FUNCTIONAL ANALYTIC PROPERTIES OF EXTREMELY AMENABLE SEMIGROUPS

BY
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Introduction. Let S be a semigroup $m(S)$ the Banach space of all bounded real functions on S with the norm $\|f\| = \sup\{|f(s)|; s \in S\}$ and $m(S)^*$ the conjugate Banach space of $m(S)$. $\phi \in m(S)^*$ is a mean if $\phi \geqq 0$ (i.e. $\phi(f) \geqq 0$ if $f \geqq 0$, $f \in m(S)$) and $\phi(1) = 1$ (1 stands also for the constant one function on S). The semigroup S is left amenable (LA) if there exists a mean $\phi \in m(S)^*$ which is left invariant i.e. $\phi(f_a) = \phi(f)$ for all $a \in S$, $f \in m(S)$ (we define $f_a(s) = f(as)$, $f^a(s) = f(sa)$, $l_a f = f_a$, $r_a f = f^a$, $L_a = l_a^*: m(S)^* \to m(S)^*$, $R_a = r_a^*$, for all $a, s \in S$ and $f \in m(S)$). The semigroup S is extremely left amenable (ELA) if there is a left invariant mean ϕ on $m(S)$ which is multiplicative (i.e. such that $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in m(S)$). In anology one defines (extremely) right amenable semigroups (ERA) RA.

ELA semigroups have been studied at first by T. Mitchell in [22] (under different terminology) and afterwards by this author in [15], [16]. They are known to be characterized by the following equivalent properties:

1. S is ELA.
2. S has the common fixed point property on compacta (i.e. for any compact Hausdorff space and any homomorphic representation S' of S as a semigroup of continuous maps from X to X (under functional composition), there is some $x_0 \in X$ such that $s'(x_0) = x_0$ for all $s' \in S'$ (Mitchell [22]).
3. For any $a, b \in S$ there is some $c \in S$ with $ac = bc = c$ [16].
4. For any function h of the ideal

$$H = \left\{\sum_1^n f_j(g_j - l_{s_j} g_j);\ s_j \in S, f_j, g_j \in m(S), n = 1, 2, \ldots\right\}$$

there is some $a \in S$ such that $h(a) = 0$.

(There still may be some h in the closure $\bar{H}$ of H with $h(s) \neq 0$ for all $s \in S$ even though S is ELA [16].)

5. For any $f \in m(S)$, the pointwise closure of the right orbit $\{r_s f; s \in S\}$ of f contains some constant function [16].
6. H is not dense in $m(S)$ [16].

Several other characterizations of ELA semigroups are given in [22], [15], [16].

The class of ELA semigroups is indeed immense and as shown in [15] *any* left cancellation semigroup can be imbedded in a *left cancellation* ELA semigroup. Nevertheless no right cancellation ELA semigroup (except the identity semigroup) exists.

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It is the main purpose of this paper to display and prove several functional analytic properties of ELA semigroups some of which turn out to be new characterizations of ELA semigroups (§2). Theorem 5 (§2) yields the following beautiful (we think) geometric characterization of ELA semigroups:

A semigroup S is ELA if and only if for any normed space X and any antirepresentation $\{T_s; s \in S\}$ of S as linear maps from X to X with $\|T_s\| \leqq 1$

$$\text{dist}\,(0, O(z)) = \text{dist}\,(0, \text{Co}\, O(z)) \quad \text{for all } z \in X$$

where $O(z) = \{T_s z; s \in S\}$, Co B is the convex hull of $B \subset X$ and

$$\text{dist}\,(0, B) = \inf \{\|x\|; x \in B\}.$$

This theorem as well as Theorem 6, on the representations of S as algebra homomorphisms of certain Banach algebras are inspired by a result of I. Glicksberg [30, pp. 99, 104].

Theorem 8 of this section is an analogue for extremely amenable semigroups of the Alaoglu-Birkhoff ergodic theorem, (see Dixmier [6, p. 223]) the main difference being that the convex hull of the orbit of an element is replaced by the orbit itself.

Theorem 7 shows that the class of ELA semigroups coincides with the class of semigroups possessing a kind of "multiplicative invariant extension property" for Banach algebras. It is analogous to a property of amenable semigroups obtained by R. J. Silverman in [24] and turns out to be a characterization of ELA semigroups. The proofs of Theorems 7 and 8 are different from the proofs given in the literature for the amenable case.

Theorem 4 is a generalization of a theorem of Bonsall, Lindenstrauss, and Phelps [1] on the extreme points of the set of linear nonnegative normalized operators between two function algebras and turns out again to be a characterization of ELA semigroups. It does not have an analogue to the left amenable case.

Corollary 3 provides a new purely algebraic characterization of an ELA semigroup S in terms of its antirepresentations $\{T_s; s \in S\}$ as ring homomorphisms of any (not necessarily commutative) ring R into itself, S being ELA if and only if the linear span of $\{\bigcup_{s \in S} (I - T_s)R\}$ coincides with $\bigcup_{s \in S} T_s^{-1}\{0\}$ (where $I: R \to R$ is the identity) for any R and each antirepresentation. If S is ELA then the linear span of $\{\bigcup_{s \in S} (I - T_s)R\}$ will hence be a two-sided ideal of R (and in particular $H = K$ in [16]).

In §3 of this paper, we study the support functional $q(f) = \sup \phi(f)$ of the set of left invariant means for an ELA semigroup S (where the sup is over all multiplicative left invariant means). We furthermore give some results on left almost convergence. ($f \in m(S)$ is left almost convergent to c iff $\phi(f) = c$ for any left invariant mean $\phi \in m(S)^*$.)

In Theorem 6 we give several expressions for $q(f)$ which have partial analogues for the left amenable case. In Theorem 8 we give a characterization of left almost convergence which again has a partial analogue for the left amenable case, the main difference being that the convex hull of the orbit of an element is replaced here by the orbit itself. We also give in this section a new characterization of an ELA semigroup S in terms of the functional $p(f)=\lim\sup_s f(s)$ on $m(S)$, where the lim sup is with respect to a certain partial order which renders any S, such that $aS \cap bS \neq \varnothing$ for all $a, b \in S$, a directed set. This result does not have an analogue for the amenable case.

We give in what follows an example of a semigroup S which is ELA and ERA for which there is a function f_0 which is left almost convergent to 1 (say) but not right almost convergent. It will follow, partly from Theorem 8, that the *norm* closure of $\{l_s f_0; s \in S\}$ contains a unique constant function which is 1. 1 is also the unique constant function in the *pointwise* closure of $\{r_s f_0; s \in S\}$ while the *norm* closure of $\{r_s f_0; s \in S\}$ does not contain any constant. Furthermore the *pointwise* closure of $\{l_s f_0; s \in S\}$ contains at least two different constant functions.

We end this paper with more examples of ELA semigroups. They should convince the reader that even the class of semigroups which are ELA and ERA and do not possess a zero is immense.

I. **Some notations.** If S is a semigroup, we denote by LIM (RIM) the set of left (right) invariant means on $m(S)$. LIM (RIM) will also stand as an abbreviation for "left (right) invariant mean."

If a semigroup S is extremely left amenable (ELA) and extremely right amenable (ERA) (i.e. there are multiplicative ϕ, $\psi m(S)^*$ such that $\phi \in$ LIM, $\psi \in$ RIM) then S is extremely amenable (EA) i.e. there is some multiplicative $\psi_0 \in$ LIM $\cap$ RIM (for example $\psi_0=\phi \odot \psi \in$ LIM $\cap$ RIM, see Day [4, p. 529], and ψ_0 is multiplicative).

We shall abbreviate left (right) amenable by LA (RA).

If S is a semigroup and $A \subset S$ we shall sometimes identify A with the set of point measures $\{1_a; a \in A\} \subset \beta(S) \subset m(S)^*$ and then $\bar{A}$ the w^*-closure of A in $\beta(S)$ (or for that matter in $m(S)^*$) coincides with $\{\phi \in \beta(S); \phi(A)=1\}$ as readily seen and well known. ($1_a(f)=f(a)$ for all $f \in m(S)$, $a \in S$.)

If X is a Banach space with conjugate Banach space X^* then the w-topology on X (the w^*-topology on X^*) is the weakest topology on X (on X^*) which renders all $x^* \in X^*$ (all $x \in X$) continuous linear functionals on X (on X^*).

The relation $\geqq$ renders the set D a directed set [28, p. 65] if: (a) $x \geqq x$ for all x in D. (b) If $y \geqq x$ and $z \geqq y$ with $x, y, z \in D$ then $z \geqq x$. (c) For any $x, y \in D$ there is some $z \in D$ such that $z \geqq x$ and $z \geqq y$.

If S is any semigroup such that $sS \cap tS \neq \varnothing$ for all $s, t \in S$ then the relation $t \geqq s$ iff $t \in sS \cup \{s\}$ renders S a directed set ((a) and (c) are clear and if $u \geqq t$, $t \geqq s$ then $u \in tS \cup \{t\}$ and $t \in sS \cup \{s\}$. So $u \in sS \cup \{s\}$ i.e. $u \geqq s$).

In particular if S is an amenable semigroup, then $Ss \cap St \neq \varnothing$ and $sS \cap tS \neq \varnothing$ for all $s, t \in S$. Two partial order relations are thus defined in S. The right partial order: $t \geqq s$ iff $t \in sS \cup \{s\}$. And the left partial order: $t \geqq s$ iff $t \in Ss \cup \{s\}$. Both these partial orders render S a directed set. If then $\{y_s; s \in S\}$ is a net with values in a topological space Y then $(l)-\lim_s y_s$ $[(r)-\lim_s y_s]$ will denote the limit of the net y_s with respect to the left [right] partial order of S. For any other notation we refer to [15], [16].

II. **Some functional analytic properties of ELA semigroups.** We will need in what follows the following easy lemma:

LEMMA 1. *Let S be ELA and $\phi \in m(S)^*$ be a multiplicative mean such that $\phi(sS)=1$ for all $s \in S$. If $\{1_{s_\alpha}\} \subset m(S)^*$ is any net such that $w^*-\lim_\alpha 1_{s_\alpha}=\phi$, then* $\lim_\alpha \|L_s 1_{s_\alpha}-1_{s_\alpha}\|=0$ *for all $s \in S$ (i.e. for any $s \in S$ there is some α_0 with $ss_\alpha=s_\alpha$ if $\alpha \geqq \alpha_0$).*

Proof. Let $s \in S$ and choose $t \in S$ such that $st=t$. Then $1=\phi(tS)=\lim_\alpha 1_{tS}(s_\alpha)$. Since $1_{tS} \in m(S)$ is 0, 1, valued there is some α_0 such that $s_\alpha \in tS$ if $\alpha \geqq \alpha_0$. Thus $ss_\alpha=s_\alpha$ if $\alpha \geqq \alpha_0$ or $\|L_s 1_{s_\alpha}-1_{s_\alpha}\|=\|1_{ss_\alpha}-1_{s_\alpha}\|=0$ for $\alpha \geqq \alpha_0$.

REMARKS. 1. Necessarily ϕ is a LIM (see Day [4, p. 520(A)]). This lemma yields an easier proof to Theorem 2 on p. 187 of [15].

2. In particular the set of multiplicative LIM's coincides with the set of multiplicative means on $m(S)$ which "live" on right ideals of S (i.e. such that $\phi(sS)=1$ for all $s \in S$). This identification of the set of extreme LIM's is due to J. Sorenson and is given in his thesis [25] with different proof.

3. If S is EA and ϕ is a multiplicative left and right invariant mean on $m(S)$ and if $\{s_\alpha\} \subset S$ is such that w^*-lim $1_{s_\alpha}=\phi$ then for any $s \in S$ there is some α_0 such that $ss_\alpha=s_\alpha s=s_\alpha$ if $\alpha \geqq \alpha_0$. In particular if S is an EA semigroup of linear bounded maps from the normed space X to X then there is a net $\{s_\alpha\} \subset S$ such that $\lim_\alpha \|s_\alpha s-s_\alpha\|=\lim_\alpha \|ss_\alpha-s_\alpha\|=0$ for all $s \in S$ (i.e. S is uniformly ergodic under a net $\{s_\alpha\}$ of *elements of S*. Compare with M. Day [4, p. 536]).

