IDEALS IN CHEVALLEY ALGEBRAS

BY
JAMES F. HURLEY[1]

1. **Introduction.** In Chevalley's fundamental paper [2], a procedure is given for obtaining a very special canonical basis for a finite dimensional split semisimple Lie algebra L over a field of characteristic zero. With respect to this basis, all the constants of structure of L turn out to be integers, thus enabling the original field to be replaced by an arbitrary field. In this paper we replace the original field by an arbitrary commutative ring R with identity, and we call the resulting structure the Chevalley algebra L_R of L over R. Our attention focuses on the structure of L_R and we obtain theorems characterizing its ideals, under the assumption that 2 and 3 are not zero divisors in R.

The main result is that for simple algebras L a necessary and sufficient condition for every ideal in L_R to arise from an ideal of R is that the Cartan matrix of L be invertible over R and that the integer m be invertible in R, where m is defined as the ratio of the square of the length of any long root of L to the square of the length of any short root of L. Along the way toward the proof we obtain results characterizing the ideals of L_R when that condition does not necessarily hold in R. A special case of note occurs when R is a field of finite characteristic not equal to 2 or 3. Then our main theorem contains certain results of Dieudonné [3] for exceptional algebras, as well as implicit results alluded to in that paper for the classical algebras.

In §2 we give definitions of the principal concepts and list for future reference certain computational properties. (For a detailed discussion of the structure theory of semisimple Lie algebras, the reader is referred to [4], [6], or [7]. An excellent summary of the basic results used here can be found in [1].) §3 contains statements of the principal theorems. In §4 and §5 the general results needed for the sufficiency part of the main theorem are proved, including the decomposition lemmas which are also used in proving necessity. §§6, 7, and 8 contain discussions of the general structure of L_R for algebras with one root length, nonsymplectic algebras with two root lengths, and symplectic algebras respectively, as well as the proof of the necessity part of the main theorem for those respective cases. In §9 we give a general theorem on generators of ideals in L_R together with some consequences for principal ideal rings R. We note in §8 that our main theorem specialized to the algebra of

Presented to the Society, January 25, 1967 under the title *Scalar replacement in Lie algebras*; received by the editors April 15, 1967 and, in revised form, February 8, 1968.

(1) This research was supported by a National Science Foundation Fellowship. The results published here are contained in the author's doctoral dissertation submitted to the University of California, Los Angeles, in 1966. The author wishes to express his gratitude to Professor Robert Steinberg for his guidance throughout the course of this research.

type C_n yields a recent result of W. Jehne [5] which he obtained using the concrete model of C_n as a Lie algebra of matrices. The abstract approach we employ throughout is that of [2] and [9].

2. **The Chevalley algebras.** Let L be a finite dimensional split semisimple Lie algebra over a field of characteristic zero. Let H be an n-dimensional Cartan subalgebra, Σ the (ordered) set of nonzero roots determined by H, and Π the set of simple roots. If r and s are nonzero roots, then we define $c(r, s)$ to be $2(r, s)/(s, s)$, where the inner product is that derived from the Killing form of L in the usual way. We define p_{rs} to be 0 if $r+s$ is not a root and to be the greatest integer i such that $s-ir$ is a root in case $r+s$ is a root. We define $N_{rs}=p_{rs}=0$ if $r+s$ is not a root, and $N_{rs}=p_{rs}+1$ if $r+s$ is a root. By the length of a root r, we simply mean $(r, r)^{1/2}$. This is a departure from the terminology of Chevalley [2, p. 17]. It is known (see [7, p. V–9]) that if two roots r and s of a simple algebra L have unequal lengths, then the lengths are in the ratio $\sqrt{3}$ to 1 or $\sqrt{2}$ to 1.

Chevalley [2, Theorem 1] constructs a basis for L consisting of certain elements e_r from each one-dimensional root space L_r as r varies through Σ, together with n elements $h_1, h_2, \ldots, h_n$ of H suitably obtained from the corresponding simple roots $r_1, \ldots, r_n$ in Π. This basis has the property that if h_r is obtained from r in Σ in the same way that h_i is obtained from r_i in Π, then h_r is an integral linear combination of the h_i's. With respect to this basis, the equations of structure of L are:

(1) $e_r e_{-r}=h_r$,

(2) $h_i h_j=0$,

(3) $e_r e_s= \pm N_{rs} e_{r+s}$ if $r+s \in \Sigma$ or if $r+s$ is not a root,

(4) $h_r e_s=c(s, r) e_s$.

These equations reveal the premier property of a Chevalley basis, namely, the product of any two of its elements is an integral linear combination of basis elements. Now let us define L_R to be the R-module generated by the elements of a fixed Chevalley basis of L, if R is any commutative ring with identity. Equations (1) through (4) show that L_R is closed under multiplication, as we need only interpret the integers of structure as integers in R. We call L_R with this multiplication the Chevalley algebra of L over R, and it is our purpose here to study its structure. We mention that in [2] the specific Chevalley algebra L_Z over the ring of integers arises for the first time. Before proceeding with our study of L_R, we give the following result which enables us to reduce our considerations to Chevalley algebras arising from simple algebras.

2.1. *If L decomposes into the direct sum of simple ideals $I_1, I_2, \ldots, I_k$, then L_R decomposes into the direct sum of the Chevalley algebras generated over R by $I_1, I_2, \ldots, I_k$.*

Proof. Clearly L_R contains this direct sum. Furthermore, Σ can be partitioned into k mutually disjoint orthogonal systems comprising the nonzero roots of the simple algebras I_j [4, Theorem 4.4]. Thus the Chevalley basis breaks into k corresponding disjoint pieces. Hence forming the direct sum of the Chevalley algebras generated by these pieces yields all of L_R.

It is apparent that the Chevalley algebras generated by the I_j are still ideals in L_R. But they need no longer be simple, as we see by considering the ideal generated in L_Z by $2e_1$ if L is simple. (We denote by e_i the Chevalley basis element chosen from the root space of the simple root r_i.)

In view of 2.1 we shall henceforth consider only simple algebras L. In this case it is known that L has at most two root lengths [7, p. V–9], so in the sequel we shall feel free to refer to "short roots" or "long roots" of L.

