ON ALGEBRAIC EXTENSIONS AND ORDER-PRESERVING ISOMORPHISMS OF CERTAIN PARTIALLY ORDERED FIELDS([1,2])

BY
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Introduction

Let $X=(\mathscr{K}, \prec, \mathscr{U}, \mathscr{C})$ be a loosely closed graduated field; in [5, §8, Theorem I], Strodt showed that the algebraic closure of any field $\mathscr{F}$ asymptotically constrained over X is realized by an extension field in $\mathscr{K}$ which is also asymptotically constrained over X (see [5, §§1–7] for definitions and terminology). In certain applications of this result the asymptotically constrained field $\mathscr{F}$ is subjected to isomorphisms λ (onto other such subfields of $\mathscr{K}$) satisfying $\lambda(f)\sim f$ for each nonzero f in $\mathscr{F}$, and $\lambda(m)=m$ for each $m \in \mathscr{M}$. We call such isomorphisms $\mathscr{M}$-stable; their action is often connected with dependence of the members of $\mathscr{F}$ on parameters. The existence of $\mathscr{M}$-stable isomorphisms and their role in the theory developed by Strodt and the author [4]–[8] are recent discoveries. [5] provides the basic context for discussion of $\mathscr{M}$-stable isomorphisms, but provides no techniques for attacking certain problems arising in their study. Two particular questions arise in connection with an $\mathscr{M}$-stable isomorphism λ of the asymptotically constrained field $\mathscr{F}$:

(a) Does λ extend, preferably uniquely, to an $\mathscr{M}$-stable isomorphism σ of the algebraic closure in $\mathscr{K}$ of $\mathscr{F}$ onto that of $\lambda(\mathscr{F})$?

(b) If such a σ exists, can its action be related to that of λ with sufficient precision that we can identify and study effectively the action induced by σ of certain parameters associated with λ?

Affirmative answers to these questions, especially the second, are crucial to our applications. In this paper we obtain affirmative answers to both of these questions.

Part I contains two main results, Theorems A and B. In Theorem A we show that the elements $\prec 1$ in algebraic extensions in $\mathscr{K}$ of an asymptotically constrained field $\mathscr{F}$ have simple representations in terms of the elements $\prec 1$ in $\mathscr{F}$ and special algebraic elements whose behavior is easily determined without "multiplicity problems" from certain equations they satisfy. This theorem provides the constructions which clinch both the existence and uniqueness arguments in Theorem B,

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where we obtain the unique extending isomorphism σ desired in question (a). Once σ is produced, its action is easily documented in Theorem B via the analysis in Theorem A. This documentation constitutes a satisfactory answer to question (b).

We postpone most of the proof of Theorem A until §§10–20, so as to give a quicker airing of the role of Theorem A in the proof of Theorem B.

§§10–20 involve an excursion into valuation theory. We consider an asymptotically constrained field $\mathscr{F}$, and a finitely generated normal extension (in $\mathscr{K}$), $\mathscr{G}$ of $\mathscr{F}$. We study $\bar{\mathcal{O}}$, the integral closure in $\mathscr{G}$ of the valuation ring $\mathcal{O}=\{f\in\mathscr{F}: f\lesssim 1\}$. The crux of the discussion is our construction of an element η in $\bar{\mathcal{O}}$, all of whose algebraic conjugates are distinct *and* are $\sim$ to distinct elements of $\mathscr{C}$. Using η with its strong "nonmultiplicity" properties, we perform an analysis of the structure of $\bar{\mathcal{O}}$ and $\mathscr{G}$ which results in the proof of Theorem A.

In part II we consider systems of asymptotically constrained fields connected by $\mathscr{M}$-stable isomorphisms. We apply Theorem B to show, in Theorem C, that the algebraic closures (in $\mathscr{K}$) of these fields, together with the unique extending $\mathscr{M}$-stable isomorphisms between them, form systems with the same properties as the original ones.

In part III we specialize the results in part II to the context of the class of function theoretic graduated fields of [5, §§51–61]. The fields in part II become essentially fields of meromorphic functions with certain asymptotic behavior. The isomorphisms are given "locally" by analytic dependence of the functions on certain parameters; the basic asymptotic behavior of each function is uniform with respect to this dependence. Theorem D, in a direct application of Theorems B and C, shows that the extending $\mathscr{M}$-stable isomorphisms are given locally by the analytic dependence on these parameters of the functions in the algebraic closures, and that the basic asymptotic behavior of each of these functions is uniform with respect to this dependence.

In part IV we sketch an application of the above to a problem in algebraic differential equations.

In the sequel we employ the definitions and notations of [5, §§1–7, 15], with the exception that we do *not* adhere to the convention in [5] of denoting sets containing zero by letters with zero subscript. Throughout this paper the graduated field $X=(\mathscr{K}, \prec, \mathscr{U}, \mathscr{C})$ is fixed and loosely closed in any given discussion. Accordingly we will employ the following conventions:

(c) all extensions of rings and fields we discuss are tacitly understood to lie in the relevant $\mathscr{K}$, with exception of those in §16;

(d) the phrase "asymptotically constrained over X" in [5] will usually be abbreviated to "a.cn. over X" or simply "a.cn." Thus all algebraic elements and extensions relative to a given a.cn. field $\mathscr{F}$ are understood to lie in the relevant field $\mathscr{K}$; the fact that X is loosely closed assures that all such constructions can be realized within a.cn., algebraically closed subfields of $\mathscr{K}$.

We introduce the following notations:

(e) the set $\{f \in \mathscr{K} : f \prec 1\}$ is denoted by $\{\prec 1\}$, and the set $\{f \in \mathscr{K} : f \precsim 1\}$ is denoted by $\{\precsim 1\}$ (where $(\mathscr{K}, \prec, \mathscr{U}, \mathscr{C})$ is the ambient graduated field).

I. Quasilinear Generators and $\mathscr{M}$-Stable Isomorphisms

1. In part I (§§1–20) all discussions except in §16 take place relative to a given, loosely closed graduated field $X = (\mathscr{K}, \prec, \mathscr{U}, \mathscr{C})$.

2. Definition. Let $Q(Y) = q_0 + Y + q_2 Y^2 + \cdots + q_n Y^n$, where $\{q_0, q_2, \ldots, q_n\} \subset \{\prec 1\}$. Then Q is called a *quasilinear polynomial*. If in addition, $\{q_0, q_2, \ldots, q_n\} \subset \mathscr{F}$, where $\mathscr{F}$ is a subfield of $\mathscr{K}$, then Q is said to be a *quasilinear polynomial over* $\mathscr{F}$. If ν is a root $\prec 1$ of some quasilinear polynomial Q over a subfield $\mathscr{F}$ of $\mathscr{K}$, ν is said to be *quasilinear over* $\mathscr{F}$.

3. Remark. A given quasilinear polynomial Q has at most one root $\prec 1$; if $Q(\nu_1) = Q(\nu_2) = 0$ with $\nu_1, \nu_2 \in \{\prec 1\}$, then after some manipulation we obtain $Q(\nu_2) - Q(\nu_1) = (\nu_2 - \nu_1)(1 + e) = 0$ with $e \prec 1$, which implies $\nu_2 = \nu_1$.