THEOREM 2. (a) *Let G be a group with identity e and $\{T_s; s \in S\}$ be an antirepresentation of the semigroup S as homomorphisms from G to G. Let K_G be the subgroup of G generated by all elements $\{g(T_s g)^{-1}; g \in G, s \in S\}$.*

If S is ELA then

$$(*) \qquad K_G = \bigcup_{s \in S} T_s^{-1}\{e\}$$

and the set $\{g(T_s g)^{-1}; g \in G, s \in S\}$ coincides with the group K_G.

(b) *If S is a semigroup whose antirepresentation $\{l_s; s \in S\}$ on the additive group $m(S)$ satisfies* (*) *then S is ELA.*

REMARK. If G is a linear space then $K_G=\{\sum_1^n (g_i-T_{s_i} g_i); g_i \in G, s_i \in S, n>0\}$ and K_G coincides with the *set* $\{g-T_s g; s \in S, g \in G\}$.

Proof. (a) If $T_s g=e$ for some $s \in S$ then $g=g(T_s g)^{-1} \in K_G$. Conversely if $f \in K_G$ then $f=[g_1(T_{s_1} g_1)^{-1}]^{k_1} \cdots [g_n(T_{s_n} g_n)^{-1}]^{k_n}$ with $g_i \in G$, $s_i \in S$ and integers k_i. If $s \in S$ satisfies $s_i s=s$, $i \leqq n$, then

$$T_s[g_i(T_{s_i} g_i)^{-1}]^{k_i} = [(T_s g_i)(T_s T_{s_i} g_i)^{-1}]^{k_i} = [(T_s g_i)(T_{s_i s} g_i)^{-1}]^{k_i} = e$$

and so $T_s f = e$. Consequently (*) holds and even $f = f(T_s f)^{-1}$ i.e.

$$K_G = \{g(T_s g)^{-1};\ g \in G,\ s \in S\}.$$

(b) In this case, $K = K_{m(S)} = \bigcup_{s \in S} \{f \in m(S);\ l_s f = 0\}$ and if $f \in K$, $g \in m(S)$, and $l_s f = 0$ then $l_s(fg) = (l_s f)(l_s g) = 0$. Hence K is an ideal of the algebra $m(S)$ and so $K = H = \{\sum_1^n f_i(g_i - l_{s_i} g_i);\ f_i, g_i \in m(S), s_i \in S, 1 \leqq n < \infty\}$. But moreover, H is not dense in $m(S)$. Otherwise there would be some $h \in K$ with $\|1 - h\| < \frac{1}{2}$. Then $h(s) > \frac{1}{2}$ for all $s \in S$. But then $(l_t h)(s) = h(ts) > \frac{1}{2}$ for all $t, s \in S$. Thus $l_t h \neq 0$ for all $t \in S$ which cannot be. By Theorem 2 of [16], S is ELA.

[One cannot replace "antirepresentation" by "representation" in this theorem. Since take $S = \{e_1, \ldots, e_n\}$ with $e_i e_j = e_i$ for $1 \leqq i, j \leqq n$ and $n > 1$. Then the representation $\{r_s;\ s \in S\}$ as (even algebra) homomorphisms from $m(S)$ to $m(S)$ will satisfy $\{0\} = K_{m(S)} = \bigcup_{s \in S} r_s^{-1}\{0\}$ as readily checked, but S is not ELA, so (b) does not hold, (here $K_{m(S)}$ is the linear span of $\{f - r_s f;\ f \in m(S), s \in S\}$).

For (a) consider the ELA semigroup $S' = \{e_1, \ldots, e_n\}$ with $e_i e_j = e_j$ for $1 \leqq i, j \leqq n$ (where $n > 1$) and its representation $\{r_s;\ s \in S'\}$. Let $f \in m(S')$ be defined by $f(e_j) = j$ for $1 \leqq j \leqq n$. Then $r_{e_2} f - r_{e_1} f \in K_{m(S')}$ and $(r_{e_j} f)(s) = f(e_j)$ for all $s \in S'$. Thus $r_a(r_{e_2} f - r_{e_1} f) = 1 \in m(S')$ for all $a \in S'$. Thus $K_{m(S')} \neq \bigcup_{s \in S'} r_s^{-1}\{0\}$ so (a) does not hold. It is interesting though that the *set* $\{f - r_s f;\ f \in m(S'), s \in S'\}$ coincides with $\bigcup_{s \in S'} r_s^{-1}\{0\}$ (and in particular both these sets are not linear subspaces of $m(S')$ as would be the case for antirepresentations). Since $[r_a(f - r_a f)](s) = (f - r_a f)(a) = f(a) - f(a^2) = 0$ for any $s \in S'$. Conversely if $r_b f = 0$ then $f = f - r_b f \in K_{m(S')}$, (here $K_{m(S')}$ stands for the linear span of $\{f - r_s f;\ f \in m(S'), s \in S'\}$).

Notation. In all that follows $K(H)$ will always stand for the linear span (ideal) generated by $\{f - l_s f;\ f \in m(S), s \in S\}$.

REMARK. Let $S = S_n = \{e_1, \ldots, e_n\}$ with $e_i e_j = e_i$ for $i, j \leqq n$. Consider the antirepresentation $R_s\colon m(S)^* \to m(S)^*$ of S where $R_s \phi(f) = \phi(f^s)$. Since $f = f^a$ for all $a \in S$, $f \in m(S)$ one has that $\phi - R_a \phi = 0$ for all $\phi \in m(S)^*$. Thus $K_{m(S)^*} = \{0\} = \bigcup_{s \in S} \{\phi \in m(S)^*;\ R_s \phi = 0\}$ but S is *not* ELA. This shows that the antirepresentation $\{l_s;\ s \in S\}$ over $m(S)$ is distinguished in a sense. It is an antirepresentation of S as algebra homomorphisms from $m(S)$ to $m(S)$.

COROLLARY 3. *Let A be a ring (not necessarily commutative) and $\{T_s;\ s \in S\}$ an antirepresentation of the ELA semigroup S as ring homomorphisms of A into A. Then $K_A = \{\sum_1^n (a_i - T_{s_i} a_i);\ a_i \in A, s_i \in S, n = 1, 2, \ldots\}$ satisfies*

$$(**) \qquad K_A = \{a - T_s a;\ s \in S, a \in A\} = \bigcup_{s \in S} T_s^{-1}\{0\}.$$

Furthermore K_A is a two-sided ideal of the ring A which coincides with $H_A = \{\sum_1^n b_i(a_i - T_{s_i} a_i) c_i;\ s_i \in S, a_i, b_i, c_i \in A$ (some of b_i or c_i need not appear), $n = 1, 2, \ldots\}$.

Consequently, if T is an additive map from A to the abelian group B which satisfies $TT_s a = Ta$ for all $s \in S$, $a \in A$ then $T[b(T_s a)c] = T(bac)$ for all $s \in S$, a, b, $c \in A$, (where b or c need not appear).

Proof. (**) is just (*) of Theorem 2 applied to the additive group A. If $g \in K_A$ and $T_s g = 0$ and $f \in A$ then $T_s(fg) = (T_s f)(T_s g) = 0 = T_s(gf)$ thus K_A is a two-sided ideal of A and so $K_A = H_A$. If $T: A \to B$ is as above then $T(H_A) = T(K_A) = \{0\}$. Hence $T[b(a - T_s a)c] = 0$ for all $s \in S$, $a, b, c \in A$ (where b or c need not appear).

Remark. In particular, if S is ELA not only is $\bar{K} = \bar{H}$ as shown in Theorem 6 of [16] (there $A = m(S)$, $H = H_A$, $K = K_A$) but even $K = H$. Furthermore any linear operator T on $m(S)$ which satisfies $T(g_s) = T(g)$ for all $g \in m(S)$ and $s \in S$ will also satisfy $T(fg_s) = T(fg)$ for all $f, g \in m(S)$, $s \in S$.

Examples. Let $\mathscr{F}$ be a σ-field of subsets of X and S be an ELA semigroup of $\mathscr{F}$-measurable maps $s: X \to X$. (i.e. $s^{-1}(F) \in \mathscr{F}$, if $F \in \mathscr{F}$.) If ϕ is an S-invariant finitely additive set function on $\mathscr{F}$ (i.e. $\phi(s^{-1}P) = \phi(P)$ for all $P \in \mathscr{F}$, $s \in S$) then $\phi(Q \cap s^{-1}P) = \phi(Q \cap P)$ for all $P, Q \in \mathscr{F}$ and $s \in S$. Since let A be the algebra of all simple measurable functions on X. Then $(T_s f)(x) = f(sx)$ is an antirepresentation of S as ring homomorphisms of A to A. ϕ can be viewed as a linear functional on A such that $\phi T_s f = \phi f$ for $s \in S$, $f \in A$. Thus $\phi(f(T_s g)) = \phi(fg)$. Upon taking $f = 1_P$, $g = 1_Q$ one gets the result.

We bring now a generalization of a theorem of Bonsall, Lindenstrauss and Phelps in [1] which will turn out to be a new characterization of ELA semigroups. Let A, B be algebras of bounded real functions on the sets X and Y, containing the constants. Let S be a semigroup of maps which acts from X to X i.e. such that $(st)(x) = s(tx)$ for $x \in X$, $s, t \in S$. Denote by $K_S(A, B)$ the set of linear transformations T from A to B such that $Tf \geqq 0$ if $f \geqq 0$ and $T1 = 1$ (the constant one functions on X, Y resp.) and such that $T(f_s) = T(f)$ for any $f \in A$, $s \in S$ where $f_s(x) = f(sx)$.

Theorem 4. *If S is ELA then $K_S(A, B)$ has extreme points (a fortiori $K_S(A, B) \neq \varnothing$) and any such extreme point is multiplicative.*

Remark. It is interesting, even though trivial, that this theorem holds *only* for ELA semigroups. In fact if S is not ELA take $X = S$ with $s(x) = sx$, $A = m(S)$, and Y to contain just one point, i.e. B to be the field of reals. Then $K_S(A, B)$ coincides with the set of LIM's. So, either $K_S(A, B) = \varnothing$ or if not then $K_S(A, B)$ has an extreme point (by the Kreĭn-Mil'man theorem) which cannot be multiplicative since S is not ELA.

Proof. $\beta(X) \subset m(X)^*$ with the w^* topology is compact Hausdorff and if $(L_s\phi)(f) = \phi(f_s)$, for $\phi \in \beta X$, $f \in m(X)$ and $s \in S$, then $L_s[\beta(X)] \subset \beta(X)$ and L_s are w^*-continuous and $L_s L_t = L_{st}$ for $s, t \in S$. Thus, by Mitchell's fixed point theorem, there is some $\phi \in \beta X$ such that $L_s \phi = \phi$ for $s \in S$. The restriction ϕ_0 of ϕ to A is a multiplicative mean on A with $\phi(f_s) = \phi f$ for $s \in S$, $f \in A$. Define $T: A \to B$ by $(Tf) = [\phi_0(f)]1$ where $1 \in B$ is the constant one function on Y. Then $T \in K_S(A, B)$ is multiplicative and thus even an extreme point of the set of *all* linear $T: A \to B$ which satisfy $T(1) = 1$ and $Tf \geqq 0$ if $f \geqq 0$ (see [1, Proposition 15]) and a fortiori of $K_S(A, B)$. To show that any extreme $T \in K_S(A, B)$ is multiplicative we use the idea in Lemma 1 of Bonsall, Lindenstrauss and Phelps in [1] and the previous corollary.