The example just considered indicates that the invertibility of certain integers will be of great importance in studying the structure of L_R, and indeed the integers we shall need to study most closely are N_{rs} and $c(r, s)$. We take the opportunity now to list a series of elementary computational properties of N_{rs} in the various simple algebras, properties we shall need to make reference to later. Concrete realizations of the root systems of the simple Lie algebras may be found in [4, pp. 135–145] or in [7, pp. V–27 to V–30]. These realizations make possible the identifications given below between certain subsets of root systems of one type and entire systems of another type. In (2.2)–(2.10) we suppose that r and s are roots such that $r+s\in\Sigma$.

2.2. If L is of one of the types A_n, D_n, E_6, E_7, or E_8, then since r and s are not orthogonal, they must be part of a system of type A_2. Since $r+s$ is a root, $s-r$ is not and so $N_{rs}=1$.

2.3. If r and s are long and L is of type B_n, $n\geqq 3$, or F_4, then since the long roots constitute a system of type D_n, $n\geqq 3$, or D_4 respectively, we have $N_{rs}=1$.

2.4. If L is of type G_2 and r and s are long, then since the long roots form a system of type A_2, $N_{rs}=1$.

2.5. If r, s, and $r+s$ are all short and L is of type C_n, $n\geqq 3$, then as in 2.3, $N_{rs}=1$.

2.6. If r and s are short, but $r+s$ is long and L is of type C_n, then imbedding r and s in a system of type B_2, we see that $s-r$ is also a root, $s-2r$ is not, and so $N_{rs}=2$.

2.7. If r, s, and $r+s$ are all short and L is of type F_4, then they are seen to generate a system of type A_2, so that $N_{rs}=1$.

2.8. If r and s are short and L is of type B_n, $n\geqq 2$, then imbedding r and s in a system of type B_2, we see that $r+s$ must be long, $s-r$ is a root, but $s-2r$ is not. Thus $N_{rs}=2$. The same reasoning applies if r and s are short but $r+s$ is long for L of type F_4.

2.9. If r and s are of unequal length, then consideration of types B_n, C_n, F_4, and G_2 separately shows that $r+s$ must be short, $s-r$ is not a root, and so $N_{rs}=1$.

2.10. If r and s are short and L is of type G_2, then $r+s$ also short implies that $s-r$ is a root, $s-2r$ is not, and $N_{rs}=2$. But $r+s$ long implies that both $s-r$ and $s-2r$ are roots, and hence $N_{rs}=3$.

2.11. If S is a set of roots which is itself a system of roots of some simple algebra, then $\{e_r \mid r \in S\}$ generates a simple subalgebra of L which will have a Chevalley basis consisting of $\{e_r \mid r \in S\}$ together with suitably chosen elements from H.

3. **Principal results.** Throughout the sequel we assume that neither 2 nor 3 is a zero divisor in R, where we consider 0 to be a zero divisor. We shall denote by C the Cartan matrix of L as well as the transformation on H represented by this matrix, relative to the basis $\{h_1, h_2, \ldots, h_n\}$ of H given in §2. We shall use s and t as generic symbols for long and short roots of L respectively. We define the integer $m=(t, t)/(s, s)$. We denote by E_R the R-module generated by the Chevalley basis elements e_r, $r \in \Sigma$, and by H_R the R-module generated by $\{h_1, \ldots, h_n\}$. The symbol E_L (respectively H_L) will denote the R-module generated by $\{e_t \mid t$ a long root in $\Sigma\}$ (respectively $\{h_t \mid t$ a long root in $\Sigma\}$), and E_S (respectively H_S) will denote the corresponding R-modules with short roots s replacing t. If $A \subseteq R$, then $m^{-1}A$ will denote the set of $x \in R$ such that $mx \in A$.

3.1. *For every ideal I of L_R such that $I \nsubseteq H_R$, there is a nonzero ideal J in R for which $I \supseteq L_J$.*

3.2. *If L is not of type A_1 or C_n, $n \geqq 2$, and I is an ideal in L_R, then $I=(I \cap E_R) \oplus (I \cap H_R)$.*

3.3. MAIN THEOREM. *A necessary and sufficient condition for every ideal in L_R to have the form L_J for some ideal J in R is that m and* det C *be invertible in R.*

3.4. *Suppose that L is of one of the types A_n, $n \geqq 2$, D_n, $n \geqq 3$, E_6, E_7, or E_8. Let I be an ideal in L_R. Then $I \cap E_R = E_J$ where J is the ideal in R generated by the scalar coefficients of the elements in $I \cap E_R$. Also $H_J \subseteq I \cap H_R \subseteq C^{-1}(H_J)$. Conversely, if $\tilde{H}$ is any R-module such that $H_J \subseteq \tilde{H} \subseteq C^{-1}(H_J)$ for some ideal J in R, then $I=\tilde{H} \oplus E_J$ is an ideal in L_R.*

3.5. *Suppose that L is of one of the types B_n, $n \geqq 3$, F_4, or G_2. Let I be an ideal in L_R. Let J be the ideal in R generated by the scalar coefficients of elements in $I \cap E_L$. Then $I \cap E_R = JE_L \oplus J'E_S$ where J' is an ideal of R such that $J \subseteq J' \subseteq m^{-1}J$; also, $J'H_S + JH_L \subseteq I \cap H_R \subseteq C^{-1}(J'H_S+JH_L)$. Conversely, given any $E=JE_L+J'E_S$ with J and J' ideals of R such that $J \subseteq J' \subseteq m^{-1}J$, then $I=E \oplus J'H_S$ is an ideal in L_R; also, given any R-module $\tilde{H}$ such that $J'H_S+JH_L \subseteq \tilde{H} \subseteq C^{-1}(J'H_S+JH_L)$, then $I=\tilde{H} \oplus JE_L \oplus J'E_S$ is an ideal in L_R.*

3.6. *Suppose L is of type C_n, $n \geqq 2$. Let I be an ideal in L_R. Then $I \cap E_S = JE_S$ where J is the ideal in R generated by the scalar coefficients of elements in $I \cap E_S$; also,*

$$2JE_L + 2JH_L + JH_S \subseteq I \cap (E_L \oplus H_R) \subseteq JE_L \oplus C^{-1}(JH_R).$$

Conversely, given any R-module N of the form $N=J'E_L \oplus \tilde{H}$ where J' is an ideal of R such that $2J \subseteq J' \subseteq J$ and where $J'H_L+JH_S \subseteq \tilde{H} \subseteq C^{-1}(J'H_L+JH_S)$, then $N \oplus JE_S$ is an ideal in L_R.