4. Definition. Let $\mathscr{G}$ be an extension field of an a.cn. field $\mathscr{F}$. Suppose that for some ν which is quasilinear over $\mathscr{F}$, we have

(4.1) $\mathscr{G} = \mathscr{F}(\nu)$ ($\mathscr{G}$ is generated over $\mathscr{F}$ by ν),

(4.2) if g in $\mathscr{G}$ is $\prec 1$, then $g = R(\nu)[1 + S(\nu)]^{-1}$ holds, where R and S are polynomials whose coefficients lie in $\mathscr{F}$ and are $\prec 1$.

Then $\mathscr{G}$ is said to be *quasilinearly generated over* $\mathscr{F}$, and ν is called a *quasilinear generator for* $\mathscr{G}$ *over* $\mathscr{F}$.

5. Theorem A. *Let* $\mathscr{F}$ *be an a.cn. field.*

(5.1) *Every finitely generated normal algebraic extension of* $\mathscr{F}$ *is quasilinearly generated over* $\mathscr{F}$.

(5.2) *Let* $\{g_1, g_2, \ldots, g_n\}$ *be a finite set of elements, each* $\prec 1$ *and algebraic over* $\mathscr{F}$. *Then there is an element* ν, *quasilinear over* $\mathscr{F}$, *and there are polynomials* $R_1, S_1, R_2, S_2, \ldots, R_n, S_n$, *with their coefficients in* $\mathscr{F}$ *and* $\prec 1$, *such that*

$$g_i = R_i(\nu)[1 + S_i(\nu)]^{-1}$$

holds, $i = 1, 2, \ldots, n$. ν *may be assumed to be a quasilinear generator for a finitely generated normal algebraic extension of* $\mathscr{F}$ *containing* $\{g_1, g_2, \ldots, g_n\}$.

Proof. (5.1) is proven in §§10–20. To prove (5.2), we find (as we may by standard field theory) a finitely generated normal algebraic extension of $\mathscr{F}$ (in $\mathscr{K}$, as always), containing $\{g_1, g_2, \ldots, g_n\}$ and apply (5.1) and the conditions of §4.

6. Definition. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be a.cn. fields. Let λ be an isomorphism of $\mathscr{F}_1$ onto $\mathscr{F}_2$ satisfying

(6.1) $\lambda(f) \sim f$ for every f in $\mathscr{F}_1 - \{0\}$;

(6.2) $\lambda(m) = m$ for every m in $\mathscr{M}$.

Then λ is called an $\mathscr{M}$*-stable isomorphism*.

7. REMARK. The identity automorphism of any a.cn. field is $\mathscr{M}$-stable. We consider nontrivial examples of $\mathscr{M}$-stable isomorphisms in §35.

8. LEMMA. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be a.cn. fields. Let λ be an isomorphism of $\mathscr{F}_1$ onto $\mathscr{F}_2$ such that $\lambda(m)=m$ holds for each $m \in \mathscr{M}$. Then λ is $\mathscr{M}$-stable if and only if $\lambda(e) \prec 1$ holds for each e in $\mathscr{F}_1$ and $\prec 1$.*

Proof. Let $f \in \mathscr{F}_1 - \{0\}$. Then $f = m(1+e)$ where $m \in \mathscr{M}$, $e \in \mathscr{F}_1$ with $e \prec 1$. From this the conclusions are obvious.

9. THEOREM B. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be a.cn. fields, λ an $\mathscr{M}$-stable isomorphism of $\mathscr{F}_1$ onto $\mathscr{F}_2$. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be the algebraic closures of $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively. The following are true.*

(9.1) *There is an $\mathscr{M}$-stable isomorphism σ of $\mathscr{G}_1$ onto $\mathscr{G}_2$, extending λ.*

(9.2) *σ is the only such $\mathscr{M}$-stable isomorphism.*

(9.3) *Let $\{g_1, g_2, \ldots, g_n\}$ be a finite set of elements in $\mathscr{G}_1$ and $\prec 1$. Then there is an element ν, quasilinear over $\mathscr{F}_1$, and there are polynomials R_i, S_i, $i=1, 2, \ldots, n$, all with coefficients in $\mathscr{F}_1$ and $\prec 1$, such that*

$$g_i = R_i(\nu)[1+S_i(\nu)]^{-1},$$
$$\sigma(g_i) = \lambda R_i(\sigma(\nu))[1+\lambda S_i(\sigma(\nu))]^{-1}$$

hold, $i=1, 2, \ldots, n$, where λR_i and λS_i are the polynomials obtained by applying λ to the coefficients of R_i and S_i, $i=1, 2, \ldots, n$, and where $\sigma(\nu)$ is the unique root $\prec 1$ of the quasilinear polynomial λQ obtained by applying λ to the coefficients of any quasilinear polynomial Q over $\mathscr{F}_1$ of which ν is a root.

(9.4) *Let $\{h_1, \ldots, h_n\}$ be a finite set of nonzero elements in $\mathscr{G}_1$. We have $h_i = m_i(1+g_i) \sim m_i$ with m_i in $\mathscr{M}$ and g_i in $\mathscr{G}_1$ and $\prec 1$, $i=1, 2, \ldots, n$. Moreover $\sigma(h_i) = m(1+\sigma(g_i))$ holds, where g_i and $\sigma(g_i)$ may be given as in (9.3), $i=1, 2, \ldots, n$.*

Proof. Zorn's lemma yields, after a routine discussion, a triple $(\mathscr{H}_1, \mathscr{H}_2, \sigma)$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are algebraic extensions of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively, σ is an $\mathscr{M}$-stable isomorphism of $\mathscr{H}_1$ onto $\mathscr{H}_2$ extending λ, and $(\mathscr{H}_1, \mathscr{H}_2, \sigma)$ is maximal in the sense that if $\mathscr{J}_1$ and $\mathscr{J}_2$ are algebraic extensions of $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, and τ is an $\mathscr{M}$-stable isomorphism of $\mathscr{J}_1$ onto $\mathscr{J}_2$ extending σ, then $\mathscr{J}_1 = \mathscr{H}_1$ and $\mathscr{J}_2 = \mathscr{H}_2$.

We claim $\mathscr{H}_1 = \mathscr{G}_1$. Suppose this claim false. Then $\mathscr{H}_1$ is not algebraically closed, and neither is its isomorph $\mathscr{H}_2$. We may, then, and do choose $\mathscr{J}_1$ a *proper*, finitely generated normal algebraic extension of $\mathscr{H}_1$. By Theorem A, §5, (5.1), $\mathscr{J}_1$ has a quasilinear generator, say ν, over $\mathscr{H}_1$; since $\mathscr{J}_1$ is a proper extension, $\nu \neq 0$ must hold.

