# ON TOPOLOGICAL EQUIVALENCE OF $\aleph_0$-DIMENSIONAL LINEAR SPACES(1)

BY

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1. **Introduction.** In this paper, $l_2$ will denote the separable Hilbert space of all square summable real sequences with the usual norm $\| \ \|$. $s$ will denote the countable infinite product of real lines $R$. $l_F$ and $s_F$ will denote the subspaces of $l_2$ and $s$ respectively consisting of all points $x=(x_1, x_2, \ldots)$ such that $x_i=0$ for all but finitely many $i$. In 1957 Long and Klee in [5] showed that all $\aleph_0$-dimensional (algebraic) normed linear spaces are homeomorphic. Thus, in particular, they are all homeomorphic to $l_F$. Klee pointed out, however, that the assumption of normability cannot be completely abandoned, since there are $\aleph_0$-dimensional locally convex topological linear spaces which are not metrizable. In 1963, C. Bessaga in [0] generalized Klee and Long's result (using their method) to some $\aleph_0$-dimensional linear metric space which need not be normed. In particular, he shows that all $\aleph_0$-dimensional locally convex metric linear spaces having a radially bounded neighborhood of zero (or equivalently, containing no subspace which is linearly homeomorphic with $s_F$) are homeomorphic [0, Proposition 3]. Therefore the ultimate question (classification problem) is whether all $\aleph_0$-dimensional locally convex linear metric spaces are homeomorphic.

The purpose of this paper is to settle a special case of the above question raised by Fréchet [2, p. 83] in 1928, also by Klee, Bessaga [0, p. 163] and Pełczynski [6]. We shall prove

THEOREM I. *$l_F$ is homeomorphic with $s_F$.*

COROLLARY I. $\bar{s}_p \sim \bar{s}_f \sim l_F$,
*where "$\sim$" means homeomorphic to; see §2 for definitions of $\bar{s}_p$ and $\bar{s}_f$.*

(The proof in this paper is rather self-contained.)

2. **Definition and notation.** (All subsets inherit the subspace topology.)

(0) $s = \prod_{i=1}^{\infty} R_i$ where $R_i = R =$ reals;

$s' = \{x \in s : x_1 < 0\}$.

Received by the editors January 3, 1968.

(1) This research was supported in part by the National Research Council and the Office of Naval Research Grant ONR:432:LDB:lcd.

(2) The author is indebted to H. Corson and V. Klee for their helpful conversation associated with this paper.

(1) $l_+ = \{(x_1, x_2, \ldots) \in l_F : x_i = 0$ for all $i$ or $x_n > 0$ when $n$ is the largest nonzero index of $x\}$;

$l_- = \{x \in l_F : -x \in l_+\}$.

(2) $s_+ = \{(x_1, x_2, \ldots) \in s_F : x_i = 0$ for all $i$ or $x_n > 0$ when $n$ is the largest nonzero index of $x\}$;

$s_- = \{x \in s_F : -x \in s_+\}$.

(3) $S_F = \{x \in l_F : \|x\| = 1\}$;

$S_- = (S_F \cap l_-) \cap s'$.

(4) $Q, Q'$ = Hilbert cubes represented by $\prod_{i=1}^{\infty} J_i$ and $\prod_{i=1}^{\infty} I_i$ respectively where each $J_i = [-1, 1] \subset R_i$ and each $I_i = [0, 1] \subset R_i$.

(5) $s_f = \{x \in s_F : x_i \in (-1, 1)$ for all $i\}$;

$$\bar{s}_f = \bigcup_{n=1}^{\infty} \left( \prod_{i=1}^{n} J_i \right).$$

(6) $s'_+ = s_+ \cap s_f$.

(7) $$s_p = \{0\} \cup \bigcup_{n=1}^{\infty} \left[ \operatorname{Int} \left( \prod_{i=1}^{n} I_i \right) \right];$$

$$\bar{s}_p = \bigcup_{n=1}^{\infty} \left( \prod_{i=1}^{n} I_i \right).$$

Let $\pi_n$, $\tau_n$ denote the projection functions defined by $\pi_n(x_1, x_2, \ldots) = x_n$ and $\tau_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n)$. Let $R^n = \prod_{i=1}^{n} R_i$ denote the Euclidean $n$-space with the usual norm. $R^n$ is also considered as $R^n \times 0 \subset R^{n+1}$ and as $R^n \times 0 \times 0 \cdots \subset s$. Let $R^n_+$ and $R^n_-$ denote respectively the subspaces $\{x \in R^n : x_n \geqq 0\}$ and $\{x \in R^n : x_n \leqq 0\}$. If $f$ is an imbedding of $R^n$ into $R^n$, then $f'$ is said to be the imbedding of $s$ into $s$ induced by $f$ if $f'(x) = (f(\tau_n(x)), x_{n+1}, x_{n+2}, \ldots)$ for all $x \in s$.

**3. Proof of Theorem** I (§3–§6).

3.1 *A basic lemma.* Before introducing the lemma, we need the following definition.

DEFINITION. Let $(X, d)$ be a metric space and $A \subset X$. Let $f$, $g$ be imbeddings of $A$ into $X$. For any $\varepsilon > 0$, let

$$\alpha(f, \varepsilon) = \inf \{d(f(x), f(y)) : x, y \in A \text{ and } d(x, y) > \varepsilon\},$$
$$\lambda(f, g) = \sup \{d(f(x), g(x)) : x \in A\},$$

where $\lambda(f, g)$ may be infinite. $f$ is called an *$\alpha$-imbedding* if $\alpha(f, \varepsilon) > 0$ for any $\varepsilon > 0$.

