FUNCTIONS OF EXPONENTIAL TYPE NOT VANISHING IN A HALF-PLANE AND RELATED POLYNOMIALS

BY

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1. **Introduction and statement of results.** If $f(z)$ is an entire function of exponential type τ and $|f(x)| \leqq 1$ for real x, then according to a well-known theorem of S. N. Bernstein [2, p. 206]

$$|f'(x)| \leqq \tau, \qquad -\infty < x < \infty. \tag{1.1}$$

Besides, it is a simple consequence of the Phragmén-Lindelöf principle that [2, 6.2.4, p. 82]

$$|f(x+iy)| \leqq e^{\tau|y|}, \qquad -\infty < x < \infty, -\infty < y < \infty. \tag{1.2}$$

It has been proved by Boas [3] that if $h_f(\pi/2)=0$

$$\left(h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} \text{ is the indicator function of } f(z)\right)$$

and $f(x+iy\neq 0)$ for $y>0$, then (1.1) can be replaced by

$$|f'(x)| \leqq \tau/2, \qquad -\infty < x < \infty, \tag{1.3}$$

and (1.2) by

$$|f(x+iy)| \leqq (e^{\tau|y|}+1)/2, \qquad -\infty < x < \infty, -\infty < y \leqq 0. \tag{1.4}$$

An entire function $f(z)$ of exponential type which is bounded on the real axis, does not vanish in the upper half-plane, and for which $h_f(\pi/2)=0$ is called asymmetric.

The class of asymmetric entire functions of exponential type τ includes all functions $p(e^{iz})$ where $p(z)$ is a polynomial of degree $n \leqq [\tau]$ and $p(z)\neq 0$ in $|z|<1$. Thus the above results of Boas are generalizations of the following theorems about polynomials.

THEOREM A [5]. *If $p(z)$ is a polynomial of degree n such that $|p(z)|\leqq 1$ for $|z|\leqq 1$ and $p(z)$ does not vanish in $|z|<1$, then*

$$|p'(z)| \leqq n/2 \tag{1.5}$$

for $|z|\leqq 1$.

THEOREM B [1]. *Under the conditions of Theorem* A

$$|p(z)| \leqq (R^n+1)/2 \tag{1.6}$$

for $|z|=R>1$.

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About ten years ago the following problem was proposed by Professor R. P. Boas, Jr.

If $f(z)$ is an entire function of exponential type τ such that $|f(x)| \leqq 1$ for real x, $h_f(\pi/2)=0$ and $f(x+iy)\neq 0$ for $y>k$, what are the bounds for $|f'(x)|$ and $|f(x+iy)|$? For $k=0$ these should reduce to (1.3) and (1.4) respectively.

While seeking the desired extension of (1.4) we have only been able to prove the following

THEOREM 1. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leqq 1$ for real x, $h_f(\pi/2)=0$ and $f(x+iy)\neq 0$ for $y>k$ where $k\leqq 0$. If, in addition, $f(z)$ is periodic on the real axis with period 2π, then for $2k\leqq y<0$*

$$|f(x+iy)| \leqq ((e^{|y|}+e^{|k|})/(1+e^{|k|}))^{[\tau]}. \tag{1.7}$$

In the problem proposed by Professor Boas the hypothesis "$f(x+iy)\neq 0$ for $y>k$" is stronger than the hypothesis "$f(x+iy)\neq 0$ for $y>0$" if $k<0$ but surprisingly enough we cannot improve upon (1.3) as the following example shows.

Let τ be rational, i.e. it is the quotient of two positive integers N_1, N_2. Let $a=1/N_3N_2$ where N_3 is a positive integer. Then

$$f_a(z) = ((e^{iaz}-e^{-ak})/(1+e^{-ak}))^{\tau/a}$$

is an entire function of exponential type τ such that $h_{f_a}(\pi/2)=0$,

$$\max_{-\infty<x<\infty} |f_a(x)| = 1$$

and $|f_a(x+iy)|$ has all its zeros on $y=k$. But $\varepsilon>0$ being given we can make

$$\begin{aligned}\max_{-\infty<x<\infty} |f_a'(x)| &= \max_{-\infty<x<\infty} \left|\left(\frac{e^{iax}-e^{-ak}}{1+e^{-ak}}\right)^{\tau/a-1}\right| \frac{\tau}{1+e^{-ak}} \\ &= \tau/(1+e^{-ak}) \\ &> \tau/2-\varepsilon\end{aligned}$$

by making a sufficiently small. Thus the bound in (1.3) cannot in general be improved by assuming that $f(z)\neq 0$ in a larger half-plane. However, we prove the following

THEOREM 2. *Let $f(z)$ be an entire function of exponential type τ having all its zeros on $\operatorname{Im} z=k\leqq 0$. If $h_f(\pi/2)=0$, $h_{f'}(\pi/2)\leqq -c_1<0$, and $|f(x)|\leqq 1$ for real x, then*

$$|f'(x)| \leqq \tau/(1+\exp(c_1|k|)), \qquad -\infty < x < \infty. \tag{1.8}$$

If τ/c_1 is a positive integer, then the function

$$f_{c_1}(z) = \left(\frac{\exp(ic_1z)-\exp(-c_1k)}{1+\exp(-c_1k)}\right)^{\tau/c_1}$$

satisfies the hypotheses of Theorem 2 and

$$\max_{-\infty<x<\infty} |f'_{c_1}(x)| = \tau/(1+\exp(c_1|k|)).$$

Hence the bound in (1.8) is sharp.

We also prove

THEOREM 3. *Let $f(z)$ be an entire function of exponential type τ having all its zeros in* $\operatorname{Im} z \leqq k \leqq 0$. *If $h_f(\pi/2)=0$, $h_{f'}(\pi/2) \leqq -c_1<0$, and also $h_{g'}(\pi/2) \leqq -c_1<0$ where $g(z)=e^{i\tau z}$* con $\{f(\bar{z})\}$, *then $|f(x)| \leqq 1$ for real x implies* (1.8). *The result is sharp and $f_{c_1}(z)$ is extremal.*

Theorem 3 includes as a special case the following result recently proved by M. A. Malik [7].

THEOREM C. *Let $p(z)$ be a polynomial of degree n which does not vanish in $|z|<K$ where $K \geqq 1$. Then $|p(z)| \leqq 1$ for $|z| \leqq 1$ implies $|p'(z)| \leqq n/(1+K)$ for $|z| \leqq 1$. The bound is attained for $p(z)=((z+K)/(1+K))^n$.*

In fact, $p(e^{iz})$ satisfies the hypotheses of Theorem 3 with $\tau=n$ and $c_1=1$.