Let P be the minimal polynomial of ν over $\mathscr{H}_1$, and denote by σP the polynomial obtained by applying σ to the coefficients of P. Using [5, §24], we see that ν is $\sim$ to some point of instability of P (cf. [5, §5]). In [5, §§28–36], an algorithm is discussed for finding all the points of instability of P. This algorithm depends only on the members of $\mathscr{M}$ to which the coefficients of P are $\sim$; since σ is $\mathscr{M}$-stable, the algorithm and its results are invariant under σ. We conclude that P and σP have the same points of instability. Since $\mathscr{G}_2$ is algebraically closed, we may write $\sigma P(Y) = (Y-r_1)(Y-r_2)\cdots(Y-r_n)$ where the r_i all lie in $\mathscr{G}_2$, and see from this representation that every point of instability for σP is $\sim$ to at least one r_i. Hence some r_i must be $\sim\nu$; we fix such a root r_i and call it $\tau(\nu)$.

From standard field theory, we obtain an isomorphism τ of $\mathscr{J}_1=\mathscr{H}_1(\nu)$ onto $\mathscr{H}_2(\tau(\nu))$, extending σ and taking ν to $\tau(\nu)$. We claim τ is $\mathscr{M}$-stable. Evidently $\tau(m)=\sigma(m)=m$ for each $m\in\mathscr{M}$. Hence, by §8, τ will be $\mathscr{M}$-stable if $\tau(g)\prec 1$ holds whenever $g\prec 1$. We choose such a g in $\mathscr{H}_1(\nu)=\mathscr{J}_1$ and conclude from Theorem A, §5, that since ν is a quasilinear generator for $\mathscr{J}_1$ over $\mathscr{H}_1$,

$$g = R(\nu)[1+S(\nu)]^{-1}$$

holds where R and S are polynomials with their coefficients in $\mathscr{H}_1\cap\{\prec 1\}$. Since τ is simply the $\mathscr{M}$-stable σ on these coefficients and since $\tau(\nu)\prec 1$, $\tau(g)\prec 1$ holds.

Letting $\mathscr{J}_2=\mathscr{H}_2(\tau(\nu))$, we see that the triple $(\mathscr{J}_1, \mathscr{J}_2, \tau)$ contradicts the maximality of $(\mathscr{H}_1, \mathscr{H}_2, \sigma)$. Hence our original claim that $\mathscr{H}_1=\mathscr{G}_1$ is true. It follows immediately that $\mathscr{H}_2=\mathscr{G}_2$.

To show σ is the unique $\mathscr{M}$-stable isomorphism of $\mathscr{G}_1$ onto $\mathscr{G}_2$ extending λ, we suppose φ is any other such isomorphism. Then $\varphi^{-1}\sigma$ is an $\mathscr{M}$-stable automorphism of $\mathscr{G}_1$, fixing $\mathscr{F}_1$. We will show $\varphi^{-1}\sigma$ is the identity automorphism, which will prove the uniqueness.

Since automorphisms of $\mathscr{G}_1$ fixing $\mathscr{F}_1$ carry each normal extension of $\mathscr{F}_1$ onto itself, and since each element of $\mathscr{G}_1$ lies in a finitely generated normal algebraic extension of $\mathscr{F}_1$, it suffices to let $\mathscr{E}_1$ be such an extension, to let θ be an $\mathscr{M}$-stable automorphism of $\mathscr{E}_1$ fixing $\mathscr{F}_1$, and to show that θ is the identity. By Theorem A, §5, (5.1), $\mathscr{E}_1=\mathscr{F}_1(\mu)$ where μ is a quasilinear generator of $\mathscr{E}_1$ over $\mathscr{F}_1$ and is a root of some quasilinear polynomial Q over $\mathscr{F}_1$. $\theta(\mu)$ is a root $\prec 1$ of Q since θ is $\mathscr{M}$-stable; hence $\theta(\mu)=\mu$ (cf. §3) holds. Hence θ is the identity, as claimed.

The assertion in (9.3) follows immediately from Theorem A, §5, (5.1), from the fact that σ is $\mathscr{M}$-stable extending λ, and from the fact that quasilinear polynomials have at most one root $\prec 1$. To prove (9.4) we write $g_i=(1/m_i)(h_i-m_i)$, where $h_i\sim m_i\in\mathscr{M}$, and apply (9.3) and the fact that σ fixes $\mathscr{M}$.

10. In §§11–20, $\mathscr{F}$ denotes an a.cn. field, and $\mathscr{G}$ denotes a finitely generated normal algebraic extension of $\mathscr{F}$. Since $\mathscr{F}$ is of characteristic zero, $\mathscr{G}$ is also (automatically) a separable, hence a finite Galois extension of $\mathscr{F}$; G will denote the Galois group of $\mathscr{G}$ over $\mathscr{F}$. We will denote $[\mathscr{G} : \mathscr{F}]$, the degree of $\mathscr{G}$ over $\mathscr{F}$, by n. By well-known Galois theory we have $[\mathscr{G} : \mathscr{F}]=$ cardinality of G.

11. In [5, §15], there is defined the "gauge" function, $]\,[$, from $\mathscr{A}\cup\{0\}$ to $\mathscr{U}\cup\{0\}$, which takes each f in $\mathscr{A}$ to $]f[=u$, the unique u in $\mathscr{U}$ such that $f\approx u$, and takes 0 to 0.

We see that $]\,[$ induces a valuation on $\mathscr{F}$, which we denote by $\boldsymbol{v}$; its value group is $\mathscr{U}$, its valuation ring is $\mathscr{O}=\mathscr{F}\cap\{\precsim 1\}$, and the maximal ideal of $\mathscr{O}$ is $\mathscr{E}=\mathscr{O}\cap\{\prec 1\}$ (cf. [3, pp. 296–298] for general discussion of these terms; on p. 298, line 8, $x<1$ should be replaced by $x \leqq 1$). $\mathscr{F}$ is clearly the field of fractions of $\mathcal{O}$. If f in $\mathscr{F}$ satisfies an equation of integral dependence over $\mathcal{O}$, it is evident that $f \lessapprox 1$ holds, so that $\mathcal{O}$ is integrally closed in $\mathscr{F}$.

Likewise,] [induces a valuation V on $\mathscr{G}$, with value group $\mathscr{U}$, valuation ring $\mathcal{O}_1=\mathscr{G} \cap \{\lessapprox 1\}$, and maximal ideal $\mathscr{E}_1=\mathcal{O}_1 \cap \{\prec 1\}$. Evidently $\mathcal{O}=\mathcal{O}_1 \cap \mathscr{F}$ and $\mathscr{E}=\mathscr{E}_1 \cap \mathscr{F}$. Any valuation on $\mathscr{G}$ restricting to $\boldsymbol{v}$ on $\mathscr{F}$ is said to extend $\boldsymbol{v}$; V is of course such a valuation. If $\mathcal{O}_i \subset \mathscr{G}$ is the valuation ring of any valuation on $\mathscr{G}$ extending $\boldsymbol{v}$, we will say that $\mathcal{O}_i$ *extends* $\mathcal{O}$.

12. Let $\bar{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\mathscr{G}$. By definition, $\bar{\mathcal{O}}$ is the set of all elements x in $\mathscr{G}$ satisfying some polynomial equation (which depends on x) with leading coefficient 1 and all coefficients in $\mathcal{O}$. Since such elements x are clearly $\lessapprox 1$, it follows that $\bar{\mathcal{O}} \subset \mathscr{G} \cap \{\lessapprox 1\}=\mathcal{O}_1$. That $\bar{\mathcal{O}}$ is a subring of $\mathscr{G}$ is a standard result.

