ integrations by parts, and the result follows.

LEMMA 2.11. *Let q be a nonnegative integer and suppose $a+\nu+1>0$ and $b=ma+(\nu-\mu)(m-1)-1/2\geqq 0$. Then, as $\xi\to\infty$,*

$$\int_0^\xi u^a K_\nu(u;\mu;m)\log^q u\,du$$
$$= O(1)+\xi^{b/m}\left(\sum_{n=0}^{[b]}\xi^{-n/m}\sum_{k=0}^{\min(n,q)}\log^{q-k}\xi(c_k\exp(im\xi^{1/m})+c_k'\exp(-im\xi^{1/m}))\right),$$

where $[b]$ denotes the greatest integer less than or equal to b.

Proof. In view of Lemma 2.5 it is sufficient to examine

$$\int_1^\xi u^{(b-m+1)/m}\log^q u\exp(imu^{1/m})\,du$$
$$= c\int_m^{m\xi^{1/m}} u^b\log^q u\,e^{iu}\,du$$
$$= c+\exp(im\xi^{1/m})\sum_{j=0}^{n-1}\xi^{(b-j)/m}\sum_{k=0}^{\min(j,q)}c_k\log^{q-k}\xi$$
$$+\int_m^{m\xi^{1/m}} u^{b-n}\sum_{j=0}^{\min(n,q)}c_j'\log^{q-j}u\,e^{iu}\,du$$
$$= \exp(im\xi^{1/m})\sum_{j=0}^{n-1}\xi^{(b-j)/m}\sum_{k=0}^{\min(j,q)}c_k\log^{q-k}\xi+O(1),$$

upon $n=[b]+1$ integrations by parts. The result now follows.

3. The identities for $S(x,q)$.

THEOREM 1. *Let ϕ satisfy Definition 1 and choose $\gamma>r$. Let $x>0$, $\xi=2^m(\mu_n x)^{1/2}$, and q a positive integer such that $q>2m\sigma_a^*-mr-1/2$.*

(i) *Suppose $q<m$. If $m\geqq 2$, assume $r>-1/2$ unless $q+1=m$ in which case we only require $r>-3/2$. Then,*

$$S(x;q) = R(x;q)+2^q\sum b(n)\left(\frac{x}{\mu_n}\right)^{r/2}K_r(\xi;r-1;m_{q+1}) \tag{3.1}$$

where the series on the right-hand side converges absolutely and

$$R(x;q) = \frac{1}{2\pi i}\int_C\frac{\phi(s)x^s}{s^{q+1}}\,ds,$$

where C is a curve, or curves, encircling all of the singularities of $\phi(s)/s^{q+1}$.

(ii) *Suppose $q\geqq m$. If $m\geqq 2$, suppose $r>-3/2$. Then, if $q>mr-\frac{1}{2}$,*

$$S(x;q) = R(x;q)-2^{2q-mr-m}$$
$$\cdot\sum\frac{b(n)d^{q-m}}{\mu_n^r dx^{q-m}}\left(x^{q-m}\int_\xi^\infty u^{r-q+m-1}K_{r+q-m}(u;r;m)s(u,\xi;q-m)\,du\right), \tag{3.2}$$

where the series on the right-hand side converges absolutely.

Note that if $\sigma_a^* \geqq r$ in (ii), the condition $q > mr - 1/2$ is redundant.

Proof. Let $\gamma > r$ be as given in Definition 1. By Lemma 2.3 for $q \geqq 0$,

$$\frac{1}{2\pi i}\int_{(\gamma)} \frac{\phi(s)x^s}{s^{q+1}}\,ds = S(x;q). \tag{3.3}$$

Consider the integrand on the left-hand side of (3.3) and integrate around the rectangle with vertices $\gamma \pm iT$ and $r-\gamma \pm iT$, where T is chosen large enough so that S is entirely contained within the rectangle. Since $\gamma > \sigma_a$, on $s = \gamma + it$, $\phi(s)/s^{q+1} = o(1)$ as $|t| \to \infty$. Using (1.1) and Lemma 2.1, we have on $s = r - \gamma + it$, since $\gamma > \sigma_a^*$,

$$\frac{\phi(s)}{s^{q+1}} = \frac{\Gamma^m(r-s)\psi(r-s)}{\Gamma^m(s)s^{q+1}} = O(|t|^{2m\gamma - mr - q - 1}) = o(1),$$