It is clearly of interest to determine a bound for the sth derivative. Here we restrict ourselves to polynomials and prove the following

THEOREM 4. *Let $p(z)$ be a polynomial of degree n which does not vanish in $|z|<K$ where $K \geqq 1$. Then $|p(z)| \leqq 1$ for $|z| \leqq 1$ implies*

$$|p^{(s)}(z)| \leqq \frac{n(n-1)\cdots(n-s+1)}{1+K^s} \quad \textit{for } |z| \leqq 1. \tag{1.9}$$

We also consider the analogous problem for rational functions, i.e. quotient of two polynomials.

Let $f(z)=p_1(z)/p_2(z)$ be a rational function of degree n in the sense that $f(z)=w$ has n roots for general w. Suppose that $f(z)$ has neither zeros nor poles in $|z|<K$ where $K>1$. If $|f(z)| \leqq 1$ for $|z| \leqq 1$ what is the bound for $|f'(z)|$?

The function $u(z)=f(Kz)$ has all its zeros and poles in $|z| \geqq 1$. Let $|z_0|<1$ and consider the function $v(z)=u((z+z_0)/(\bar{z}_0 z+1))$ which is also a rational function of degree n and its zeros z_ν, poles ζ_ν lie in $|z| \geqq 1$. Hence

$$|v'(0)/v(0)| = \left|\sum(-1/z_\nu)-\sum(-1/\zeta_\nu)\right| \leqq 2n.$$

Since $v'(0)/v(0)=(1-|z_0|^2)u'(z_0)/u(z_0)$ we conclude that $|u'(z_0)/u(z_0)| \leqq 2n/(1-|z_0|^2)$. Thus $|f'(Kz_0)/f(Kz_0)|=(1/K)|u'(z_0)/u(z_0)| \leqq (2n/K)/(1-|z_0|^2)$. Taking $z_0=(1/K)\exp(i\theta_0)$ we get $|f'(\exp(i\theta_0))/f(\exp(i\theta_0))| \leqq 2nK/(K^2-1)$. Hence

$$\max_{|z|\leqq 1} |f'(z)| \leqq 2nK/(K^2-1). \tag{1.10}$$

Next we consider the following more general problem.

Let $f(z)$ be a rational function of degree n. Assume that $f(z)$ does not have poles in the annulus $1-d<|z|<1+d$. If $|f(z)|\leqq 1$ for $|z|=1$, what is the bound for $|f'(z)|$?

We prove

THEOREM 5. *Let $f(z)=p_1(z)/p_2(z)$ where $p_1(z), p_2(z)$ are polynomials of degree n_1 and n_2 respectively, $|f(z)|\leqq 1$ for $|z|=1$, and let $f(z)$ have no poles in $1-d<|z|<1+d$. Then*

$$\max_{|z|=1} |f'(z)| \leqq n_2(3+4/d) \quad \text{if } n_1 \leqq n_2, \tag{1.11}$$

whereas

$$\max_{|z|=1} |f'(z)| \leqq 4n_1(1+1/d)-n_2 \quad \text{if } n_1 \geqq n_2. \tag{1.12}$$

One might wish to determine a bound for the sth derivative of the $f(z)$ of Theorem 5. To this effect we prove the following

THEOREM 6. *Let $f(z)=p_1(z)/p_2(z)$ where $p_1(z), p_2(z)$ are polynomials of degree n_1, n_2 respectively with $n_1\leqq n_2$. If $|f(z)|\leqq 1$ for $|z|=1$ and $p_2(z)$ has no zeros in $1-d<|z|<1+d$, then*

$$\max_{|z|=1} |f^{(s)}(z)| < e^3 s!\left(\frac{n_2}{d}\right)^s \tag{1.13}$$

if $n_2\geqq 3$.

We also prove

THEOREM 7. *Let $f(z)=f_1(z)/f_2(z)$ where $f_1(z), f_2(z)$ are entire functions of exponential type τ_1, τ_2 respectively. If the zeros z_n of $f_2(z)$ do not lie in $|\text{Im } z|<d$ and $\sum_{n=1}^{\infty} 1/|\text{Im } z_n|\leqq A<\infty$, then $|f(x)|\leqq 1$ for $-\infty<x<\infty$ implies*

$$\sup_{-\infty<x<\infty} |f^{(s)}(x)| < \frac{s!}{\pi}\int_0^{\pi}\left(\frac{\tau_1+\tau_2+1}{d}\right)^s \exp\left\{d\sin\theta+\frac{2Ad\sin\theta}{(\tau_1+\tau_2+1-\sin\theta)}\right\}d\theta. \tag{1.14}$$

L^p inequalities. The following theorem which is analogous to (1.3) is proved in [8].

THEOREM D. *If $f(z)$ is an entire function of exponential type τ belonging to L^p, $1\leqq p<\infty$, on the real axis, $h_f(\pi/2)=0$ and $f(z)\neq 0$ for $y>0$, then for $p\geqq 1$ we have*

$$\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} \leqq c_p^{1/p}\tau\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}, \tag{1.15}$$

where

$$c_p = \frac{2\pi}{\int_0^{2\pi} |1+e^{i\alpha}|^p\,d\alpha} = \frac{2^{-p}\pi^{1/2}\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}.$$

If we use Lemma 11 below, the argument of Rahman [8] yields the following extension of Theorem D.

THEOREM 8. *If $f(z)$ is an entire function of exponential type τ belonging to L^p, $1 \leqq p < \infty$, on the real axis, $h_f(\pi/2)=0$, $h_{f'}(\pi/2) \leqq -c_1 < 0$, $h_{g'}(\pi/2) \leqq -c_1 < 0$ and $f(z) \neq 0$ for* Im $z > k$ *where $k \leqq 0$, then for $p \geqq 1$ we have*

$$\left(\int_{-\infty}^{\infty} |f'(x)|^p \, dx\right)^{1/p} \leqq D_p^{1/p} \tau \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx\right)^{1/p}, \tag{1.16}$$

where

$$D_p = \frac{2\pi}{\int_0^{2\pi} |\exp(c_1|k|) + e^{i\alpha}|^p \, d\alpha}.$$

We do not know if inequality (1.16) is sharp.

