ON HANDLE DECOMPOSITIONS AND DIFFEOMORPHISMS

BY

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In this paper we are concerned with two questions which arise naturally in the study of manifolds. One has to do with whether certain types of handle decompositions of a manifold are unique; and the other, under what circumstances they are preserved by diffeomorphisms. The corollaries to Theorem 1 and 2 are partial answers to these questions. Corollary 2 may prove useful for the general study of diffeomorphisms. We shall begin with some notation.

$\mathbf{R}^k$ shall denote real k-dimensional vector space with unit disc D^k and unit sphere S^{k-1}. Let $I=[0, 1]$. All manifolds are assumed to be compact and C^∞. Homology groups are taken with integer coefficients. $\approx$ means either diffeomorphic or isotopic. By a *nice Morse function* on a manifold M^n (see [5, p. 44]) we shall mean a function $\eta\colon M^n \to \mathbf{R}$ which has only nondegenerate critical points, and if p is a critical point, then $\eta(p)=\text{index } p$. It is well known that in this case $M_k(\eta)\equiv\eta^{-1}[0, k+1/2]$ is a n-submanifold of M for $0\leqq k\leqq n$, and

$$M_k(\eta) = M_{k-1}(\eta) \cup_{\phi_1} (D^k\times D^{n-k}) \cup\cdots\cup_{\phi_p} (D^k\times D^{n-k}),$$

where $\phi_j\colon \partial D^k\times D^{n-k}\to\eta^{-1}[k-1/2]$ are imbeddings. Call η *minimal* if the number of critical points of index k is equal to $p(k)+q(k)+q(k-1)$, where $p(k)$ is the torsion-free rank of H_kM and $q(k)$ is the rank of the torsion subgroup of H_kM (see [6, §6]).

From now on M^n will be a closed simply-connected manifold with $n\geqq 6$.

THEOREM 1. *Let η_j be minimal nice Morse functions on M, $j=1, 2$. Assume that $H_{i_s}M$ is torsion-free for $0=i_1<i_2<\cdots<i_{t-1}<i_t=n$. Then there is a diffeomorphism $H\colon M\to M$ satisfying*

(i) *$H\approx$ identity,*

(ii) *$H(M_{i_s}(\eta_1))=M_{i_s}(\eta_2)$, for $1\leqq s\leqq t$.*

THEOREM 2. *Let $F\colon M\to M$ be a diffeomorphism and η_1 any minimal nice Morse function on M. Assume that $F_*|\text{torsion }(H_{i_s}M)=\text{identity}$ for $0=i_1<i_2<\cdots<i_{t-1}<i_t=n$. Then F is isotopic to a diffeomorphism F' such that $F'(M_{i_s}(\eta_1))=M_{i_s}(\eta_1)$, for $1\leqq s\leqq t$.*

REMARK. The existence of minimal nice Morse functions on M is guaranteed by [6, §6].

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As an immediate consequence of Theorems 1 and 2 we get the following corollaries.

COROLLARY 1. *Assume that H_*M is torsion-free. Let η_j be minimal nice Morse functions on M, $j = 1, 2$. Then $M_i(\eta_1) \approx M_i(\eta_2)$, for $0 \leqq i \leqq n$ (i.e., minimal handle decompositions of M are unique up to diffeomorphisms).*

COROLLARY 2. *Let $F: M \to M$ be a diffeomorphism such that $F_* | \text{torsion } H_*M = \text{identity}$. Let η be a minimal nice Morse function on M. Then $F \approx F'$, where $F'(M_i(\eta)) = M_i(\eta)$, for $0 \leqq i \leqq n$ (i.e., up to isotopy, most diffeomorphisms of M are "level-preserving").*

In order to prove Theorems 1 and 2 we begin with a lemma.

LEMMA. *Let η_1 and η_2 be minimal nice Morse functions on M. Define $M_i^j \equiv \eta_j^{-1}[0, i+1/2]$ and suppose that $M_k^1 = M_k^2 = A$ for some fixed k, $0 \leqq k < n$. Let $k < l < n-2$.*

(a) *If H_lM is torsion-free, then there is a diffeomorphism H of M such that*

(i) *$H \approx \text{identity}$,*

(ii) *$H | A = \text{identity}$,*

(iii) *$H(M_l^1) = M_l^2$.*

(b) *Assume $\eta_2 = \eta_1 F^{-1}$, where F is a diffeomorphism of M with $F_* | \text{torsion } (H_lM) = \text{identity}$. Then there is a diffeomorphism H of M satisfying* (i)–(iii) *in part* (a).