LEMMA (CONVERGENCE CRITERION). *Let $(X, d)$ be a metric space, $A \subset X$. If $\{f_i\}_{i \geq 1}$ is a sequence of imbedding of $A$ into $X$ satisfying* (1) *For each $x \in A$, $\lim_{n \geq 1} f_n(x)$ exists and belongs to $X$ and* (2) *For all $n$, $\lambda(f_{n+1}, f_n)$*

$\leqq \min(1/3^{n+1}, 1/3^{n+1}\cdot\alpha(f_n, 1/3^n))$. *Then the function f defined by* $f(x)=\lim_{n\geqq 1} f_n(x)$ *is an imbedding of A into X.*

**Proof.** If $x, y \in A$ such that $d(x, y) > 1/3^n$ for some $n$, it is easy to see from (2) that $d(f_m(x), f_m(y)) \geqq 1/3\cdot d(f_n(x), f_n(y))$ for all $m \geqq n$. This shows $f$ is 1-1. We claim this also shows $f^{-1}$ is continuous for the following reason. If $\{x_i\}_{i\geqq 1}$ is a sequence in $X$ such that $d(x_i, x) > 1/3^n$ for some fixed $n$. Then $d(f_n(x_i), f_n(x)) \not\to 0$. Hence without loss of generality, we may assume that for all $i$, $d(f_n(x_i), f_n(x)) > \varepsilon$ for some $\varepsilon > 0$. But this implies for all $i$, $d(f(x_i), f(x)) \geqq \varepsilon/3$. Thus $f(x_i) \not\to f(x)$. Finally $f$ is continuous as a simple consequence of the fact $\lambda(f_{n+1}, f_n) \leqq 1/3^{n+1}$ for all $n$.

3.2 *Strategy for the proof of Theorem* I. We shall prove Theorem I by constructing homeomorphisms $h_1, \ldots, h_6$, where

$$l_F \overset{h_1}{\sim} l_- \overset{h_2}{\sim} S_- \overset{h_3}{\sim} s_p \overset{h_4}{\sim} s'_+ \overset{h_5}{\sim} s_- \overset{h_6}{\sim} s_F.$$

We point out, first of all, that $h_2$ and $h_5$ are rather trivial; namely, define $h_2: l_- \to S_-$ by

$$h_2(x) = \left(-\left(1-\left(\frac{\|x\|}{1+\|x\|}\right)^2\right)^{1/2}, \frac{x_1}{1+\|x\|}, \frac{x_2}{1+\|x\|}, \ldots\right)$$

and define $h_5: s'_+ \to s_-$ by

$$h_5(x) = \left(\frac{-x_1}{1-|x_1|}, \frac{-x_2}{1-|x_2|}, \ldots\right).$$

To see $h_2$ is a homeomorphism of $l_-$ onto $S_-$, we decompose $h_2$ into $g_2\cdot f_2$, where $f_2$ is the homeomorphism between $l_-$ and the subspace $B=\{x \in l_- : \|x\| < 1\}$ defined by $f_2(x) = x/(1+\|x\|)$, and $g_2$ is the homeomorphism of $B$ onto $S_-$ defined by $g_2(y) = (-(1-\|y\|^2)^{1/2}, y_1, y_2, \ldots)$. To see $h_5$ is a homeomorphism, note that

$$h'_5(y_1, y_2, \ldots) = \frac{-y_1}{1+|y_1|}, \frac{-y_2}{1+|y_2|}, \ldots$$

is the inverse of $h_5$.

## 4. $h_1$ and $h_6$.

4.1 Lemma. *If for some fixed* $n \geqq 1$, $f_{n+1}$ *is a mapping of* $R^n$ *into* $R$ *satisfying* $f_{n+1}(b_j) \to 0$ *whenever* $\|b_j\| \to \infty$. *Then the function* $F_{n+1}$ *defined by*

$$F_{n+1}(x_1, X_2, \ldots) = (x_1, \ldots, x_n, x_{n+1} + f_{n+1}(x_1, \ldots, x_n), x_{n+2}, \ldots)$$

*induces an* $\alpha$*-imbedding of* $l_F$ *onto* $l_F$.