From Lemma 11 below it follows that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \geqq K \geqq 1$, $q(z) = z^n \operatorname{con}\{p(1/\bar{z})\}$, then

$$K|p'(z)| \leqq |q'(z)| \quad \text{for } |z| = 1. \tag{1.17}$$

Hence if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ then

$$\begin{aligned}
(1+K^2) \int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta &= \int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta + \int_0^{2\pi} K^2 |p'(e^{i\theta})|^2 \, d\theta \\
&\leqq \int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta + \int_0^{2\pi} |q'(e^{i\theta})|^2 \, d\theta \\
&= 2\pi \left\{ \sum_{\nu=0}^{n} \nu^2 |a_\nu|^2 + \sum_{\nu=0}^{n} (n-\nu)^2 |a_\nu|^2 \right\} \\
&= 2\pi \left\{ \sum_{\nu=0}^{n} (\nu^2 + (n-\nu)^2) |a_\nu|^2 \right\} \\
&\leqq n^2 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 \\
&= n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta
\end{aligned}$$

and we get

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqq \frac{n^2}{1+K^2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{1.18}$$

We can prove the following more general

THEOREM 9. *If the polynomial $p(z)$ of degree n has no zeros in $|z| < K$ where $K \geqq 1$, then for $\delta \geqq 1$*

$$\int_0^{2\pi} |p'(e^{i\theta})|^\delta \, d\theta \leqq E_\delta n^\delta \int_0^{2\pi} |p(e^{i\theta})|^\delta \, d\theta,$$

where $E_\delta = 2\pi / \int_0^{2\pi} |K + e^{i\alpha}|^\delta \, d\alpha$.

For $K=1$, Theorem 9 reduces to a theorem of De Bruijn [4]. Our method of proof is analogous to a proof of De Bruijn's theorem given in [8] and so we shall omit it.

Theorem 9 is not precise. The sharp inequality does not seem to be easily obtainable even for $\delta=2$. To substantiate this remark we take the simple case of cubic polynomials and prove the following

THEOREM 10. *Let* $p(z)=a_3z^3+a_2z^2+a_1z+a_0$ *be a cubic polynomial having all its zeros on* $|z|=K\geqq 1$. *Then*

$$\int_0^{2\pi} |p'(e^{i\theta})|^2\, d\theta \leqq \frac{9}{1+K^6}\int_0^{2\pi} |p(e^{i\theta})|^2\, d\theta$$

or

$$\int_0^{2\pi} |p'(e^{i\theta})|^2\, d\theta \leqq \frac{9(1+4K^2+K^4)}{1+9K^2+9K^4+K^6}\int_0^{2\pi} |p(e^{i\theta})|^2\, d\theta$$

according as $K\leqq K_3$ *or* $K\geqq K_3$ *where* K_3 *is the (only) root of the equation* $K^8+4K^6-8K^2-5=0$ *in* $(1,\infty)$.

We have considerable evidence in favour of the following

CONJECTURE. *Let* $p(z)$ *be a polynomial of degree n having all its zeros in* $|z|\geqq K\geqq 1$. *Then*

$$\frac{\int_0^{2\pi} |p'(e^{i\theta})|^2\, d\theta}{\int_0^{2\pi} |p(e^{i\theta})|^2\, d\theta} \leqq \frac{n^2}{\int_0^{2\pi} |e^{in\theta}+K^n|^2\, d\theta}$$

or

$$\frac{\int_0^{2\pi} |p'(e^{i\theta})|^2\, d\theta}{\int_0^{2\pi} |p(e^{i\theta})|^2\, d\theta} \leqq \frac{n^2\int_0^{2\pi} |e^{i\theta}+K|^{2n-2}\, d\theta}{\int_0^{2\pi} |e^{i\theta}+K|^{2n}\, d\theta}$$

according as $K\leqq K_n$ *or* $K\geqq K_n$ *where* K_n *is the (only) root of the equation*

$$\frac{1}{\int_0^{2\pi} |e^{in\theta}+K^n|^2\, d\theta} = \frac{\int_0^{2\pi} |e^{i\theta}+K|^{2n-2}\, d\theta}{\int_0^{2\pi} |e^{i\theta}+K|^{2n}\, d\theta}$$

in $(1,\infty)$.

2. **Lemmas.**

LEMMA 1. *Let* $f(z)$ *be an entire function of exponential type having no zeros for* $y>0$ *and having* $h(\alpha)\geqq h(-\alpha)$ *for some* α, $0<\alpha<\pi$. *Then* $|f(z)|\geqq|f(\bar{z})|$ *for* $y>0$.

Lemma 1 is due to Levin [6]. For a proof see [2, p. 129].

DEFINITION 1. An entire function $f(z)$ of exponential type having no zeros for $y<0$ and satisfying one of the conditions (equivalent, by Lemma 1) $h(-\alpha)\geqq h(\alpha)$ for some α, $0<\alpha<\pi$, or $|f(z)|\geqq|f(\bar{z})|$ for $y<0$ is said to belong to class P.

DEFINITION 2. An additive homogeneous operator $B[f(z)]$ which carries entire functions of exponential type into entire functions of exponential type and leaves the class P invariant is called a B-operator.

LEMMA 2. *Differentiation is a B-operator.*

For a proof of Lemma 2 see [2, pp. 226–227].

Lemma 3 below is due to Levin [6]. For proof see [2, p. 226].

LEMMA 3. *If $f(z)$ is an entire function of exponential type τ, B is a B-operator, $\omega(z)$ is an entire function of class P and of order 1, type $\sigma \geqq \tau$, then*

$$|f(x)| \leqq |\omega(x)|, \qquad -\infty < x < \infty,$$

implies

$$|B[f(x)]| \leqq |B[\omega(x)]|, \qquad -\infty < x < \infty.$$

LEMMA 4. *If $f(z)$ is an entire function of exponential type τ such that $h_f(\pi/2)=0$, $f(x+iy)\neq 0$ for $y>k\geqq 0$, then*

$$|f^{(s)}(x)| \leqq e^{-\tau k}|g^{(s)}(x-2ik)|, \qquad -\infty < x < \infty. \tag{2.1}$$

Here and elsewhere $g(z)$ stands for $e^{i\tau z}$ con $\{f(\bar{z})\}$.

Proof of Lemma 4. Let $F(z)=f(z+ik)$ and consider $G(z)=e^{i\tau z}$ con $\{F(\bar{z})\}=e^{-\tau k}g(z-ik)$ which is an entire function of order 1, type $\geqq \tau$. Since $f(z)$ has no zeros for $\operatorname{Im} z>k$, $h_f(\pi/2)=0$ and $h_f(-\pi/2)\leqq \tau$, the function $G(z)$ has no zeros for $\operatorname{Im} z<0$, $h_G(-\pi/2)=\tau$ and $h_G(\pi/2)\leqq 0$. The function $e^{-i\tau z/2}G(z)$ therefore belongs to the class P mentioned above, and by Lemma 1

$$|\exp(-i\tau z/2)G(z)| \geqq |\exp(-i\tau \bar{z}/2)G(\bar{z})|$$

for $\operatorname{Im} z<0$. Thus for $\operatorname{Im} z\leqq 0$ we have

$$|F(z)| = |e^{i\tau z}\operatorname{con}\{G(\bar{z})\}| \leqq |G(z)|.$$

From Lemma 3 it follows that $|F^{(s)}(z)|\leqq |G^{(s)}(z)|$ for $\operatorname{Im} z\leqq 0$. In particular, $|F^{(s)}(x-ik)|\leqq|G^{(s)}(x-ik)|$. Since $|F^{(s)}(x-ik)|=|f^{(s)}(x)|$,

$$|G^{(s)}(x-ik)| = e^{-\tau k}|g^{(s)}(x-2ik)|$$

the lemma follows.