Proof. (a) The following easy general position argument shows that there is a diffeomorphism h_1 of M satisfying (i), (ii) and $h_1(M_l^2) \subset \text{int}\,(M_l^1) \equiv \text{interior of } M_l^1$: If $M_l^2 \subset \text{int}\, M_q^1$, for $q > l$, write $M_q^1 = M_{q-1}^1 \cup_{\psi_1} (D^q \times D^{n-q}) \cup \cdots \cup_{\psi_p} (D^q \times D^{n-q})$, with respect to imbeddings $\psi_j: \partial D^q \times D^{n-q} \to \partial M_{q-1}^1$. Now as a cell complex M_l^2 has dimension at most l, so that $q > l$ implies that we may deform M_l^2 so that $M_l^2 \cap (0 \times D^{n-q})_j = \varnothing$; but once we have this, it is easy to push M_l^2 into int M_{q-1}^1. Repeating this process for each $q > l$ we get h_1.

Let $B \equiv h_1(M_l^2)$ and let $f: B \to M$ be the inclusion map. Consider

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{j+1}(M, B) & \xrightarrow{\partial_{j+1}} & H_jB & \longrightarrow & H_jM & \longrightarrow & H_j(M, B) & \longrightarrow \cdots \\
& \downarrow & & \downarrow f_* & & \downarrow \text{identity} & & \downarrow & \\
\cdots \longrightarrow & H_{j+1}(M, M_l^1) & \xrightarrow{\partial'_{j+1}} & H_jM_l^1 & \longrightarrow & H_jM & \longrightarrow & H_j(M, M_l^1) & \longrightarrow \cdots
\end{array}$$

Since $H_j(M, B) = 0 = H_j(M, M_l^1)$ for $j \leqq l$, f_* is an isomorphism for $j < l$. But $\partial_{l+1} = 0 = \partial'_{l+1}$, because H_lM is torsion-free. Thus f_* is an isomorphism for all j. By the Whitehead theorem (see [3, Theorem 10.1]), f is a homotopy equivalence.

CLAIM. $\pi_1 Y = \pi_1\, \partial M_l^1 = \pi_1\, \partial B = 0$, where $Y = \text{closure of } M_l^1 - B$.

Assuming this for the moment, we get by [5, Theorem 9.1] that $Y \approx \partial B \times I$. Thus, using the I-coordinate it should be clear how to define a diffeomorphism h_2 of M satisfying (i), (ii) and $h_2(B) = M_l^1$. Let $H = h_2 h_1$. Then H is the desired map.

To prove the claim, let $g\colon S^1 \to Y$. Since $\pi_1 M_l^1 = 0 = \pi_1 M$, there is a $\tilde{g}\colon D^2 \to M_l^1$ with $\tilde{g}|S^1 = g$. But $l+2<n$ and so we can push $\tilde{g}$ into Y by a general position argument similar to the one at the beginning of this proof. Therefore, $g \simeq 0$ in Y, i.e., $\pi_1 Y = 0$. To see that $\pi_1\,\partial M_l^1 = 0$ we use induction ($\pi_1\,\partial B = 0$ by an analogous argument). Clearly, $\pi_1\,\partial M_0^1 = \pi_1\,\partial M_1^1 = \pi_1 S^{n-1} = 0$, because $n>2$ and $\pi_1 M = 0$. Now ∂M_j^1 is obtained from ∂M_{j-1}^1 by cutting out a certain number of disjoint $S^{j-1}\times D^{n-j}$ and adding an equal number of $D^j \times S^{n-j-1}$. If $\pi_1\,\partial M_{j-1}^1 = 0$, for $1<j<n$, then Van Kampen's theorem [2, Theorem 6.4.3] shows that $\pi_1\,\partial M_j^1 = 0$. This proves that $\pi_1\,\partial M_j^1 = 0$ for $0 \leqq j \leqq n$, and finishes the proof of part (a).