**Proof.** $F_{n+1}$ is clearly an imbedding of $l_F$ onto $l_F$. To show $F_{n+1}$ is an $\alpha$-imbedding, let us assume the contrary. So for some $\varepsilon > 0$ and some sequence of pairs of points $\{(a_j, b_j)\}_{j\geqq 1}$ we have $\|a_j - b_j\| > \varepsilon$ for all $j$ but $\|F_{n+1}(a_j) - F_{n+1}(b_j)\| \to_j 0$. Thus for each $i \leqq n$, $|\pi_i(a_j) - \pi_i(b_j)| \to_j 0$. We consider two cases. (1) $\{\|\tau_n(a_j)\|\}_{j\geqq 1}$ is unbounded. Without loss of generality, we assume $\|\tau_n(a_j)\| \to_j \infty$. This implies $\|\pi_n(b_j)\| \to_j \infty$. Then by the definition of $f_{n+1}$, $f_{n+1}(\tau_n(a_j)) \to_j 0$ and $f_{n+1}(\tau_n(b_j)) \to_j 0$. This means $\|F_{n+1}(a_j)-F_{n+1}(b_j)\| - \|a_j-b_j\| \to 0$ contradicting the assumption. (2) $\{\|\tau_n(a_j)\|\}_{j\geq 1}$ is bounded. This implies $\{\|\tau_n(b_j)\|\}_{j\geq 1}$ is also bounded. Without loss of generality we may assume $\tau_n(a_j) \to_j C$ and $\tau_n(b_j) \to_j C$ for some $C \in R^n$ since $|\pi_i(a_j)-\pi_i(b_j)| \to_j 0$ for all $i \leqq n$. Hence $f_{n+1}(\tau_n(a_j)) - f_{n+1}(\tau_n(b_j)) \to_j 0$. This again implies $\|F_{n+1}(a_j)-F_{n+1}(b_j)\| - \|a_j-b_j\| \to 0$, a contradiction.

4.2 COROLLARY. *If for each* $n \geqq 1$, $f_{n+1}$ *is a mapping of* $R^n$ *into* $R$ *such that* $f_{n+1}(a_i) \to 0$ *whenever* $\|a_i\| \to \infty$. *Then for each* $n \geqq 1$, *the function* $F_{n+1}$ *defined by*

$$F_{n+1}(x) = (x_1, x_2+f_2(x_1), \ldots, x_{n+1}+f_{n+1}(x_1, \ldots, x_n), x_{n+2}, \ldots)$$

*is an* $\alpha$*-imbedding of* $l_F$ *onto* $l_F$.

**Proof.** It is clear that each $F_{n+1}$ is a homeomorphism of $l_F$ onto $l_F$. For $n=1$, $F_2$ is an $\alpha$-imbedding according to §4.1. Suppose for $n \geqq 1$, $F_{n+1}$ is an $\alpha$-imbedding. Clearly $F_{n+2}F_{n+1}^{-1}$ satisfies §4.1, hence is an $\alpha$-imbedding. Thus

$$F_{n+2} = (F_{n+2}F_{n+1}^{-1})F_{n+1}$$

is an $\alpha$-imbedding. This completes the induction.

4.3 Let $G$ be a mapping of $R$ into $R$ defined by

$$G(x) = (|x|+x)/2(1+x^2).$$

For each fixed $n>1$, define a sequence of mappings $\{G_i^n\}_{1<i\leqq n}$ as follows: Each

$$G_i^n : R^{i-1} \to R,$$

$$G_2^n(x_1) = G(x_1)/n \text{ and inductively, if } 1 < k < n,$$

$$G_{k+1}^n((x_1, \ldots, x_k)) = 1/2^{n-k}\cdot G(x_k + G_k^n((x_1, \ldots, x_{k-1}))).$$

The following lemma is evident.

LEMMA. (1) $0 \leqq G(x) \leqq 1$ *for all* $x \in R$; $G(x)>0$ *if and only if* $x>0$; $G(x_i) \to 0$ *whenever* $|x_i| \to \infty$.

(2) *For each* $n>1$ *and each* $1<i\leqq n$, $0 \leqq G_i^n(x) \leqq 1$ *for all* $x \in R^{i-1}$.

(3) *If for some* $1<k<n$, $G_k^n(x)=0$, *then* $G_n^n(x')=0$ *where* $x'=(x, 0, \ldots, 0) \in R^{n-1}$.

(4) $G_k^n(x)>0$ *if* $\pi_{k-1}(x)>0$.

4.4 For each $n>1$, let $\{G_i^n\}_{1<i\leqq n}$ be defined as in §4.3. Define a sequence of mapping $\{Z_n\}_{n>1}$ of $R^{n-1}$ into $R$ as follows:

$$Z_2(x_1) = G_2^2(x_1) \text{ and inductively for } k > 2,$$

$$(*) \qquad Z_{k+1}((x_1, \ldots, x_k)) = G_{k+1}^{k+1}((x_1, \ldots, x_k))G(x_k - \tfrac{1}{2}Z_k((x_1, \ldots, x_{k-1}))).$$

The following lemma is evident.

LEMMA. (1) $1 \geqq Z_n(x) \geqq 0$ *for all* $x \in R^{n-1}$,

(2) *for all* $n>1$, $Z_n(x)=0$ *if* $\pi_{n-1}(x) \leqq 0$.

4.5 *Homeomorphism* $h_6$. Let $\{Z_n\}_{n>1}$ be defined as above. Define $H: s \to s$ by

$$H(x) = (x_1, x_2+Z_2(x_1), x_3+Z_3(x_1, x_2), \ldots).$$

THEOREM. *$h_6 = H|_{s_-}$ is a homeomorphism of $s_-$ onto $s_F$.*

**Proof.** It is clear that $h_6$ is 1-1, continuous and $h_6^{-1}$ is continuous. To show $h_6(s_-) \supset s_f$, suppose $x \in s_f$. If $x=0$, it is trivial since $h_6(0)=0$ and $0 \in s_-$. If $x \neq 0$, let $n$ be the largest nonzero index of $x$. Let