Let $T \in K_S(A, B)$ be extreme. For $g \in A$ with $0 \leqq g \leqq 1$ define $U_g: A \to B$ by $U_g(f) = T(gf) - (Tg)(Tf)$. Then $U_g(f_s) = T(gf_s) - (Tg)(Tf_s) = U_g(f)$ by Corollary 3. Furthermore $U_g(1) = 0$ thus $(T \pm U_g)(1) = T(1) = 1$ and $(T \pm U_g)(f_s) = (T \pm U_g)(f)$ for $f \in m(S)$, $s \in S$. If now $f \in A, f \geqq 0$ then $(T + U_g)f = T(f) + T(fg) - (Tf)(Tg) = (Tf)[1 - Tg] + T(fg) \geqq 0$ since $0 \leqq g \leqq 1$. Now $(T - U_g)(f) = T[f(1-g)] + (Tf)(Tg) \geqq 0$ which shows that $T \pm U_g \in K_S(A, B)$. Since T is extreme $U_g = 0$ i.e. $(Tf)(Tg) = T(fg)$ for all $f, g \in A$ with $0 \leqq g \leqq 1$ (thus for any $g \geqq 0$). Since A is a lattice one gets that T is multiplicative. (We assume that A is norm closed.)

REMARK. Taking S as the identity semigroup one gets a theorem of Bonsall, Lindenstrauss and Phelps in [1]. Other results of [1] could be generalized in this direction. Some boundedness condition on the functions in A is needed though. Since, if S denotes the semigroup of positive integers with the multiplication $i \vee j = \max \{i, j\}$ and A is the set of *all* real functions on S and $B = R$ is the field of reals then $K_S(A, R) = \varnothing$. For any linear ϕ on A such that $\phi(f) \geqq 0$ if $f \geqq 0$ and $\phi(1) = 1$ is of the form $\phi(f) = \sum_{i=1}^k \alpha_i f(n_i)$ where $\alpha_i \geqq 0$, $\sum_1^k \alpha_i = 1$. But if $n > n_1$ then $l_n 1_{\{n_1\}} = 0$. Thus $\alpha_1 = \phi(1_{\{n_1\}}) = \phi\{0\} = 0$. In the same way $\alpha_i = 0$ for all i. Thus $\phi(1) = 0$ which cannot be (see also [1, Theorem 8]). Note that this S is ELA.

We give now a geometric characterization of ELA semigroups inspired by a result of Glicksberg [30, p. 99, 104]. The proof is entirely different from [30].

Let $\{T_s; s \in S\}$ be a (anti)representation of the semigroup S as bounded linear maps on the normed space X. Denote by $O(x) = \{T_s x; s \in S\}$ Co $O(x)$ the convex hull of $O(x)$ and $\text{dist}(x, A) = \inf\{\|x - y\|; y \in A\}$ for all $x \in X$ and $A \subset X$. K_X as above is the linear span of $\{x - T_s x, x \in X; s \in S\}$. But $\text{Co } O(x) \subset x + K_X$ (if $y = \sum_1^n \alpha_i T_{s_i} x$ with $\sum_1^n \alpha_i = 1$. Then $y = x + \sum_1^n \alpha_i (T_{s_i} x - x) \in x + K_X$). Thus for any semigroup S:

$$\text{dist}(0, O(x)) \geqq \text{dist}(0, \text{Co } O(x)) \geqq \text{dist}(0, x + K_X) = \text{dist}(x, K_X).$$

THEOREM 5. (I) *Let $\{T_s; s \in S\}$ be an antirepresentation of the semigroup S as linear maps on the normed space X to X, with $\|T_s\| \leqq 1$ for all $s \in S$.*

(a) *If S is LA then* $\text{dist}(0, \text{Co } O(x)) = \text{dist}(x, K_X)$ *for all $x \in X$ (due to Glicksberg* [30]).

(b) *If S is ELA then even* $\text{dist}(0, O(x)) = \text{dist}(0, \text{Co } O(x)) = \text{dist}(x, K_X)$ *for all $x \in X$.*

(II) *Let S be a semigroup and consider the antirepresentation $\{l_s; s \in S\}$ over $m(S)$.*

(a) *If $\{l_s; s \in S\}$ satisfies* $\text{dist}(0, \text{Co } O\{1\}) = \text{dist}(1, K_{m(S)})$ *then S is LA and consequently* $\text{dist}(0, \text{Co } O(x)) = \text{dist}(x, K_{m(S)})$ *for all $x \in m(S)$ (compare with* [18, p. 235]).

(b) *If $\{l_s; s \in S\}$ satisfies* $\text{dist}(0, O(x)) = \text{dist}(0, \text{Co } O(x))$ *for all $x \in m(S)$ then S is ELA and consequently even* $\text{dist}(0, O(x)) = \text{dist}(x, K_{m(S)})$ *for all $x \in m(S)$.*

Proof I (a). We could refer to [30, p. 99 and p. 104]. For the sake of completeness we give a different simple proof based on Day's strong amenability theorem [4,

p. 524] (see also p. 538). For any $\phi \in m(S)^*$ which belongs to the linear span of point measures (i.e. $\phi=\sum_1^n \alpha_i 1_{s_i}$ for some $\{s_1,\dots,s_n\}\subset S$) define $T_\phi x=\sum_1^n \alpha_i T_{s_i} x$, for all $x\in X$. Then $\|T_\phi x\|\leqq \sum_1^n |\alpha_i|\,\|x\|=\|\phi\|\,\|x\|$. Now $\|T_\phi(x-T_a x)\|=\|(\sum \alpha_i T_{s_i}-\sum \alpha_i T_{as_i})x\|$ $=\|T_{\phi-L_a\phi}x\|\leqq \|\phi-L_a\phi\|\,\|x\|$, for all $x\in X$.

Let now $\{\phi_a\}\subset m(S)^*$ be a net of finite means (i.e. $\phi_\alpha\in \text{Co}\,\{1_s;\, s\in S\}$) strongly converging to left invariance (i.e. such that $\|L_a\phi_\alpha-\phi_\alpha\|\to 0$ for all $a\in S$ [4, p. 524]. Then $\|T_{\phi_\alpha}(x-T_a x)\|\to 0$ for all $x\in X$ and $a\in S$. Thus $\|T_{\phi_\alpha} y\|\to 0$ for $y\in K_X$. Notice also that $T_\psi x\in \text{Co}\, O(x)$ for any finite mean ψ.

Let now $z\in X$, $\varepsilon>0$ and $h\in K_X$ satisfy $\|z+h\|<\text{dist}\,(z, K_X)+\varepsilon$. Choose β such that $\|T_{\phi_\beta}h\|<\varepsilon$. Then

$$\|T_{\phi_\beta}z\| \leqq \|T_{\phi_\beta}(z+h)\|+\|T_{\phi_\beta}h\| < \text{dist}\,(z, K_X)+2\varepsilon.$$

Hence dist (0, Co $O(z))\leqq$ dist (z, K_X) and I(a) holds.

I(b). If $y\in K_X$ then $T_s y=0$ for some $s\in S$ by Theorem 2(a). Thus $\|x+y\|$ $\geqq \|T_s(x+y)\|=\|T_s x\|$ so dist $(x, K_X)\geqq$ dist $(0, O(x))\geqq$ dist (x, K_X).

II(a). If S is not LA then $\inf\{\|1-h\|;\, h\in K_{m(S)}\}<1$ [18, p. 235] while

$$\text{dist}\,(0, \text{Co}\, O\{1\}) = 1.$$

Invoke now I(a).

II(b). We show at first that $A\cap B\neq\varnothing$ for any two nonvoid right ideals A, B of S. If not let A, B be two nonvoid right ideals with $A\cap B=\varnothing$. Let $a\in A$, $b\in B$ and $f=1_A-1_B$. Then $f_a=1$, $f_b=-1$ so $0=\frac{1}{2}(f_a+f_b)$. Thus $0=$ dist (0, Co $O(f)$) $=$ dist (0, $O(f)$). So there is some $c\in S$ such that $\|l_c f\|<\frac{1}{2}$. Since f takes only the values ± 1 and 0 this shows that $l_c f=0$ i.e. that $1_A(cs)=1_B(cs)$ for all $s\in S$. If $t\in S$ satisfies $ct\in A$ (or $ct\in B$) then $1=1_A(ct)=1_B(ct)$ so $ct\in B$ $(ct\in A)$ which cannot be since $A\cap B=\varnothing$ therefore there is a nonvoid right ideal C (namely cS) such that $C\cap(A\cup B)=\varnothing$.

Let now A_0, B_0 be two fixed disjoint nonvoid right ideals. By an easy application of Zorn's lemma there is a family of pairwise disjoint right ideals $\mathscr{R}$ to which A_0 and B_0 belong which is maximal with respect to the property of pairwise disjointness (i.e. any other right ideal of S intersects some element of $\mathscr{R}$).

Let $A=A_0$ and B be the union of all other right ideals of $\mathscr{R}$. Then $B_0\subset B$ and A, B are nonvoid right ideals with $A\cap B=\varnothing$. Let $C\neq\varnothing$ be a right ideal with $C\cap(A\cup B)=\varnothing$. Then $\mathscr{R}_1=\mathscr{R}\cup\{C\}$ properly includes $\mathscr{R}$ and all elements of $\mathscr{R}_1$ are pairwise disjoint, which contradicts the maximality of $\mathscr{R}$. This contradiction shows that any two (and hence any finite collection of) right ideals have nonvoid intersection.

If now $f\in m(S)$, $a\in S$ and $\varepsilon>0$ then

$$\left\|\frac{1}{n}\left(\sum_1^n l_{a^i}\right)(f-l_a f)\right\| = \frac{1}{n}\,\|l_a f-l_{a^{n+1}}f\| < \frac{2}{n}\,\|f\| < \varepsilon$$

if n is big. Thus dist (0, Co $O(f-f_a))=0=$ dist (0, $O(f-f_a)$). So, $\|l_{cs}(f-f_a)\|$ $\|l_s l_c(f-f_a)\|\leqq \|l_c(f-f_a)\|<\varepsilon$ for some $c\in S$ and all $s\in S$. Let now $h=\sum_1^n g_i(f_i-l_{a_i}f_i)$

for some $f_i, g_i \in m(S)$, $a_i \in S$. Choose $c_i \in S$ with $\|l_{c_i}(f_i - l_{a_i} f_i)\| < \varepsilon/nk$ where $k = \max(\|g_1\|, \ldots, \|g_n\|)$. Choose $s_1, \ldots, s_n \in S$ such that $c_1 s_1 = c_2 s_2 = \cdots = c_n s_n = c$. Then

$$\|l_c h\| < \left(\max_{1 \leqq i \leqq n} \|g_i\|\right) \sum_1^n \|l_c(f_i - l_{a_i} f_i)\| < \varepsilon.$$

So $\|1-h\| \geqq \|l_c(1-h)\| = \|1 - l_c h\| > 1 - \varepsilon$ i.e. $\|1-h\| \geqq 1$ for all $h \in H = H_{m(S)}$. Hence $\inf\{\|1-h\|; h \in H\} = 1$. Theorem 2 of [16] implies now that S is ELA.

REMARKS. 1. If the hypothesis of II(b) would be replaced by the stronger assumption that $\operatorname{dist}(0, O(x)) = \operatorname{dist}(x, K_{m(S)})$ for all $x \in m(S)$ then the following easier proof can be provided: Since $1 = \operatorname{dist}(0, O\{1\}) = \inf\{\|1-h\|; h \in K_{m(S)}\}$, $K_{m(S)}$ is not dense in $m(S)$. If now $f \in K_{m(S)}$ and $g \in m(S)$ is arbitrary then

$$\operatorname{dist}(fg, K_{m(S)}) = \inf_s \|l_s f l_s g\| \leqq \|g\| \inf \|l_s f\| = \|g\| \operatorname{dist}(f, K_{m(S)}) = 0.$$

Thus $fg \in \bar{K}_{m(S)}$ and $\bar{K}_{m(S)}$ is an ideal which is not dense in $m(S)$ and contains $H_{m(S)}$. By Theorem B in [16] S is ELA.

