CONCERNING CELLULAR DECOMPOSITIONS OF 3-MANIFOLDS WITH BOUNDARY

BY

STEVE ARMENTROUT[1]

1. **Introduction.** In [2], we proved that if G is a cellular decomposition of a 3-manifold M such that the associated decomposition space is a 3-manifold N, then M and N are homeomorphic. In this paper we shall establish a related result for 3-manifolds with boundary. We shall show that if G is a cellular decomposition of a 3-manifold with boundary M such that the associated decomposition space is a 3-manifold with boundary N, then M and N are homeomorphic.

The techniques of this paper have applications to the study of embeddings of curves and surfaces in 3-manifolds. Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that the associated decomposition space is a 3-manifold with boundary N. By Theorem 2 of this paper, M and N are homeomorphic. Suppose K is a surface or a curve in M such that no nondegenerate element of G intersects K. If P denotes the projection map from M onto N, then $P|K$ is a homeomorphism. It is natural to ask the following: Is $P[K]$ embedded in N the same way that K is embedded in M? In §5, we give an affirmative answer to this question. In particular, K is tame if and only if $P[K]$ is tame.

In §6 we shall show that if G is a cellular decomposition of a 3-manifold with boundary into a 3-manifold with boundary, then the projection map can be approximated arbitrarily closely by homeomorphisms.

In [2], we established a theorem of basic importance in the study of cellular decompositions of 3-manifolds that yield 3-manifolds as their decomposition spaces. The main result, Theorem 1, of this paper is a useful corollary of the results of [2]. The results mentioned in the preceding two paragraphs are applications of Theorem 1. In [4], we shall apply Theorem 1 to the study of shrinkability conditions which are satisfied by certain cellular decompositions of E^3 that yield E^3 as their decomposition space.

2. **Terminology and notation.** The statement that M is a *3-manifold with boundary* means that M is a separable metric space such that each point of M has a neighborhood in M which is a 3-cell. If M is a 3-manifold with boundary, a point p of M is an *interior point* of M if and only if p has an open neighborhood in M which is an open 3-cell. The *interior* of M, Int M, is the set of all boundary points,

Received by the editors January 25, 1968.

(1) This research was supported in part by the National Science Foundation under Grant No. GP-3857.

and the *boundary* of M, Bd M, is $M-\text{Int } M$. The statement that M is a 3-*manifold* means that M is a 3-manifold with boundary such that Bd M is void.

If M is a 3-manifold with boundary, a set X in M is a *cellular* subset of M if and only if there exists a sequence $C_1, C_2, C_3, \ldots$ of 3-cells in M such that (1) for each positive integer n, $C_{n+1} \subset \text{Int } C_n$, and (2) $X=\bigcap_{i=1}^{\infty} C_i$. Note that a cellular subset of a 3-manifold with boundary M necessarily lies in the interior of M.

If M is a 3-manifold with boundary, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M such that each element of G is a cellular subset of M.

If X is a topological space and G is an upper semicontinuous decomposition of X, then X/G denotes the associated decomposition space, P denotes the projection map from X onto X/G, and H_G denotes the union of all the nondegenerate elements of G.

If A is a set in a topological space X, let βA denote the (topological) boundary of A, and let Cl A denote the closure of A. If X is a metric space, $A \subset X$, and ε is a positive number, then $V(\varepsilon, A)$ denotes the open ε-neighborhood of A.

3. **The main result.** The purpose of this section is to establish the main result of the paper. We shall depend heavily on the following result from [2].

THEOREM 1 OF [2]. *Suppose that M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. Suppose that N has a triangulation T such that if σ is a simplex of T, $P^{-1}[\sigma]$ lies in an open 3-cell U_σ in M. Then there exists a triangulation Σ of M and an isomorphism ϕ from T onto Σ such that for each simplex σ of T, $\phi(\sigma) \subset U_\sigma$.*

THEOREM 1. *Suppose M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose U is an open set in* Int N *such that βU misses $P[H_G]$. Then there is a homeomorphism h from* Cl $P^{-1}[U]$ *onto* Cl U *such that $h|\beta P^{-1}[U]=P|\beta P^{-1}[U]$.*

Proof. We shall apply Theorem 1 of [2], and we have some preliminary steps to take before we can apply Theorem 1 of [2].