LEMMA 5. *If $f(z)$ is an entire function of exponential type τ such that $h_f(-\pi/2)=\tau$ and $f(z)$ has all its zeros in $\operatorname{Im} z\geqq k\geqq 0$ then*

$$e^{-\tau k}|g^{(s)}(x-2ik)| \leqq |f^{(s)}(x)|, \qquad -\infty < x < \infty. \tag{2.2}$$

Proof of Lemma 5. Since $f(z)$ has all its zeros in $\operatorname{Im} z\geqq k\geqq 0$ the function $g(z)=e^{i\tau z}$ con $\{f(\bar{z})\}$ has all its zeros in $\operatorname{Im} z\leqq -k\leqq 0$. Hence $g(z-2ik)$ does not vanish in $\operatorname{Im} z>k\geqq 0$. Since $h_g(\pi/2)=0$ the function $g(z-2ik)$ satisfies the conditions of Lemma 4. Hence

$$|g^{(s)}(x-2ik)| \leqq e^{-\tau k}|(d^s/dx^s)\{e^{2\tau k}f(x)\}|, \qquad -\infty < x < \infty,$$

which is equivalent to (2.2).

LEMMA 6. *If $f(z)$ is an entire function of exponential type τ such that $h_f(\pi/2)=0$ and $f(x+iy)\neq 0$ for $y>k$ where $k\leqq 0$, then*

$$|g^{(s)}(x)| \geqq e^{\tau k}|f^{(s)}(x+2ik)|, \qquad -\infty < x < \infty. \tag{2.3}$$

Proof of Lemma 6. Since $f(z)\neq 0$ for $y>k$ where $k\leqq 0$, the function $g(z) = e^{i\tau z}\operatorname{con}\{f(\bar{z})\}$ has all its zeros in $\operatorname{Im} z\geqq -k\geqq 0$. Hence by Lemma 5

$$|g^{(s)}(x)| \geqq e^{\tau k}|f^{(s)}(x+2ik)|, \qquad -\infty < x < \infty. \tag{2.4}$$

LEMMA 7. *If $f(z)$ is an entire function of order 1 type τ such that $h_f(\pi/2)\leqq -c<0$, $|f(x)|$ is bounded for real x and $f(x+iy)$ has all its zeros in $y\geqq k$ where $k\leqq 0$, then*

$$|f(x+2ik)| \geqq \exp((\tau+c)|k|)|f(x)|, \qquad -\infty < x < \infty. \tag{2.5}$$

Proof of Lemma 7. Note that $h_f(-\pi/2)=\tau$, otherwise $f(z)$ cannot be of type τ. If $F(z)=f(z+ik)$ then the function $e^{-i(\tau+c)z/2}F(z)$ belongs to the class P. Hence by Lemma 1

$$|\exp(-i(\tau+c)z/2)F(z)| \geqq |\exp(-i(\tau+c)\bar{z}/2)F(\bar{z})|$$

for $\operatorname{Im} z\leqq 0$. Thus for real x

$$|f(x+2ik)| = |F(x+ik)| \geqq \exp(-(\tau+c)k)|F(x-ik)| = \exp((\tau+c)|k|)|f(x)|.$$

Hence Lemma 7 is proved.

LEMMA 8. *If $f(z)$ is an entire function of order 1 and type τ such that $h_f(\pi/2)=0$, $h_{f^{(s)}}(\pi/2)\leqq -c_s<0$, $|f(x)|$ is bounded for real x, $f(x+iy)$ has all its zeros on $y=k\leqq 0$, then*

$$\exp(c_s|k|)|f^{(s)}(x)| \leqq |g^{(s)}(x)|, \qquad -\infty < x < \infty. \tag{2.6}$$

Proof of Lemma 8. By Lemma 6

$$|g^{(s)}(x)| \geqq e^{\tau k}|f^{(s)}(x+2ik)|, \qquad -\infty < x < \infty. \tag{2.7}$$

Now we note that $f^{(s)}(z)$ has all its zeros in $\operatorname{Im} z\geqq k$. This follows from the fact that $f(z+ik)$ belongs to the class P and differentiation leaves the class P invariant. Hence by Lemma 7

$$|f^{(s)}(x+2ik)| \geqq \exp((\tau+c_s)|k|)|f^{(s)}(x)|.$$

This together with (2.7) gives Lemma 8.

LEMMA 9. *If $f(z)$ is an entire function of exponential type τ such that $h_f(\pi/2)=0$, $f(x+iy)$ has all its zeros in $y\leqq k$, then for every real γ the function $f(z)+e^{i\gamma}\mathscr{F}(z)$, where $\mathscr{F}(z)=e^{i\tau(z-ik)}\operatorname{con}\{f(\bar{z}+2ik)\}$, has all its zeros on $\operatorname{Im} z=k$.*

Proof of Lemma 9. It is clear that $|\mathscr{F}(z)|=|f(z)|$ for $\operatorname{Im} z=k$. Besides it can be proved with the help of Lemma 1 that

$$|\mathscr{F}(z)| \geqq |f(z)| \tag{2.8}$$

for Im $z<k$, whereas

$$|\mathscr{F}(z)| \leqq |f(z)| \tag{2.9}$$

for Im $z>k$. The argument is contained in the proof of Lemma 4 so we omit it. Equality is possible in (2.8) or (2.9) only if $f(z)$ is a constant multiple of $\mathscr{F}(z)$. This follows easily from the maximum modulus principle. Hence unless $f(z)$ is a constant multiple of $\mathscr{F}(z)$, $|\mathscr{F}(z)|>|f(z)|$ for Im $z<k$, whereas $|\mathscr{F}(z)|<|f(z)|$ for Im $z>k$. Therefore $f(z)+e^{i\gamma}\mathscr{F}(z)$ can vanish only when Im $z=k$. In case $f(z)$ is a constant multiple of $\mathscr{F}(z)$ the function $f(z)$ cannot have a zero in Im $z<k$; otherwise $\mathscr{F}(z)$ which has all its zeros in Im $z\geqq k$ cannot be a constant multiple of $f(z)$. Hence in this case $f(z)$ has all its zeros on Im $z=k$ and so does $f(z)+e^{i\gamma}\mathscr{F}(z)$, being a constant multiple of $f(z)$.