3.7. *Let L be nonsymplectic of rank at least* 2 *and let I be an ideal in L_R. Let J be the ideal in R defined in* 3.4 *and* 3.5, *and suppose $I \cap E_R = E_J$. Let g_I be the minimal* *number of generators of the R-module* $(I \cap H_R)/H_J$. *Suppose every ideal in R is generated by no more than yz elements where* $y = \min\{g_I \mid I \cap E_R = E_J\}$ *and z is the number of roots of L (respectively, long roots of L) if L has a single (respectively two) root length(s). Then* g_I *is the minimum number of generators of the ideal I.*

4. **Lower bounds for ideals.** In writing elements of L_R in terms of the Chevalley basis, we shall use the following lexicographical ordering. If $r = \sum_{i=1}^{n} k_i r_i$, then $\operatorname{ht} r = \sum_{i=1}^{n} k_i$. We shall write e_r before e_u if $\operatorname{ht} r < \operatorname{ht} u$ or $\operatorname{ht} r = \operatorname{ht} u$ and the first simple root whose coefficient is different for r and u has smaller integral coefficient in the expression for r. The h_i's will occur before the e_r's corresponding to positive roots, and after those corresponding to negative roots, and will be arranged in increasing order of subscripts. The root d attached to the largest e_r in this ordering will be called the highest root.

4.1. *If* $r \neq u$ *are roots of L, then there is a sequence of roots* $t_0 = r, t_1, t_2, \ldots, t_k = u$ *such that* $t_{i+1} - t_i$ *is a root for* $0 \leqq i \leqq k-1$. (Note that some t_i may be zero. See 5.1 below where the assumption that rank $L \geqq 2$ is needed to avoid this possibility.)

Proof. Since L is simple, e_r will generate all of L as an ideal, and so multiplications by a suitable sequence of root elements and field elements will bring us to e_u. The subscripts of the intermediate root elements furnish us with the required sequence.

4.2. **Proof of 3.1.** Let I be an ideal in L_R with $I \nsubseteq H_R$. Then I contains an element x whose expression in terms of basis elements involves at least one e_r with a nonzero coefficient c_r. Now calling upon the sequence of roots in 4.1 which begins at r and ends at d, we multiply x successively by root elements attached to $t_{i+1} - t_i$ until we obtain the element $x' \in I$ with final component $c_r' e_d$. Here in view of 2.2 through 2.10 and our assumptions on R, $c_r' \neq 0$. Next we form a sequence of roots beginning at d and ending at $-d$ and proceed in the same way to obtain $x'' = c_r'' e_{-d} \in I$. Then given any root u we can obtain $n_u c_r'' e_u \in I$ and given any i we can also obtain $n_i c_r'' h_i \in I$, where the n's are products of powers of 2 or 3. If n is the least common multiple of all the n_u and n_i, then $nc_r'' \neq 0$, and letting J be the principal ideal generated by nc_r'', we have $I \supseteq JL_R = L_J$.

Theorem 3.1 shows that most ideals in L_R are large in that they contain a complete Chevalley algebra of L over a smaller ring. The reason for requiring $I \nsubseteq H_R$ is shown by the following example.

Consider the algebra L of type A_4 over the ring $\boldsymbol{Z}_5$ of integers modulo 5. Then the Cartan matrix of L has rank 3 since it diagonalizes to diag (1, 1, 1, 0). So the transformation C on H_R defined by this matrix has a 1-dimensional kernel. We claim that $I = \operatorname{Ker} C$ is an ideal in L_R. Clearly $H_R I = 0$. If $h = \sum_{i=1}^{n} n_i h_i \in I$ and $r = \sum_{j=1}^{n} k_j r_j$, then

$$he_r = \sum_{j=1}^{n} k_j \left[\sum_{i=1}^{n} n_i c(r_j, r_i) \right] e_r = 0$$

since the bracketed part is the jth component of the vector $C(h)$. Thus $L_R I=0$, so I is a nonzero ideal in L_R and $I \cap E_R=0$.

5. **Decomposition lemmas and the main theorem.** The main result we use in studying ideals in nonsymplectic L_R is 3.2.

5.1. **Proof of 3.2.** We may suppose that $I \neq 0$. It is clear that $I \supseteq (I \cap E_R) \oplus (I \cap H_R)$. To establish the converse inclusion we distinguish the cases in which L has one root length and in which L has two root lengths. In the first case let $x=\sum_{r\neq 0} c_r e_r+\sum_{i=1}^n n_i h_i$ be a nonzero element of I. We need to show that each partial sum is also in I. First suppose $c_u e_u$ occurs in this expression for x. Then we proceed as in 4.2. Owing to the fact that rank $L \geqq 2$, we see from 2.2 that we can so proceed as to obtain $x' \in I$ with last component $c_u e_d$. In the same manner we can finally obtain $c_u e_u$ in I. Hence $\sum_{r\neq 0} c_r e_r$ belongs to I, as does $x-\sum_{r\neq 0} c_r e_r = \sum_{i=1}^n n_i h_i$.

The second case is more complicated in that the sequences of roots of 4.2 must be more carefully chosen. These choices are dictated by the following two lemmas.