$$x' = (x_1, x_2-Z_2(x_1), \ldots, -Z_{n+1}(x_1, \ldots, x_n), 0, 0, \ldots).$$

By Lemma 4.4, $H(x')=x$. So $h_6$ is onto $s_f$ if $x' \in s_-$. We consider four cases. (1) If $n=1$ and $x_1>0$, then $-Z_2(x_1) = -G_2^2(x_1) < 0$ by Lemma 4.3(4), hence $x' \in s_-$. (2) if $n=1$ and $x_1<0$, then by Lemma 4.4(2), $-Z_2(x_1)=0$, hence $x' \in s_-$. (3) If $n>1$ and $x_n<0$, then $-Z_{n+1}(x_1, \ldots, x_n)=0$ by 4.4(2) and $x_n - Z_n(x_1, \ldots, x_{n-1}) < 0$, hence $x' \in s_-$. (4) If $n>1$ and $x_n>0$, we have two cases. (1) If $-Z_{n+1}(x_1, \ldots, x_n)<0$, then we are finished since this implies $x' \in s_-$. If $-Z_{n+1}(x_1 \cdots x_n)=0$, by (*) of §4.4, and by §4.3(4), $G(x_n - \frac{1}{2}Z_n((x_1, \ldots, x_n)))=0$. By §4.3(1), $x_n - \frac{1}{2}Z_n((x_1, \ldots, x_n)) \leqq 0$. Hence $x_n - Z_n((x_1, \ldots, x_n)) < 0$. This implies $x' \in s_-$.

Finally we have to show $h_6(s_-) \subset s_F$. Suppose $x\ (\neq 0) \in s_-$, let $x_n$ be the largest nonzero index of $x$. Hence $x_n<0$. If $n=1$, then clearly $Z_{k+1}(x_1, \ldots, x_k)=0$ for all $k \geqq 1$, thus $h_6(x) \in s_F$. If $n>1$, for any $m>n$, consider

$$G_{n+1}^m(\tau_n(x)) = \frac{1}{2^{m-n}} G\left(x_n + \frac{1}{2^{m-n+1}} G(x_{n-1} + G_{n-1}^m(\tau_{n-1}(x)))\right).$$

Since $x_n<0$ and

$$x_n + \frac{1}{2^{m-n+1}} G(x_{n-1} + G_{n-1}^m(\tau_{n-1}(x))) \leqq x_n + \frac{1}{2^{m-n+1}}$$

by §4.3(1), there is a large enough $m$ such that $G_{n+1}^{m'}(\tau_n(x))=0$ for all $m' \geqq m$. By §4.3(3), $G_{m'}^{m'}(\tau_{m'-1}(x))=0$ for all $m' \geqq m$. This implies $Z_{m'}(\tau_{m'-1}(x))=0$ for all $m' \geqq m$. Hence $h_6(x) \in s_F$.

4.6 For each $n>1$, let $\{G_i^n\}_{1<i\leqq n}$ and $\{Z_n\}_{n>1}$ be defined as in §4.3 and §4.4. Define a sequence of mapping $\{Z_n'\}_{n>1}$ of $R^{n-1}$ into $R$ and a sequence of $\alpha$-imbedding $\{F_n\}_{n>1}$ of $l_F$ onto $l_F$ as follows:

$$Z_2'(x_1) = Z_2(x_1);\ F_2(x) = (x_1, x_2+Z_2'(x_1), x_3, \ldots).$$

Inductively suppose for $k>1$, $\{Z_i'\}_{1<i\leqq k}$ and $\{F_i\}_{1<i\leqq k}$ have been defined. Define

$$(**)\qquad Z_{k+1}'(x_1, \ldots, x_k) = Z_{k+1}(x_1, \ldots, x_k) \cdot G(1+x_1^2+\cdots+x_k^2) \cdot \min\left(\frac{1}{3^{k+1}}, \frac{1}{3^{k+1}}\alpha\left(F_k, \frac{1}{3^k}\right)\right),$$

and

$$F_{k+1}(x) = (x_1, x_2+Z_2'(x_1), \ldots, x_{k+1}+Z_{k+1}'(x_1, \ldots, x_k), x_{k+2}, x_{k+3}, \ldots).$$

LEMMA. (1) *Each $F_n$ is an $\alpha$-imbedding.*

(2) $\lambda(F_{n+1}, F_n) \leqq \min(1/3^{k+1}, 1/3^{k+1} \cdot \alpha(F_k, 1/3^k))$.

(3) *If* $x \in l_2$, *then* $x' = (x_1, x_2 + Z_2'(x_1), x_3 + Z_3'(x_1, x_2), \ldots) \in l_2$.

**Proof.** (1) If $a_i \in R^{n-1}$ such that $\|a_i\| \to \infty$, then $G(1+\|a_i\|^2) \to 0$ by §4.3(1). Hence $Z_n'(a_i) \to 0$. By Corollary 4.2, $F_n$ is an $\alpha$-imbedding. (2) Follows from §4.4(1) and §4.3(1). (3) Follows from (**). Specifically:

$$|Z_{k+1}'(x_1, \ldots, x_k)| \leqq 1/3^{k+1}.$$

4.7 *Homeomorphism* $h_1$. Define $H': l_2 \to l_2$ by

$$H'(x) = (x_1, x_2 + Z_2'(x_1), x_3 + Z_3'(x_1, x_2), \ldots).$$

THEOREM. $h_1 = H'|_{l_-}$ *is a homeomorphism of* $l_-$ *onto* $l_F$.