13. Let $\bar{\mathscr{E}}_1=\mathscr{E}_1 \cap \bar{\mathcal{O}}=\{e \in \bar{\mathcal{O}} : e \prec 1\}$. Since $\mathscr{E}_1$ is a prime ideal, so is $\bar{\mathscr{E}}_1$. Since $\mathscr{C} \subset \mathcal{O} \subset \bar{\mathcal{O}}$, it follows that $\bar{\mathcal{O}}-\bar{\mathscr{E}}_1=\{h \in \bar{\mathcal{O}} : h \approx 1\}=\{c+e : c \in \mathscr{C}-\{0\}, e \in \bar{\mathscr{E}}_1\}$. Now $\mathcal{O}_1$ is the localization of $\bar{\mathcal{O}}$ at $\bar{\mathscr{E}}_1$ (cf. [3, p. 302, Proposition 17]); that is, $\mathcal{O}_1$ is the ring of fractions with numerators in $\bar{\mathcal{O}}$ and denominators in $\bar{\mathcal{O}}-\bar{\mathscr{E}}_1$. Nonzero elements of $\mathscr{C}$ are units in $\mathcal{O}$, thus in $\bar{\mathcal{O}}$; it follows that

$$(13.1) \qquad \mathcal{O}_1 = \{(c+e_1)(1+e_2)^{-1} : c \in \mathscr{C}, (e_1, e_2) \in \bar{\mathscr{E}}_1 \times \bar{\mathscr{E}}_1\},$$

$$(13.2) \qquad \mathscr{E}_1 = \{e_1(1+e_2)^{-1} : (e_1, e_2) \in \bar{\mathscr{E}}_1 \times \bar{\mathscr{E}}_1\}.$$

14. The value group of V on $\mathscr{G}$ is $\mathscr{U}$. The range of the restriction of V to $\mathscr{F}$, that restriction being $\boldsymbol{v}$, clearly is all of $\mathscr{U}$. Hence the quantity e in [9, p. 68 (2)] must be 1. Furthermore, the residue field $\mathcal{O}/\mathscr{E}$ is isomorphic to $\mathscr{C}$, hence is algebraically closed; thus the quantity f in [9, p. 68 (2)] is 1 also. (We note that [9, Chapter VI, §12] applies since $\mathscr{G}$ is finite normal and separable over $\mathscr{F}$.) Since $\mathcal{O}/\mathscr{E}$ is evidently of characteristic zero, [9, p. 77, Corollary to Theorem 24] applies, and in view of the above, shows that $g=n$, so that

(14.1) there are precisely $n=[\mathscr{G} : \mathscr{F}]$ distinct valuation rings in $\mathscr{G}$ extending $\mathcal{O}$.

15. According to [2, §8, n° 6, Proposition 6], there is a bijection of the set of valuation rings $\mathcal{O}_i$ in $\mathscr{G}$ extending $\mathcal{O}$ onto the set of maximal ideals of $\bar{\mathcal{O}}$, given by $\mathcal{O}_i \mapsto \mathscr{E}_i \cap \bar{\mathcal{O}}$, where $\mathscr{E}_i$ is the unique maximal ideal of $\mathcal{O}_i$. We apply [3, p. 244, Proposition 11] to conclude that the maximal ideals of $\bar{\mathcal{O}}$ may be given as the set of conjugates $\sigma(\bar{\mathscr{E}}_1)$ of $\bar{\mathscr{E}}_1=\mathscr{E}_1 \cap \bar{\mathcal{O}}$, as σ runs through G. Because of (14.1) and the fact that the cardinality of G is n, we may conclude from this that

(15.1) there is a *bijection* of G onto the set of maximal ideals of $\bar{\mathcal{O}}$, given by $\sigma \mapsto \sigma(\bar{\mathscr{E}}_1)$.

16. Remarks. §§17–19 are capable of presentation in the general context of a field Φ containing a valuation ring $\mathcal{O}$ with maximal ideal $\mathscr{E}$, and lying in a finite normal and separable (Galois) extension Γ in which the integral closure of $\mathcal{O}$ is $\bar{\mathcal{O}}$ and in which a valuation ring lying over $\mathcal{O}$ is $\mathcal{O}_1$ with maximal ideal $\mathscr{E}_1$. We have presented these sections in the more special context of [5] and the present paper so as to maintain continuity of exposition and so as to enjoy substantial technical

convenience arising from the calculus of the $\prec$ relation, and from the canonical representations which our context allows for the various valuation rings, their maximal ideals, and their unit groups in terms of the field $\mathscr{C}$ and the $\prec$ relation. In general terms, our program requires us to find n distinct units $c_1, c_2, \ldots, c_n$ in $\mathcal{O}$ (where $n=[\Gamma : \Phi]$), and an $\eta \in \bar{\mathcal{O}}$ such that $\bar{\mathcal{O}}=\mathcal{O}[\eta]$ holds and such that with $\eta=\eta_1, \eta_2, \ldots, \eta_n$ the n algebraic conjugates of η, $\eta_i - c_i \in \mathscr{E}_1$ holds, $i=1, 2, \ldots, n$. To do this in the general context we would hypothesize that

(16.1) $\bar{\mathcal{O}}$ has $n=[\Gamma : \Phi]$ maximal ideals (the number is known by general valuation theory to be $\leqq n$; equality for our case follows from (15.1) which in turn is a consequence of special properties of our value groups and residue fields),

(16.2) $n=[\Gamma : \Phi]$ units $c_1, c_2, \ldots, c_n$ in $\mathcal{O}$ exist whose images in $\mathcal{O}/\mathscr{E}$ are distinct (true in our case since $\mathcal{O}/\mathscr{E}$ is canonically isomorphic to $\mathscr{C}$ which is infinite and whose nonzero elements are units in $\mathcal{O}$). Then an argument analogous to §18 yields an $\eta \in \bar{\mathcal{O}}$ whose conjugates satisfy the desired congruences to the c_i, while an argument along the lines of §19 uses these properties of η, together with the distinctness modulo $\mathscr{E}$ of the c_i, to show that $\bar{\mathcal{O}}=\mathcal{O}[\eta]$.

17. Definition. Let $\eta \in \mathscr{G}$ be such that

(17.1) $\eta=c+e$ where $c \in \mathscr{C}-\{0\}$ and $0 \prec e \prec 1$,

(17.2) $\mathscr{G}=\mathscr{F}(\eta)$,

(17.3) the roots η_i of the minimal polynomial of η over $\mathscr{F}$ satisfy $\eta_i \approx \eta_j$, all i, j, but $\eta_i \sim \eta_j$ only if $\eta_i=\eta_j$.