(b) The method of part (a) breaks down only at one point; namely, we do not know that $\partial_{l+1} = 0 = \partial'_{l+1}$. Hence we must give a special argument to make f into a homotopy equivalence. Specifically, assume that $F(M_l^1) \subset \operatorname{int} M_l^1$ and let $f \equiv F \mid M_l^1 : M_l^1 \to M_l^1$. Again, f_* is an isomorphism for $j<l$. Consider

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_{l+1}(M, M_l^1) & \xrightarrow{\partial_{l+1}} & H_l M_l^1 & \xrightarrow{i_*} & H_l M & \longrightarrow 0 \\
& \downarrow F_* & & \downarrow f_* & & \downarrow F_* & \\
\cdots \longrightarrow & H_{l+1}(M, M_l^1) & \xrightarrow{\partial_{l+1}} & H_l M_l^1 & \xrightarrow{i_*} & H_l M & \longrightarrow 0.
\end{array}$$

Write

$$H_l M_l^1 = \oplus \sum_{s=1}^{p(l)} \{x_s\} \oplus \sum_{s=1}^{q(l)} \{y_s\},$$

$$H_l M = \oplus \sum_{s=1}^{p(l)} \{x'_s\} \oplus \sum_{s=1}^{q(l)} \{y'_s\}, \qquad k_s \equiv \text{order of } y'_s.$$

Note that $\ker f_* \subset \ker i_* \subset \sum_{s=1}^{q(l)} \{y_s\}$ and $f_*(\sum_{s=1}^{q(l)} \{y_s\}) \subset \sum_{s=1}^{q(l)} \{y_s\}$. Let $f_* y_s = \sum_{j=1}^{q(l)} c_{sj} y_j$, and set $C \equiv (c_{sj})$. We may assume without loss of generality that $i_* x_s = x'_s$ and $i_* y_s = y'_s$. Then our hypothesis shows that $y'_s = i_* y_s = F_* i_* y_s = i_* f_* y_s = i_* \sum_{j=1}^{q(l)} c_{sj} y_j = \sum_{j=1}^{q(l)} c_{sj} y'_j$, i.e., $c_{sj} \equiv \delta_{sj}(k_j)$. Our task will be to modify F so that $c_{sj} = \delta_{sj}$.

Put $c_{sj} = \delta_{sj} + a_{sj} k_j$, and let

$$M_j^1 = M_{j-1}^1 \cup_{\psi_{1,j}} (D^j \times D^{n-j}) \cup \cdots \cup_{\psi_{t_j,j}} (D^j \times D^{n-j}),$$

$$H_j(M_j^1, M_{j-1}^1) = \oplus \sum_{s=1}^{t_j} \{z_s^j\},$$

where $t_j = p(j) + q(j) + q(j-1)$ and z_s^j corresponds to $(D^j \times 0)_{\psi_{s,j}}$. Consider

$$\begin{array}{ccc}
H_{l+1}(M_{l+1}^1, M_l^1) & \longrightarrow & H_l(M_l^1, M_{l-1}^1) \\
& \partial' \searrow \quad \nearrow i'_* & \\
& H_l M_l^1 &
\end{array}$$

We may assume (see [6, §6]) that $i'_* x_s = z_s^l$, $i'_* y_s = z_{s+p(l)}^l$, and $\partial' z_{s+p(l+1)+q(l+1)}^{l+1} = k_s y_s$.

Let

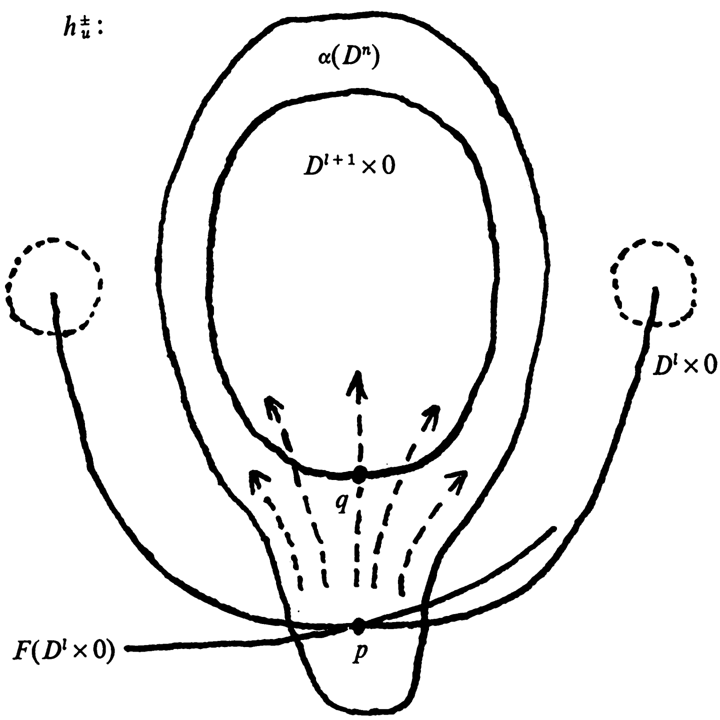
$$T^{s,t} \equiv F[(D^l \times 0)_{\psi_{s+p(l),l}}] \cap (0 \times D^{n-l})_{\psi_{t+p(l),l}},$$