**Proof.** It is clear that $h_1(x) = \lim_{n>2} F_n(x)$, where $\{F_n\}$ as defined in §4.6. By Lemma 4.6(2) and Lemma 3.1, $h_1$ is an imbedding of $l_-$ into $l_2$. The proof that $h_1(l_-) = l_F$ is similar (the fact $F_n$ is an $\alpha$-imbedding is needed here) to the proof that $h_6(s_-) = s_F$ in §4.5.

5[3]. $h_4$.

5.1 LEMMA. *Let X be a compact Hausdorff space and* $A \subset X$. *If g is a mapping of X into X such that* $f|_A$ *is* 1-1 *and* $f(X-A) \subset X - f(A)$, *then* $f|_A$ *is an imbedding.*

**Proof.** We have to show $f^{-1}: f(A) \to A$ is continuous. Suppose $x, \{x_\alpha\}_\alpha \subset A$ such that $f(x_\alpha) \to f(x)$. Since $X$ is compact, we may assume $x_\alpha \to y \in X$. Hence $f(x_\alpha) \to f(y) (= f(x)) \in f(A)$. By hypothesis $y \in A$. Hence $y = x$ since $f$ is 1-1 on $A$.

5.2 Let $D_n = J_n \times J_{n+1}$. Let $a = (1/n, 1)$, $b = (0, 1)$, $c = (-1, 0)$, $d = (0, -1)$ be points in $D_n$ and let $K$ = closure of the component of Bd $(D_n) - \{a, d\}$ that does not include $b$. For each $n \geqq 1$, let $f_n$ be a homeomorphism of $D_n$ onto $D_n$ such that $f_n|_K$ = identity, $f_n(b) = c$ and $f_n(tx) = t(f_n(x))$ for each $0 \leqq t \leqq 1$ and each $x \in$ Bd $(D_n)$.

LEMMA. *If* $x_{n+1} > 0$ *and* $\pi_n(f_n((x_n, x_{n+1}))) = 0$, *then* (1) $0 < x_n < 1/n$ (2) $x_{n+1} = \pi_{n+1} f_n((x_n, x_{n+1}))$ *and* (3) $x_{n+1} > x_n'$.

**Proof.** Clear.

5.3 *Homeomorphism* $h_4$. Let $\{f_n\}_{n \geqq 1}$ be defined as in §5.2. Define a sequence of homeomorphisms $\{F_n\}_{n \geqq 1}$ of $Q$ onto $Q$ by $F_1(x) = (f_1(x_1, x_2), x_3, x_4, \ldots)$ and for $n > 1$, $F_n(x) = (x_1, \ldots, x_{n-1}, f_n(x_n, x_{n+1}), x_{n+2}, \ldots)$. Let $F$ be the mapping of $Q$ into $Q$ defined by $F(x) = \lim_{n \geqq 1} F_n \cdots F_2 F_1(x)$.

THEOREM. $h_4 = F|_{s_p}$ *is a homeomorphism of* $s_p$ *onto* $s'_+$.

**Proof.** It is easy to check that $F(s_p) = s'_+$, $F(\bar{s}_p) = \bar{s}_f$, $s_p \subset \bar{s}_p$, $s'_+ \subset \bar{s}_f$ and $F$ is a 1-1,

[3] The method employed in this section is due to Corson [1].

continuous mapping of $\bar{s}_p$ onto $\bar{s}_f$. Therefore by means of §5.1, it is sufficient to show $F(Q-\bar{s}_p)\subset Q-\bar{s}_f$. Suppose $x\in Q$ and $x\notin \bar{s}_p$.

*Case* 1. For some $i$, $x_i<0$. Then $\pi_j(F(x))<0$ for all $j>i$. Hence $F(x)\notin \bar{s}_f$.

*Case* 2. For all $i$, $x_i\geqq 0$ and for some $i$, $x_i=0$. Hence for infinite many $i$, $x_i>0$. It is easy to check $\pi_j(F(x))=-\pi_{j+1}(x)$ for all $j\geqq i$. Hence $F(x)\notin \bar{s}_f$.

*Case* 3. For each $i$, $x_i>0$. Let us assume $F(x)\in \bar{s}_f$. Hence there exists an $n>1$ such that $\pi_m(F(x))=0$ for all $m\geqq n$. It follows from Lemma 5.2(2) $x_{m+1}=\pi_{m+1}(F_mF_{m-1}\cdots F_1(x))=x'_{m+1}$ for all $m\geqq n$. By §5.2(1), $0<x'_n=\pi_n(F_{n-1}\cdots F_1(x))<1/n$ and by repeating §5.2(3), $0<x'_n<x_{n+1}=x'_{n+1}<x_{n+2}=x'_{n+2}<\cdots$. This is clearly impossible since $x'_{n+k}<1/(n+k)$ for all $k$.

6. $h_3$.