as $|t| \to \infty$, provided $q > 2m\gamma - mr - 1$. Hence, from (2.1) and a Phragmén-Lindelöf Theorem [9, p. 109], the integrals along the horizontal edges tend to 0 as $T \to \infty$. Thus,

$$\frac{1}{2\pi i}\int_{(\gamma)} \frac{\phi(s)x^s}{s^{q+1}}\,ds = I(x) + R(x;q), \tag{3.4}$$

where

$$I(x) = \frac{1}{2\pi i}\int_{(r-\gamma)} \frac{\phi(s)x^s}{s^{q+1}}\,ds.$$

Replacing s by $r-s$, using the functional equation, and interchanging the order of summation and integration by absolute convergence if $q > 2m\gamma - mr$, we arrive at

$$I(x) = \sum \frac{b(n)}{\mu_n^r}\frac{1}{2\pi i}\int_{(\gamma)} \frac{\Gamma^m(s)(\mu_n x)^{r-s}}{\Gamma^m(r-s)(r-s)^{q+1}}\,ds. \tag{3.5}$$

Suppose $q < m$. Then, using Lemma 2.4, we have

$$\begin{aligned} I(x) &= \sum \frac{b(n)}{\mu_n^r}\frac{1}{2\pi i}\int_{(\gamma)} \frac{\Gamma^m(s)(\mu_n x)^{r-s}}{\Gamma^{m-q-1}(r-s)\Gamma^{q+1}(r+1-s)}\,ds \\ &= 2^q \sum b(n)\left(\frac{x}{\mu_n}\right)^{r/2} K_r(2^m(\mu_n x)^{1/2}; r-1; m_{q+1}), \end{aligned} \tag{3.6}$$

provided $r > -1/2$ if $m \geqq 2$. However, if $q+1 = m$, we only need $r > -3/2$ since $K_r(z; r-1; m_m) = K_r(z; r; m)$. Combining (3.3), (3.4) and (3.6), we have shown (3.1) for $q > 2\gamma - r$ and $q \geqq 0$.

But, by Lemma 2.5,

$$\begin{aligned} I(x) &= O(x^{(mr-q-1/2)/2m} \sum |b(n)|\mu_n^{-(mr+q+1/2)/2m}) \\ &= O(x^{(mr-q-1/2)/2m}), \end{aligned}$$

provided $q > 2m\sigma_a^* - mr - 1/2$. Thus, $I(x)$ converges absolutely and uniformly for $q > 2m\sigma_a^* - mr - 1/2$. By $2mp$ differentiations with the help of Lemma 2.9, (3.1) is then upheld for $q > 2m\sigma_a^* - mr - 1/2$ and $q \geqq 0$. However, the uniform convergence of $I(x)$ for $q > 2m\sigma_a^* - mr - 1/2$ implies that the $2mp$th derivative is continuous.

Hence, $S(x;q)$ is continuous, and $q>0$, since $S(x;q)$ is discontinuous on $(0,\infty)$ for $q\leqq 0$.

Assume now that $q\geqq m$. From Lemma 2.4,

$$(3.7)\quad u^{-r-q+m}K_{r+q-m}(u;r;m)=\frac{1}{2\pi i}\int_{(c)}\frac{2^{2ms-mr-q+1}\Gamma^m(s)u^{-2s}}{\Gamma^{m-1}(r+1-s)\Gamma(r+q+1-m-s)}\,ds,$$

provided $0<c<r/2+3/4$. But, by using Lemma 2.1 and Cauchy's Theorem we may move the line of integration to $\gamma+it$, $-\infty<t<\infty$, provided $q>2m\gamma-mr$. Multiply both sides of (3.7) by u^{2r-1} and integrate over (ξ_1,∞), $\xi_1>0$, provided that $q>mr-\frac{1}{2}$ so that the resulting integral on the left-hand side converges. Using a standard theorem (e.g. see [8, p. 349]) to invert the order of integration, we find

$$(3.8)\quad \begin{aligned}&\int_{\xi_1}^{\infty}u^{r-q+m-1}K_{r+q-m}(u;r;m)\,du\\ &\qquad=-\frac{1}{2\pi i}\int_{(\gamma)}\frac{2^{2ms-mr-q}\Gamma^m(s)\xi_1^{2r-2s}}{\Gamma^{m-1}(r+1-s)\Gamma(r+q+1-m-s)(r-s)}\,ds,\end{aligned}$$

provided $\gamma>r$, $q>2m\gamma-mr$ and $q>mr-\frac{1}{2}$. Now multiply both sides of (3.8) by $1/\xi_1$ and integrate over (ξ_2,∞), $\xi_2>0$. After $q-m+1$ such integrations and the application of Lemma 2.7, we find