LEMMA 10. *Let $f(z)$ be an entire function of exponential type τ with $|f(x)|\leqq M$ on the real axis. If $h_f(\pi/2)=0$ then*

$$|f'(x)|+|g'(x)| \leqq M\tau, \qquad -\infty < x < \infty. \tag{2.10}$$

The proof of this lemma is contained in ([8], see (3.12)). For sake of completeness we include a brief outline of the proof. From $h_f(\pi/2)=0$, $|f(x)|\leqq M$, it follows that $|f(z)|\leqq M$ for Im $z>0$. Hence for $|\lambda|>1$ the function $\phi(z)=f(z)-\lambda M$ is asymmetric. From Lemma 4 it follows that

$$|f'(x)| = |\phi'(x)| \leqq |\psi'(x)|, \qquad -\infty < x < \infty, \tag{2.11}$$

where $\psi(z)=e^{i\tau z}\operatorname{con}\{\phi(\bar{z})\}$. Noting that $\psi(z)=g(z)-M\bar{\lambda}e^{i\tau z}$ we can easily deduce (2.10) from (2.11).

LEMMA 11. *Let $f(z)$ be an entire function of order 1 type τ having all its zeros in Im $z\leqq k\leqq 0$. If $|f(x)|$ is bounded for real x, $h_f(\pi/2)=0$, $h_{f'}(\pi/2)\leqq -c_1<0$, $h_{g'}(\pi/2)\leqq -c_1<0$, then*

$$\exp(c_1|k|)|f'(x)| \leqq |g'(x)|, \qquad -\infty < x < \infty. \tag{2.12}$$

Proof of Lemma 11. If $f(z)$ has all its zeros in Im $z\leqq k\leqq 0$ and γ is real, then by Lemma 9 the function

$$L_\gamma(z) = f(z)+e^{i\gamma}e^{i\tau(z-ik)}\operatorname{con}\{f(\bar{z}+2ik)\}$$

has all its zeros on Im $z=k$. Besides we claim that $h_{L_\gamma}(\pi/2)=0$ except possibly for one value of γ mod 2π. The fact that $h_{L_\gamma}(\pi/2)\leqq 0$ for every γ is obvious. We need to prove that $h_{L_\gamma}(\pi/2)\geqq 0$ except possibly for one value of γ mod 2π. From $h_f(\pi/2)=0$ it follows that for a given $\varepsilon>0$ there exists a sequence $\{y_n\}$ tending to infinity such that $|f(iy_n)|>\exp(-\varepsilon y_n)$. If $\mathscr{F}(iy_n)/f(iy_n)=\rho_n\exp(i\phi_n)$, then

$$\begin{aligned}|L_\gamma(iy_n)| &= |f(iy_n)|\,|1+e^{i\gamma}(\mathscr{F}(iy_n)/f(iy_n))| \\ &= |f(iy_n)|\,|1+\rho_n\exp(i(\gamma+\phi_n))|.\end{aligned}$$

Now unless $\gamma+\phi_n$ tends to π, there exists a positive number δ and a subsequence $\{\gamma+\phi_{n_j}\}_{j=1}^{\infty}$ of the sequence $\{\gamma+\phi_n\}_{n=1}^{\infty}$ such that $|\gamma+\phi_{n_j}|\leqq\pi-\delta$ for $j=1, 2, \ldots$. Since $\rho_n\leqq 1$ we have

$$\begin{aligned}|L_\gamma(iy_{n_j})| &= |f(iy_{n_j})|\ |1+\rho_n \exp(i(\phi_{n_j}+\gamma))|\\ &\geqq \exp(-\varepsilon y_{n_j}) \sin\delta.\end{aligned}$$

Hence $h_{L_\gamma}(\pi/2)\geqq 0$ except possibly for one value of γ mod 2π.

In an analogous manner it can be verified that $L_\gamma(z)$ is of order 1 and type τ except possibly for one value of γ mod 2π.

Thus the conditions of Lemma 8 are satisfied for $L_\gamma(z)$ except possibly for two values of γ mod 2π. Therefore

$$\begin{aligned}\exp(c_1|k|)|f'(x)+e^{i\gamma}\{e^{i\tau(x-ik)} \operatorname{con}\{f(x+2ik)\}\}'|&\\ \leqq |g'(x)+e^{-i\gamma}e^{\tau k}f'(x+2ik)|&\end{aligned} \tag{2.13}$$

for real x and for every γ in $[0, 2\pi)$ except possibly two.

By continuity the above inequality holds for every γ. Now we first choose γ such that the left-hand side of (2.13) is

$$\exp((c_1|k|)|f'(x)|+\exp(c_1|k|)|\{e^{i\tau(x-ik)} \operatorname{con}\{f(x+2ik)\}\}'|.$$

Thus

$$\begin{aligned}\exp(c_1|k|)|f'(x)|+\exp(c_1|k|)|\{e^{i\tau(x-ik)} \operatorname{con}\{f(x+2ik)\}\}'|&\\ \leqq |g'(x)|+e^{\tau k}|f'(x+2ik)|.&\end{aligned} \tag{2.14}$$

Next choose γ such that the right-hand side of (2.13) is equal to $|g'(x)| -e^{\tau k}|f'(x+2ik)|$ which is possible by Lemma 6. We get

$$\begin{aligned}\exp(c_1|k|)|f'(x)|-\exp(c_1|k|)|\{e^{i\tau(x-ik)} \operatorname{con}\{f(x+2ik)\}\}'|&\\ \leqq |g'(x)|-e^{\tau k}|f'(x+2ik)|.&\end{aligned} \tag{2.15}$$

Adding the corresponding sides of (2.14) and (2.15) the lemma follows.

3. **Proofs of the theorems.**

Proof of Theorem 1. Since $f(z)$ is periodic on the real axis with period 2π, and $h_f(\pi/2)=0$, we have [2, p. 109]

$$f(z) = \sum_{j=0}^{n} a_j e^{ijz} = p(e^{iz})$$

where $n=[\tau]$. The polynomial $p(w)$ does not vanish in $|w|<e^{|k|}$ since $f(z)$ has all its zeros in $\operatorname{Im} z\leqq k\leqq 0$. Thus

$$p(w) = a_n \prod_{j=1}^{n} (w-R_j \exp(i\theta_j))$$

where $R_j\geqq e^{|k|}$ for $j=1, 2, \ldots, n$.