6.1 Lemma. *Let $X$ be a metric space, $A\subset X$ and $f$ an imbedding of $A$ into $X$. Then $f$ is an $\alpha$-imbedding if either* (1) *$A$ is compact, or* (2) *$A=X$ and $f$ is supported on a compact subset $K$ of $X$; i.e., $f|_{X-K}=$identity, or* (3) *$X=A=R^{n+m}$ and $f=f_n\times e$ where $f_n$ is an $\alpha$-imbedding of $R^n$ into $R^n$ and $e$ is the identity function on $\prod_{i=n+1}^{m+n} R_i$.*

**Proof.** Well known.

6.2 Lemma. *If for each $n$, $\{f_n\}_{n\geqq 1}$ is an imbedding of $R^n$ into $R^n$ such that for $n>1$ and any $x\in R^n$, $\pi_i(f_n(x))=\pi_i(x)$ for all $i<n$. Then the function $f$ defined by $f(x)=\lim_{n\geqq 1} f'_1\cdots f'_n(x)$ is an imbedding of $s$ into $s$, where each $f'_i$ is the imbedding of $s$ into $s$ induced by $f_i$.*

**Proof.** It is evident that $f(x)$ exists for each $x\in s$ and is a 1-1, continuous mapping of $s$ into $s$. To show $f^{-1}$ is continuous, let us suppose $\{a_i\}_{i\geqq 1}$ is a sequence in $s$ such that $f(a_i)\to f(a)\in f(s)$. We want to show $\pi_j(a_i)\to_i \pi_j(a)$ for each $j$. For $j=1$, it is obvious since $f_1$ is an imbedding of $R_1$ into $R_1$. Now suppose it is true for all $j\leqq k$. From the definition of $f_j$, we observe that $\pi_j(f(x))=\pi_j(f'_j(x))$ for any $x\in s$. Hence

$$\pi_{k+1}(f'_{k+1}(a_i)) = \pi_{k+1}(f(a_i)) \to_i \pi_{k+1}(f(a)).$$

On the other hand, since $\pi_j(a_i)\to_i\pi_j(a)$ for all $j\leqq k$ by assumption, $\pi_j(f'_{k+1}(a_i))=\pi_j(a_i)\to_i\pi_j(a)$ for all $j\leqq k$. This shows that $f_{k+1}(\tau_{k+1}(a_i))\to_i(\pi_1(a),\ldots,\pi_k(a),\pi_{k+1}(f(a)))$. Therefore $\pi_{k+1}(a_i)\to\pi_{k+1}(a)$. This completes the induction.

6.3 Let $B^n$, $S^{n-1}$ denote respectively the closed unit ball, unit sphere of $R^n$. Recall that $R^n$ is also regarded as $R^n\times 0\subset R^{n+1}$ and as $R^n\times 0\times 0\cdots\subset s$.

Lemma. *For $n>1$ and any $\varepsilon>0$, there is an $\varepsilon$-imbedding $g_n$ of $R^n$ onto $R^n$ satisfying the following conditions*:

(1) $g_n|_{R^n_-}=$*identity*,

(2) $\tau_{n-1}(g_n(B^n_+))=B^{n-1}$,

(3) $\pi_1(g_n(x))=0$ iff $\pi_1(x)=0$,

(4) *each $g_n$ is an $\alpha$-imbedding of $R^n$ onto $R^n$ and*

(5) *for each* $x \in g_n(B^{n-1}) = B^{n-1}$, $\{(x, t) : t \in R_n\} \cap g_n(B^n)$ *is a nondegenerate closed line segment* $[a, b]$ *such that* (1) *if* $x \in \operatorname{Int}(B^{n-1})$, *then* $b \in g_n(S_+^{n-1})$, $a \in g_n(S_-^{n-1}) = S_-^{n-1}$ *and* $\operatorname{Int}[a, b] \subset \operatorname{Int} g_n(B^n)$, (2) *if* $x \in \operatorname{Bd}(B^{n-1}) = S^{n-2}$, *then* $\|a-b\|$ *is a constant* $r$ *and* $[a, b] \subset g_n(S_+^{n-1})$.

**Proof.** We shall prove this by actually constructing $g_n$. We may assume $g_2$ has been constructed for the given $\varepsilon$ since $g_2$ obviously exists. To define $g_n$, write $R^n$ as $R^{n-1} \times R_n$. For each $x \in S^{n-2}$, let $\mathscr{L}_x = \{tx : t \in R\}$ and $E_x = \mathscr{L}_x \times R_n$. We consider $E_x$ as $R^2$ by identifying $(tx, r) \in \mathscr{L}_x \times R_n$ with $(t\|x\|, r) \in R^2$. Hence $g_2$ induces a homeomorphism $g_x$ of $E_x$ onto $E_x$. We assume $g_2$ is so chosen that the motion of $g_2$ is symmetric with respect to the $R_2$-axis; that is, if $g_2(x_1, x_2) = (x_1', x_2')$, then $g_2(-x_1, x_2) = (-x_1', x_2')$. This implies $g_{(-x)} = g_x$. It is evident that defining $g_x$ on each $\mathscr{L}_x \times R_n$ induces a homeomorphism $g_n$ of $R^n$ onto $R^n$ and satisfies all the required conditions.

6.4 Let $\{g_n\}_{n>1}$ be inductively defined as in §6.3 subject to the following $\varepsilon$-condition.

For $k \geqq 1$, let $g_n^k$ denote the homeomorphism of $R^{n+k}$ onto $R^{n+k}$ induced by $g_n$; i.e., $g_n^k(x_1, \ldots, x_{n+k}) = (g_n(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{n+k})$. First we define $g_2$ as in §6.3 for $\varepsilon = \frac{1}{2}$. Suppose $\{g_i\}_{i=2}^n$ has been defined, let $\tilde{g}_n = g_2^{n-2} \cdots g_{n-1}^1 g_n$ and let $\tilde{g}_n^1$ be defined as $g_n^1$. We then choose $\varepsilon$ to be so small for $g_{n+1}$ so that

$$(\#) \qquad \lambda(\tilde{g}_n^1 \cdot g_{n+1}, \tilde{g}_n^1) \leqq \min\left(\frac{1}{3^{n+1}}, \frac{1}{3^{n+1}}\,\alpha\left(\tilde{g}_n^1, \frac{1}{3^n}\right)\right).$$

Let $G_n$ denote the homeomorphism of $s$ onto $s$ induced by $g_n$ and let $G_n' = G_2, \ldots, G_n$.