2. One can now state the following beautiful geometric characterization of ELA semigroups:

THEOREM. *A semigroup S is ELA if and only if it enjoys the following geometric property.*

(G) *For any normed space X and any antirepresentation $\{T_s; s \in S\}$ of S as linear maps from X to X with $\|T_s\| \leqq 1$, for all $s \in S$,* $\operatorname{dist}(0, O(x)) = \operatorname{dist}(0, \operatorname{Co} O(x))$ *holds for all $x \in X$. For S to be ELA, it suffices that* (G) *holds for the antirepresentation $\{T_s = l_s; s \in S\}$ over $m(S)$.*

We find it striking that the geometric property (G) is equivalent to the algebraic property that each $s, t \in S$ admit a common right zero, or to the topological property that S has the common fixed point property on compacta.

3. Let G be any amenable group with identity e and $G \neq \{e\}$. We show that any left translation invariant subalgebra $A \subset m(G)$ which contains a nonconstant function f also contains some function h with $\operatorname{dist}(0, O(h)) \neq \operatorname{dist}(0, \operatorname{Co} O(h))$. Since if $f \in A$ satisfies $f(a) \neq f(b)$ where $a, b \in G$ and $c = ab^{-1}$ then $h = f - f_c$ will satisfy $h(b) \neq 0$ so $0 \neq \|h\| = \|l_s h\|$ for all $s \in G$. Thus $0 \neq \operatorname{dist}(0, O(h))$. But $h \in K_{m(S)}$ so $0 = \operatorname{dist}(h, K_{m(S)}) = \operatorname{dist}(0, \operatorname{Co} O(h))$ by I(a).

Consider the group Z of additive integers. Let $x \in m(Z)$ be defined by $x(n) = (-1)^n$. Then $\|l_n x\| = \|x\| = 1$ for all $n \in Z$ so $\operatorname{dist}(0, O(x)) = 1$. But $\frac{1}{2}(x + l_1 x) = 0$ so $\operatorname{dist}(0, \operatorname{Co} O(x)) = 0$.

4. It follows under the assumptions of I(a) that for any $x \in X$ with $\operatorname{dist}(0, \operatorname{Co} O(x)) = c > 0$, there is some $x^* \in X^*$ with $\|x^*\| = 1$, $x^*(K_X) = 0$ and $|x^*(x)| = c = \operatorname{dist}(x, K_X)$. (Invoke the Hahn-Banach theorem.) This is again a result of I. Glicksberg [30]. In partial analogy to it one has

THEOREM 6. *Let S be ELA and $\{T_s; s \in S\}$ be an antirepresentation of S as linear multiplicative maps from the complex commutative Banach algebra X to X. Assume* *also that X has identity and that $\|x\| = \sup_{x^* \in \mathcal{M}} |x^*(x)|$ for all $x \in X$ where $\mathcal{M}$ is the set of all multiplicative $x^* \in X^*$.*

Then for all $x \in X$

$$\text{dist}\,(0, O(x)) = \max_{x^* \in \mathcal{M}_S} |x^*(x)|$$

where $\mathcal{M}_S = \{x^* \in \mathcal{M};\, T_s^* x^* = x^*$ for all $s \in S\}$.

This property characterizes ELA semigroups.

Proof. Our conditions imply that $\|T_s\| \leqq 1$ for all $s \in S$, and that $\mathcal{M} \subset X^*$ and hence $\mathcal{M}_S$ are compact hausdorff in the w^* topology.

Let $\psi \in m_c(S)^*$ be multiplicative on $m_c(S)$ (all complex bounded functions on S) and $x^* \in \mathcal{M}$. Then $\psi \odot x^* \in X^*$ defined by $(\psi \odot x^*)(x) = \psi[x^*(T_s x)]$ belongs to $\mathcal{M}_S$, if ψ is left invariant (i.e. if $L_s\psi = \psi$ for all $s \in S$). This is readily checked. Then

$$\begin{aligned}\text{dist}\,(0, O(x)) &= \inf_s \sup_{x^* \in \mathcal{M}} |x^*(T_s x)| \leqq \inf_s \sup_t \sup_{x^* \in \mathcal{M}} |x^*(T_{st} x)| \\ &= \sup_{x^* \in \mathcal{M}} \left[\inf_s \sup_t |F(st)|\right]\end{aligned}$$

where $F(u) = x^*(T_u x)$ for $u \in S$.

By Theorem 6 of the next section $\inf_s \sup_t |F(st)| = \sup_\phi \phi(|F|)$ where ϕ ranges over all multiplicative LIM's ϕ on $m(S)$ (see Theorem A. 5 in the introduction of [16]). But if ϕ is a multiplicative LIM on $m(S)$ then $\phi_c \in m_c(S)^*$ defined by $\phi_c(f+ig) = \phi(f) + i\phi(g)$ for $f, g \in m(S)$ is multiplicative on $m_c(S)$, $\phi_c(l_s G) = \phi_c(G)$ for all $s \in S$, $G \in m_c(S)$ and $\phi_c(G) = \phi(G)$ if $G \in m(S)$. Since any multiplicative $\psi \in m_c(S)$ satisfies $|\psi(G)| = \psi(|G|)$ one has

$$\begin{aligned}\text{dist}\,(0, O(x)) &= \sup_{x^* \in \mathcal{M}} \sup_\phi \phi(|F|) = \sup_{x^* \in \mathcal{M}} \sup_\phi |\phi_c(F)| \\ &= \sup_{x^* \in \mathcal{M}} \sup_\phi |(\phi_c \odot x^*)(x)| \leqq \sup_{y^* \in \mathcal{M}_S} |y^*(x)| \\ &= \max_{y^* \in \mathcal{M}_S} |y^*(x)| \leqq \text{dist}\,(0, O(x))\end{aligned}$$

(where $\sup_\phi$ denotes sup over all multiplicative LIM's ϕ on $m(S)$), since $y^*(x) = y^*(y)$ for any $y^* \in \mathcal{M}_S$ and $y \in O(x)$. Hence for any $x \in X$ there is some $x^* \in \mathcal{M}_S$ with $|x^*(x)| = \text{dist}\,(0, O(x))$. (Notice that $0 \in \mathcal{M}_S$ according to our definition of $\mathcal{M}$ and $\mathcal{M}_S$.)

Considering the representation $\{l_s;\, s \in S\}$ over $m_c(S)$ and assuming the theorem to hold for $X = m_c(S)$ one has

$$1 = \text{dist}\,(0, O\{1\}) = \max_{\phi \in \mathcal{M}_S} |\phi(1)| = |\phi_0(1)| \quad \text{for some } 0 \neq \phi_0 \in \mathcal{M}_S$$

where $\mathcal{M}_S$ is the set of multiplicative left invariant $\phi \in m_c(S)^*$. ϕ_0 restricted to $m(S)$ will be a multiplicative LIM as readily seen.

Remarks. 1. Upon taking S as the identity semigroup in Theorem 6 one has $O(x) = \{x\}$ so $\|x\| = \text{dist}\,(0, O(x)) = \max_{x^* \in \mathcal{M}} |x^*(x)| (\mathcal{M} = \mathcal{M}_S$ in this case). Hence, the assumption $\|x\| = \sup_{x^* \in \mathcal{M}} |x^*(x)|$ for all $x \in X$, cannot be relaxed.

2. Under the assumptions of Theorem 6, consider the linear span K_X of $\{x-T_s x;\, s\in S,\, x\in X\}$. Then $K_X=H_X$ is an ideal of the commutative Banach algebra X (Corollary 3). The space of maximal ideals of the quotient Banach algebra $A=X/\overline{H}_X$ coincides with the set of multiplicative $\phi\in X^*$ which satisfy $\phi(H_X)=0$ [20, p. 63] which is just $\mathcal{M}_S$. Denoting by $\tilde{x}=x+\overline{H}_X$ one has for the norm in A,

$$\|\tilde{x}\| = \inf\{\|x+h\|,\, h\in H_X\} = \operatorname{dist}(0, O(x)) = \operatorname{dist}(0, \operatorname{Co} O(x)) = \max_{\phi\in\mathcal{M}_S} |\phi(x)|.$$

Thus the Gelfand homomorphism of A onto its image $\hat{A}$ is an isometry and $\hat{A}=C_c(\mathcal{M}_S)=$ all complex continuous functions on $\mathcal{M}_S\subset X^*$ with the w^* topology.

In particular let $X=m_c(S)$, $T_s=l_s\colon m_c(S)\to m_c(S)$. Then

$$H_X = H_c = \{f-l_s f;\, s\in S,\, f\in m_c(S)\}$$

(Corollary 3). Hence the image of $m_c(S)/\overline{H}_c$ under the Gelfand homomorphism coincides with $C_c(\mathcal{M}_S)$ where $\mathcal{M}_S$ can be identified with the set LIM $\cup\, \{0\}\subset m(S)^*$ in the w^* topology (via the extension to $m_c(S)$ given by $\phi(f+ig)=\phi(f)+i\phi(g)$). Furthermore for any $f\in m_c(S)$ one has

$$\inf_s \sup_t |f(ts)| = \inf_{h\in H_c} \sup_t |f(t)+h(t)| = \max |\psi(f)| = \inf_{\mathcal{A}} \sup_t \left|\sum_1^n \alpha_i f(a_i t)\right|$$

where the max is over all multiplicative left invariant $\psi\in m_c(S)^*$ and

$$\mathcal{A} = \left\{(\alpha_1,\ldots,\alpha_n, a_1,\ldots,a_n),\, \alpha_i \geqq 0, \sum \alpha_i = 1, a_i\in S, n = 1, 2, \ldots\right\}.$$

Another characterization of ELA semigroups is given in terms of a "multiplicative invariant extension property" below. It is partly analogous to a property of amenable semigroups given by R. J. Silverman [24, p. 75]:

Theorem 7. *Let $\{T_s;\, s\in S\}$ be an antirepresentation of the ELA semigroup S as continuous algebra homomorphisms from the Banach algebra B into B. Let $A\subset B$ be a subalgebra such that $T_s(A)\subset A$ for all $s\in S$. Let $\phi\in A^*$ be multiplicative and satisfy $\phi(T_s x)=\phi(x)$ for all $x\in A$ and $s\in S$. If there is some multiplicative $\psi\in B^*$ which extends ϕ then there is a multiplicative extension $\psi_0\in B^*$ of ϕ for which $\psi_0(T_s y)=\psi_0(y)$ for all $y\in B$ and $s\in S$. Conversely, any semigroup which has this "multiplicative invariant extension property" in necessarily ELA.*

Proof. Let $\mathcal{K}=\{\psi\in B^*;\, \psi$ is a multiplicative extension of $\phi\}$. Then $\mathcal{K}$ is a w^*-closed subset of the unit ball of B^* and is hence w^*-compact. Each $T_s^*\colon B^*\to B^*$ is w^*-continuous and $T_s^*T_t^*=T_{st}^*$ so $\{T_s^*;\, s\in S\}$, as a homomorphic image of S, is ELA. Furthermore $T_s^*(\mathcal{K})\subset\mathcal{K}$ for all $s\in S$. Since if $\psi\in\mathcal{K}$, $x\in A$, $y, z\in B$ then $(T_s^*\psi)(x)=\psi(T_s x)=\phi(T_s x)=\phi(x)$ and $(T_s^*\psi)(yz)=\psi[(T_s y)(T_s z)]=(T_s^*\psi)(y)(T_s^*\psi)(z)$. Thus $T_s^*\psi\in\mathcal{K}$. By Mitchell's fixed point theorem there is some $\psi_0\in\mathcal{K}$ such that $T_s^*\psi_0=\psi_0$ for all $s\in S$. ψ_0 is the required extension.

Let now S be a semigroup, $B=m(S)$, and A be the algebra of all constant functions in B. Let $T_s=l_s\colon B\to B$ for $s\in S$ and define ϕ on A by $\phi(c1)=c$. If $a\in S$

is fixed, define $\psi \in m(S)^*$ by $\psi(f)=f(a)$ for $f \in m(S)$. Then ψ is a multiplicative extension of ϕ. If S has the multiplicative invariant extension property then there is a multiplicative $\psi_0 \in m(S)^*$ which extends ϕ and such that $\psi_0(f_s)=\psi_0(f)$ for $s \in S, f \in m(S)$. $\psi_0 \neq 0$ since $\psi_0(1)=\phi(1)=1$. ψ_0 is necessarily a mean and so S is ELA.