If $1 \leqq \rho \leqq e^{2|k|}$, $0 \leqq \phi < 2\pi$, then clearly

$$\left|\frac{\rho e^{i\phi}-R_j \exp(i\theta_j)}{e^{i\phi}-R_j \exp(i\theta_j)}\right| = \left\{\frac{\rho^2+R_j^2-2\rho R_j \cos(\phi-\theta_j)}{1+R_j^2-2R_j\cos(\phi-\theta_j)}\right\}^{1/2}$$

$$\leqq \frac{\rho+R_j}{1+R_j} \leqq \frac{\rho+e^{|k|}}{1+e^{|k|}}.$$

Hence

$$\left|\frac{p(\rho e^{i\phi})}{p(e^{i\phi})}\right| \leqq \left(\frac{\rho+e^{|k|}}{1+e^{|k|}}\right)^n$$

for $1 \leqq \rho \leqq e^{2|k|}$, $0 \leqq \phi < 2\pi$. This implies that

$$\max_{(|w|=\rho \leqq \exp(2|k|))} |p(w)| \leqq \left(\frac{\rho+e^{|k|}}{1+e^{|k|}}\right)^n \max_{|w|=1} |p(w)|,$$

which is equivalent to the desired result.

Proof of Theorem 2. The theorem follows immediately from Lemmas 10 and 8.

Proof of Theorem 3. This theorem follows from Lemmas 10 and 11.

Proof of Theorem 4. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z|<1$. If $Q(z)=z^n \operatorname{con}\{P(1/\bar{z})\}$, then the function $f(z)=P(z)/Q(z)$ is analytic in $|z| \leqq 1$ and $|f(z)|=1$ for $|z|=1$. By the maximum modulus principle $|f(z)| \leqq 1$ for $|z| \leqq 1$. Replacing z by $1/\bar{z}$ we conclude that $|Q(z)| \leqq |P(z)|$ for $|z| \geqq 1$. Hence for every λ such that $|\lambda|>1$ the polynomial $Q(z)-\lambda P(z)$ has all its zeros in $|z|<1$. By Gauss-Lucas theorem the polynomial $Q^{(s)}(z)-\lambda P^{(s)}(z)$ does not vanish in $|z| \geqq 1$ for any λ with $|\lambda|>1$. This implies that

$$|Q^{(s)}(z)| \leqq |P^{(s)}(z)| \quad \text{for } |z| \geqq 1. \tag{3.1}$$

If $p(z)$ is a polynomial of degree n such that $|p(z)| \leqq M$ for $|z| \leqq 1$, then by Rouché's theorem the polynomial $p(z)-\lambda M z^n$ has all its zeros in $|z|<1$ if $|\lambda|>1$. Applying (3.1) to the polynomial $p(z)-\lambda M z^n$ we conclude that if $q(z) = z^n \operatorname{con}\{p(1/\bar{z})\}$, then

$$|p^{(s)}(z)|+|q^{(s)}(z)| \leqq Mn(n-1)\cdots(n-s+1)|z|^{n-s} \quad \text{for } |z| \geqq 1. \tag{3.2}$$

Inequality (3.2), which holds for all polynomials $p(z)$ of degree n satisfying $|p(z)| \leqq M$ for $|z| \leqq 1$, is a result of independent interest.

Now we wish to prove that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \geqq K \geqq 1$, then

$$K^s|p^{(s)}(e^{i\theta})| \leqq |q^{(s)}(e^{i\theta})|, \qquad 0 \leqq \theta < 2\pi. \tag{3.3}$$

To start with let us suppose that all the zeros of $p(z)$ lie on $|z|=K \geqq 1$. All the zeros of the polynomial $P^*(z)=p(Kz)$ lie on $|z|=1$ and so do the zeros of $Q^*(z) = z^n \operatorname{con}\{P^*(1/\bar{z})\} = K^n q(z/K)$. For every λ with $|\lambda|>1$ the polynomial $P^*(z)$

$-\lambda Q^*(z)$ has all its zeros on $|z|=1$. By Gauss-Lucas theorem all the zeros of the sth derivative $P^{*(s)}(z)-\lambda Q^{*(s)}(z)$ lie in $|z|\leqq 1$. This implies that

$$K^s|p^{(s)}(Kz)| = |P^{*(s)}(z)| \leqq |Q^{*(s)}(z)| = K^{n-s}|q^{(s)}(z/K)|$$

for $|z|\geqq 1$. In particular we have

$$|p^{(s)}(K^2e^{i\theta})| \leqq K^{n-2s}|q^{(s)}(e^{i\theta})|, \qquad 0 \leqq \theta < 2\pi. \tag{3.4}$$

The polynomial $p^{(s)}(Kz)$ is a polynomial of degree $n-s$ having all its zeros in $|z|\leqq 1$. Considering the quotient

$$z^{n-s}\operatorname{con}\{p^{(s)}(K/\bar{z})\}/p^{(s)}(Kz)$$

in $|z|\geqq 1$, it is an easy consequence of the maximum modulus principle that $|z^{n-s}\operatorname{con}\{p^{(s)}(K/\bar{z})\}|\leqq|p^{(s)}(Kz)|$ for $|z|\geqq 1$.

This gives

$$K^{n-s}|p^{(s)}(e^{i\theta})| \leqq |p^{(s)}(K^2e^{i\theta})|, \qquad 0 \leqq \theta < 2\pi. \tag{3.5}$$

Combining this with (3.4) we get (3.3) for polynomials having all their zeros on $|z|=K\geqq 1$.

If the zeros of $p(z)$ lie in $|z|\geqq K\geqq 1$ but not necessarily on $|z|=K$, then for every real γ the polynomial $p(z)+e^{i\gamma}Q^*(z/K)$ has all its zeros on $|z|=K\geqq 1$. This follows from the fact that either $p(z)$ is a constant multiple of $Q^*(z/K)$ or else $|p(z)| <|Q^*(z/K)|$ for $|z|>K$, and $|Q^*(z/K)|<|p(z)|$ for $|z|<K$. Since (3.3) has already been proved to be true for polynomials having all their zeros on $|z|=K\geqq 1$, we obtain

$$K^s|p^{(s)}(e^{i\theta})+e^{i\gamma}(1/K^s)Q^{*(s)}(e^{i\theta}/K)|$$
$$\leqq |q^{(s)}(e^{i\theta})+e^{-i\gamma}(1/K^{n-s})p^{*(s)}(Ke^{i\theta})|, \qquad 0 \leqq \theta < 2\pi.$$

Arguing in the same way as for Lemma 11 we can now complete the proof of (3.3).

Theorem 4 follows from (3.2) and (3.3).