5.2. *Suppose L is of type B_n, $n \geqq 3$, or F_4. If $ke_t \in I$, then $ke_r \in I$ for every root r. If $ke_s \in I$, then $2ke_t \in I$ for every long root t, and k times every short root element also belongs to I.*

Proof. Let $ke_t \in I$. Consider t to be imbedded in a system of type D_n, $n \geqq 3$, or D_4, as the case may be. Then given a long root t', we proceed to multiply ke_t by long root elements (considered likewise to be in a system of type D_n or D_4), which we choose judiciously as in 5.1. By 2.2, we can thus obtain $ke_{t'} \in I$. Now also, if s' is any short root, then from 2.9, $ke_{t'} \cdot e_{s'} = \pm ke_{t+s'} \in I$, where $t+s'$ is short. So at least I contains k times one short root element whenever $ke_t \in I$. Presently we show that then I contains k times every short root element. Before turning to that however, let us suppose that $ke_s \in I$. Choose a short root u such that $s+u$ is a root. One sees that this is always possible from a study of the previously cited lists of roots in [4] or [7]. Then $s+u$ is long (by 2.8), and $ke_s \cdot e_u = \pm 2ke_{s+u} \in I$. But by the above, then I contains $2k$ times every long root element. All that remains now is to prove that if $ke_s \in I$, then k times every short root element is in I. So let r be any short root element. Suppose first that $(r, s) \geqq 0$. Then $r-s=u$ is a root, and $r=s+u$. Since r and s are short, it is easily seen that $s-u=2s-r$ cannot be a root. Hence $ke_s \cdot e_u = \pm ke_r \in I$. Now if $r=-s$, we need only pick a short root r' orthogonal to s, and we can conclude from the preceding that $ke_{r'} \in I$ since $(r', s)=0$. Since $(r, r')=0$, $ke_r \in I$ also. Finally, if $(r, s)<0$ and r and s are linearly independent, then we can obtain $ke_{-s} \in I$ as we just showed, and then $(r, -s)>0$ allows us to conclude that $ke_r \in I$, completing the proof.

The following simple example may clarify the implementation in practice of the procedures in 5.2. Using the notation of [4], the roots of B_n consist of $\pm\omega_i$, $\pm\omega_i \pm \omega_j$, $i \neq j$, where $\{\omega_1, \ldots, \omega_n\}$ is the canonical basis for $\boldsymbol{R}^n$. If the short root element

$ke_{\omega i} \in I$, then $ke_{\omega i} \cdot e_{\omega j} = \pm 2ke_{\omega i+\omega j} \in I$ since $\omega_i - \omega_j$ is also a root. Once we have $2ke_{\omega i+\omega j} \in I$, we can obtain $2ke_{\omega k+\omega m} \in I$ as well, for any long root $\omega_k + \omega_m$. For,

$$2ke_{\omega.+\omega j} \cdot e_{\omega k-\omega j} = \pm 2ke_{\omega k+\omega i} \in I$$

since $\omega_i + \omega_j - (\omega_k - \omega_j)$ is not a root. Then

$$2ke_{\omega k+\omega i} \cdot e_{\omega m-\omega.} = \pm 2ke_{\omega k+\omega m} \in I,$$

since $\omega_k + \omega_i - (\omega_m - \omega_i)$ is not a root.

5.3. *Suppose L is of type G_2. If $ke_t \in I$, then $ke_r \in I$ for every root r. If $ke_s \in I$, then $3ke_t \in I$ for every long root t, and k times every short root element belongs to I.*

Proof. If $ke_t \in I$, then we consider t to be imbedded in a system of type A_2 (see 2.4). Then the reasoning of 5.2 shows that k times all long root elements are in I, as well as k times at least one short root element, using again 2.9. Also, if $ke_s \in I$, then 2.10 and 2.4 show that $3ke_t \in I$ for all long roots t. So as in 5.2, we are reduced to showing that $ke_s \in I$ implies $ke_r \in I$ for all short roots r. We know that r and s are not orthogonal. If $(r, s) > 0$, then $r-s$ is a root u, and $r=s+u$. If $s-u$ is not a root, then $ke_s \cdot e_u = \pm ke_r \in I$. If $s-u$ is a root, then $s-2u=3s-2r$ is not, so $ke_s \cdot e_u = \pm 2ke_r \in I$. But we can also obtain $3ke_t$ and then $3ke_r$ in I by the preceding, thus enabling us to obtain $ke_r \in I$. If $(r, s) < 0$, then $r+s$ is a root. We use e_r and e_{-s} as multipliers and unwanted coefficients of 2 once more are eliminated using a suitable $3ke_t \in I$, giving us $ke_r \in I$.

Now, continuing with 5.1, let $x = \sum c_s e_s + \sum c_t e_t + \sum_{i=1}^n n_i h_i$ be a nonzero element in I. If r is any long root whose coefficient c_r is nonzero, then the above proofs show us how to obtain $c_r e_r \in I$, confining our operations to the set of long root elements. If u is a short root of L and $c_u \neq 0$ in the expression for x, then the preceding proofs again show that we can find an element x' in I with last component $c_u e_v$, where v is the largest short root in our lexicographical ordering. Similarly, we can obtain $x'' \in I$ with last component $c_u e_{-v}$. Any earlier components will be of the form $c_w e_t$ and so may be subtracted off leaving $c_u e_{-v} \in I$. Then it is a simple matter to finally get $c_u e_u \in I$. Hence $\sum c_s e_s + \sum c_t e_t$ belongs to I, as therefore does $\sum_{i=1}^n n_i h_i$, completing the proof. We note that in the second case our proof actually shows that $I = (I \cap E_L) \oplus (I \cap E_S) \oplus (I \cap H_R)$, a fact we shall find useful in §7.

We have need of a lemma similar to 3.2 for symplectic algebras.

5.4. *Suppose L is of type C_n, $n \geqq 2$. If I is an ideal in L_R, then*

$$I = (I \cap E_S) \oplus (I \cap [E_L \oplus H_R]).$$

Proof. Let $x = \sum c_s e_s + \sum c_t e_t + \sum_{i=1}^n n_i h_i$ be a nonzero element of I. Suppose u is short and $c_u \neq 0$. As in 4.2 we can, in view of 2.9, obtain an element $x' \in I$ with component $c_u e_v$, where v is as in 5.1, using multipliers that are long root elements. Later components will all have been eliminated in the process since the sum of two linearly independent long roots is never a root. Similarly, we can obtain $c_u e_{-v}$ in I and hence $c_u e_u \in I$. Then $\sum c_s e_s \in I$ and the lemma follows immediately.

We are now in a position to prove the sufficiency part of the main theorem.