$$(3.9)\quad \begin{aligned}&\int_{\xi_{q-m+1}}^{\infty}\frac{d\xi_{q-m}}{\xi_{q-m}}\cdots\int_{\xi_2}^{\infty}\frac{d\xi_1}{\xi_1}\int_{\xi_1}^{\infty}u^{r-q+m-1}K_{r+q-m}(u;r;m)\,du\\ &\qquad=(-1)^{q-m}\int_{\xi_{q-m+1}}^{\infty}u^{r-q+m-1}K_{r+q-m}(u;r;m)s(u,\xi_{q-m+1};q-m)\,du\\ &\qquad=\frac{(-1)^{q-m+1}}{2\pi i}\int_{(\gamma)}\frac{2^{2ms-mr-2q+m}\Gamma^m(s)\xi_{q-m+1}^{2r-2s}}{\Gamma^{m-1}(r+1-s)\Gamma(r+q+1-m-s)(r-s)^{q-m+1}}\,ds.\end{aligned}$$

Letting $\xi_{q-m+1}=\xi$ and multiplying both sides of (3.9) by x^{q-m}, we have

$$(3.10)\quad \begin{aligned}&x^{q-m}\int_{\xi}^{\infty}u^{r-q+m-1}K_{r+q-m}(u;r;m)s(u,\xi;q-m)\,du\\ &\qquad=-\frac{2^{mr-2q+m}}{2\pi i}\int_{(\gamma)}\frac{\Gamma^m(s)\mu_n^{r-s}x^{r+q-m-s}}{\Gamma^{m-1}(r+1-s)\Gamma(r+q+1-m-s)(r-s)^{q-m+1}}\,ds.\end{aligned}$$

Now differentiate both sides of (3.10) $(q-m)$ times to obtain

$$(3.11)\quad \begin{aligned}&\frac{d^{q-m}}{dx^{q-m}}\left(x^{q-m}\int_{\xi}^{\infty}u^{r-q+m-1}K_{r+q-m}(u;r;m)s(u,\xi;q-m)\,du\right)\\ &\qquad=-\frac{2^{mr-2q+m}}{2\pi i}\int_{(\gamma)}\frac{\Gamma^m(s)(\mu_n x)^{r-s}}{\Gamma^m(r+1-s)(r-s)^{q-m+1}}\,ds.\end{aligned}$$

By a well-known theorem (see e.g. [13, p. 59]) these differentiations are easily justified if $q>2m\gamma-mr$. Substituting (3.11) into (3.5) and combining it with (3.3) and (3.4), we arrive at (3.2), provided $q>2m\gamma-mr$ and $q\geqq 0$.

However, with the aid of (2.1), (2.3) and Lemma 2.10, we find

$$\frac{d^{q-m}}{dx^{q-m}}\left(x^{q-m}\int_{\xi}^{\infty} u^{r-q+m-1}K_{r+q-m}(u;r;m)s(u,\xi;q-m)\,du\right)$$
$$= \int_{\xi}^{\infty} u^{r-q+m-1}K_{r+q-m}(u;r;m)\sum_{k=0}^{q-m} c_k s(u,\xi;k)\,du$$
$$= O(\xi^{(mr-q-1/2)/2m}\log^{q-m}\xi).$$

Hence,

$$I(x) = O(x^{(mr-q-1/2)/2m}\log^{q-m} x),$$

provided $q>2m\sigma_a^*-mr-\frac{1}{2}$. Since $I(x)$ converges uniformly for $q>2m\sigma_a^*-mr-\frac{1}{2}$, by $2mp$ differentiations the validity of (3.2) is upheld for $q>2m\sigma_a^*-mr-\frac{1}{2}$. Again, the uniform convergence implies $q>0$. This completes the proof of Theorem 1.

In many cases the identity (3.2) can be greatly simplified. We illustrate this in the following

THEOREM 2. *Let the hypotheses of Theorem* 1 *be satisfied for* $q \geqq m$, *except that the condition* $q>mr-\frac{1}{2}$ *is replaced by* $r<1+1/2m$. *Then,*

$$S(x;q) = R(x;q)-2^{q-mr}\sum\frac{b(n)}{\mu_n^r}\int_{\xi}^{\infty} u^{r-1}K_r(u;r;m)s(u,\xi;q-m)\,du, \tag{3.12}$$

where the series on the right-hand side converges absolutely.