Then η is called a *simple generator of $\mathscr{G}$ over $\mathscr{F}$.*

18. Lemma. *There is an $\eta \in \mathscr{G}$ such that η is a simple generator of $\mathscr{G}$ over $\mathscr{F}$.*

Proof. If $n=[\mathscr{G} : \mathscr{F}]=1$, the conclusion is obvious. We suppose henceforth in the proof that $n>1$. We enumerate the elements of G; $G=(\sigma_1, \sigma_2, \ldots, \sigma_n)$, with σ_1 the identity (since $\mathscr{G}$ is finite Galois over $\mathscr{F}$, cardinality $(G)=n$). In view of (15.1), the distinct maximal ideals of $\bar{\mathcal{O}}$ are given as $\tau_1(\bar{\mathscr{E}}_1), \tau_2(\bar{\mathscr{E}}_1), \ldots, \tau_n(\bar{\mathscr{E}}_1)$, where τ_i is the inverse of σ_i, $i=1, 2, \ldots, n$, and where $\bar{\mathscr{E}}_1=\mathscr{E}_1 \cap \bar{\mathcal{O}}$.

Let $c_1, c_2, \ldots, c_n$ be *distinct* elements of $\mathscr{C}-\{0\}$. Then $c_i \in \mathscr{C} \subset \mathcal{O} \subset \bar{\mathcal{O}}$, $i=1, 2, \ldots, n$. Since the $\tau_i(\bar{\mathscr{E}}_1)$ are maximal, we know that the ideal in $\bar{\mathcal{O}}$ generated by $\tau_i(\bar{\mathscr{E}}_1)$ and $\tau_j(\bar{\mathscr{E}}_1)$ is just $\bar{\mathcal{O}}$ whenever $i \neq j$. Hence by the Chinese Remainder Theorem ([3, p. 63]), there is an η in $\bar{\mathcal{O}}$ satisfying the simultaneous congruences

$$\eta \equiv c_i \pmod{\tau_i(\bar{\mathscr{E}}_1)}, \qquad i = 1, 2, \ldots, n. \tag{18.1}$$

We will show that η is a simple generator for $\mathscr{G}$ over $\mathscr{F}$.

As i runs from 1 to n we apply σ_i to the ith relation in (18.1), obtaining the simultaneous congruences

$$\sigma_i(\eta) \equiv \sigma(c_i) \pmod{\bar{\mathscr{E}}_1}, \qquad i = 1, 2, \ldots, n.$$

These relations, the definition of $\bar{\mathscr{E}}_1$, and the fact that $\mathscr{C}$ lies in the fixed field of G, imply that

$$\sigma_i(\eta)-c_i \prec 1, \qquad \text{i.e.,} \quad \sigma_i(\eta) \sim c_i, \qquad i = 1, 2, \ldots, n. \tag{18.2}$$

Since the c_i are distinct, (18.2) implies that the n conjugates of η under G are distinct; it follows immediately that $\mathscr{G}=\mathscr{F}(\eta)$, and (17.2) is verified. Since $n>1$, η cannot lie in $\mathscr{C}\subset\mathscr{F}$; (17.1) follows from this and from (18.2). (17.3) is an obvious consequence of (18.2) and the distinctness of the c_i.

The lemma is proven.

19. LEMMA. *Let $\eta\in\mathscr{G}$ be a simple generator for $\mathscr{G}$ over $\mathscr{F}$. Then $\bar{\mathcal{O}}=\mathcal{O}[\eta]$ ($\bar{\mathcal{O}}$ is the ring of polynomials in η with coefficients in $\mathcal{O}$).*

Proof. Let $\eta=\eta_1, \eta_2, \ldots, \eta_n$ be the n roots of the minimal polynomial P of η over $\mathscr{F}$. It follows easily from (17.3) and the definition of minimal polynomial that the leading coefficient of P is 1 and that all coefficients of P are $\lesssim 1$, i.e., lie in $\mathcal{O}$. This means all the η_i are integral over $\mathcal{O}$, hence lie in $\bar{\mathcal{O}}$. Therefore $\mathcal{O}[\eta]\subset\bar{\mathcal{O}}$.

Now let $x\in\bar{\mathcal{O}}$. η generates $\mathscr{G}$ over $\mathscr{F}$ and is algebraic of degree n over $\mathscr{F}$. Thus we may and do find elements $b_0, b_1, \ldots, b_{n-1}$ in $\mathscr{F}$ such that

$$x = b_0+b_1\eta+\cdots+b_{n-1}(\eta)^{n-1}. \tag{19.1}$$

To show $x\in\mathcal{O}[\eta]$, it suffices to show that $b_j\in\mathcal{O}$, i.e., $b_j\lesssim 1, j=0, 1, \ldots, n-1$.

We may and do enumerate G as $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ in such a way that $\sigma_i(\eta)=\eta_i$ holds, $i=1, 2, \ldots, n$. This done, we apply the σ_i in succession to (19.1), obtaining the system

$$\sigma_i(x) = b_0+b_1\eta_i+\cdots+b_{n-1}(\eta_i)^{n-1} \tag{19.2}$$

of linear equations in the $b_j, j=0, 1, \ldots, n-1$ (we note that $\sigma_i(b_j)=b_j$, all i, j since G fixes $\mathscr{F}$). The determinant Δ of (19.2) is the Vandermondian of the η_i. If $n=1$, $\Delta=1$, while if $n>1$, we have

$$\Delta = \Pi\{(\eta_k-\eta_i) : 1 \leqq i < k \leqq n\},$$

in which case it follows from (17.3) that $\Delta\approx 1$. Thus, in any case, we have

$$\Delta\approx 1. \tag{19.3}$$

We have seen that each η_i lies in $\bar{\mathcal{O}}$; furthermore each $\sigma_i(x)$ lies in $\bar{\mathcal{O}}$ since $x\in\bar{\mathcal{O}}$ and since $\sigma_i(\bar{\mathcal{O}})\subset\bar{\mathcal{O}}$ holds, $i=1, 2, \ldots, n$. (See [3, p. 240, Proposition 5]; note that each σ_i fixes $\mathscr{F}$.) We solve (19.2), obtaining

(19.4) $b_j=(1/\Delta)\Gamma_j$, $j=0, 1, \ldots, n-1$, where each Γ_j is a polynomial, with integer coefficients, in the η_i and $\sigma_i(x)$, $i=1, 2, \ldots, n$. Clearly each Γ_j lies in $\bar{\mathcal{O}}$; as noted in §12, $\bar{\mathcal{O}}\subset\mathcal{O}_1$ holds, so each Γ_j is $\lesssim 1$. This result, together with (19.4) and (19.3), implies $b_j\lesssim 1, j=0, 1, \ldots, n-1$.

We have thus proven $x\in\mathcal{O}[\eta]$; since $x\in\bar{\mathcal{O}}$ was arbitrary, it follows that $\bar{\mathcal{O}}\subset\mathcal{O}[\eta]$. But $\mathcal{O}[\eta]\subset\bar{\mathcal{O}}$ is also known; the lemma is proven.

20. **Proof of Theorem A §5, (5.1).** By §18, $\mathscr{G}$ has a simple generator, say η, over $\mathscr{F}$. By §19, $\bar{\mathcal{O}}=\mathcal{O}[\eta]$. We know $\eta\sim c$ for some $c\in\mathscr{C}-\{0\}$. Let $\mu=\eta-c$. By (17.1), $0\prec\mu\prec 1$ holds; in particular, μ is nonzero. Let $\nu=(\mu/v)$, where $v=]\mu[^{1/2}$. Then $0\prec\mu\prec\nu\prec 1$ holds.