5.5. *If m and* det C *are invertible in R, then every ideal in L_R is of the form L_J for some ideal J in R.*

Proof. We may suppose that $I \neq 0$. Let J be the ideal generated in R by all coefficients of elements in $I \cap E_R$. Since C is invertible, the reasoning of §4 shows that I cannot be confined to H_R. For if $h = \sum_{i=1}^n n_i h_i \in I$, then

$$he_j = \sum_{i=1}^{n} n_i c(r_j, r_i) e_j.$$

If $h \neq 0$, then $C(h) \neq 0$ and so for some j, $\sum_{i=1}^n n_i c(r_j, r_i) \neq 0$, whence $he_j \neq 0$. So we may suppose $x = \sum c_r e_r \in I$, with the r's short in case C_n, arbitrary otherwise. Now in attempting to isolate a given $c_u e_u \in I$, the multiplications used in 4.2 will introduce at worst some powers of 2 or 3. But the invertibility of m will permit us to obtain $c_u e_u \in I$ in every case except A_1, where the invertibility of det $C = 2$ can be used. [Note that judicious procedure in case G_2 will avoid introducing any powers of 2, in view of Lemma 5.3.] Then $c_u e_u \in I$ quickly gives $c_u e_r \in I$ for all roots r. Thus $I \cap E_R = JE_R$. Hence $I \cap H_R \supseteq JH_R$. Furthermore, if $h = \sum_{i=1}^n n_i h_i \in I \cap H_R$, then $he_j = \sum_{i=1}^n n_i c(r_j, r_i) e_j \in I \cap E_R$ showing that $C(h) \in JH_R$. Otherwise put, $I \cap H_R \subseteq C^{-1}(JH_R) = JH_R$. Thus $I = JL_R = L_J$.

5.6. Corollary. *If L is of type E_8, then every ideal in L_R is of the form L_J for some ideal J in R.*

5.7. Corollary. *If m and* det C *are invertible in R, then the maximal ideals in L_R consist precisely of the ideals L_M, where M is maximal in R.*

In studying L_R more closely, we shall have occasion to use bases for H_R other than the given Chevalley basis. Over the original ground field of L we can find a basis $\{h_i'\}$ of H which is dual to the system Π of simple roots. We denote by H_R' the R-module generated by $\{h_i'\}$. Over the original field we can express the elements of $\{h_i\}$ in terms of the elements of $\{h_i'\}$, $h_i = \sum_{j=1}^n n_{ij} h_j'$. Evaluation of each side at the simple root r_k shows that $n_{ik} = c(r_k, r_i)$, and so $h_i = \sum_{j=1}^n c(r_j, r_i) h_i'$ is an expression valid over R also, and it shows that $H_R \subseteq H_R'$.

We can diagonalize C using elementary matrices and so can view the diagonalization as taking place over R. This process gives us new bases $\{\bar{h}_i\}$ and $\{\bar{h}_i'\}$ of H_R and H_R' respectively, with $\bar{h}_i = d_i \bar{h}_i'$, where the d_i are the elementary divisors of C. For reference we list here the values of the d_i for each of the simple algebras.

A_n: 1, 1, . . . , 1, $n+1$; B_n and C_n: 1, 1, . . . , 1, 2; D_n, n even: 1, 1, . . . , 1, 2, 2; D_n, n odd: 1, 1, . . . , 1, 4; E_6: 1, 1, . . . , 1, 3; E_7: 1, 1, . . . , 1, 2; E_8: 1, 1, . . . , 1; F_4: 1, 1, 1, 1; G_2: 1, 1.

6. **Algebras of one root length.** In this section we complete our main theorem and also the characterization of ideals in L_R when L has a single root length.

6.1. **Proof of 3.4.** The first two assertions follow directly from the proof of 5.5. Conversely, suppose $\tilde{H}$ is any R-module such that $JH_R \subseteq \tilde{H} \subseteq C^{-1}(JH_R)$ where J is an ideal in R. Then relations (1), (3), and (4) of §2 show at once that $L_R \cdot E_J \subseteq \tilde{H} \oplus E_J$. It is equally clear from (2) that $H_R \cdot \tilde{H} = 0$, so we are reduced to considering $E_R \cdot \tilde{H}$. If $h \in \tilde{H}$ and r are as in the discussion following 4.2, then $he_r \in E_J$, since $\sum_{i=1}^{n} n_i c(r_j, r_i) \in J$ follows from $\tilde{H} \subseteq C^{-1}(H_J)$. Hence $\tilde{H} \oplus JE_R$ is an ideal in L_R.

6.2. *If I is an ideal in L_R with $I \cap E_R = E_J$ for some ideal J in R, then the possibilities for $I \cap H_R$ in* 3.4 *are as follows. If L is of type A_n, $n \geqq 2$, D_n, n odd $\geqq 3$, E_6, or E_7 and k is respectively $n+1$, 4, 3, or 2, then*

$$I \cap H_R = J\bar{h}_1 \oplus \cdots \oplus J\bar{h}_{n-1} \oplus J'\bar{h}_n$$

where J' is an ideal in R such that $J \subseteq J' \subseteq k^{-1}J$. If L is of type D_n, n even >3, then

$$I \cap H_R = J\bar{h}_1 \oplus \cdots \oplus J\bar{h}_{n-2} \oplus J'\bar{h}_{n-1} \oplus J''\bar{h}_n$$

where J' and J'' are ideals of R lying between J and $\frac{1}{2}J$.

Proof. Let $h \in I \cap H_R$, $h = \sum n_i h_i = \sum c_i h_i' = \sum \bar{n}_i \bar{h}_i = \sum \bar{c}_i \bar{h}_i'$. Since $he_j = c_j e_j$, we see that every $c_j \in J$. Thus all the $\bar{c}_j \in J$ also. We have further $(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n) = D(\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_n)$ where D is the diagonal matrix of elementary divisors of C. From the lists of elementary divisors, we read off the asserted possibilities for the $\bar{n}_i$.