LEMMA. (1) $G(x) = \lim_{n \geqq 2} G_n'(x)$ *is an imbedding of* $s$ *into* $s$, (2) *for* $n > 1$, $G(B^n) = G_n'(B^n)$, (3) $G|_{B^1} = $ *identity*.

**Proof.** (1) $G(x)$ exists since for each $i$, $\{\pi_i(G_n'(x))\}_{n \geqq 2}$ is a Cauchy sequence in $R_i$. The fact that $G$ is an imbedding is a rather straightforward consequence of (#). (The proof can be reconstructed similar to that in §3.1.)

(2) If $x \in B^n$ then $x \in R_-^{n+k}$ for all $k \geqq 1$, hence by §6.3(1), $g_{n+k}(x) = x$. This implies $G_{n+k}'(x) = G_n'(x)$ for all $k \geqq 1$. Thus $G(x) = G_n'(x)$.

(3) By (2), $G|_{B^1} = G_2'|_{B^1} = G_2|_{B^1} = g_2|_{B^1} = $ identity.

6.5 Let $G$, $\{G_n'\}_{n \geqq 2}$ be defined as in §6.4. For each $A \subset s$, we shall denote $G(A)$ by $\tilde{A}$. Hence $\tilde{B}^n = G(B^n) = G_n'(B^n) = \tilde{g}_n(B^n) = g_2^{n-2} \cdots g_{n-1}^1(g_n(B^n))$ by §6.4. Now by applying Lemma 6.3 and the known properties of $R^n$, we have

LEMMA. *There is a homeomorphism* $\phi_n$ *of* $R^n$ *onto* $R^n$, $n > 1$, *such that*

(1) $\phi_n(\tilde{B}^n) = \tilde{B}^{n-1} \times I_n$, *where* $I_n = [0, 1] \subset R_n$,

(2) $\phi_n(\tilde{S}_-^{n-1}) = \tilde{B}^{n-1}$,

(3) $\phi_n|_{\tilde{S}^{n-2}} = $ *identity and*

(4) *for any* $x \in R^n$, $\pi_i(\phi_n(x)) = \pi_i(x)$ *for all* $i < n$.

6.6 Let $\{\phi_n\}_{n>1}$ be defined as above. Let $\phi'_n$ be the homeomorphism of $s$ onto $s$ induced by $\phi_n$. Define $\phi: s \to s$ by $\phi(x)=\lim_{n>1} \bar{\phi}_n(x)$ where $\bar{\phi}_n=\phi'_1\phi'_2\cdots\phi'_n$. Then according to §6.2, $\phi$ is an imbedding of $s$ into $s$. Furthermore, for $n>1$, we have the following Lemma:

LEMMA. (1) $\bar{\phi}_n(\tilde{B}^n)=B^1\times\prod_{j=2}^n I_j$, *where* $I_j=[0, 1]\subset R_j$,

(2) $\phi(\tilde{S}^{n-1})=\text{Bd}\,(B^1\times\prod_{j=2}^n I_j)$,

(3) $\phi(\tilde{S}^1_-)=B^1$ *and* $\phi(\tilde{S}^{n-1}_-)=B^1\times\prod_{j=2}^{n-1} I_j$ *if* $n>2$,

*and*

(4) $\phi(\text{Int}\,\tilde{S}^1_-)=\text{Int}\,B^1$ *and* $\phi(\text{Int}\,(\tilde{S}^{n-1}_-))=\text{Int}\,(B^1\times\prod_{j=2}^{n-1} I_j)$ *if* $n>2$.

**Proof.** (1) For $n=2$, this follows from Lemma 6.5(1) and Lemma 6.4(3). Suppose it is true for all $i\leqq k$. $\bar{\phi}_{n+1}(\tilde{B}^{n+1})=\bar{\phi}_n(\phi'_{n+1}(\tilde{B}^{n+1}))$. By Lemma 6.5(1), $\phi'_{n+1}(\tilde{B}^{n+1})=\tilde{B}^n\times I_{n+1}$. Hence

$$\bar{\phi}_{n+1}(\tilde{B}^{n+1}) = \bar{\phi}_n(\tilde{B}^n\times I_{n+1}) = \bar{\phi}_n(\tilde{B}^n)\times I_{n+1} = B^1\times\prod_{j=2}^{n+1} I_j.$$

This completes the induction.

(2) By (1), $\bar{\phi}_n(\tilde{S}^{n-1})=\text{Bd}\,(B^1\times\prod_{j=2}^n)$. Hence we only have to show that for each $x\in\tilde{S}^{n-1}$, $\phi(x)=\bar{\phi}_n(x)$. $x\in\tilde{S}^{n-1}$ implies $x\in\tilde{S}^{k-2}$ for all $k-2\geqq n-1$. Hence by Lemma 6.5(3), $\phi'_k(x)=x$ for all $k\geqq n+1$. Thus $\bar{\phi}_n(x)=\bar{\phi}_n\phi'_{n+1}\cdots\phi'_{n+j}(x)=\bar{\phi}_{n+j}(x)$ for all $j\geqq 1$. This shows $\bar{\phi}_n(x)=\phi(x)$.