Proof. We derive (3.5) as before, but now we consider

$$u^{-r}K_r(u;r;m) = \frac{1}{2\pi i}\int_{(c)}\frac{2^{2ms-mr-m+1}\Gamma^m(s)u^{-2s}}{\Gamma^m(r+1-s)}\,ds, \tag{3.13}$$

provided $0<c<r/2+3/4$. Proceeding as before, we find

$$\int_{\xi}^{\infty} u^{r-1}K_r(u;r;m)s(u,\xi;q-m)\,du = -\frac{2^{mr-q}}{2\pi i}\int_{(c)}\frac{\Gamma^m(s)(\mu_n x)^{r-s}}{\Gamma^m(r+1-s)(r-s)^{q-m+1}}\,ds, \tag{3.14}$$

provided $c>r$, $q>2mc-mr$, and $r<1+1/2m$ so that the integral on the left-hand side converges. By Lemma 2.1 and Cauchy's Theorem we can move the line of integration of the integral on the right-hand side of (3.14) to $\gamma+it$, $-\infty<t<\infty$, provided $q>2m\gamma-mr$. Substituting (3.14) into (3.5) and combining it with (3.3) and (3.4), we have established (3.12) if $q>2m\gamma-mr$.

Now integrate by parts $(q-m)$ times with the aid of Lemma 2.9 and (2.3). The integrated terms always vanish because of (2.4) and the fact that $r<1+1/2m$. Thus, we find that

$$\int_{\xi}^{\infty} u^{r-1}K_r(u;r;m)s(u,\xi;q-m)\,du$$
$$= \int_{\xi}^{\infty} u^{r-q+m-1}K_{r+q-m}(u;r;m)\sum_{k=0}^{q-m} c_k s(u,\xi;k)\,du$$
$$= O(\xi^{(mr-q-1/2)/2m}\log^{q-m}\xi),$$

by Lemma 2.10. Hence,

$$I(x) = O(x^{(mr-q-1/2)/2m}\log^{q-m} x),$$

provided that $q > 2m\sigma_a^* - mr - \frac{1}{2}$, and the validity of (3.12) follows as before.

The following theorem gives an analogous identity to (3.2) in the case that $q \leqq mr - \frac{1}{2}$.

THEOREM 3. *In the same notation as Theorem* 1, *assume that* $q > 2m\sigma_a^* - mr - \frac{1}{2}$ *and* $m \leqq q \leqq mr - \frac{1}{2}$. *Suppose also that* $0 < \gamma < r$. *Then,*

$$S(x; q) = R(x; q) + 2^{q-mr} \sum \frac{b(n)}{\mu_n^r} \int_0^\xi u^{r-1} K_r(u; r; m) s(u, \xi; q-m)\, du, \tag{3.15}$$

where the series on the right-hand side converges absolutely.

Proof. Consider again (3.13) and multiply both sides by u^{2r-1}. However, now we integrate over $(0, \xi_1)$, $\xi_1 > 0$, to obtain

$$\int_0^{\xi_1} u^{r-1} K_r(u; r; m)\, du = \frac{1}{2\pi i} \int_{(c)} \frac{2^{m(2s-r-1)}\Gamma^m(s)\xi_1^{2r-2s}}{\Gamma^m(r+1-s)(r-s)}\, ds,$$

provided $0 < c < r$. After $q-m+1$ integrations and the application of Lemma 2.8, we have

$$\begin{aligned} &\int_0^\xi \frac{d\xi_{q-m}}{\xi_{q-m}} \cdots \int_0^{\xi_2} \frac{d\xi_1}{\xi_1} \int_0^{\xi_1} u^{r-1} K_r(u; r; m)\, du \\ &\qquad = \int_0^\xi u^{r-1} K_r(u; r; m) s(u, \xi; q-m)\, du \\ &\qquad = \frac{1}{2\pi i} \int_{(c)} \frac{2^{2ms-mr-q}\Gamma^m(s)\xi^{2r-2s}}{\Gamma^m(r+1-s)(r-s)^{q-m+1}}\, ds, \end{aligned} \tag{3.16}$$

provided $0 < c < r$. Again, we move the line of integration to $\gamma + it$, $-\infty < t < \infty$, $\gamma < r$, with the aid of Lemma 2.1 and Cauchy's Theorem, provided that $q > 2m\gamma - mr$. Substituting (3.16) into (3.5) and combining the result with (3.3) and (3.4), we have proven (3.15) for $q > 2m\gamma - mr$.