Let $e \in \bar{\mathscr{E}}_1 = \bar{\mathcal{O}} \cap \{\prec 1\}$. Since $c \in \bar{\mathcal{O}}$, we have $\bar{\mathcal{O}} = \mathcal{O}[\eta] = \mathcal{O}[\mu]$. Hence we may and do write e as a polynomial in μ with coefficients in $\mathscr{F}$ and $\precsim 1$. Since e and μ are $\prec 1$, the zero-degree coefficient must be $\prec 1$. Now $\mu = \nu v$, and $v \in \mathscr{U} \cap \{\prec 1\} \subset \mathscr{F} \cap \{\prec 1\}$. It follows that we may write $e = r_0 + r_1\nu + \cdots + r_k(\nu)^k$ where $\{r_0, r_1, \ldots, r_k\} \subset \mathscr{F} \cap \{\prec 1\}$.

It follows from this and (13.2) that if $g \in \mathscr{G}$ is $\prec 1$ (i.e., if $g \in \mathscr{E}_1$), then

$$g = R(\nu)[1+S(\nu)]^{-1}$$

where R and S are polynomials in ν with coefficients in $\mathscr{F}$ and $\prec 1$.

We will show ν is quasilinear over $\mathscr{F}$. Let P be the minimal polynomial for η over $\mathscr{F}$. It is clear from (17.1) and (17.3) that the coefficients of P are all $\precsim 1$. With this, and the fact that $\eta \approx 1$, in mind, expanding $P(c) = P(\eta - \mu)$ in powers of μ, we see that $P(c) \precsim \mu \prec v$. P is of course of degree n, and its leading coefficient is 1. It is an easy consequence of (17.3) and of [5, §24], that, in the terminology of [5], all the points of instability of P lie in $\mathscr{C}$ and are *simple* points of instability; in particular, c is such. We apply [5, §23] to conclude that $P^{(i)}(c) \precsim 1$, $i = 1, 2, \ldots, n$, and that $P^{(1)}(c) \approx 1$. Let

$$Q(Y) = [vP^{(1)}(c)]^{-1}[P(c+vY)].$$

Obviously $Q(\nu) = 0$. Using the above information on $P^{(i)}(c)$, $i = 0, 1, \ldots, n$, it is routine to verify that $Q(Y) = q_0 + Y + q_2Y^2 + \cdots + q_nY^n$ where $q_i \prec 1$ holds, $i = 0, 2, \ldots, n$. Since c and v lie in $\mathscr{F}$, so do the coefficients of Q. Hence ν is quasilinear over $\mathscr{F}$.

Finally, it is clear that $\mathscr{G} = \mathscr{F}(\nu)$, since $\mathscr{G} = \mathscr{F}(\eta)$ and c and v lie in $\mathscr{F} - \{0\}$.

Hence, ν is a quasilinear generator of $\mathscr{G}$ over $\mathscr{F}$. Throughout this discussion, $\mathscr{G}$ was an arbitrary, finitely generated normal algebraic extension of $\mathscr{F}$. Hence, Theorem A, §5, (5.1) is proven.

II. $\mathscr{M}$-Stable Systems

21. In part II, all discussions take place relative to a given, loosely closed graduated field $X = (\mathscr{K}, \prec, \mathscr{U}, \mathscr{C})$.

22. Definition. Let Φ be a nonempty set of a.cn. fields. Let Λ be a set of $\mathscr{M}$-stable isomorphisms between the members of Φ. Suppose that

(22.1) for each $(\mathscr{F}_1, \mathscr{F}_2) \in \Phi \times \Phi$ there is at least one $\lambda \in \Lambda$ mapping $\mathscr{F}_1$ onto $\mathscr{F}_2$,

(22.2) whenever the composition of two members of Λ is defin^ed, that composition lies in Λ,

(22.3) Λ contains the identity automorphism of each memb^ of Φ. We then call the ordered pair (Φ, Λ) an *$\mathscr{M}$-stable system.*

23. Theorem C. *Let (Φ, Λ) be an $\mathscr{M}$-stable system. For each λ in Λ, there is an $\mathscr{M}$-stable isomorphism σ of the algebraic closure of the domain of λ onto that of the range of λ, extending λ; σ is unique with these properties. Let Γ be the set of algebraic closures (in $\mathscr{K}$!) of the members of Φ. Let Σ be the set of extending isomorphisms σ. Then (Γ, Σ) is an $\mathscr{M}$-stable system.*

Proof. The existence and uniqueness of each extension σ follows from Theorem B, §9, (9.1) and (9.2). It is routine, using in particular the uniqueness of each σ, to show that each of the three properties in §22 is implied for (Γ, Σ) by the fact that it holds for (Φ, Λ). For example, (22.3) holds since Λ contains the identity on each $\mathscr{F}$ in Φ, and since the identity on the algebraic closure of each $\mathscr{F}$ in Φ extends that on $\mathscr{F}$.

24. Definition. Let (Φ, Λ) be an $\mathscr{M}$-stable system. Let Γ and Σ be the sets defined in §23. Then the $\mathscr{M}$-stable system (Γ, Σ) is said to be the *algebraic closure* of (Φ, Λ).

III. Analytic $\mathscr{M}$-Stable Systems

25. Let $\bar{N}$ be a domain system, E a subconstant class over $\bar{N}$, such that the subconstant pair $(\bar{N}, E)$ is quasilinearly closed (see [5, §§51–57] for definitions). If f and g are meromorphic over $\bar{N}$, we write $f \prec g$ if $g \neq 0$ and $fg^{-1} \in E$ (see [5, §§52, 58]). Let V be a real // rational vector space ([5, §59]), L a nonempty set of functions analytic over $\bar{N}$, such that $Y = (\bar{N}, E, V, L)$ is a logarithmic quadruple (see [5, §60]). Let $\mathscr{U}(Y)$ be defined as in [5, §61] in terms of Y.

26. Let $\mathscr{K}$ be the field of classes of functions meromorphic over $\bar{N}$. (Each f meromorphic over $\bar{N}$ determines the class $f{:}f=g$ over $\bar{N}$; see [5, §54]. The construction is that of [5, §§63, 64], although we do not here use the notation of [5].) Let $\mathscr{U}$ denote the multiplicative subgroup of $\mathscr{K}$ consisting of the classes of the elements of $\mathscr{U}(Y)$. Let $\boldsymbol{C}$ denote the natural copy of the complex numbers contained in $\mathscr{K}$. We denote by $\prec$ the partial order on $\mathscr{K}$ induced in the obvious way by the relation $\prec$ defined above for functions meromorphic over $\bar{N}$.

By [5, §64], $X = (\mathscr{K}, \prec, \mathscr{U}, \boldsymbol{C})$ is a graduated field, and is loosely closed, since we assume $(\bar{N}, E)$ quasilinearly closed.