For each positive integer k, let A_k be the union of all the sets of G lying in $P^{-1}[U]$ and of diameter at least $1/k$. Then by upper semicontinuity of G, A_k is closed. It follows that there exists a sequence $V_1, V_2, V_3, \ldots$ of open sets in M such that for each positive integer i, V_i contains $\beta P^{-1}[U]$, $\overline{V}_{i+1} \subset V_i$, $\overline{V}_i$ and A_i are disjoint, and $V_i \subset V(1/i, \beta P^{-1}[U])$. There is an open covering $\mathscr{W}$ of $P^{-1}[U]$ such that (1) if $W \in \mathscr{W}$, W is an open 3-cell and $\overline{W} \subset P^{-1}[U]$, (2) if n is any positive integer, W is a set of $\mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam $W) < 1/n$, and (3) if $g \in G$ and $g \subset P^{-1}[U]$, then there is a set W of $\mathscr{W}$ such that $g \subset W$. Such a set $\mathscr{W}$ may be constructed in the following way. If $g \in G$ and $g \subset A_1$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, and W_g and $\overline{V}_1$ are disjoint. If $g \in G$ and $g \subset A_2 - A_1$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, (diam $W_g) < 1$, and W_g and $\overline{V}_2$ are disjoint. If n is any positive integer, $n > 1$, $g \in G$, and $g \subset A_n - A_{n-1}$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, (diam $W_g) < 1/(n-1)$, and W_g and $\overline{V}_n$ are disjoint. Let $\mathscr{W}$ be the collection of all such sets W_g for the elements g of G lying in $P^{-1}[U]$.

We shall show that if n is any positive integer, W is a set of $\mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam $W) < 1/n$. If W intersects $\overline{V}_{n+1}$ and g is any set of G lying in A_{n+1}, then W is distinct from W_g. It follows that (diam $W_g) < 1/n$. The remaining properties of $\mathscr{W}$ are evident.

There is a triangulation T of U such that (1) if n is a positive integer, $\sigma \in T$, and σ intersects $P[V_n]$, then (diam $\sigma) < 1/n$, and (2) if $\sigma \in T$, then $P^{-1}[\sigma]$ lies in some set of $\mathscr{W}$. For each simplex σ of T, let W_σ be some open 3-cell of $\mathscr{W}$ such that $P^{-1}[\sigma] \subset W_\sigma$.

Let G_0 be the set of all elements of G contained in $P^{-1}[U]$. Then G_0 is a cellular decomposition of the 3-manifold $P^{-1}[U]$. By Theorem 1 of [2], there exist a triangulation Σ of $P^{-1}[U]$ and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset W_\sigma$.

Since ϕ^{-1} is an isomorphism from Σ onto T, by the proof of Lemma 8 of [2], there is a homeomorphism h_0 from $P^{-1}[U]$ onto U such that if σ is a simplex of Σ, then $h_0[\sigma] = \phi^{-1}(\sigma)$. Define a function h as follows: (1) If $x \in P^{-1}[U]$, then $h(x) = h_0(x)$. (2) If $x \in \beta P^{-1}[U]$, then $h(x) = P(x)$. Since $P^{-1}[\beta U] = \beta P^{-1}[U]$, and $\beta P^{-1}[U]$ and $P^{-1}[U]$ are disjoint, h is well defined. Clearly h is from Cl $P^{-1}[U]$, and since h_0 is onto U and $P|\beta P^{-1}[U]$ is onto βU, h is onto Cl U. By definition of h, $h|\beta P^{-1}[U] = P|\beta P^{-1}[U]$. In order to complete the proof of Theorem 1, we need only to show that h is a homeomorphism.

Since βU misses $P[H_G]$, P is one-to-one on $\beta P^{-1}[U]$. Since h_0 is one-to-one, it follows that h is one-to-one.