6.3. Corollary. *If L is of type A_n, $n \geqq 2$, D_n, n odd $\geqq 3$, E_6, or E_7, then the maximal ideals of L_R have the form*

$$ME_R \oplus M\bar{h}_1 \oplus \cdots \oplus M\bar{h}_{n-1} \oplus R\bar{h}_n$$

where M is maximal in R. If L is of type D_n, n even >3, then the maximal ideals of L_R have the form

$$ME_R \oplus M\bar{h}_1 \oplus \cdots \oplus M\bar{h}_{n-2} \oplus R\bar{h}_{n-1} \oplus R\bar{h}_n.$$

6.4. Corollary. (*Necessity part of* 3.3 *for algebras of rank at least two and one root length.*) *If L has one root length and rank at least* 2, *then every ideal in L_R is of the form L_J for some ideal J in R only if* det C *is invertible in R.*

Proof. In case det C is not invertible, then the maximal ideals of Corollary 6.3 provide us with examples of ideals in L_R which do not have the form L_J for any ideal J in R.

6.5. *The preceding corollary holds also for L of type A_1.*

Proof. Here det $C=2$. The ideal generated by e is maximal, but is not L_J for any J if 2 is not invertible, since it consists precisely of the R-module generated by e, h, and $2f$.

7. **Nonsymplectic algebras with two root lengths.** Our attention is now turned to the simple algebras of type B_n, $n \geqq 3$, F_4, and G_2.

7.1. **Proof of 3.5.** The reasoning of §5 shows that $I \cap E_L = JE_L$ and thus $I \cap E_R \supseteq JE_R$. If the inclusion is strict, then there is some short root element e_s occurring in an expression for an element of I whose coefficient c_s is not in J. From §5 again we see that $c = mc_s \in J$, so that $c_s \in m^{-1}J$. Then also $c_s e_u \in I$ for each short root u and $mc_s e_t \in I$ for each long root t. Let J' denote the collection of all coefficients c_s such that $c_s e_s$ is a component of an element of I and s is short. Then J' is an ideal in R, and $J' \subseteq m^{-1}J$, and thus $I \cap E_R = JE_L \oplus J'E_S$. Again $I \cap H_R \supseteq J'H_S + JH_L$ is immediate and $he_j \in I \cap E_R$ for every $h \in I \cap H_R$ shows that $C(h) \in J'H_S + JH_L$, completing the first half of 3.5.

Now let $E = JE_R \oplus J'E_S$ be of the hypothesized sort. Since $c(t, s) = m$ if t is long and s is short, we have $L_R \cdot J'H_S \subseteq J'E_S + JE_L$. Also

$$L_R E = (E_S \oplus E_L \oplus H_R) \cdot (JE_L \oplus J'E_S) \subseteq JE_R + JH_L + JE_R + JE_L + J'E_S + J'H_S$$

as follows from 2.3, 2.4, 2.7, 2.8, 2.9, and 2.10. Thus $E \oplus J'H_S$ is an ideal in L_R.

Finally let $\tilde{H}$ satisfy the hypotheses of the last statement. Then we assert that $I = \tilde{H} \oplus JE_L \oplus J'E_S$ is an ideal in L_R. Clearly $H_R \cdot I \subseteq I$. From the above,

$$E_R \cdot (JE_L \oplus J'E_S) \subseteq I.$$

If $h = \sum n_i h_i \in \tilde{H}$, then $he_j = \sum n_i c(r_j, r_i) e_j$ and $n_i c(r_j, r_i)$ belongs to J' if r_j is short, to J if r_j is long. Thus $he_j \in J'E_S \oplus JE_L \subseteq I$. If $r = \sum_{j=1}^n k_j r_j$, then

$$he_r = \sum_{j=1}^n k_j \left(\sum_{i=1}^n n_i c(r_j, r_i) \right) e_r.$$

If r is short, then clearly $he_r \in J'E_S \supseteq JE_S$. If r is long, then consideration of separate cases shows that all coefficients k_i of short roots in the expression for r are multiples of m. Specifically, in type B_n we take for a simple system $r_1 = \omega_1$, $r_i = \omega_{i+1} - \omega_i$ for $1 < i < n$, where $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is the standard basis for $\boldsymbol{R}^n$. Thus there is only one short simple root, and in this case the preceding assertion follows from 2.3 and 2.9. In type F_4, we take for a simple system $r_1 = \frac{1}{2}(\omega_1 - \omega_2 - \omega_3 - \omega_4)$, $r_2 = \omega_4$, $r_3 = \omega_3 - \omega_4$, and $r_4 = \omega_2 - \omega_3$. If one lists the 24 long roots in the manner of [8, p. 500], the assertion is readily seen to be true. Finally, in type G_2, we take r_1 long and r_2 short, so the long roots consist of r_1, $r_1 + 3r_2$, $2r_1 + 3r_2$, and their negatives, so the assertion is verified immediately. Since $\sum n_i c(r_j, r_i)$ belongs to J if r_j is long and to $m^{-1}J$ if r_j is short, we see in either case that $he_r \in JE_L$, completing the proof.

7.2. Corollary. *If L is of type F_4 or G_2, then in* 3.5 *the only possibility for $I \cap H_R$ is $J'H_S + JH_L$ if $I \cap E_R = J'E_S \oplus JE_L$.*

In order to give a parallel to 6.2, more needs to be said about the diagonalization of C. To do this we divide the algebras of type B_n, F_4, and G_2 into separate cases.

For B_n, $n \geqq 3$, we take $\{r_i\}$ to be the above standard simple system, where r_1 is the only short simple root. One can diagonalize C in this case by adding its second column to its first, then the first column of the new matrix to its second, and then repeating these steps on the $(n-1)$ by $(n-1)$ submatrix of B_{n-1} obtained by adding the first row of this matrix to its third row. What is of interest is that the first row and column affect and are in turn affected by only the two following rows and columns. Now if I is an ideal in L_R with $I \cap E_R = JE_L \oplus J'E_S$, then we see that, if $h = \sum c_i h_i' \in I \cap H_R$, we have $c_1 \in J'$ and $c_i \in J$ for $j = 2, \ldots, n$. In view of the above, we can thus restrict attention to B_3 in studying the nature of the $\bar{n}_i$. We have $\operatorname{diag}(1, 1, 2) = Q^{-1}CP$ where

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Thus $\bar{c}_1 = c_1$, $\bar{c}_2 = c_2$, and $\bar{c}_3 = c_3 + c_1$. In general, then c_1 and c_3 belong to $\frac{1}{2}J$, while the other c_i belong to J. Then $(\bar{c}_1, \ldots, \bar{c}_n) = D(\bar{n}_1, \ldots, \bar{n}_n)$ shows that in general $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_n$ belong to $\frac{1}{2}J$, while the other $\bar{n}_i$ belong to J. Similar reasoning in case $I \cap E_R = JE_R$ provides the rest of the following result.