In the sequel we will ignore the distinction between functions meromorphic over $\bar{N}$ and the members of $\mathscr{K}$ (classes) they determine. When we ascribe function-theoretic properties to a member of $\mathscr{K}$ (e.g., being $< \varepsilon$ in modulus in some $T \in \bar{N}$), we will be tacitly referring to the properties of an appropriately chosen member of the class in question. This abuse of terminology is harmless since if f and g belong to the same class, then $f(x) = g(x)$ holds for each x in each member of a cofinal subfamily of $\bar{N}$.

27. In §§28–32, $\mathscr{K}$, $\prec$, $\mathscr{U}$, and $\boldsymbol{C}$ denote the items defined in §26, and all discussion takes place with respect to the loosely closed graduated field $X = (\mathscr{K}, \prec, \mathscr{U}, \boldsymbol{C})$ which they constitute.

28. Definition. Let (Φ, Λ) be an $\mathscr{M}$-stable system. Let k be a positive integer. Let Δ be a compact polydisc in $\boldsymbol{C}^k$ with nonvoid interior, and centered at $0 = (0, 0, \ldots, 0)$. Suppose that for each $\mathscr{F}_0$ in Φ, there is associated a function π_0 from Δ to Λ satisfying:

(28.1) the domain of $\pi_0(c)$ is $\mathscr{F}_0$, for each c in Δ;

(28.2) $\pi_0(0)$ is the identity automorphism of $\mathscr{F}_0$.

Let $\Pi=\{\pi_0 : \mathscr{F}_0 \in \Phi\}$ be the set of all the functions π_0. Π is called a *k-parametrization of* Λ.

29. NOTATION. Let (Φ, Λ) be an $\mathscr{M}$-stable system. Let k be a positive integer. Let Π be a k-parametrization of Λ. When a given $\mathscr{F}_0$ in Φ is under discussion, we will often denote each isomorphism $\pi_0(c)$ $(c \in \Delta)$ simply by λ_c.

30. DEFINITION. Let (Φ, Λ) be an $\mathscr{M}$-stable system. Let (Γ, Σ) be its algebraic closure. Let k be a positive integer. Let Π be a k-parametrization of Λ. For each $\mathscr{G}_0$ in Γ, define ψ_0 from Δ to Σ by setting, for each c in Δ, $\psi_0(c)=\sigma$, where σ is the unique extension in Σ of that isomorphism $\pi_0(c)$ whose domain is the field $\mathscr{F}_0$ of which $\mathscr{G}_0$ is the algebraic closure. The set $\Psi=\{\psi_0 : \mathscr{G}_0 \in \Gamma\}$ is called the *natural extension of* Π *to* Σ.

It is easy to verify that Ψ is a k-parametrization of Σ.

31. DEFINITION. Let (Φ, Λ) be an $\mathscr{M}$-stable system. Let k be a positive integer. Let Π be a k-parametrization of Λ. Suppose that

(31.1) for each $\mathscr{F}_0$ in Φ, each finite set $\{f_1, \ldots, f_n\}$ of elements in $\mathscr{F}_0$ and $\prec 1$, and each $\varepsilon>0$, there is a member T of $\bar{N}$ such that $\lambda_c(f_i)$, as a function of $(c, x) \in \Delta \times T$, is continuous and bounded in modulus by ε in $\Delta \times T$, analytic in its $k+1$ complex variables in the interior of $\Delta \times T$, $i=1, 2, \ldots, n$. The ordered quadruple $(\Phi, \Lambda, \Delta, \Pi)$ is called a *k-analytic $\mathscr{M}$-stable system*, or simply, an *analytic $\mathscr{M}$-stable system*, when the context makes reference to k unnecessary.

32. THEOREM D. *Let* $(\Phi, \Lambda, \Delta, \Pi)$ *be a k-analytic $\mathscr{M}$-stable system, for some positive integer k. Let* (Γ, Σ) *be the algebraic closure of* (Φ, Λ). *Let* Ψ *be the natural extension of* Π *to* Σ. *Then* $(\Gamma, \Sigma, \Delta, \Psi)$ *is a k-analytic $\mathscr{M}$-stable system.*

Proof. Let $\mathscr{G}_0 \in \Gamma$. Let $\{g_1, \ldots, g_n\} \subset \mathscr{G}_0 \cap \{\prec 1\}$. $\mathscr{G}_0$ is the algebraic closure of some $\mathscr{F}_0 \in \Phi$. Using Theorem B, §9, (9.3), we may and do write

$$g_i = [R_i(\nu)][1+S_i(\nu)]^{-1}$$

where R_i and S_i are polynomials with coefficients in $\mathscr{F}_0 \cap \{\prec 1\}$, $i=1, 2, \ldots, n$, and where ν is quasilinear over $\mathscr{F}_0$. Let $Q(Y)=q_0+Y+q_2Y^2+\cdots+q_nY^n$ be a quasilinear polynomial over $\mathscr{F}_0$ of which ν is a root. By the classical implicit function theorem, the equation $X_0+Y+X_2Y^2+\cdots+X_nY^n=0$ has a solution θ, analytic in the variables $X_0, X_2, \ldots, X_n$ at $(0, 0, \ldots, 0)$ and satisfying $\theta(0, 0, \ldots, 0)=0$. We now use the hypothesis on $(\Phi, \Lambda, \Delta, \Pi)$ to see that given any $\delta>0$, there is a T in $\bar{N}$ such that the function μ given by

$$\mu(c, x) = \theta(\lambda_c(q_0(x)), \lambda_c(q_2(x)), \ldots, \lambda_c(q_n(x)))$$

is continuous and bounded by δ in $\Delta \times T$, analytic in the interior of $\Delta \times T$ (here λ_c and σ_c denote the isomorphisms $\pi_0(c)$ and $\psi_0(c)$, for each $c \in \Delta$; (cf. §29)).

For each c in Δ, we denote by $\lambda_c Q$ the polynomial over $\lambda_c(\mathscr{F}_0)$ gotten by applying λ_c to the coefficients of Q. For each such c, the function μ_c defined from μ by "freezing c" is a solution of $\lambda_c Q(Y)=0$, analytic and tending to zero over $\bar{N}$. It is easy to verify that $\lambda_c Q(Y)=0$ has at most one such solution (one uses the same manipulations as in the proof of the assertion in §3, substituting "$\to 0$" for "$\prec 1$"). $\sigma_c(\nu)$ is such a solution, hence $\sigma_c(\nu)=\mu_c$ holds for each c in Δ.

Now given any $\delta>0$, we may use the hypothesis on $(\Phi, \Lambda, \Delta, \Pi)$ to obtain a T in $\bar{N}$ such that all the functions $\lambda_c(r_{ij})$ and $\lambda_c(s_{ij})$, where r_{ij} and s_{ij} run through the coefficients of the R_i and S_i, $i=1, 2, \ldots, n$, are simultaneously continuous and bounded by δ in modulus in $\Delta\times T$, analytic in the interior of $\Delta\times T$.