The next theorem is the analogue to EA semigroups of the Birkhoff-Alaoglu ergodic theorem (see Dixmier [6, pp. 223–224]). The proof given is short and straightforward. We remind the reader that, if y_s is a net on S with values in the Banach space Y the norm $(l)-\lim_s y_s$ [norm $(r)-\lim_s y_s$] denotes the limit, in the norm topology, of the net y_s, with respect to the left [right] partial order in S.

THEOREM 8. *Let X be a (real or complex) Banach space and $\{T_s; s \in S\}$ be a [anti]representation of the EA semigroup S as linear bounded maps from X to X, such that $\{T_s x; s \in S\}$ is w-conditionally compact for all $x \in X$. Then* norm (l)-$\lim_s T_s x = Px$ [norm (r)-$\lim_s T_s x=Px$] *exists for all $x \in X$, P is a linear bounded projection onto $F=\{x \in X; T_s x=x$ for all $s \in S\}$ and $P^{-1}\{0\}=\bar{K}_X$=norm closure of*

$$\{x-T_s x;\ x \in X,\ s \in S\}.$$

(Thus $X=F \oplus \bar{K}_X$.) Furthermore, for any $x \in X$, the closed convex hull of $\{T_s x; s \in S\}$ (denoted by $C(x)$) intersects F in the unique point Px.

Proof. Assume that $\{T_s; s \in S\}$ is a representation and X is complex. Then, since each $T_s: X \to X$ is w-w continuous [7, p. 422] T_a[w-closure of $\{T_s x; s \in S\}$] $\subset w$-closure of $\{T_s x; s \in S\}$, which is w-compact. By Mitchell's fixed point theorem there is some $x_0 \in w$-closure of $\{T_s x; s \in S\} \subset C(x)$ such that $T_s x_0 = x_0$ for all $s \in S$. Hence $C(x) \cap F \neq \varnothing$ for all $x \in X$. Let now $y \in C(x) \cap F$ be arbitrary, $\beta = \sup_s \|T_s\| < \infty$ [7, p. 53] and $\varepsilon > 0$. Let $\sum_1^n \beta_i(T_{s_i}x)$ be a convex combination such that $\|y-\sum_1^n \beta_i T_{s_i}x\| < \varepsilon/\beta$. Let $a \in S$ satisfy $as_i = a$, $1 \leqq i \leqq n$. Then for any $s \in Sa \cup \{a\}$, $T_s T_{s_i} = T_s$ and so:

$$\|T_s x - y\| = \|T_s x - T_s y\| = \left\|T_s\left(\sum \beta_i T_{s_i} x - y\right)\right\| < \beta\varepsilon/\beta = \varepsilon.$$

By definition then norm (l)-$\lim_s T_s x$ exists and equals y where $y \in C(x) \cap F$ is arbitrary. Hence $C(x) \cap F$ contains a unique element, denoted by Px, and Px=norm $(l)-\lim_s T_s x$ for $x \in X$. Thus P is a linear map $\|P\| \leqq \sup_s \|T_s\|$ and $Px=x$ if $x \in F$. But $P(T_a x) \in C(T_a x) \cap F \subset C(x) \cap F = \{Px\}$ and $T_a(Px) \subset T_a[C(x)] \cap T_a(F) \subset C(x) \cap F = \{Px\}$. So $T_a P = P T_a = P$ for all $a \in S$. Hence $PX=F$ (so $P^2=P$) and $F=\{x \in X; Px=x\}$. Furthermore $P(x-T_s x)=0$ so $\bar{K}_X \subset P^{-1}\{0\}$. If now x is such that $Px=0$ then $x=x-Px$ is in the norm closure $\bar{K}_X$ of the set $K_X=\{x-T_s x; s \in S\}$. Thus $P^{-1}\{0\}=\bar{K}_X$.

If now $y \in X$ then $y=Py+(y-Py)$. Thus $X=PX \oplus P^{-1}\{0\}=F \oplus \bar{K}_X$ since $P(X) \cap P^{-1}\{0\}=\{0\}$.

REMARKS. 1. Let $S=\{e_1, \ldots, e_n\}$, $n>1$, with $e_i e_j = e_j$ for $i, j \leqq n$. Let $f_0 \in m(S)$ be defined by $f_0(e_j)=j$, $1 \leqq j \leqq n$. Then $r_{e_j} f_0(s)=j$ for $s \in S$. Thus for the representation $\{r_s; s \in S\}$ over $X=m(S)$ one has that $\{r_s f_0; s \in S\} \cap F$ coincides with the $n>1$ constant functions $\{1, 2\cdot 1, \ldots, n\cdot 1\}$. Thus $C(f_0) \cap F$ contains *more* than one point. This comes about, since S is ELA but not ERA (not even RA). We see that the assumptions on S in the above theorem cannot be entirely dropped. They can though be relaxed to: S is ERA and LA. We conjecture though, that any ERA and LA semigroup is necessarily ELA. In fact if S is ERA and LA but not ELA then the semigroup $S/(r)$ (see [14, p. 371]) would be a right cancellation ERA and LA semigroup which contains at least two elements. It would thus be a counterexample to a conjecture of Sorenson (and also to one made by us in Bull. Amer. Math. Soc. **72** (1966), p. 1031 which we believe to hold true).

THEOREM 9. *Let X be a Banach algebra, $\{T_s; s \in S\}$ a representation of the EA semigroup S as algebra homomorphisms from X to X. Assume that $\{T_s x; s \in S\}$ is w-conditionally compact for any $x \in X$. Then $Px=$*norm (l)-$\lim_s$ *$T_s x$ exists for all $x \in X$, P is a bounded multiplicative projection onto the closed subalgebra*

$$F = \{x; T_s x = x, \textit{for all } s \in S\},$$

$P^{-1}\{0\}=\bar{K}_X=$norm closure of $\{x-T_s x; x \in X, s \in S\}$ is a closed two-sided ideal and $X=F \oplus \bar{K}_X$.

Proof. By Corollary 3, K_X is a two-sided ideal and since T_s are bounded and multiplicative F is a closed subalgebra. Furthermore, if $x, y \in X$ then $P(xy)=$ norm $(l)-\lim_s T_s(xy)=$norm $(l)-\lim_s (T_s x)(T_s y)=(Px)(Py)$. (Notice that X need not be commutative.)

An extension of the concept of a multiplicative invariant mean which is analogous to the result of Dixmier [6, p. 219] for the left amenable case is given in the following theorem.

THEOREM 10. *Let S be a family of maps from X to X. Let $A \subset m(X)$ be an S-invariant subalgebra with $1 \in A$, which admits an S-invariant multiplicative mean. Let B be a semisimple complex commutative Banach algebra. Let $\mathscr{A}_A^B(X)=\mathscr{A}$ be a S-invariant algebra of functions $f\colon X \to B$ for which:*

(a) *$\psi[f(x)] \in A_c=A+iA$ for all $\psi \in B^*$ and $f \in \mathscr{A}$,*

(b) *The w-closure of $\{f(x); x \in X\} \subset B$ is w-compact. Then there exists a linear operator $T\colon \mathscr{A}_A^B(X) \to B$ with satisfies for any $f, g \in \mathscr{A}$, $s \in S$:* 1. *$Tf \in$ w-closure $\{f(x); x \in X\}$.* 2. *$T(fg)=(Tf)(Tg)$ and* 3. *$Tf_s=Tf$ (where $f_s(x)=f(s(x))$, $x \in X$, $s \in S$).*

Proof. Let μ_1 be a multiplicative S-invariant mean on $A \subset m(X)$ and define μ on $A_c=A+iA \subset m_c(X)$ by: $\mu(f+ig)=\mu_1(f)+i\mu_1(g)$ for $f, g \in A$. Then $\mu \in A_c^*$ is a multiplicative S-invariant extension of μ_1. Let now $\{x_\alpha\} \subset X$ be a net such that $\mu(F)=\lim_\alpha F(x_\alpha)$ for all $F \in A_c$. (If A_c would separate the points of X this would follow from Corollary 19, p. 276 and Lemma 25 on p. 278 in [7]. If A_c does not separate points then A_c can be viewed as a point separating algebra on the set of equivalence classes of X by the relation: $x \sim y$ iff $F(x)=F(y)$ for all $F \in A_c$. One readily gets that a net $\{x_\alpha\} \subset X$ as above again exists.) Hence $\mu[\psi(f(x))] = \lim_\alpha \psi(f(x_\alpha))$ for any $\psi \in B^*$ and $f \in \mathscr{A}$.

Now for fixed $f \in \mathscr{A}$ there is a subnet x_{α_β} such that $w\text{-}\lim f(x_{\alpha_\beta}) = x_f$, for some $x_f \in B$. Hence

$$\lim_\beta \psi(f(x_{\alpha_\beta})) = \psi(x_f) = \lim_\alpha \psi(f(x_\alpha)) \quad \text{for } \psi \in B^*,$$

i.e. $w\text{-}\lim f(x_\alpha) = x_f$ exists for all $f \in \mathscr{A}$. Define now $Tf = x_f$. Then

$$\psi(Tf) = \lim_\alpha \psi(f(x_\alpha)) = \mu[\psi(f(x))]$$

and

$$\psi(Tf_s) = \mu[\psi(f_s(x))] = \mu[\psi(f(x))] = \psi(Tf) \quad \text{for all } \psi \in B^*,\ s \in S,\ f \in \mathscr{A}.$$

Furthermore T is linear (even $\|Tf\| \leqq \sup_x \|f(x)\|$) hence 1 and 3 hold. But T is even multiplicative. Let $\phi \in B^*$ be multiplicative, then

$$\phi(T(fg)) = \mu[\phi(f(x)g(x))] = \mu[\phi(f(x))]\mu[\phi(g(x))] = \phi(Tf)\phi(Tg) = \phi(Tf\,Tg)$$

for $f, g \in \mathscr{A}$. But B is commutative and semisimple and so the multiplicative $\phi \in B^*$ separate the points of B. Hence $T(fg) = TfTg$ and 3 holds.

Remarks. 1. If S is ELA one can take $A = m(X)$ hence condition (a) can be dropped.

2. If S is generated by one element s and $X = S$ and if $\lim_{n\to\infty} \psi[f(s^n)]$ exists for all $f \in \mathscr{A}$ and $\psi \in B^*$ then one can take as $A = c$ the space of real convergent sequences and condition (a) of the theorem holds since $c \subset m$ is ELA.

3. Take $X = S$, $A = m(S)$, and $B = R$ the field of reals. Then $\mathscr{A} = m(S)$. If an operator T satisfying 1, 2, 3 exists then T is a multiplicative LIM on S and S is ELA.

4. If B in this theorem is only a locally convex linear topological space then, under the same conditions, there exists a linear operator T for which 1 and 3 hold.

III. **The support functional for the set of LIM's and left almost convergence.** Let S be a semigroup having the finite intersection property for right ideals (f.i.p.r.i.) i.e. such that $\bigcap_1^n s_i S \neq \varnothing$ for any finite subset $\{s_1, \ldots, s_n\} \subset S$. Then the (right) partial order defined by $b \geqq a$ iff $b \in aS \cup \{a\}$ renders S a directed set (see introduction). We shall deal in what follows *only* with this (right) partial order, even when S will be amenable. In the whole of this section

$$\limsup_s f(s) = \inf_s \left[\sup_{t \geqq s} f(t)\right], \quad \liminf_s f(s) = \sup_s \left[\inf_{t \geqq s} f(t)\right] \quad \text{and} \quad \lim_s f(s)$$

are defined with respect to this *right* partial order.