7.3. *If L is of type B_n, $n \geqq 3$, I is an ideal in L_R, and the basis $\{\bar{h}_i\}$ is obtained as above, then the possibilities for $I \cap H_R$ in* 3.5 *are as follows.*

(a) *If $I \cap E_R = JE_R$, then*

$$I \cap H_R = J\bar{h}_1 \oplus \cdots \oplus J\bar{h}_{n-1} \oplus J'\bar{h}_n$$

with J' an ideal of R satisfying $J \subseteq J' \subseteq \frac{1}{2}J$.

(b) *If rank $L > 3$ and $I \cap E_R = JE_L \oplus J'E_S$ with J and J' as in* 3.6, *then*

$$I \cap H_R = J''\bar{h}_1 \oplus Jh_2 \oplus J^{(\mathrm{iii})}\bar{h}_3 \oplus J\bar{h}_4 \oplus \cdots \oplus J\bar{h}_{n-1} \oplus J^{(\mathrm{iv})}\bar{h}_n,$$

where J'', $J^{(\mathrm{iii})}$, and $J^{(\mathrm{iv})}$ are ideals of R in the same range as J'.

(c) *If rank $L = 3$ and $I \cap E_R$ is as in* (b), *then*

$$I \cap H_R = J''h_1 \oplus J\bar{h}_2 \oplus J^{(\mathrm{iii})}\bar{h}_3$$

where $J^{(\mathrm{iii})}$ is an ideal of R lying between J and $\frac{1}{4}J$, and J'' is as in (b).

7.4. Corollary. *If L is of type B_n, $n \geqq 3$, then the maximal ideals of L are of the form*

$$ME_L \oplus RE_S \oplus R\bar{h}_1 \oplus M\bar{h}_2 \oplus R\bar{h}_3 \oplus M\bar{h}_4 \oplus \cdots \oplus M\bar{h}_{n-1} \oplus R\bar{h}_n,$$

where M is a maximal ideal in R, and $\{\bar{h}_i\}$ is the basis of 7.3.

For the algebra of type F_4 we again take the above standard simple system with r_1 and r_2 short. If $I \cap E_R = JE_L \oplus J'E_S$ as before and $h \in I \cap H_R$ is written as before, then we see that c_1 and c_2 are in $\frac{1}{2}J$, with c_3 and c_4 in J. Here we have $(\bar{n}_1, \ldots, \bar{n}_4) = (\bar{c}_1, \ldots, \bar{c}_4)$, and $I_4 = I_4CP$ where P is a suitable 4 by 4 matrix. Thus each $\bar{c}_i = c_i$ and hence $\bar{n}_1$ and $\bar{n}_2$ belong to $\frac{1}{2}J$, while $\bar{n}_3$ and $\bar{n}_4$ belong to J. Similar reasoning shows that all $\bar{n}_i \in J$ if $I \cap E_R = JE_R$. We have the following results then.

7.5. *If L is of type F_4, I is an ideal in L_R, and the basis $\{\bar{h}_i\}$ is that obtained above, then*

(a) *$I \cap E_R = JE_R$ implies $I \cap H_R = JH_R$, and $I = L_J$;*

(b) *if $I \cap E_R = JE_L \oplus J'E_S$ as in* 3.5, *then*

$$I \cap H_R = J''\bar{h}_1 \oplus J^{(iii)}\bar{h}_2 \oplus J\bar{h}_3 \oplus J\bar{h}_4,$$

where J'' and $J^{(iii)}$ are ideals in R lying between J and $\frac{1}{2}J$.

7.6. Corollary. *If L is of type F_4, then the maximal ideals of L_R are of the form*

$$ME_L \oplus RE_S \oplus R\bar{h}_1 \oplus R\bar{h}_2 \oplus M\bar{h}_3 \oplus M\bar{h}_4$$

where M is a maximal ideal in R, and $\{\bar{h}_i\}$ is the elementary divisor basis for H_R obtained above.

If L is of type G_2, then we take r_1 long and r_2 short. If $I \cap E_R$ is as in 3.5 and $h \in I \cap H_R$ is written as above, then $c_1 \in J$ and $c_2 \in \frac{1}{3}J$. Also $\bar{n}_i = \bar{c}_i$, $i = 1, 2$, and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = Q^{-1}CP.$$

Thus $\bar{c}_1 = c_1$ and $\bar{c}_2 = c_2 \in \frac{1}{3}J$.

7.7. *If L is of type G_2, I is an ideal of L_R, and $\{\bar{h}_i\}$ is the basis obtained above, then* (a) *$I \cap E_R = JE_R$ implies that $I \cap H_R = JH_R$, and $I = L_J$;* (b) *if $I \cap E_R = Je_1 \oplus J'e_2$ as in* 3.5, *then $I \cap H_R = J\bar{h}_1 \oplus J''\bar{h}_2$, where J'' is an ideal of R lying between J and $\frac{1}{3}J$.*

7.8. Corollary. *If L is of type G_2, then the maximal ideals in L_R are of the form $Me_1 \oplus Re_2 \oplus M\bar{h}_1 \oplus R\bar{h}_2$, where M is a maximal ideal of R and $\bar{h}_1$ and $\bar{h}_2$ are as in* 7.7.

Combining 7.4, 7.6, and 7.8, we obtain the necessity part of the main theorem.

7.9. Corollary. *If L is nonsymplectic and has two root lengths, then every ideal in L_R is of the form L_J for some ideal J in R only if m and* det *C are invertible in R.*

8. **The symplectic algebra.** In this section the proof of the main theorem is completed, with the aid of 3.6 whose proof we now give.