Now we shall prove that h is continuous. Clearly, it is sufficient to show that if $x_1, x_2, x_3, \ldots$ is a sequence of points in $P^{-1}[U]$ and converging to the point x_0 of $\beta P^{-1}[U]$, then $h(x_1), h(x_2), h(x_3), \ldots$ converges to $h(x_0)$, or, in view of the definition of h, to $P(x_0)$.

For each positive integer i, let τ_i be a 3-simplex of Σ containing x_i, and let σ_i be $\phi^{-1}(\tau_i)$. By construction of h, $h(x_i) \in \sigma_i$. Now $x_1, x_2, x_3, \ldots$ converges to x_0 and $x_0 \in \beta P^{-1}[U]$. Further, for each positive integer i, $\tau_i \subset W_{\sigma_i}$. It follows from these facts and properties of $\mathscr{W}$ that (diam τ_1), (diam τ_2), (diam τ_3), ... converges to 0.

Let Q be a neighborhood of $h(x_0)$. Since $h(x_0) = P(x_0)$ and $\{x_0\}$ is an element of G, $P^{-1}[Q]$ is a neighborhood of x_0. From facts mentioned in the preceding paragraph, it follows that there is a positive integer s such that if $n > s$, $\tau_n \subset P^{-1}[Q]$. It follows from facts about the construction of $\mathscr{W}$ that there is a positive integer t greater than s such that if $n > t$, $W_{\sigma_n} \subset P^{-1}[Q]$. Hence if $n > t$, $P[W_{\sigma_n}] \subset Q$. The open covering $\mathscr{W}$ has the property that if $\sigma \in T$, $P^{-1}[\sigma] \subset W_\sigma$. Hence for each positive integer n, $P^{-1}[\sigma_n] \subset W_{\sigma_n}$, and if $n > t$, then both $\sigma_n \subset P[W_{\sigma_n}] \subset Q$ and $h(x_n) \in Q$. It follows that $h(x_1), h(x_2), h(x_3), \ldots$ converges to $h(x_0)$. Consequently h is continuous.

Now we shall prove that h^{-1} is continuous. It is sufficient to show that if

$y_1, y_2, y_3, \ldots$ is a sequence of points in U converging to the point y_0 of βU, then $h^{-1}(y_1), h^{-1}(y_2), h^{-1}(y_3), \ldots$ converges to $h^{-1}(y_0)$, or equivalently, to $P^{-1}[y_0]$.

For each positive integer i, let σ_i be a 3-simplex of T containing y_i, and let τ_i be $\phi(\sigma_i)$. By construction of h, for each i, $h^{-1}(y_i) \in \tau_i$.

Suppose R is a neighborhood of $P^{-1}[y_0]$. Since $\{P^{-1}[y_0]\}$ is an element of G, there is a neighborhood R_0 of $P^{-1}[y_0]$ such that $R_0 \subset R$, R_0 is a union of elements of G, and if $W \in \mathscr{W}$ and W intersects R_0, then $W \subset R$. Notice that $P[R_0]$ is a neighborhood in N of y_0.

There is a positive integer t such that if $n > t$, $y_n \in P[R_0]$. If $n > t$, $P^{-1}[y_n] \in R_0$. Since for each i, $P^{-1}[\sigma_i] \subset W_{\sigma_i}$ and $y_i \in \sigma_i$, then if $i > t$, W_{σ_i} intersects R_0 and hence lies in R. By construction of Σ, it follows that for each i, $\tau_i \subset W_{\sigma_i}$, and hence if $i > t$, $\tau_i \subset R$ and therefore $h^{-1}(y_i) \in R$. It follows that $h^{-1}(y_1), h^{-1}(y_2), h^{-1}(y_3), \ldots$ converges to $P^{-1}[y_0]$, or to $h^{-1}(y_0)$. Hence h^{-1} is continuous.

Therefore h is a homeomorphism from Cl $P^{-1}[U]$ onto Cl U such that $h|\beta P^{-1}[U] = P|\beta P^{-1}[U]$. This establishes Theorem 1.

4. **Application to 3-manifolds with boundary.** We are now prepared to extend Theorem 2 of [2] to the case of cellular decompositions of 3-manifolds with boundary.