Denote by $Q(f) = \limsup_s f(s)$. Then $Q(f+g) \leqq Q(f) + Q(g)$, $Q(\lambda f) = \lambda Q(f)$ if $\lambda \geqq 0$ and $-Q(-f) = \liminf_s f(s)$. Furthermore $Q(f) = \inf_{s\in S} [\sup_{t\in S} f(st)]$. Since $\sup_{u \geqq s} f(u) \geqq \sup_t f(st)$, ($u \geqq s$ iff $u \in sS \cup \{s\}$), one has $Q(f) \geqq \inf_s \sup_t f(st)$. But if $t_0 \in S$ then $\sup_{u \geqq st_0} f(u) \leqq \sup_t f(st)$. Hence $Q(f) \leqq \sup_t f(st)$ for any $s \in S$ thus

$Q(f)=\inf_s \sup_t f(st)$. Moreover $Q(f_a)=Q(f)$ for any $a \in S$. Clearly $Q(f_a)=\inf_s \sup_t f(ast) \geqq Q(f)$. But if $s \in S$ then $\sup_t f(st) \geqq \sup_t f(ut)$ for any $u \in sS \cap aS$. Hence $\sup_t f(st) \geqq \inf_v \sup_t f(avt)=Q(f_a)$. Hence $Q(f)=Q(f_a)$. We have thus proved the

PROPOSITION 1. *Let S have the f.i.p.r.i. and let $Q(f)=\lim \sup_s f(s)$ for $f \in m(S)$. Then $Q(f)=\inf_s \sup_t f(st)$, $Q(\lambda f)=\lambda Q(f)$ if $\lambda \geqq 0$, $Q(f+g) \leqq Q(f)+Q(g)$ and $Q(f_a)=Q(f)$ for any $a \in S$ and $f, g \in m(S)$.*

REMARK. We did not assume that S is LA or ELA but only that S has the f.i.p.r.i. For example any group S has this property and in this case

$$Q(f) = \sup_{s \in S} f(s).$$

Denote by $M \subset m(S)^*$ the set of means and by $M_r=\{\phi \in M;\ \phi(sS)=1$ for each s in $S\}$ (i.e. the set of means which "live" on right ideals). Then M_r is convex and w^*-compact. We show in the next proposition that Q is the "support functional" of the convex set M_r.

PROPOSITION 2. *Let S have the f.i.p.r.i. Then $Q(f)=\sup_{\phi \in M_r} \phi(f)$. Consequently a linear functional ψ on $m(S)$ satisfies $\psi(f) \leqq Q(f)$ for all $f \in m(S)$ if and only if $\psi \in M_r$.*

Proof. Let $\phi \in M_r$ and assume that $f, g \in m(S)$ are such that $f(t)=g(t)$ for $t \in s_0S$. Then $|\phi(f-g)| \leqq \phi(|f-g|)=\phi(|f-g|1_A) \leqq \|f-g\|\phi(A)=0$, where A is the complement of s_0S in S. Thus $\phi(f)=\phi(g)$ if $f=g$ on some s_0S. If $f \in m(S)$ let $s_0 \in S$ and define $f_1 \in m(S)$ by $f_1(t)=f(t)$ for $t \in s_0S$ and $f_1(s)=\sup_t f(s_0t)$ if $s \notin s_0S$. Then $\phi(f)=\phi(f_1) \leqq \sup_s f_1(s)=\sup_t f(s_0t)$. Thus $\phi(f) \leqq \inf_s \sup_t f(st)=Q(f)$ and so $\sup_{\phi \in M_r} \phi(f) \leqq Q(f)$. Let now $f_0 \in m(S)$ be fixed and $Q(f_0)=\alpha$. Then there is a linear functional ψ on $m(S)$ such that $\psi(f) \leqq Q(f)$ for $f \in m(S)$ and $\psi(f_0)=Q(f_0)$, by the Hahn-Banach theorem. But ψ necessarily will satisfy $-Q(-f) \leqq \psi(f) \leqq Q(f)$ for any $f \in m(S)$. If $f \geqq 0$ then $-Q(-f)=\lim \inf_s f(s) \geqq 0$ and $Q(1)=-Q(-1)=1$. This implies that ψ is a mean. Let $a \in S$. Then $\inf_t 1_{aS}(at)=1$. Thus $\sup_s \inf_t 1_{aS}(st)=1=-Q(-1_{aS})$. Thus $\psi(aS) \geqq 1$. Since ψ is a mean and $\psi(aS)=1$ for any $a \in S$, $\psi \in M_r$ and $\psi(f_0)=Q(f_0)$. Thus $\sup_{\psi \in M_r} \psi(f_0) \geqq Q(f_0)$. Since any linear ψ on $m(S)$ dominated by Q is necessarily a mean (and hence in $m(S)^*$) and by Corollary 2 on p. 22 of [2] we get this proposition.

REMARK. Let S have the f.i.p.r.i. then any extreme point ϕ of M_r is an extreme point of the set of means M (i.e. multiplicative). Since if $\phi=\frac{1}{2}(\phi_1+\phi_2)$ with $\phi_1, \phi_2 \in M$ then $\phi_1(aS)+\phi_2(aS)=2$ and $\phi_i(aS) \leqq 1$. Hence $\phi_1(aS)=\phi_2(aS)=1$ for $a \in S$. Thus $\phi_1, \phi_2 \in M_r$ and $\phi_1=\phi_2$. In fact the set of extreme points of M_r coincides with the set $\bigcap_{s \in S}(sS)^-$, where bar denotes w^*-closure in βS. (This intersection is not empty since $\bigcap_{i=1}^n s_iS \neq \varnothing$ for $\{s_1, \ldots, s_n\} \subset S$.) Semigroups with the f.i.p.r.i. and in particular the set $\bigcap_{s \in S}(sS)^-$ have been considered in more detail by J. Sorenson in [25]. The above expression for $Q(f)$ does though not appear there.

We get as a corollary the following characterization of ELA semigroups:

COROLLARY 3. *Let S be a semigroup.*

(a) *If S is ELA then* $\limsup_s (f-f_a)(s) = \liminf_s (f-f_a)(s) = 0$ *for any* $f \in m(S)$ *and* $a \in S$.

(b) *If S has the f.i.p.r.i. and* $\limsup_s (f-f_a)(s) \leqq 0$ *for each* $f \in m(S)$ *and* $a \in S$ *then S is ELA.*

Proof (a). If S is ELA and $a \in S$ let $b \in S$ satisfy $ab=b$. If $c \geqq b$ then $c=b$ or $c=bc'$ with $c' \in S$. In both cases $f(c)-f(ac)=0$.

(b) If S has the f.i.p.r.i. and $Q(f-f_a) \leqq 0$ for $f \in m(S)$ and $a \in S$ then let $\phi \in \bigcap_{s \in S} (sS)^-$ (where bar denotes w^*-closure in βS). Then $\phi \in M_r$ and $\phi(f) \leqq Q(f)$ for all $f \in m(S)$. Thus $\phi(f-f_a) \leqq 0$ and $\phi((-f)-(-f_a)) \leqq 0$ i.e. $\phi(f-f_a)=0$ for all $a \in S$. Thus ϕ is a multiplicative LIM and S is ELA.

COROLLARY 4. *If S is ELA then the set of LIM's coincides with* M_r *and consequently the support functional of the set of LIM's is*

$$Q(f) = \inf_s \sup_t f(st) = \limsup_s f(s) = \sup_{\phi \in M_r} \phi(f).$$

Proof. From Lemma 1 of §2, M_r and the set of LIM's have the same extreme points and are both w^*-compact and convex. By the Kreĭn-Mil'man theorem they coincide. $Q(f)$ being the support functional of M_r is also the support functional of the set of LIM's.

REMARK. If S is LA and M_r coincides with the set of LIM's then S is ELA (since $\varnothing \neq \bigcap_{s \in S} (sS)^- \subset M_r \cap \beta S$).

LEMMA 5. *If S is ELA then* $H=K$ *and*

$$\inf_{h \in H} \sup_s [f(s)+h(s)] = \inf_s \sup_t f(st) \quad \textit{for } f \in m(S).$$

Proof. That $H=K$ is immediate from Corollary 3 of §2.

Let $f \in m(S)$ be fixed. For any LIM ϕ and any $h \in H=K$ one has $\phi(f)=\phi(f+h) \leqq \sup_s [f(s)+h(s)]$. Thus $\phi(f) \leqq \inf_{h \in H} \sup_s [f(s)+h(s)]$. Taking sup over ϕ in the set of LIM's one gets $Q(f) \leqq \inf_{h \in H} \sup_{s \in S} [f(s)+h(s)]$. Let now $a \in S$ and define $h \in H$ by $h(aS)=0$ and $h(s)=-f(s)+\sup_{t \in S} f(at)$ if $s \notin aS$. Then $h=h-h_a \in H$

$$\sup_{s \in S} [f(s)+h(s)] \leqq \sup_{t \in S} f(at).$$

Hence $\inf_{h \in H} \sup_{s \in S} [f(s)+h(s)] \leqq \sup_{t \in S} f(at)$ for any $a \in S$ which by Corollary 4 implies this lemma.

REMARKS. 1. It has been shown by Følner in [10, pp. 5–6] that if G is an amenable *group* then

$$\inf_{h \in H} \sup_{s \in G} [f(s)+h(s)] = \inf_{\mathscr{A}} \sup_{s \in G} \sum_1^N \alpha_n f(a_n s)$$

where

$$\mathscr{A} = \left\{(\alpha_1, \ldots, \alpha_N; a_1, \ldots, a_N),\ \alpha_n \geqq 0,\ \sum_1^N \alpha_n = 1,\ a_n \in G,\ N = 1, 2, \ldots\right\}$$

and this common value is the support functional for the set of LIM's. He uses in his proof his deep characterization of amenable groups (see [9]). The above lemma could also be proved by first generalizing Følner's result in [10] to semigroups, using the result of Namioka in [22] and then showing that for ELA semigroups S one has $\inf_s \sup_t f(st) = \inf_{\mathscr{A}} \sup_t \sum_1^N \alpha_n f(a_n t)$. This equality is readily seen: for any $(\alpha_1, \ldots, \alpha_N, a_1, \ldots, a_N)$ let $s_0 \in S$ satisfy $a_i s_0 = s_0$ for $1 \leqq i \leqq N$. Then

$$\sup_t \sum \alpha_n f(a_n t) \geqq \sup_t \sum \alpha_n f(a_n s_0 t) = \sup_t f(s_0 t) \geqq \inf_s \sup_t f(st)$$

which readily implies the above equality.

2. Let Z denote the additive group of integers and let $A \subset m(Z)$ be the subalgebra of $x = \{x(n)\} \in m(Z)$ for which $\phi(x) = \lim_{n \to +\infty} x(n)$ exists. Then ϕ is a multiplicative translation invariant mean on A i.e. A is ELA. For $x \in A$ let $p(x) = \inf_{n \in Z} \sup_{k \in Z} x(n+k) = \sup_{n \in Z} x(n)$. Then $p(x)$ is *not* the support functional of the set of LIM's on A. Since $\psi(x) = x(0)$ for $x \in A$ is dominated by p but is *not* translation invariant. This shows that one should be careful when trying to generalize the above results to ELA subalgebras of $m(S)$ if S is not ELA (see [8, p. 14]). Summarizing part of the above results and denoting by $K(f)$ the set of reals c for which $c \cdot 1$ is in the pointwise (or equivalently the w^*) closure of $\{r_s f;\ s \in S\}$. One has

THEOREM 6. *Let S be ELA and $Q(f) = \sup\{\phi(f);\ \phi \in LIM\}$. Then*

$$Q(f) = \limsup_s f(s) = \inf_s \sup_t f(st) = \inf_{\mathscr{A}} \sup_s \left[\sum_1^n \alpha_i f(a_i s)\right]$$
$$= \inf_{h \in H} \sup_s [f(s) + h(s)] = \sup\{c;\ c \in K(f)\}.$$

Furthermore $f \in m(S)$ is left almost convergent to α if and only if $\lim_s f(s)$ exists and equals α or equivalently if and only if $K(f) = \{\alpha\}$. (Compare with Mitchell [21, p. 253].)