However, upon $q-m$ integrations by parts with the use of Lemma 2.9, Lemma 2.6, (2.3) and (2.4), we arrive at

$$\begin{aligned} &\int_0^\xi u^{r-1} K_r(u; r; m) s(u, \xi; q-m)\, du \\ &\qquad = \int_0^\xi u^{r-q+m-1} K_{r+q-m}(u; r; m) \sum_{k=0}^{q-m} c_k s(u, \xi; k)\, du \\ &\qquad = O(\xi^{mr-q-1/2)/2m} \log^{q-m} \xi), \end{aligned}$$

by Lemma 2.11, provided that $q \leqq mr - \frac{1}{2}$. Thus,

$$I(x) = O(x^{(mr-q-1/2)/2m}\log^{q-m} x),$$

provided $q > 2m\sigma_a^* - mr - \frac{1}{2}$. As before, (3.15) is then valid for $q > 2m\sigma_a^* - mr - \frac{1}{2}$.

COROLLARY 1. *Under the hypotheses of Theorems* 1–3, *if* $q<m$,

$$S(x;q) = R(x;q)+O(x^{(mr-q-1/2)/2m});$$

if $q \geqq m$,

$$S(x;q) = R(x;q)+O(x^{(mr-q-1/2)/2m}\log^{q-m} x).$$

Richert [12] has obtained big O-estimates for $S(x;q)$ for a large class of Dirichlet series. The functional equation satisfied by Richert's series is more general than ours, but he requires that $\phi=\psi$ and that ϕ be entire.

We now wish to establish identities (3.1), (3.2), (3.12) and (3.15) for the smallest possible value of q.

THEOREM 4. (i) *Let* $q<m$ *and suppose that for* $\sigma>\sigma_a^*$,

$$\sup_{0\leqq h\leqq 1}\left|\sum_{k^{2m}\leqq \mu_n\leqq (k+h)^{2m}} b(n)\mu_n^{-\sigma+1/2m}\right| = o(1)$$

as $k\to\infty$. *Then*, (3.1) *is valid for* $q>2m\sigma_a^*-mr-3/2$. *The series on the right-hand side of* (3.1) *converges uniformly on any interval for* $x>0$ *when* $q>0$. *The convergence is bounded on any interval* $0<x_1\leqq x\leqq x_2<\infty$ *when* $q=0$.

(ii) *Let* $q\geqq m$ *and suppose that for* $\sigma>\sigma_a^*$,

$$\sup_{0\leqq h\leqq 1}\left|\sum_{k^{2m}\leqq \mu_n\leqq (k+h)^{2m}} b(n)\mu_n^{-\sigma+1/2m}\log^{q-m}\mu_n\right| = o(1)$$

as $k\to\infty$. *Then*, (3.2), (3.12) *and* (3.15) *are valid for* $q>2m\sigma_a^*-mr-3/2$, *and the respective series converge uniformly on any interval*, $x>0$.

The proof of Theorem 4 is similar to the proof of Theorem 4 in [2], and we omit the details. We use Lemma 2.5 in the extension of (3.1), Lemma 2.10 in the extensions of (3.2) and (3.12), and Lemma 2.11 in the extension of (3.15), and then proceed in exactly the same manner as in [2].

4. Examples.

In the following we do not concentrate on examples for $q=0$, because these identities are identical to the ones given in [5] and [2] for $q=0$ there. Most of the identities given in the examples are new.

EXAMPLE 1. Consider $\pi^{-s}\zeta(2s)$, where $\zeta(s)$ is the Riemann zeta-function, which satisfies Definition 1 for $m=1$, $r=\frac{1}{2}$, $\lambda_n=\mu_n=\pi n^2$ and $\sigma_a=\frac{1}{2}$. $\zeta(s)$ has a simple pole of residue 1 at $s=1$. Replacing x by πx^2 and noting that $J_{1/2}(z)=(2/\pi z)^{1/2}\sin z$, we have from Theorem 2 for $q>0$,

$$\sum_{n\leqq x}\log^q(x/n) = q!x+\sum_{j=0}^{q}\binom{q}{j}\zeta^{(j)}(0)\log^{q-j}x$$
$$-\frac{q!}{\pi}\sum\frac{1}{n}\int_{2\pi nx}^{\infty}\frac{\sin u}{u}\,s(u,2\pi nx;q-1)\,du.$$