THEOREM 2. *Suppose M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Then there is a homeomorphism h from M onto N such that $h|$Bd $M = P|$Bd M.*

Proof. Int N is an open subset of N lying in Int N, and β(Int N) = Bd N. Further, P^{-1}[Int N] = Int M, and $P^{-1}[\beta$(Int N)] = Bd M. We also have that Cl Int $M = M$ and Cl Int $N = N$. With the aid of Theorem 1, it follows that there exists a homeomorphism h from M onto N such that $h|$Bd $M = P|$Bd M.

5. **Applications to embeddings.** In this section we establish some results concerning embeddings of surfaces and curves in manifolds. Our first result is a slightly more general theorem.

THEOREM 3. *Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose that K is a closed nowhere dense subset of M such that K and H_G are disjoint. Then there is a homeomorphism h from M onto N such that $h|K = P|K$.*

Proof. This follows by applying Theorem 1 to the open subset (Int M) $- K$ of M. Since K is nowhere dense in M, Cl [(Int M) $- K$] $= M$, and it follows that there exists a homeomorphism h from M onto N such that $h|(K \cup$ Bd $M) = P|(K \cup$ Bd $M)$. In particular, $h|K = P|K$.

COROLLARY 1. *Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose K is a* *manifold with boundary, of dimension* 1 *or* 2, *contained in* M, *and missing* H_G. *Then there is a homeomorphism* h *from* M *onto* N *such that* $h|K=P|K$.

The next two corollaries may be regarded as extensions of Theorem 1 of [1].

COROLLARY 2. *Suppose that* M *is a* 3*-manifold with boundary and* G *is a cellular decomposition of* M *such that* M/G *is a* 3*-manifold with boundary. Suppose* K *is a* 2*-manifold with boundary in* M *such that* K *misses* H_G. *Then* $P[K]$ *is tame in* N *if and only if* K *is tame in* M.

A compact connected set X in E^3 is *pointlike* in E^3 if and only if E^3-X is homeomorphic to $E^3-\{0\}$. It is well known (see [6]) that in E^3, "pointlike" and "cellular" are equivalent. By a *pointlike decomposition* of E^3 is meant an upper semicontinuous decomposition of E^3 into pointlike compact connected sets.

COROLLARY 3. *Suppose that* G *is a pointlike decomposition of* E^3 *such that* E^3/G *is homeomorphic to* E^3. *Suppose* K *is a* 2*-manifold in* E^3 *such that* K *misses* H_G. *Then* $P[K]$ *is tame if and only if* K *is tame.*

COROLLARY 4. *Suppose that* G *is a pointlike decomposition of* E^3 *such that* E^3/G *is homeomorphic to* E^3. *Suppose that* J *is an arc or a simple closed curve in* E^3 *such that* J *misses* H_G. *Then* $P[J]$ *is tame if and only if* J *is tame.*

It follows from results of [3] and [5] that under the hypothesis of Corollary 4, if J is a simple closed curve, $\pi_1(E^3-J)$ and $\pi_1(E^3-P[J])$ are isomorphic. Corollary 1 gives a considerably stronger result in this case.

6. Approximating the projection map.

The result of this section shows that if G is a cellular decomposition of a 3-manifold with boundary into a 3-manifold with boundary, then the projection map can be approximated arbitrarily closely by homeomorphisms. D. R. McMillan, Jr. raised the question as to whether such approximations are possible.

THEOREM 4. *Suppose that* M *is a* 3*-manifold with boundary and* G *is a cellular decomposition of* M *such that* M/G *is a* 3*-manifold with boundary* N. *Suppose* U *is an open set in* Int M *containing* H_G *and* ε *is a positive number. Then there exists a homeomorphism* h *from* M *onto* N *such that* (1) *if* $x\in M-U$, $h(x)=P(x)$ *and* (2) *if* $x\in M$, $d(h(x), P(x))<\varepsilon$.