Proof of Theorem 5. Let us consider the case $n_1\leqq n_2$. If $\xi_1, \xi_2, \dots \xi_m$ are the zeros of $p_2(z)=b_{n_2}\prod_{j=1}^{n_2}(z-\xi_j)$ lying in $|z|\leqq 1-d$, we write

$$f(z) = \frac{p_1(z)}{b_{n_2}\prod_{j=1}^{m}(z-\xi_j)\prod_{j=m+1}^{n_2}(\bar{\xi}_jz-1)}\cdot\prod_{j=m+1}^{n_2}\frac{(\bar{\xi}_jz-1)}{(z-\xi_j)} = f_1(z)\cdot f_2(z).$$

On $|z|=1$, $|f_2(z)|=1$, $|f_1(z)| = |f(z)/f_2(z)|\leqq 1$, and hence

$$|f'(z)| = |f_1(z)f_2'(z)+f_2(z)f_1'(z)| \leqq |f_1'(z)|+|f_2'(z)|. \tag{3.6}$$

Since the polynomial

$$q_1(z) = b_{n_2}\prod_{j=1}^{m}(z-\xi_j)\prod_{j=m+1}^{n_2}(\bar{\xi}_jz-1)$$

has all its zeros in $|z|<1$ and

$$|f_1(z)| = |p_1(z)/q_1(z)| \leqq 1 \quad \text{on } |z| = 1,$$

it follows from a theorem of De Bruijn [4, p. 1265] that $|p_1'(z)| \leqq |q_1'(z)|$ on $|z|=1$. Hence for $|z|=1$

$$\begin{aligned} |f_1'(z)| &= \left|\frac{p_1'(z)}{q_1(z)} - \frac{p_1(z)}{q_1(z)} \cdot \frac{q_1'(z)}{q_1(z)}\right| \\ &\leqq \left|\frac{p_1'(z)}{q_1'(z)}\right| \left|\frac{q_1'(z)}{q_1(z)}\right| + \left|\frac{p_1(z)}{q_1(z)}\right| \cdot \left|\frac{q_1'(z)}{q_1(z)}\right| \\ &\leqq 2\left|\frac{q_1'(z)}{q_1(z)}\right| \qquad (3.7) \\ &= 2\left|\sum_{j=1}^{m} \frac{1}{z-\xi_j} + \sum_{j=m+1}^{n_2} \frac{1}{z-1/\bar{\xi}_j}\right| \\ &\leqq 2n_2(1+d)/d. \end{aligned}$$

On the other hand, it is fairly easy to verify that for $|z|=1$

$$\left|\frac{f_2'(z)}{f_2(z)}\right| \leqq \sum_{j=m+1}^{n_2} \frac{|\xi_j|+1}{|\xi_j|-1} \leqq \frac{n_2(2+d)}{d}. \tag{3.8}$$

Inequalities (3.7) and (3.8) imply (1.11).

If the degree n_1 of the polynomial $p_1(z)$ is greater than the degree n_2 of the polynomial $p_2(z)$, we write

$$f(z) = z^{n_1-n_2}(p_1(z)/z^{n_1-n_2}p_2(z)).$$

Then by (1.11)

$$\max_{|z|=1} \left| \frac{d}{dz} \left\{ \frac{p_1(z)}{z^{n_1-n_2}p_2(z)} \right\} \right| \leqq n_1(3+4/d),$$

and obviously enough

$$\max_{|z|=1} \left| \frac{d}{dz} (z^{n_1-n_2}) \right| = n_1 - n_2.$$

Hence for $|z|=1$

$$|f'(z)| \leqq n_1 - n_2 + n_1(3+4/d) = 4n_1(1+1/d) - n_2$$

and (1.12) follows.

Proof of Theorem 6. With the notation used in the proof of Theorem 5, the function $f_1(z)=p_1(z)/q_1(z)$ is analytic in $|z|\geqq 1$, $|f_1(z)|\leqq 1$ on $|z|=1$. Hence by the maximum modulus principle, $|f_1(z)|\leqq 1$ for $|z|\geqq 1$. Hence for $|z|\geqq 1$

$$
\begin{aligned}
|f(z)| &= |f_1(z)|\,|f_2(z)| \\
&\leqq |f_2(z)| \\
&= \prod_{j=m+1}^{n_2} \left|\frac{\bar{\xi}_j z-1}{z-\xi_j}\right| \\
&\leqq \prod_{j=m+1}^{n_2} \frac{|\xi_j|\,|z|-1}{|\xi_j|-|z|} \quad \text{for } 1+d > |z| \geqq 1 \\
&\leqq \left\{\frac{(1+d)|z|-1}{(1+d)-|z|}\right\}^{n_2-m} \quad \text{for } 1+d > |z| \geqq 1.
\end{aligned} \tag{3.9}
$$

In particular, for $1<|z|\leqq 1+d/n_2$

$$|f(z)| < e^3 \tag{3.10}$$

if $n_2\geqq 3$.

In order to estimate $|f(z)|$ for $1-d<|z|<1$, we write

$$f(z) = \frac{p_1(z)}{b_{n_2}\prod_{j=1}^{m}(\bar{\xi}_j z-1)\prod_{j=m+1}^{n_2}(z-\xi_j)}\prod_{j=1}^{m}\frac{(\bar{\xi}_j z-1)}{(z-\xi_j)} = f_3(z)\cdot f_4(z).$$

On $|z|=1$,

$$|f_4(z)| = 1, \qquad |f_3(z)| = |f(z)/f_4(z)| \leqq 1.$$

Since $f_3(z)$ is analytic in $|z|\leqq 1$, inequality $|f_3(z)|\leqq 1$ holds inside the unit circle as well. Hence for $|z|<1$

$$
\begin{aligned}
|f(z)| &\leqq |f_4(z)| \\
&= \prod_{j=1}^{m}\left|\frac{\bar{\xi}_j z-1}{z-\xi_j}\right| \\
&\leqq \prod_{j=1}^{m}\frac{1-|\xi_j|\,|z|}{|z|-|\xi_j|} \quad \text{for } 1-d < |z| < 1 \\
&\leqq \left\{\frac{1-(1-d)|z|}{|z|-(1-d)}\right\}^{m} \quad \text{for } 1-d < |z| < 1.
\end{aligned} \tag{3.11}
$$

In particular (3.10) holds for $1-d/n_2\leqq |z|<1$ if $n_2\geqq 3$.

By Cauchy's integral formula

$$|f^{(s)}(1)| \leqq \frac{s'}{2\pi}\int_{|z-1|=d/n_2}\frac{|f(z)|}{|z-1|^{s+1}}\,|dz|.$$

Using (3.9) for $|z|>1$, (3.11) for $|z|<1$, we easily get

$$|f^{(s)}(1)| \leqq \frac{s\,!}{\pi}\int_0^{\pi-\cos^{-1}d/2n_2}\left\{\frac{(1+d)(1+d^2/n_2^2+(2d/n_2)\cos\theta)^{1/2}-1}{(1+d)-(1+d^2/n_2^2+(2d/n_2)\cos\theta)^{1/2}}\right\}^{n_2-m}\left(\frac{n_2}{d}\right)^s d\theta$$

$$+\frac{s\,!}{\pi}\int_{\pi-\cos^{-1}d/2n_2}^{\pi}\left\{\frac{1-(1-d)(1+d^2/n_2^2+(2d/n_2)\cos\theta)^{1/2}}{(1+d^2/n_2^2+(2d/n_2)\cos\theta)^{1/2}-(1-d)}\right\}^m\left(\frac{n_2}{d}\right)^s d\theta.$$

This last inequality is a refined version of (1.13).