$$T_0^{s,t} \equiv (\partial D^{l+1} \times 0)_{\psi_{s+p(l+1)+q(l+1),l+1}} \cap (0 \times \partial D^{n-l})_{\psi_{t+p(l),l}}.$$

By a general position argument we can make $T^{s,t}$ and $T_0^{s,t}$ consist of only a finite number of points. In fact, using methods of Whitney (e.g., see [1, §7]), we may suppose that $|T^{s,t}| = |c_{st}|$, $|T_0^{s,s}| = k_s$, and $T_0^{s,t} = \varnothing$, for $s \neq t$ ($|X|$ denotes the cardinality of X).

We shall now describe isotopies $h_u^{\pm}$, with $h_0^{\pm} = \text{identity} = h_u^{\pm} | A$, so that if we let $F' = h_1^{\pm} F$ and define $C' = (c'_{st})$ similarly to C, then $c'_{s_1 t_1} = c_{s_1 t_1}$, for $s_1 \neq s$, $t_1 \neq t$, and $c'_{st} = c_{st} \pm k_t$. Using such isotopies it should be clear how to deform F so that $c_{st} = \delta_{st}$.

THE CONSTRUCTION OF $h_u^{\pm}$. Let $p \in T^{s,t}$ and $q \in T_0^{t,t}$. Let $\alpha: D^n \to M^n$ be an



imbedding with the following properties:

(1) $(D^{l+1} \times 0)_{\psi_{t+p(l+1)+q(l+1),l+1}} \subset \text{int}\, \alpha(D_0^{l+1})$, where

$$D_0^{l+1} \equiv \{(x_1, \ldots, x_{l+1}) \in D^{l+1} \mid x_{l+1} \geqq 0\},$$

(2) $\alpha(D^{l+1}) \cap F[(D^l \times 0)_{\psi_{s+p(l),l}}] = \alpha(D^l)$,

(3) If $U \equiv F(M_l^1) \cap \alpha(D^n)$, then U is a ball neighborhood of p with $U \subset (D^l \times D^{n-l})_{\psi_{t+p(l),l}}$ and $U \cap T^{s,t} = \{p\}$,

(4) $\alpha(D^n) \cap (D^l \times D^{n-l})_{\psi_{j+p(l)},l} = \varnothing$, for $j \neq t$,

(5) $\alpha(\partial D^{l+1}) \subset \text{int } M_l^1$.

h_u^+ is then the isotopy which pushes $\alpha(D^l)$ to $\alpha(D_+^l)$, where $D_+^l \equiv D_0^{l+1} \cap S^l$. h_u^- is defined in essentially the same manner, except that one must give $\alpha(D^l)$ first a small twist in order to change the orientation.

Thus, assume now that $c_{st} = \delta_{st}$. By previous observations we get that f_* is a monomorphism. A little diagram chasing shows that it is also onto, since it is already onto $\sum_{s=1}^{q(l)} \{y_s\}$. Therefore f_* is an isomorphism and f is a homotopy equivalence. The rest of the proof goes as in part (a). This finishes the lemma. □

We are now ready to prove our theorems, keeping the notation of the lemma.

Proof of Theorem 1. By [4] we can find a diffeomorphism H_{i_1} of M isotopic to the identity and $H_{i_1}(M_0^1) = M_0^2$. Suppose inductively that we have defined a sequence of diffeomorphisms, $\{H_{i_s} \mid s = 1, 2, \ldots, k\}$, of M satisfying

(1) $H_{i_s} \approx H_{i_{s-1}}$,

(2) $H_{i_s} \mid M_{i_{s-1}}^1 = H_{i_{s-1}} \mid M_{i_{s-1}}^1$,

(3) $H_{i_s}(M_{i_s}^1) = M_{i_s}^2$.