By Corollary 1, the series on the right-hand side is $O(x^{-q}\log^{q-1}x)$, but this term may be easily replaced by an asymptotic series. We shall work out the most important case of $q=1$, but the method is applicable for any positive integer. Since $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\frac{1}{2}\log 2\pi$, we see that after several integrations by parts,

$$\sum_{n\leqq x}\log(x/n)\sim x-\frac{1}{2}\log 2\pi x$$
$$-\frac{1}{\pi}\sum\frac{1}{n}\left(\frac{\cos 2\pi nx}{2\pi nx}+\frac{\sin 2\pi nx}{(2\pi nx)^2}-\frac{2!\cos 2\pi nx}{(2\pi nx)^3}-\frac{3!\sin 2\pi nx}{(2\pi nx)^4}+\cdots\right).$$

But,

$$(-1)^j\sum\frac{\cos 2\pi nx}{n^{2j}}=\frac{(2\pi)^{2j}}{2(2j)!}B_{2j}(x-[x]),\qquad j=1,2,\ldots,$$

and

$$(-1)^j\sum\frac{\sin 2\pi nx}{n^{2j-1}}=\frac{(2\pi)^{2j-1}}{2(2j-1)!}B_{2j-1}(x-[x]),\qquad j=2,3,\ldots,$$

where $B_j(x)$ is the jth Bernoulli polynomial [1, p. 805]. Thus,

$$\sum_{n\leqq x}\log(x/n)\sim x-\frac{1}{2}\log 2\pi x-\sum_{j=2}^{\infty}\frac{B_j(x-[x])}{(j-1)jx^{j-1}}.$$

This result may also be obtained from the Euler-Maclaurin formula.

EXAMPLE 2. Let $\phi(s)=\pi^{-2s}\zeta^2(2s)$, where here $m=2$, $\lambda_n=\pi^2n^2$ and $a(n)=d(n)$, where $d(n)$ denotes the number of divisors of n. Since [15, p. 184],

$$\int_0^{\infty}u^{-1}\sin u\sin x^2/u\,du=\frac{\pi}{2}Y_0(2x)+K_0(2x),$$

where $Y_0(z)$ and $K_0(z)$ are the Bessel functions usually so denoted, and

$$\zeta(s)=(s-1)^{-1}+\gamma+O((s-1)),$$

where γ denotes Euler's constant, we have from Theorem 1 for $q=1$ upon replacing x by π^2x^2,

$$\sum_{n\leqq x}d(n)\log(x/n)=x(\log x-2+2\gamma)+\frac{1}{4}\log 4\pi^2x$$
$$+\frac{1}{2\pi}\sum\frac{d(n)}{n}\left[Y_0(4\pi(nx)^{1/2})+\frac{2}{\pi}K_0(4\pi(nx)^{1/2})\right].$$

This identity was first proven by Oppenheim [11].

By Theorem 2 we have for $q\geqq 2$, upon replacing x by π^2x^2,

$$\frac{1}{q!}\sum_{n\leqq x}d(n)\log^q(x/n)$$
$$=x(\log x-(q+1)+2\gamma)+\frac{1}{q!}\sum_{j=0}^{q}\binom{q}{j}\left\{\frac{d^j}{ds^j}\zeta^2(s)\right\}_{s=0}\log^{q-j}x$$
$$-\frac{1}{2\pi}\sum\frac{d(n)}{n}\int_{4\pi^2nx}^{\infty}u^{-1}s(u,4\pi^2nx;q-2)\left[Y_0(2u^{1/2})+\frac{2}{\pi}K_0(2u^{1/2})\right]du.$$

We may also write down identities involving $d_k(n)$, where $d_k(n)$ denotes the number of ways of expressing n as a product of k factors.

EXAMPLE 3. The classical function $\Delta(z)$ is defined by

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1-e^{2\pi inz})^{24}, \qquad \operatorname{Im} z > 0.$$

Consider for $k>0$,

$$\Delta^{k/12}(z) = e^{2\pi ikz/12} \sum a^{(k)}(n) e^{2\pi i(n-1)z}.$$

If $k=\frac{1}{2}$, $\Delta^{1/24}(z)=\eta(z)$, the classical eta-function; if $k=12$, $\Delta(z)=\sum \tau(n)e^{2\pi inz}$, where $\tau(n)$ denotes Ramanujan's arithmetical function. The associated Dirichlet series $(2\pi)^{-s}\sum a^{(k)}(n)(n-1+k/12)^{-s}$ satisfies Definition 1 for $m=1$, $r=k$ and $\sigma_a \leqq \frac{1}{2}(k+1)$, and the analytic continuation of the series is an entire function. Replacing x by $2\pi x$, we have from Theorem 3 for $1\leqq q \leqq k-\frac{1}{2}$,