For each c in Δ, and each $i=1, 2, \ldots, n$, we have, using §9, Theorem B, (9.3),

$$\sigma_c(g_i) = [\lambda_c R_i(\sigma_c(\nu))][1+\lambda_c S_i(\sigma_c(\nu))]^{-1},$$

where $\lambda_c R_i$ and $\lambda_c S_i$ are gotten by applying λ_c to the coefficients of R_i and S_i. This representation, together with what we have proven about $\sigma_c(\nu)=\mu_c$ and about the coefficients $\lambda_c(r_{ij})$ and $\lambda_c(s_{ij})$, easily implies that given $\varepsilon>0$, a T in $\bar{N}$ can be found such that $\sigma_c(g_i)$ is continuous and bounded in modulus by ε in $\Delta\times T$, analytic in the interior of $\Delta\times T$, $i=1, 2, \ldots, n$. This proves the theorem.

IV

33. In §34 we define essentially the context of [4], [6], [7] and [8]. In §35, in this context, we sketch an application of the preceding theory.

34. Let $\beta\in(0, \pi]$. Let $\bar{N}$ be the domain system $F(-\beta, \beta)$ defined in [4, §§91–94]; its members are real-symmetric unbounded sector-like domains which miss the nonpositive real axis. Let p be a nonnegative integer. Let $\boldsymbol{Q}$ denote the rational number field. We specialize the constructions of §§25–26, so that $\bar{N}$ becomes $F(-\beta, \beta)$, so that $f\in E=E_\beta$ holds (i.e., $f\prec 1$) if and only if

$$\left(x(\log x)(\log\log x)\cdots(\log_i x)\frac{d}{dx}\right)^j f\to 0$$

holds, all $i\geqq 0$, $j\geqq 0$, and so that in the graduated field $X_{\beta,p}=(\mathscr{K}_\beta, \prec, \mathscr{U}_p, \boldsymbol{C})$ resulting from our choice of V and L, $\mathscr{U}=\mathscr{U}_p$ is the set of functions

$$\{x^\rho(\log x)^\sigma(\log\log x)^\tau\cdots(\log_p x)^\varphi : (\rho, \sigma, \tau, \ldots, \varphi)\in \boldsymbol{Q}^p\}.$$

The elements of $\mathscr{M}_p=\{cu : c\in \boldsymbol{C}-\{0\}, u\in\mathscr{U}_p\}$ are called logarithmic monomials of rank $\leqq p$. E_β is easily shown to be quasilinearly closed, so that by [5, §64], $X_{\beta,p}$ is loosely closed. The order $\prec$ defined in terms of E_β can be shown without difficulty to "survive differentiation" in the sense that if $f\sim m\in\mathscr{M}_p-\boldsymbol{C}$ holds, then $f'\sim m'$ holds.

35. In [4], [6] and [8], a central object of study is the algebraic differential equation

(35.1) $P(Y)=0$

where $P(Y)=\sum p_{ij}Y^i(Y')^j$ is a first-order differential polynomial whose coefficients (the p_{ij}) lie in $\mathscr{K}_\beta$ and are $\sim$ to logarithmic monomials (these hypotheses include the classical case where the p_{ij} are rational functions). We seek solutions $\sim$ to logarithmic monomials. Under further hypotheses on the p_{ij} which are implied by those we will impose, the algorithms in [4] and [1] produce all logarithmic monomials which could appear in this capacity; following [1] we call them critical monomials.

The farthest reaching results in [8] are obtained under the hypothesis that

(35.2) the p_{ij} lie in a field of functions $\mathscr{F}$ which is a.cn. over $X_{\beta,p}$, and which satisfies $\mathscr{F}=\mathscr{F}^*$, where $\mathscr{F}^*$, called the Schwarzian image of $\mathscr{F}$, is defined by

$$\mathscr{F}^* = \{f^* : f \in \mathscr{F}\} \qquad (f^* \text{ is defined by } f^*(x) = \overline{f(\bar{x})})$$

(since the members of $F(-\beta, \beta)$ are real-symmetric, f^* is analytic over $F(-\beta, \beta)$ when f is). An example of such a field is $C(\mathscr{U}_p)$.

In [6], under essentially the above hypothesis, Strodt shows that

(35.3) if m is a critical monomial for P but not for $\partial P/\partial Y'$, then

(35.4) there is a solution y in $\mathscr{K}_\delta$, for some $\delta \in (0, \beta]$, such that $y \sim m$ holds.

In [7], under the above hypotheses, we show further that

(35.5) y may be chosen to lie in a field $\mathscr{G}_y$ which is a.cn. over $X_{\delta,q}$ for some $q \geqq p$, and which contains $\mathscr{F}$, once $\mathscr{K}_\beta$, and hence $\mathscr{F}$, is viewed in the natural way as a subfield of $\mathscr{K}_\delta$.

Recent work of Strodt [7, pp. 359–361] shows that if, in addition, $\mathscr{G}_y=(\mathscr{G}_y)^*$ may be concluded, then in all such investigations as above, the hypothesis that m is not critical for $\partial P/\partial Y'$ may be dropped and conclusions (35.4) and (35.5) obtained without it. (This result, in particular, verifies the conjecture in [4, §1, last paragraph] for order 1, under our additional, but natural hypotheses on the "logarithmic domain" containing the coefficients.)

In [7] we are not able to conclude $\mathscr{G}_y=(\mathscr{G}_y)^*$ in all cases. In the cases in doubt, the problem can be reduced to one in which equation (35.1) has a 1-parameter family $\mathscr{Y}$ of solutions $\sim m$, with each y in $\mathscr{Y}$ such that the field $\mathscr{F}(y)$ is a.cn. over $X_{\delta,q}$: Recently we have discovered that for each y_i, y_j in $\mathscr{Y}$, the substitution of y_j for y_i induces an $\mathscr{M}$-stable isomorphism λ_{ij} of $\mathscr{F}(y_i)$ onto $\mathscr{F}(y_j)$, making $(\{\mathscr{F}(y_i)\}, \{\lambda_{ij}\})$ an $\mathscr{M}$-stable system. There is a natural parametrization of $\mathscr{Y}$ by C; we have shown that the parameter dependence is analytic and induces on $(\{\mathscr{F}(y_i)\}, \{\lambda_{ij}\})$ the structure of a 1-analytic $\mathscr{M}$-stable system. We know that in our cases in doubt, for each y in $\mathscr{Y}$ there is a unique z in $\mathscr{Y}$ (in general $z\neq y$) such that $\mathscr{F}(y, z^*)$ is a.cn. This "forcing" of z by the choice of y arises from certain relations between elements algebraic over $\mathscr{F}(y)$, and the coefficients of P^* (the differential polynomial defined from P via $P^*(Y)=\sum p_{ij}^* Y^i(Y')^j$). Using the theory, particularly Theorem D, developed in this paper, we have shown that these relations are invariant under the isomorphisms λ_{ij} and thence that they induce analytic dependence of z on y. From this we are able to show that a y in $\mathscr{Y}$ (in fact, a one-real-parameter family of them) can be chosen such that $z=y$ *does* hold, and thus such that $\mathscr{F}(y, y^*)$ is a.cn. Since $\mathscr{F}(y, y^*)$ is obviously its own Schwarzian image, the choice of $\mathscr{G}_y$ such that $\mathscr{G}_y=(\mathscr{G}_y)^*$ is achieved in all cases.

The above results will be discussed in forthcoming papers by Strodt and the author.

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