(3) Since $\tilde{S}^{n-1}_-\subset\tilde{S}^{n-1}$, by the proof of (2) above $\phi(\tilde{S}^{n-1}_-)=\bar{\phi}_n(\tilde{S}^{n-1}_-)$. Now for $n=2$, $\phi_2(\tilde{S}^1_-)=\tilde{B}^1=B^1$ by Lemma 6.5(2) and Lemma 6.4(3). For $n>2$,

$$\phi(\tilde{S}^{n-1}_-) = \bar{\phi}_n(\tilde{S}^{n-1}_-) = \bar{\phi}_{n-1}(\phi'_n(\tilde{S}^{n-1}_-)) = \bar{\phi}_{n-1}(\tilde{B}^{n-1})$$

by Lemma 6.5(2). By (1), $\phi_{n-1}(B^{n-1})=B^1\times\prod_{j=2}^{n-1} I_j$. Thus $\phi(\tilde{S}^{n-1}_-)=B^1\times\prod_{j=2}^{n-1} I_j$. This completes the proof.

(4) This is the consequence of (3).

6.7 *Homeomorphism* $h_3$. Let $G$ be defined as in §6.4 and $\phi$ as in §6.6. It is easy to see that by means of §6.3(3) and §6.5(4), we have $G(s')\subset s'$ and $\phi(s')\subset s'$ (see §2(0) for definition of $s'$). Furthermore they also imply $\pi_1(x)=0$ iff $\pi_1(G(x))=0$ and $\pi_1(x)=0$ iff $\pi_1(\phi(x))=0$. Let $a=(-1, 0, 0, \ldots)\in s$. By §6.4(3), $G(a)=a$. By the proof of §6.6(2), $\phi(a)=\bar{\phi}_2(a)=\phi'_2(a)=\phi_2(a)$. By §6.5(3) $\phi_2(a)=a$. Hence $\phi(a)=a$. As a consequence of §6.6(3) and (4),

$$\phi\left(\bigcup_{n>1}\tilde{S}^{n-1}_-\right) = \bigcup_{n>1}\left(B^1\times\prod_{j>1}^{n} I_j\right);$$

$$\phi\left(\bigcup_{n>1}\text{Int}\,(\tilde{S}^{n-1}_-)\right) = \text{Int}\,(B^1)\cup\left(\bigcup_{n>2}\text{Int}\left(B^1\times\prod_{j=2}^{n-1} I_j\right)\right).$$

Evidently,

$$S_- = (S_F\cap l_-)\cap s' = \{a\}\cup\left(\left(\bigcup_{n>1}\text{Int}\,(S^{n-1}_-)\right)\cap s'\right).$$

Hence

$$\phi G(S_-) = \{a\} \cup \operatorname{Int}(I_1') \cup \left( \bigcup_{n>2} \operatorname{Int}\left(I_1' \times \prod_{j=2}^{n-1} I_j\right)\right)$$

where $I_1' = [-1, 0]$. Let $T$ be the homeomorphism of $s$ onto $s$ defined by $T(x) = (x_1+1, x_2, x_3, \ldots)$. Obviously

$$T\phi G(S_-) = \{0\} \cup \bigcup_{n \geqq 1}^{\infty} \operatorname{Int}\left(\prod_{j \geqq 1}^{n} I_j\right) = s_p.$$

Thus we have

THEOREM. *$h_3 = T\phi G|_{S_-}$ is a homeomorphism of $S_-$ onto $s_p$.*

7. **Proof of Corollary** I. It is well known that $l_F$ is homeomorphic to its unit sphere $S_F$ [3], [4]. According to §6.7,

$$\phi G(S_F) = \phi\left(\bigcup_{n>1} \tilde{S}^{n-1}\right) = \bigcup_{n>2}^{\infty} \operatorname{Bd}\left(B^1 \times \prod_{j=2}^{n} I_j\right) = \bigcup_{n=2}^{\infty}\left(B^1 \times \prod_{j=2}^{n} I_j\right) \sim \bar{s}_p.$$

By the proof of §5.3, $\bar{s}_p \sim \bar{s}_f$. This completes the proof.

## References

0. C. Bessaga, *Some remarks on homeomorphisms of $\aleph_0$-dimensional linear spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **11** (1963), 159–163.

1. H. Corson and V. Klee, *Topological classification of convex sets*, Proc. Sympos. Pure Math., Vol. 7, Amer. Math. Soc., Providence, R. I. 1963, pp. 37–51.

2. M. Fréchet, *Espaces abstraits*, Gauthier-Villars, Paris, 1928.

3. V. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. **74** (1953), 10–43.

4. ———, *A note on topological properties of normed linear spaces*, Proc. Amer. Math. Soc. **7** (1956), 735–737.

5. V. Klee and R. Long, *On a method of mapping due to Kadeč and Bernstein*, Arch. Math. **8** (1957), 280–285.

6. A. Pełczynski, *Some open problems in "infinite dimensional topology"*, Manuscript, 1966.

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