$$\frac{1}{q!}\sum_{\lambda_n \leqq x} a^{(k)}(n) \log^q (x/\lambda_n)$$
$$= 2^q(4\pi)^{-k} \sum \frac{a^{(k)}(n)}{\lambda_n^k} \int_0^{4\pi(\lambda_n x)^{1/2}} u^{k-1} J_k(u) s(u, 4\pi(\lambda_n x)^{1/2}; q-1)\, du,$$

where $\lambda_n = n-1+k/12$. In particular, for $k=12$,

$$\frac{1}{q!}\sum_{n\leqq x} \tau(n) \log^q (x/n) = 2^q(4\pi)^{-12} \sum \frac{\tau(n)}{n^{12}} \int_0^{4\pi(nx)^{1/2}} u^{11} J_{12}(u) s(u, 4\pi(nx)^{1/2}; q-1)\, du,$$

provided $1 \leqq q \leqq 11$. Using Theorems 1 and 2, we may write down other identities as well. In particular, if $k=\frac{1}{2}$ and $q=0$,

$$\sum_{\lambda_n \leqq x}{}' a^{(1/2)}(n) = \frac{1}{2^{1/2}\pi} \sum \frac{a^{(1/2)}(n)}{\lambda_n^{1/2}} \sin (4\pi(\lambda_n x)^{1/2}),$$

where $\lambda_n = n - 23/24$.

EXAMPLE 4. Let $\zeta_k(s)$ denote the Epstein zeta-function defined by

$$\zeta_k(s) = \sum r_k(n) n^{-s}, \qquad k \geqq 2,$$

where $r_k(n)$ denotes the number of integral solutions to the equation $n_1^2+n_2^2+\cdots+n_k^2=n$. $\pi^{-s}\zeta_k(s)$ satisfies Definition 1 with $m=1$, $r=k/2$ and $\sigma_a=k/2$. $\zeta_k(s)$ has a pole at $s=k/2$ with residue $\pi^{k/2}/\Gamma(k/2)$. If $k=2$, we may apply Theorem 2 to obtain for $q\geqq 1$,

$$\frac{1}{q!}\sum_{n\leqq x} r_2(n)\log^q (x/n) = \pi x + \frac{1}{q!}\sum_{j=0}^{q} \binom{q}{j} \zeta_2^{(j)}(0) \log^{q-j} x$$
$$-\frac{2^{q-1}}{\pi} \sum \frac{r_2(n)}{n} \int_{2\pi(nx)^{1/2}}^{\infty} J_1(u) s(u, 2\pi(nx)^{1/2}; q-1)\, du.$$

Since $\zeta_2(0)=-1$, for $q=1$,

$$\sum_{n\leqq x} r_2(n) \log (x/n) = \pi x - \log x + \zeta_2'(0) - \frac{1}{\pi}\sum \frac{r_2(n)}{n} J_0(2\pi(nx)^{1/2}).$$

Müller [10] and Carlitz [3] have previously shown that

$$\sum_{n\leqq N} r_2(n) \log N/n = \pi N - \log N + c + O(N^{-1/4}).$$

The error term is a consequence of Corollary 1 in our work. Oppenheim [11] indicated that he was able to obtain an identity for $\sum_{n\leq x} r_2(n)\log(x/n)$, but he does not give it.

For $k>2$ we may apply Theorem 1 to obtain for $q>\frac{1}{2}(k-1)$,

$$\frac{1}{q!}\sum_{n\leq x} r_k(n)\log^q(x/n)$$
$$=\left(\frac{2}{k}\right)^{q+1}\frac{(\pi x)^{k/2}}{\Gamma(k/2)}+\frac{1}{q!}\sum_{j=0}^{q}\binom{q}{j}\zeta_k^{(j)}(0)\log^{q-j}x-\frac{2^{2q-1}}{(2\pi)^{k/2}}$$
$$\cdot\sum\frac{r_k(n)}{n^{k/2}}\frac{d^{q-1}}{dx^{q-1}}\left(x^{q-1}\int_{2\pi(nx)^{1/2}}^{\infty}u^{k/2-q}J_{k/2+q-1}(u)s(u,2\pi(nx)^{1/2};q-1)\,du\right).$$

By Corollary 1, the series on the right-hand side is $O(x^{(k-2q-1)/4}\log^{q-1}x)$.