8.1. **Proof of 3.6.** That $I \cap E_S = JE_S$ follows directly from the proof of 5.4. If $x = \sum c_t e_t + \sum n_i h_i \in I$, then multiplication of x by suitable short root elements in the manner of the proof of 5.4 yields $c_t e_s \in I$ for every short root s. Thus each $c_t \in J$, and $2c_t e_t \in I$ for every long root t. If r_i is long, then $2n_i h_i$ can be isolated in I similarly, and if r_i is short, then $n_i h_i$ can be isolated in I by confining work to the system of short roots since rank $L \geqq 2$, and $n_i \in J$ in either case. Thus

$$I \cap (E_L \oplus H_R) \supseteq 2JE_L \oplus 2JH_L \oplus JH_S.$$

To get the second inclusion of 3.6, note that multiplication of x by e_j maps

$$\sum_{i=1}^{n} n_i h_i \quad \text{to} \quad \sum_{i=1}^{n} n_i c(r_j, r_i) e_j \in JE_S.$$

Again the coefficient is the jth component of $C(\sum n_i h_i)$. Also, each coefficient belongs to J since if r_j is long, it is a c_t and if r_j is short, it is a c_s. Hence $\sum n_i h_i \in C^{-1}(JH_R)$, so an upper bound for $I \cap (E_L \oplus H_R)$ is $JE_L \oplus C^{-1}(JH_R)$. Next, let N be as in 3.6. Then $L_R N \subseteq N \oplus JE_S$. To see this, note first that $L_R \cdot JE_S \subseteq JE_S + 2JE_L + JH_S + 2JH_L$ in view of 2.5, 2.6, and 2.9, so we need only check $L_R \cdot \tilde{H}$. Let $h = \sum n_i h_i \in \tilde{H}$. Then $he_j = \sum_i n_i c(r_j, r_i) e_j$ and the coefficient is the jth component of $C(h)$, so belongs to J' if r_j is long, to J if r_j is short. Thus $he_j \in J'E_L$ if r_j is long, and $he_j \in JE_S$ if r_j is short. These conclusions remain true if we replace r_j by an arbitrary root r, as in 7.1. Thus $L_R H' \subseteq J'E_L \oplus JE_S$ and hence $N \oplus JE_S$ is an ideal in L_R.

8.2. Corollary. *If L is of type C_n, then every ideal in L_R is of the form L_J for some ideal J in R only if $m = 2 = \det C$ is invertible in R.*

Proof. If 2 is not invertible in R, then 3.7 tells us that $(H_R \oplus 2E_L) \oplus E_S$ is an ideal of L_R.

If we specialize 5.5 to C_n and combine it with 8.2, we have a recent result of W. Jehne [5, Theorem 4.1], which was obtained using the explicit classical model of the symplectic algebra as a Lie algebra of matrices.

9. **Generators.** Our previous results enable us to make some statements regarding generating sets for ideals.

9.1. **Proof of 3.7.** Suppose the cosets $H_1, H_2, \ldots, H_g$ constitute a minimal generating set for $(I \cap H_R)/H_J$. Then we claim that

$$G = \left\{ h^{(1)} + \sum_{j=1}^{z} n_{1j} e_{rj}, h^{(2)} + \sum_{j=1}^{z} n_{2j} e_{rj}, \ldots, h^{(g)} + \sum_{j=1}^{z} n_{gj} e_{rj} \right\}$$

is a minimal generating set for I, where the n_{ij} are chosen from a minimal generating set $\{n_1, \ldots, n_k\}$ of J and are distinct until all k generators of J have appeared, arbitrary thereafter. Also $h^{(i)} \in H_i$, and e_{rj} is a Chevalley root element. In either the single or the two root length case, we can break off each $n_{ij} e_{rj}$ by proceeding as in 5.1, 5.2, and 5.3. Hence G generates all of $I \cap E_R$. It also generates all of $I \cap H_R$ since JH_R comes from JE_R, and the remainder of $I \cap H_R$ can be generated by the elements $h^{(i)}$ and those in JH_R since each $h^{(i)}$ is a representative of a generating coset for $(I \cap H_R)/H_R$. So G generates I. Furthermore, none of the $h^{(j)}$ can be obtained from any expression involving the others by multiplications or scalar operations since they are all representatives of a minimal generating set for $(I \cap H_R)/H_J$. Hence any generating set for I must contain at least g elements in order to yield all the $h^{(j)}$. Hence G is a minimal generating set.

The exclusion of the symplectic algebra results from the chaotic relationship between long and short root elements in ideals in symplectic L_R, together with the fact that $I \cap H_R$ is not in general a direct summand of I.

9.2. COROLLARY. *If R is a principal ideal ring, L is nonsymplectic, I is an ideal in L_R, J is the ideal of R defined in* 3.4 *and* 3.5 *with $I \cap E_R = E_J$, and g is the minimum number of generators of the R-module $I \cap H_R/H_J$, then g is the minimum number of generators for I also.*

9.3. COROLLARY. *If in* 9.2 *we also have m and* det *C invertible in R, then every ideal in L_R is principal.*

BIBLIOGRAPHY

1. R. W. Carter, *Simple groups and simple Lie algebras*, J. London Math. Soc. **40** (1965), 193–240.
2. C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14–66.
3. J. Dieudonné, *Les algèbres de Lie simples associeés aux groupes simples algébriques sur un corps de caractéristique $p > 0$*, Rend. Circ. Mat. Palermo (2) **6** (1957), 198–204.
4. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
5. W. Jehne, *Die Struktur der symplektischen Gruppe über lokalen und dedekindschen Ringen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. **1962/1964**, 187–235.
6. Séminaire "Sophus Lie", Ecole Normale Supérieure, Paris, 1955.
7. J.-P. Serre, *Algèbres de Lie semisimples complexes*, Benjamin, New York, 1966.
8. T. A. Springer, *Some arithmetical results on semisimple Lie algebras*, Inst. Hautes Études Sci. Publ. Math. **30** (1966), 475–501.
9. R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific J. Math. **11** (1961), 1119–1129.

UNIVERSITY OF CALIFORNIA,
RIVERSIDE, CALIFORNIA