In case the degree n_1 of the polynomial $p_1(z)$ is greater than the degree n_2 of $p_2(z)$ we may write $f(z)$ in the form

$$(z) = z^{n_1-n_2}p_1(z)/z^{n_1-n_2}p_2(z)$$

and get a bound for $|f^{(s)}(1)|$ in the obvious way.

Proof of Theorem 7. By Hadamard's factorization theorem

$$(z) = \frac{z^m\exp(Q_1(z))\exp\left(\sum_{n=1}^{\infty} z/\xi_n\right)\prod_{n=1}^{\infty}(1-z/\xi_n)}{\exp(Q_2(z))\exp\left(\sum_{n=1}^{\infty} z/z_n\right)\prod_{n=1}^{\infty}(1-z/z_n)}$$

where ξ_n are the zeros of $f_1(z)$ and z_n those of $f_2(z)$. If α_n are the zeros of $f_2(z)$ which lie in the lower half-plane and β_n those which lie in the upper half-plane, then

$$\begin{aligned} f(z) &= \frac{z^m\exp(Q_1(z))\exp\left(\sum_{n=1}^{\infty} z/\xi_n\right)\prod_{n=1}^{\infty}(1-z/\xi_n)}{\exp(Q_2(z))\exp\left(\sum_{n=1}^{\infty} z/z_n\right)\prod_{n=1}^{\infty}(1-z/\alpha_n)\prod_{n=1}^{\infty}(1-z/\bar\beta_n)}\cdot\frac{\prod_{n=1}^{\infty}(1-z/\bar\beta_n)}{\prod_{n=1}^{\infty}(1-z/\beta_n)} \\ &= H(z)\cdot\frac{\prod_{n=1}^{\infty}(1-z/\bar\beta_n)}{\prod_{n=1}^{\infty}(1-z/\beta_n)}. \end{aligned} \tag{3.12}$$

It is easy to verify that under the hypotheses of the theorem $H(z)$ is of exponential type $\tau_1+\tau_2$ in the upper half-plane. Since $|H(x)|\leqq 1$ for real x we have [2, see 6.2.4 on p. 82]

$$|H(x+iy)| \leqq \exp((\tau_1+\tau_2)y), \qquad 0\leqq y<\infty. \tag{3.13}$$

For $0\leqq y=\operatorname{Im} z<d$

$$\begin{aligned} \left|\frac{\prod_{n=1}^{\infty}(1-z/\bar\beta_n)}{\prod_{n=1}^{\infty}(1-z/\beta_n)}\right| &= \exp\left(\sum_{n=1}^{\infty}\log\left|\frac{z-\bar\beta_n}{z-\beta_n}\right|\right) \\ &\leqq \exp\left(\sum_{n=1}^{\infty}\log\left(\frac{\operatorname{Im}\beta_n+y}{\operatorname{Im}\beta_n-y}\right)\right) \\ &\leqq \exp\left(\sum_{n=1}^{\infty}\frac{2y}{(\operatorname{Im}\beta_n-y)}\right) \\ &\leqq \exp((2d\,Ay)/(d-y)). \end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we get for $0 \leqq y < d$

(3.15) $\quad |f(x+iy)| \leqq \exp(\{(\tau_1+\tau_2)+2d\,A/(d-y)\}y), \qquad -\infty < x < \infty.$

Clearly we can infer a similar inequality for $-d<y<0$. Hence for $0 \leqq \theta < 2\pi$,

$$\left|f\left(x+\frac{de^{i\theta}}{(\tau_1+\tau_2+1)}\right)\right| \leqq \exp\left[\left\{(\tau_1+\tau_2)+\frac{2d\,A}{d-\dfrac{d|\sin\theta|}{(\tau_1+\tau_2+1)}}\right\}\frac{d|\sin\theta|}{(\tau_1+\tau_2+1)}\right],$$

$$-\infty < x < \infty.$$

Therefore by Cauchy's integral formula

$$|f^{(s)}(x)| \leqq \frac{s!}{\pi}\int_0^\pi \left(\frac{\tau_1+\tau_2+1}{d}\right)^s \exp\left[\left\{(\tau_1+\tau_2)+\frac{2A(\tau_1+\tau_2+1)}{(\tau_1+\tau_2+1-\sin\theta)}\right\}\frac{d\sin\theta}{(\tau_1+\tau_2+1)}\right]d\theta$$

$$< \frac{s!}{\pi}\int_0^\pi \left(\frac{\tau_1+\tau_2+1}{d}\right)^s \exp\left[d\sin\theta+\frac{2Ad\sin\theta}{(\tau_1+\tau_2+1-\sin\theta)}\right]d\theta,$$

$$-\infty < x < \infty$$

and (1.14) follows.

Proof of Theorem 10. Without loss of generality we may assume $a_3=1$. Since $|a_1|=K|a_2|$ we have

$$\frac{\int_0^{2\pi}|p'(e^{i\theta})|^2\,d\theta}{\int_0^{2\pi}|p(e^{i\theta})|^2\,d\theta} = \frac{|a_1|^2+4|a_2|^2+9}{K^6+|a_1|^2+|a_2|^2+1}$$

$$= \frac{(K^2+4)|a_2|^2+9}{K^6+(K^2+1)|a_2|^2+1}.$$

This last expression is at most equal to

$$\frac{9}{1+K^6} \quad \text{or} \quad \frac{9(1+4K^2+K^4)}{1+9K^2+9K^4+K^6}$$

according as

$$(K^8+4K^6-8K^2-5)|a_2|^2 \leqq 0$$

or

$$(K^8+4K^6-8K^2-5)(9K^2-|a_2|^2) \geqq 0.$$

This gives the desired result since clearly $0 \leqq |a_2|^2 \leqq 9K^2$.

References

1. N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein*, Pacific J. Math. **5** (1955), 849–852.

2. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.

3. ———, *Inequalities for asymmetric entire functions*, Illinois J. Math. **1** (1957), 94–97.

4. N. G. De Bruijn, *Inequalities concerning polynomials in the complex domain*, Nederl. Akad. Wetensch. Proc. Ser. A **50** (1947), 1265–1272.

5. P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

6. B. Ya. Levin, *On a special class of entire functions and on related extremal properties of entire functions of finite degree*, Izv. Akad. Nauk SSSR Ser. Fiz.-Mat. Nauk **14** (1950), 45–84. (Russian)

7. M. A. Malik, *On the derivative of a polymonial*, J. London Math. Soc. (to appear).

8. Q. I. Rahman, *Functions of exponential type*, Trans. Amer. Math. Soc. **135** (1969), 295-309.

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