EXAMPLE 5. Let k be an even integer and consider $f(s)=\zeta(s)\zeta(s-k+1)=\sum\sigma_{k-1}(n)n^{-s}$, where $\sigma_{k-1}(n)=\sum_{d|n}d^{k-1}$. Then $(2\pi)^{-s}f(s)$ satisfies Definition 1 with $m=1$, $r=k$, $b(n)=(-1)^{k/2}\sigma_{k-1}(n)$ and $\sigma_a=k$. By Theorems 1 and 4 we have for $q\geqq\max(1,k-3/2)$,

$$\frac{1}{q!}\sum_{n\leq x}\sigma_{k-1}(n)\log^q(x/n)$$
$$=\frac{\zeta(k)x^k}{k^{q+1}}+\frac{1}{q!}\sum_{j=0}^{q}\binom{q}{j}f^{(j)}(0)\log^{q-j}x+\frac{(-1)^{1+k/2}2^{2q-1}}{(4\pi)^k}$$
$$\cdot\sum\frac{\sigma_{k-1}(n)}{n^k}\frac{d^{q-1}}{dx^{q-1}}\left(x^{q-1}\int_{4\pi(nx)^{1/2}}^{\infty}u^{k-q}J_{k+q-1}(u)s(u,4\pi(nx)^{1/2};q-1)\,du\right).$$

EXAMPLE 6. Let K be an algebraic number field of degree r_1+2r_2, where r_1 denotes the number of real conjugates in K, and $2r_2$ the number of imaginary conjugates. The Dedekind zeta-function $\zeta_K(s)=\sum F(n)n^{-s}$, where $F(n)$ denotes the number of nonzero integral ideals of norm n in K, satisfies the functional equation $\xi(s)=\xi(1-s)$, where $\xi(s)=\Gamma^{r_1}(\frac{1}{2}s)\Gamma^{r_2}(s)B^{-s}\zeta_K(s)$. Here $B=2^{r_2}\pi^{r_1/2+r_2}|\Delta|^{-1/2}$, where Δ denotes the discriminant of K. $\zeta_K(s)$ converges absolutely for $\sigma>1$ and has a simple pole at $s=1$ with residue λh, where $\lambda=2^{r_1+r_2}\pi^{r_2}R/w|\Delta|^{1/2}$ and h denotes the class number. R denotes the regulator and w the number of roots of unity in K.

Suppose, first, that $r_1=0$, so that K is purely imaginary. If $q=r_2-1$, we have by Theorems 1 and 4,

$$\frac{1}{q!}\sum_{n\leq x}F(n)\log^q(x/n)=\lambda hx+\frac{1}{q!}\sum_{j=0}^{q}\binom{q}{j}\zeta_K^{(j)}(0)\log^{q-j}x$$
$$+2^q\sum F(n)\left(\frac{x}{n}\right)^{1/2}K_1(2^{q+1}B(nx)^{1/2};1;q+1).$$

For $q\geqq r_2$, we have by Theorem 2,

$$\frac{1}{q!}\sum_{n\leq x}F(n)\log^q(x/n)=\lambda hx+\frac{1}{q!}\sum_{j=0}^{q}\binom{q}{j}\zeta_K^{(j)}(0)\log^{q-j}x$$
$$-\frac{2^{q-m}}{B}\sum\frac{F(n)}{n}\int_{2^mB(nx)^{1/2}}^{\infty}K_1(u;1;m)s(u,2^mB(nx)^{1/2};q-m)\,du.$$

In particular, if $r_2=1$, i.e. K is an imaginary quadratic field, and $q=1$, the above identity yields

$$\sum_{n\leq x} F(n)\log(x/n) = \lambda hx+\zeta_K(0)\log x+\zeta_K'(0)-\frac{|\Delta|^{1/2}}{2\pi}\sum\frac{F(n)}{n}J_0(4\pi(nx/|\Delta|)^{1/2}).$$

If K is a real field we may also write down identities analogous to those above. In particular, for a real quadratic field and $q=1$ we obtain from Theorems 1 and 4,

$$\sum_{n\leq x} F(n)\log(x/n) = \lambda hx+\zeta_K(0)\log x+\zeta_K'(0)$$
$$+\frac{|\Delta|^{1/2}}{2\pi}\sum\frac{F(n)}{n}\left[Y_0(4\pi(nx/|\Delta|)^{1/2})+\frac{2}{\pi}K_0(4\pi(nx/|\Delta|)^{1/2})\right].$$

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