Let $\eta_3 = \eta_1 H_{i_k}^{-1}$ and $M_i^3 \equiv \eta_3^{-1}[0, i+1/2]$. Then $M_{i_k}^3 = M_{i_k}^2$. Assume that $1 \leq i_k < i_{k+1} < n-2$. Applying part (a) of the lemma to η_3 and η_2 we obtain a diffeomorphism h of M with $h \approx$ identity, $h \mid M_{i_k}^2 =$ identity, and $h(M_{i_{k+1}}^3) = M_{i_{k+1}}^2$. Define $H_{i_{k+1}} \equiv hH_{i_k}$. It is easily checked that $H_{i_{k+1}}$ satisfies (1)–(3).

If $n-2 \leq i_{k+1} \leq n$, let $\eta_j^* \equiv n - \eta_j$ and $M_i^{j*} \equiv \eta_j^{*-1}[0, i+1/2]$. (Observe that η_j^* is also a minimal nice Morse function.) Using [4] we can find a diffeomorphism h_1 of M such that $h_1 \approx$ identity, $h_1 \mid M_{i_k}^2 =$ identity, and $h_1(M_0^{3*}) = M_0^{2*}$. If $i_{k+1} = n-1$, define $H_{i_{k+1}} = h_1 H_{i_k}$. If $i_{k+1} = n-2$, apply part (a) of the lemma to $\eta_3^* h_1^{-1}$ and η_2^* with $l = 1$ to get a diffeomorphism h_2 of M with the property that $h_2 \approx$ identity, $h_2 \mid M_0^{2*} =$ identity, and $h_2(h_1(M_1^{3*})) = M_1^{2*}$. Furthermore, the proof of the lemma shows that we may suppose that $h_2 \mid M_{i_k}^2 =$ identity. Define $H_{i_{k+1}} = h_2 h_1 H_{i_k}$. Finally, if $i_{k+1} = n$, define $H_{i_{k+1}} = H_{i_k}$. In all cases, $H_{i_{k+1}}$ satisfies (1)–(3). This finishes our inductive definition of the H_{i_s} and $H \equiv H_n$ satisfies the conclusion of the theorem. □

Proof of Theorem 2. Let $\eta_2 \equiv \eta_1 F^{-1}$ and apply part (b) of the lemma in the proof of Theorem 1 instead of part (a) to obtain a diffeomorphism $H: M \to M$ such that $H \approx$ identity and $H(M_{i_s}^1) = M_{i_s}^2$. Let $F' = H^{-1}F$. □

At first glance, one might be led to conjecture that the condition, $F_* \mid \text{torsion } H_l M = \text{identity}$, in the lemma is unnecessary; however, the following example shows that it is not:

Let $M^8 \equiv$ double of $(S^2 \times D^6 \cup_\alpha (D^3 \times D^5))$, where $\alpha: S^2 \times D^5 \to S^2 \times S^5$ is an imbedding with $[\alpha(S^2 \times 0)] = 5[S^2 \times 0] \in H_2(S^2 \times D^6)$. Let $\eta: M \to \boldsymbol{R}$ be a minimal nice Morse function such that $M_2 = S^2 \times D^6$ and $M_3 = S^2 \times D^6 \cup_\alpha (D^3 \times D^5)$, where $M_i \equiv M_i(\eta)$. Construct an imbedding $h: M_2 \to M_2$ such that $[h_2(S^2 \times 0)] = 3[S^2 \times 0] \in H_2 M_2$. One can extend h to an imbedding $M_2 \cup_\alpha (D^3 \times 0) \to M_3$. But $\pi_2 SO_5 = 0$, and so we can thicken the handle to get an imbedding $h: M_3 \to M_3$.

Now $H_2M_3 \approx H_2M \approx (a)$, where a has order 5, and $h_*(a) = 3a$ by construction. Therefore h is a homotopy equivalence and we may assume h is a diffeomorphism of M_3. Let $F: M \rightarrow M$ be the double of h. Then $F_* \mid H_2M \neq$ identity. However, if $F \approx F'$ and $F'(M_2) = M_2$, then we must have $F_* \mid H_2M = F'_* \mid H_2M = \pm$ identity because $H_2M_2 \approx Z$. Thus (M, F) is the example that we wanted. In fact, this also shows that Theorem 1 is the best possible (set $\eta_1 = \eta$ and $\eta_2 = \eta F$).

Finally, although we just saw that Theorem 1 is the best possible, this still leaves open the interesting question whether, despite this, the manifolds $M_i(\eta_1)$ and $M_i(\eta_2)$ are diffeomorphic anyway.

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