Proof. The proof of this theorem is a modification of the proof of Theorem 1 above. Let $\mathscr{A}$ be an open covering of N by sets of diameter less than $\varepsilon/2$. Let $V_1, V_2, V_3, \dots$ be as in the proof of Theorem 1. There is an open covering $\mathscr{W}$ of U such that (1) if $W\in\mathscr{W}$, W is an open 3-cell and $\overline{W}\subset U$, (2) if n is any positive integer, $W\in\mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam $W)<1/n$, (3) if $g\in G$ and $g\subset P^{-1}[U]$, then g lies in some set of $\mathscr{W}$, and (4) if $W\in\mathscr{W}$, there is a set A of $\mathscr{A}$ such that $P[W]\subset A$.

Since $H_G \subset U$, $P[U]$ is open in N. There is a triangulation T of $P[U]$ such that (1) if n is any positive integer, $\sigma \in T$, and σ intersects $P[V_n]$, then $(\text{diam } \sigma) < 1/n$, and (2) if $\sigma \in T$, then $P^{-1}[\sigma]$ lies in some set of $\mathscr{W}$. For each simplex σ of T, let W_σ be some open 3-cell of $\mathscr{W}$ such that $P^{-1}[\sigma] \subset W_\sigma$.

Let G_0 be the set of all elements of G contained in U. Then G_0 is a cellular decomposition of the 3-manifold U. By Theorem 1 of [2], there exist a triangulation Σ of U and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset W_\sigma$. By the proof of Lemma 8 of [2], there is a homeomorphism h_0 from U onto $P[U]$ such that if $\sigma \in \Sigma$, $h_0[\sigma] = \phi^{-1}(\sigma)$. Define a function h as follows: (1) If $x \in M - U$, $h(x) = P(x)$. (2) If $x \in U$, $h(x) = h_0(x)$. As in the proof of Theorem 1, we may show that h is a homeomorphism from M onto N. By definition, if $x \in M - U$, $h(x) = P(x)$.

We shall show now that if $x \in M$, $d(h(x), P(x)) < \varepsilon$. If $x \in M - U$, clearly $d(h(x), P(x)) < \varepsilon$. Suppose $x \in U$. Let σ be a 3-simplex of T containing $P(x)$, and let τ be a 3-simplex of T containing $h(x)$. We shall prove that $P[W_\sigma]$ and $P[W_\tau]$ intersect. First, since $P(x) \in \sigma$, $x \in P^{-1}[\sigma]$, and since $P^{-1}[\sigma] \subset W_\sigma$, then $x \in W_\sigma$. Second, since $h(x) \in \tau$, then by the way h is defined, $x \in \phi(\tau)$. Since $\phi(\tau) \subset W_\tau$, then $x \in W_\tau$. Hence $x \in W_\sigma \cap W_\tau$, and thus $P[W_\sigma]$ and $P[W_\tau]$ intersect.

By construction of $\mathscr{W}$, if $W \in \mathscr{W}$, then for some set A of $\mathscr{A}$, $P[W] \subset A$ and so $(\text{diam } P[W]) < \varepsilon/2$. Since $x \in W_\sigma$, then $P(x) \in P[W_\sigma]$. Since $P^{-1}[\tau] \subset W_\tau$, then $\tau \subset P[W_\tau]$; since $h(x) \in \tau$, $h(x) \in P[W_\tau]$. It follows that $d(h(x), P(x)) < \varepsilon$. This completes the proof of Theorem 4.

References

1. S. Armentrout, *Upper semicontinuous decompositions of E^3 with at most countably many nondegenerate elements*, Ann. of Math. **78** (1963), 605–618.

2. ———, *Cellular decompositions of* 3-*manifolds that yield* 3-*manifolds*, (to appear).

3. ———, *Homotopy properties of decomposition spaces*, (to appear).

4. ———, *Shrinkability of certain decompositions of E^3 that yield E^3*, Illinois J. Math. (to appear).

5. S. Armentrout and T. M. Price, *Decompositions into compact sets with UV properties*, (to appear).

6. D. G. Stewart, *Cellular subsets of the* 3-*sphere*, Trans. Amer. Math. Soc. **114** (1965), 10–22.

University of Iowa,
Iowa City, Iowa
University of Wisconsin,
Madison, Wisconsin