Proof. $Q(f) = \sup \phi(f)$ where the sup is over the set of multiplicative LIM's. Now by 2 of Theorem A in the introduction of [16] one gets that $\sup\{c;\ c \in K(f)\} = Q(f)$. The other equalities for $Q(f)$ have been shown above.

By the Hahn-Banach theorem, $f_0 \in m(S)$ is left almost convergent to α if and only if $-Q(-f_0) = Q(f_0) = \alpha$ i.e. if and only if $\liminf_s f(s) = \limsup_s f(s) = \alpha$. This is equivalent to $K(f) = \{\alpha\}$ again by 2 of Theorem A in [16].

EXAMPLE. Let S be an EA semigroup. Denote by $Q_L(Q_R)$ the support functional of the set of LIM's (RIM's). Then $Q_L(f) = \inf_s \sup_t f(st)$ and $Q_R(f) = \inf_t \sup_s f(st)$. We give in what follows an example of a semigroup which is EA for which $Q_L \neq Q_R$ i.e. for which there is a multiplicative LIM which is not a RIM (and a multiplicative

RIM which is not a LIM). In the case of amenable *groups* such an example is given in Hewitt-Ross [18, p. 239].

Let G be the free group on the set of generators $\{t_\alpha; \alpha \in I\}$ where I is the set of ordinals less than the first incountable one (and $\geqq 1$). For $\alpha \in I$, $\alpha \geqq 2$ let G_α be the free subgroup generated by $\{t_\beta; 1 \leqq \beta < \alpha\}$. One has $t_\alpha^{-1} G_\alpha t_\alpha \cap G_\alpha = \{e\}$ for each $\alpha \geqq 2$. Since if $t_\alpha^{-1} g t_\alpha = g'$ for g, g' in G_α and $g' \neq e$ (the identity of G) then $g \neq e$. Thus g, g' can be written in reduced form as $g = t_{\alpha_1}^{n_1} \cdots t_{\alpha_j}^{n_j}$, $g' = t_{\beta_1}^{m_1} \cdots t_{\beta_k}^{m_k}$ where $n_i \neq 0$, $m_i \neq 0$ are integers $\alpha_n < \alpha$, $\beta_m < \alpha$ for $1 \leqq n < j$, $1 \leqq m < k$ and the α_n's are different and the β_m's are different. But then $t_\alpha^{-1} g t_\alpha = t_\alpha^{-1} t_{\alpha_1}^{n_1} \cdots t_{\alpha_j}^{n_j} t_\alpha = t_{\beta_1}^{m_1} \cdots t_{\beta_k}^{m_k}$ where both sides are given in reduced form and $\beta_m < \alpha$ for $1 \leqq m \leqq k$. This cannot be since G is a free group. Denote now $s = t_1$. Then $G_\alpha t_\alpha s \cap G_\alpha t_\alpha = \varnothing$ for any $\alpha \geqq 2$. Since if $G_\alpha t_\alpha s = G_\alpha t_\alpha$ for some $\alpha \geqq 2$ then $s \in t_\alpha^{-1} G_\alpha t_\alpha$ and $s \in G_2 \subset G_\alpha$, thus $s = t_1 = e$, which cannot be.

Let now $\mathfrak{G}$ be the semigroup consisting of all countable subsets of G with usual multiplication of subsets. Consider the net of point measures $\{1_{G_\alpha t_\alpha}; \alpha \in I\} \subset m(\mathfrak{G})^*$. If $A \in \mathfrak{G}$ then $A \subset G_{\alpha_0}$ for some α_0 and so $A G_\alpha t_\alpha = G_\alpha t_\alpha$ if $\alpha \geqq \alpha_0$. Thus

$$\lim_\alpha \|L_A 1_{G_\alpha t_\alpha} - 1_{G_\alpha t_\alpha}\| = \lim_\alpha \|1_{A G_\alpha t_\alpha} - 1_{G_\alpha t_\alpha}\| = 0$$

(the norms being in $m(\mathfrak{G})^*$). Therefore if $\phi \in \beta(\mathfrak{G})$ is any w^*-limit of a subnet $\{1_{G_\nu t_\nu}\}$ of $\{1_{G_\alpha t_\alpha}\}$ then ϕ is a multiplicative LIM on $m(\mathfrak{G})$. We claim that ϕ cannot be right invariant. If ϕ would be right invariant then by Lemma 1 in §2,

$$\lim_\nu \|R_A 1_{G_\nu t_\nu} - 1_{G_\nu t_\nu}\| = \lim_\nu \|1_{G_\nu t_\nu A} - 1_{G_\nu t_\nu}\| = 0$$

for any $A \in \mathfrak{G}$. But take $A = \{s\}$. Then $G_\nu t_\nu A \neq G_\nu t_\nu$ for any ν. Thus $\|1_{G_\nu t_\nu A} - 1_{G_\nu t_\nu}\| = 2$ for each ν and so ϕ is not right invariant. Thus $Q_L \neq Q_R$. If we denote by $H = H_l$ and H_r all functions in $m(\mathfrak{G})$ representable $\sum_1^n f_j(g_j - r_{A_j} g_j)$ where $A_j \in \mathfrak{G}$ are arbitrary and $f_j, g_j \in m(\mathfrak{G})$ then we shall have for the semigroup $\mathfrak{G}$ that $\overline{H}_l \neq \overline{H}_r$. Since

$$Q_L(f) = \inf_{h \in H_l} \left[\sup_s f(s) + h(s)\right] = \inf_{h \in \overline{H}_l} \sup_s [f(s) + h(s)]$$

and in analogy for $Q_R(f)$. Since $Q_L \neq Q_R$ one has $\overline{H}_l \neq \overline{H}_r$. Furthermore H_l and H_r are not dense in $m(\mathfrak{G})$.

We bring now some results on left almost convergence.

Definition. For $\phi \in m(S)^*$ define $T_\phi: m(S) \to m(S)$ by $(T_\phi f)(s) = \phi(r_s f) = \phi(f^s)$. T_ϕ is linear and $\|T_\phi f\| < \|\phi\| \, \|f\|$. If ϕ is a finite mean, i.e. if $\phi = \sum_1^n \alpha_i 1_{s_i}$, $\alpha_i \geqq 0$, $\sum \alpha_i = 1$ and $1_{s_i} \in m(S)^*$ are point measures, then $(T_\phi f)(x) = \sum_1^n \alpha_i f(a_i x) = \sum \alpha_i f_{a_i}(x)$. Thus, if $\mathrm{LO}(f) = \{f_s; s \in S\}$, $\mathrm{RO}(f) = \{f^s; s \in S\}$ and Co denotes convex hull, then $T_\phi f \in \mathrm{Co}\,\mathrm{LO}(f)$ for any finite mean ϕ. The following proposition is partly due to M. Day [4, p. 539] for the case that S is amenable, (where use of Eberlein's ergodic theorem is made in its proof) and partly due to K. Witz [27, p. 694]. It is a generalization of a theorem of G. G. Lorentz and is given here in slightly different form with straightforward proof. Denote by C the constant functions of $m(S)$.

THEOREM 7. *Let S be a LA semigroup. Then $C \oplus \bar{K}$ is the space of left almost convergent (l.a.c.) functions of $m(S)$, f being l.a.c. to c if and only if $f \in c1 + \bar{K}$ (thus $\bar{K} = \{f \in m(S); f$ is l.a.c. to zero$\}$). Furthermore:*

1. *If f is l.a.c. to c and $\{\phi_\alpha\}$ is any net of means such that $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$ then $\lim_\alpha \|T_{\phi_\alpha} f - c1\| = 0$.*
2. *If $c1$ is in the uniform closure of Co $LO(f)$ then f is l.a.c. to c.*

REMARK 1. One cannot replace "S is amenable" by "S is LA" in Eberlein's ergodic theorem as stated in Day [4, pp. 536–537]. The example given in the remark after Theorem 6 in §2 would violate (c) on p. 537 (see also p. 538) of [4].

Proof. Let f be left almost convergent (l.a.c.) to zero. Then $f \in \bar{K}$. Otherwise there would be some $\phi \in m(S)^*$ with $\phi(\bar{K}) = 0$ and $\phi(f) \neq 0$, by the Hahn-Banach theorem. But then ϕ is left invariant and hence $\phi = \alpha\phi_1 - \beta\phi_2$ where ϕ_1, ϕ_2 are LIM's and $\alpha, \beta \geqq 0$ [13, p. 55]. But then $\phi_1(f) = \phi_2(f) = 0$ i.e. $\phi(f) = 0$ which cannot be. Thus $f \in \bar{K}$. Since $\phi(\bar{K}) = 0$ for any LIM ϕ we get that $\bar{K}$ coincides with the space of $f \in m(S)$ which are l.a.c. to zero. If f is l.a.c. to c then $f - c1$ is l.a.c. to zero so $f \in c1 + \bar{K}$. Any $f \in c1 + \bar{K}$ is obviously l.a.c. to c. If $c1 \in \bar{K}$ and ϕ is a LIM then $c = \phi(c1) = 0$. Thus $C + \bar{K}$ is a direct sum ($1 \in \bar{K}$ for nonamenable groups S!). That 2 holds is immediate: If ϕ is a LIM then $\phi(f) = \phi(g)$ for any $g \in$ Co LO(f) and since $c1$ can be uniformly approximated by such g's one has $c = \phi(c1) = \phi(f)$ for any LIM ϕ. We show now 1: Let f be l.a.c. to c and $\{\phi_\alpha\}$ be a net of means with $\|L_s\phi_\alpha - \phi_\alpha\| \to 0$ for $s \in S$. If $g \in m(S)$ and $a \in S$ then

$$\begin{aligned} |T_{\phi_\alpha}(g - g_a)(x)| &= |\phi_\alpha(r_x g - r_x l_a g)| = |\phi_\alpha(r_x g - l_a r_x g)| = |\phi_\alpha g^x - (L_a\phi_\alpha) g^x| \\ &= |(\phi_\alpha - (L_a\phi_\alpha) g^x)| \leqq \|L_a\phi_\alpha - \phi_\alpha\| \, \|g\| \to 0. \end{aligned}$$

Thus $\|T_{\phi_\alpha}(g - g_a)\| \to 0$ for each $g \in m(S)$ and $a \in S$. Now any $g \in K$ is given as $g = \sum_1^n (g_i - r_{a_i} g_i)$ thus $\|T_{\phi_\alpha} g\| \to 0$ for each $g \in K$. If $g_0 \in \bar{K}$ let $g \in K$ be such that $\|g - g_0\| < \varepsilon$. Then $\|T_{\phi_\alpha} g_0\| \leqq \|T_{\phi_\alpha}(g_0 - g)\| + \|T_{\phi_\alpha} g\| \leqq \|\phi_\alpha\|\varepsilon + \|T_{\phi_\alpha} g\| = \varepsilon + \|T_{\phi_\alpha} g\| \to \varepsilon$. Thus $\|T_{\phi_\alpha} g_0\| \to 0$ for any $g_0 \in \bar{K}$. If f is l.a.c. to c then $f = c1 + g_0$ for some $g_0 \in \bar{K}$. Thus $T_{\phi_\alpha} f = c(T_{\phi_\alpha} 1) + T_{\phi_\alpha} g_0 = c1 + T_{\phi_\alpha} g_0 \to c1$ uniformly on S which finishes the proof.

REMARK. Define $P: C \oplus \bar{K} \to C \oplus \bar{K}$ by: $Pf = \phi(f).1$ where ϕ is any LIM and $1 \in m(S)$ (i.e. the constant $\phi(f)$ valued function). Then $P^2 = P$ and by Theorem 7, $\lim_\alpha \|(T_{\phi_\alpha} - P)f\| = 0$ for all $f \in C \oplus \bar{K}$. It is though not true in general that $\|T_{\phi_\alpha} - P\| \to 0$ in the uniform operator norm over $C \oplus \bar{K}$, even for ELA semigroups S. In fact let $S = \{1, 2, \ldots\}$ with $st = \max(s, t) = s \vee t$ let $\phi_n = 1_n \in m(S)^*$. Then $\lim_{n \to \infty} \|L_s\phi_n - \phi_n\| = 0$ for all $s \in S$ since $L_s\phi_n = \phi_{s \cdot n} = \phi_n$ if $n \geqq s$. If $\|T_{\phi_n} - P\| \to 0$ then since $P(\bar{K}) = 0$ one would have that $\|T_{\phi_n} h\| \to 0$ uniformly in $h \in \bar{K}$ with $\|h\| \leqq 2$, i.e. $\|T_{\phi_n}(f - f_s)\| \to 0$ uniformly both for $s \in S$ and $f \in m(S)$ with $\|f\| \leqq 1$. But $[T_{\phi_n}(f - f_s)](t) = -(L_s\phi_n - \phi_n)(f^t)$ so $\|T_{\phi_n}(f - f_s)\| \geqq |(L_s\phi_n - \phi_n) f^t|$ for any t and thus for $t = 1$. Thus $\|T_{\phi_n}(f - f_s)\| \geqq |(L_s\phi_n - \phi_n) f|$ for all $f \in m(S)$ with $\|f\| \leqq 1$, and all $s \in S$. Hence $\|L_s\phi_n - \phi_n\| \to 0$ uniformly for $s \in S$. But $\|L_s\phi_n - \phi_n\| = 0$ or 2. If $n_0 \in S$ is such that $n \geqq n_0$ implies $\|L_s\phi_n - \phi_n\| < 2$ for all $s \in S$ then $L_s\phi_n = \phi_n$ for all $s \in S$ and $n \geqq n_0$. But $L_s\phi_{n_0} = 1_{s \cdot n_0} = 1_s$ if $s = n_0 + 1$. Thus $\phi_{n_0+1} = \phi_{n_0}$ which is false. This example shows that a "uniform ergodic theorem" does not generally hold even for abelian EA semigroups.

THEOREM 8. *Let S be ELA. Then* $C \oplus \bar{H}$, $(\bar{H})$, *is the space of* $f \in m(S)$ *which is l.a.c.* (*to zero*). *Moreover the following are equivalent:* (1) *f is l.a.c. to c.* (2) $f \in c1 + \bar{H}$. (3) $c1$ *is in the uniform closure of LO(f).* (4) $\lim_s f(s)$ *exists and equals c.* (5) *The* w^* (*or pointwise*) *closure of* $RO(f) = \{r_s f;\ s \in S\}$ *contains a unique constant function which is* $c1$.

If $A \subset S$ and 1_A is l.a.c. then 1_A is l.a.c. to 0 or 1. Moreover 1_A is l.a.c. to $0 \Leftrightarrow A \cap aS = \varnothing$ for some $a \in S$ and 1_A is l.a.c. to $1 \Leftrightarrow aS \subset A$ for some $a \in S$. In fact $\lim_s 1_A(s) = 0$ or 1 and $Q(1_A) = \inf_s \sup_t 1_A(st) = 0 \Leftrightarrow \sup 1_A(at) = 0$ for some $a \in S$ i.e. $aS \cap A = \varnothing$ for some $a \in S$.

Proof. Let ϕ be a multiplicative LIM and $1_{s_\alpha} \to \phi$ in w^*. Then $\|L_s 1_{s_\alpha} - 1_{s_\alpha}\| \to 0$ for each $s \in S$ by Lemma 1 of §2. Let $\phi_\alpha = 1_{s_\alpha}$. If f is l.a.c. to c then $T_{\phi_\alpha} f = f_{s_\alpha} \to c1$ uniformly by 1 of the previous proposition. Theorems 6, 7 imply the equivalences (1)–(5).

REMARKS. Consider again the semigroup given after Theorem 6. We have shown there that $\bar{H}_l \neq \bar{H}_r$ and H_r, H_l are not dense in $m(\mathfrak{S})$. If $f \in \bar{H}_l$ and $f \notin \bar{H}_r$ then f is l.a.c. to zero but f is not right almost convergent at all. Since if f would be right almost convergent to c let ϕ be a left and right invariant mean. Then $\phi(f) = 0$ i.e. $c = 0$ and so $f \in \bar{H}_r$ which cannot be. Thus there exist two multiplicative right invariant means ϕ_1, ϕ_2 with $\alpha = \phi_1(f) \neq \phi_2(f) = \beta$. Thus by 2 of Theorem A in the introduction of [16] the functions $\alpha 1$ and $\beta 1$ are in the pointwise closure (which is the same as the w^* closure) of LO(f). While the only constant function which is in the *uniform* closure of LO(f) (or even of Co LO(f)) is the constant zero function. Use of Mitchell's Theorem 4 in [21, p. 253] and the fact that $[\alpha, \beta] \subset \{\psi(f);\ \psi \in \text{RIM}\}$ implies even that for any $\alpha < \gamma < \beta$, $\gamma 1$ belongs to the pointwise (or w^*) closure of Co LO(f). As to the right orbit of f, RO(f), its w^*-closure contains a unique constant function which is none other than 0. The zero function and for that matter any other constant function, is though not in the uniform closure of Co RO(f) since then f would be right almost convergent.

Some more examples of ELA semigroups. 1. Let G be a group. Denote by G_c the semigroup of all finite or countable subsets of G with the usual multiplication $AB = \{ab;\ a \in A,\ b \in B\}$. Then G_c is an EA semigroup (in which G is embedded). Since if A, $B \in G_c$ and C is the group generated by $A \cup B$ then $CA = CB = AC = BC = C$. If G is not countable then G_c does not contain a zero. If G, G' are groups with identities e, e' (resp.) and if ϕ: $G_c \to G'_c$ is an isomorphism then for any $g \in G$ $\{e'\} = \phi(\{e\}) = \phi(\{g\})\phi(\{g^{-1}\})$. Hence $\phi(\{g\}) \subset G$ contains just one element of G'. Thus ϕ maps $(\{g\};\ g \in G\}$ into $\{\{g'\};\ g' \in G'\}$ and is one to one. This readily implies that G and G' are isomorphic. Thus nonisomorphic groups G give rise to nonisomorphic EA semigroups G_c. G_c will not have left or right cancellation.

2. If G is any locally finite group (i.e. such that finite subsets generate finite subgroups), let G_f be the semigroup of all finite subsets of G with the same multiplication as above. Then G_f is a EA semigroup (which does not have a zero if G is not finite), in which the group G is embedded. As above nonisomorphic locally finite groups G give rise to nonisomorphic EA semigroups G_f. One can construct from any semigroup S the semigroup S_f and it is readily seen that if S is ELA, ERA or EA then S_f is ELA, ERA or EA resp. It may though happen that S_f is EA without S being so (as is the case if S is any nontrivial locally finite group).

4. Let $S=\{(m, n); m, n=1, 2, \ldots\}$ with the multiplication $(m_1, n_1)(m_2, n_2) =(m_2, n_2)(n_1, m_1)=(m_2, n_2)$ if $m_1<m_2$ and $(m_1, n_1)(m_1, n_2)=(m_1, n_1+n_2)$. S is an abelian EA semigroup with no idempotents. This example is taken from Ljapin [19, p. 69] (where it appears in different context). A slight generalization of this construction which yields a large class of EA semigroups is as follows: Let I be an infinite set and $\{S_\alpha; \alpha\in I\}$ a collection of semigroups. Linearly order I so that it contains no last element. Let $S=\bigcup_{\alpha\in I} S_\alpha$ and denote the elements of S as pairs (α, s) (or (β, t) etc...) where $s\in S_\alpha$ and α ranges over I. Define $(\alpha, s)(\beta, t)= (\beta, t)(\alpha, s)=(\beta, t)$ if $\alpha<\beta$ and $(\alpha, s)(\alpha, t)=(\alpha, st)$. S becomes this way an EA semigroup in which each S_α is embedded. S will be commutative iff all S_α are. S, as any other nontrivial EA semigroup, will never have left or right cancellation.

5. Let S be a semigroup generated by one element. Then S is ELA if and only if S is finite and has a zero. Since if $S=\{a^n\}^\infty$, is ELA, let $p>0$ be such that $aa^p =a^2a^p=a^p$. Then $a^n=a^p$ if $n\geqq p$. Thus S is a finite ELA semigroup with zero a^p. The multiplicative semigroup $\{0, 2, 4, 8, \ldots\}$ is, EA, abelian infinite and is generated by the two elements $\{0, 2\}$ (clearly any finitely generated ELA semigroup contains a right zero).

6. Let $S=[1, \Omega)$ be the semigroup of ordinals less than the first incountable ordinal Ω with the usual (noncommutative) addition $+$ of order types. Then S is a left cancellation semigroup (see Kamke [29, p. 60 and p. 94]) which is ELA but is *not* right amenable. In fact if $\alpha\in S$ then $\alpha+(\alpha\omega+\xi)=\alpha(1+\omega)+\xi=\alpha\omega+\xi$ for any $\xi\in S$. (See Kamke [29, p. 64].) If $\alpha, \beta\in S$ and $\gamma>\max\{\alpha\omega, \beta\omega\}$ where $\gamma\in S$ then there are $\xi, \eta\in S$ such that $\gamma=\alpha\omega+\xi=\beta\omega+\eta$ (see Kamke [29, p. 94]). Thus $\alpha+\gamma =\beta+\gamma=\gamma$. S is not RA since $(S+\omega)\cap(S+1)=\varnothing$. S contains no idempotents.

The examples given here together with those of [15], [16] should convince the reader that the class of ELA semigroups is indeed immense and worth being investigated in more detail.

Added in Proof.

1. R. R. Phelps has kindly informed the author that Theorem 4 in §2 generalizes Theorem 1.1 of R. R. Phelps, *Extreme positive operators and homomorphisms*, Trans. Amer. Math. Soc. **108** (1963), 265–274, which is probably a more precise reference than the Bonsall-Lindenstrauss-Phelps paper quoted. Furthermore, unlike pointed out in the introduction, Theorem 4 of §2 *does* have an analogue (with some restrictions, though) to the left amenable case. This analogue has been

obtained by G. Converse, I. Namioka, and R. R. Phelps in a paper which was being prepared for publication after this paper had been sent for publication.

2. Concerning Theorem 6 of §3, the following remark is of interest:

It is known that for a LA semigroup S, a set $A\subset S$ has the property that $\phi(A)=1$ for some LIM ϕ on $m(S)$ if and only if the sets $\{s^{-1}A; s\in S\}$ have the finite intersection property (where $s^{-1}A=\{t\in S; st\in A\}$), i.e. A is left thick (see Mitchell [21, p. 257]). For an ELA semigroup S we have the

PROPOSITION. *Let S be ELA and $A\subset S$. Then there is some LIM ϕ on $m(S)$ such that $\phi(A)>0$ if and only if $s^{-1}A\neq\varnothing$ for all $s\in S$.*

Proof. Since $l_s 1_A=1_{s^{-1}A}$ the "only if" part is trivial. For the "if" part let $\{1_{s_\alpha}\}\subset m(S)^*$ be a net of point measures such that $1_{s_\alpha}\to\phi$ in w^* where ϕ is a multiplicative LIM on $m(S)$. Then: (*) for any $s\in S$ there is some α_0 with $ss_\alpha=s_\alpha$ if $\alpha\geqq\alpha_0$. Choose for each α, $t_\alpha\in S$ such that $s_\alpha t_\alpha\in A$. If $u_\alpha=s_\alpha t_\alpha$ then $\{u_\alpha\}$ enjoys property (*) and $\{u_\alpha\}\subset A$. Thus $\lim \|L_s 1_{u_\alpha}-1_{u_\alpha}\|=0$ for all $s\in S$. Any w^* cluster point ψ of $\{1_{u_\alpha}\}$ will be a multiplicative LIM on $m(S)$ such that $\psi(A)=1$, since $1_{u_\alpha}(A)=1$ for all α. In particular, if $A\subset S$ satisfies $\psi_1(A)>0$ for some LIM ψ_1 on $m(S)$ then there is some multiplicative LIM ψ_2 on $m(S)$ such that $\psi_2(A)=1$.

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