LIE ALGEBRAS OF CHARACTERISTIC THREE WITH NONDEGENERATE KILLING FORM

BY
GORDON BROWN

1. **Introduction.** The classification of Lie algebras with nondegenerate trace form over algebraically closed fields of characteristic $p>3$, begun by Seligman in [6], was essentially completed by Block and Zassenhaus in [1]. In particular, it follows from these results that the Lie algebras with nondegenerate Killing form over algebraically closed fields of characteristic $p>3$ are direct sums of simple algebras of classical type, i.e. algebras satisfying the Mills-Seligman axioms [5], [7, p. 28].

The purpose of this paper is to determine all Lie algebras with nondegenerate Killing form over algebraically closed fields of characteristic 3. Beginning in §2 with a modified version of the Mills-Seligman axioms, a classification of algebras satisfying these axioms is obtained at the end of §3. Then in §4 it is shown that these axioms are satisfied by all algebras with nondegenerate Killing form, and thus a final classification is obtained from that of §3 by eliminating those algebras whose Killing forms are degenerate.

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2. **An axiom system and some consequences.** We begin with a modification of the axioms of [5].

Let L be a finite-dimensional Lie algebra over a field of characteristic 3 with a Cartan subalgebra H and root spaces L_α for roots α with respect to H such that the following six axioms hold:

(i) $LL=L$.

(ii) The center of L is 0.

(iii) H acts diagonally on L, i.e. if $a \in L_\alpha$, then $ah=\alpha(h)a$ for all $h \in H$.

(iv) If α is a nonzero root, then $L_\alpha L_{-\alpha}$ is one-dimensional.

(v) If β and α are roots and $\alpha\neq 0$, then there exists an integer m such that $L_{\beta+m\alpha}L_\alpha=0$.

(vi) If $(L_\beta L_\alpha)L_{-\alpha}=0$, then $L_\beta L_\alpha=0$.

It follows from (ii) and (iii) that H is abelian and that $H=L_0$.

A number of important consequences can be quickly deduced from these axioms.

LEMMA 2.1. *Let h_α be a fixed nonzero element of $L_\alpha L_{-\alpha}$. Then $\alpha(h_\alpha)\neq 0$.*

Proof. If $\alpha(h_\alpha)=0$, then $(L_{-\alpha}L_\alpha)L_{-\alpha}=0$, and by (vi) $L_{-\alpha}L_\alpha=0$, contradicting (iv).

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LEMMA 2.2. *If α is a nonzero root, then L_α is one-dimensional.*

Proof. $L_{-\alpha}L_\alpha \neq 0$ by (iv), and $L_0L_\alpha \neq 0$ by Lemma 2.1. Therefore, by (v), $L_\alpha L_\alpha = 0$. Let $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ be chosen so that $e_\alpha e_{-\alpha} = h_\alpha$, and suppose that $\dim L_\alpha > 1$. Then, since $\dim L_\alpha L_{-\alpha} = 1$, there exists $x \neq 0 \in L_\alpha$ such that $xe_{-\alpha} = 0$. Then, letting $J(a, b, c) = (ab)c + (bc)a + (ca)b$, $J(x, e_\alpha, e_{-\alpha}) = 0$ implies that $\alpha(h_\alpha) = 0$, contradicting Lemma 2.1. Therefore $\dim L_\alpha = 1$.

DEFINITION 2.1. Let β and α be roots with $\alpha \neq 0$. Let $A_{\beta,\alpha} = i - j$ where i is the least nonnegative integer such that $L_{\beta - i\alpha}L_{-\alpha} = 0$, and j is the least nonnegative integer such that $L_{\beta + j\alpha}L_\alpha = 0$. The numbers $A_{\beta,\alpha}$ are known as *root-integers.* $\bar{A}_{\beta,\alpha}$ denotes the corresponding element of the prime field.

LEMMA 2.3. $-2 \leqq A_{\beta,\alpha} \leqq 2$.

Proof. By (v) $0 \leqq i, j \leqq 2$. Therefore $-2 \leqq i - j \leqq 2$.

LEMMA 2.4. *If $L_\beta L_\alpha \neq 0$ and $L_{\beta+\alpha}L_\alpha \neq 0$, then $L_\beta L_{-\alpha} = 0$.*

Proof. If $L_\beta L_\alpha \neq 0$ and $L_{\beta+\alpha}L_\alpha \neq 0$, then by (v) $L_{\beta-\alpha}L_\alpha = 0$, and so $(L_\beta L_{-\alpha})L_\alpha = 0$. Thus $L_\beta L_{-\alpha} = 0$ by (vi).

COROLLARY 2.1. *If $L_\beta L_\alpha \neq 0$ and $L_\beta L_{-\alpha} \neq 0$, then $A_{\beta,\alpha} = 0$.*

Proof. By Lemma 2.4, $L_{\beta+\alpha}L_\alpha = 0$ and $L_{\beta-\alpha}L_{-\alpha} = 0$. Therefore $A_{\beta,\alpha} = 1 - 1 = 0$.

LEMMA 2.5. *$L_{-\beta}L_{-\alpha} \neq 0$ if and only if $L_\beta L_\alpha \neq 0$.*

Proof. Suppose $L_\beta L_\alpha \neq 0$. Then by repeated application of (vi), $L_{\beta+\alpha}L_{-\alpha} \neq 0$, $(L_{-\alpha}L_{\beta+\alpha})L_{-\beta-\alpha} \neq 0$, $L_\beta L_{-\beta-\alpha} \neq 0$, $(L_{-\beta-\alpha}L_\beta)L_{-\beta} \neq 0$, and $L_{-\alpha}L_{-\beta} \neq 0$.

Lemma 2.5 will be used repeatedly in the next section without specific mention.

LEMMA 2.6. *If α and β are nonzero roots, then $A_{\alpha,\beta} = 0$ if and only if $A_{\beta,\alpha} = 0$.*

Proof. Let $A_{\alpha,\beta} = 0$. Then $L_\alpha L_\beta$ and $L_\alpha L_{-\beta}$ are either both zero or both nonzero. If $L_\alpha L_\beta = L_\alpha L_{-\beta} = 0$, then by Lemma 2.5, $L_{-\alpha}L_\beta = 0$, and $A_{\beta,\alpha} = 0$. Similarly, if $L_\alpha L_\beta$ and $L_\alpha L_{-\beta}$ are both nonzero, then $L_{-\alpha}L_\beta \neq 0$ and $A_{\beta,\alpha} = 0$ by Corollary 2.1.

LEMMA 2.7. *If $L_\beta L_{-\alpha} = 0$, then $A_{\beta,\alpha} = 0$ or -1 if and only if $-\beta(h_\alpha)/\alpha(h_\alpha) = 0$ or -1 respectively.*

Proof. Select $e_\phi \neq 0 \in L_\phi$ for all roots $\phi \neq 0$, and let $h_\phi = e_\phi e_{-\phi}$. By Lemma 2.1, $\alpha(h_\alpha) \neq 0$. $J(e_\beta, e_\alpha, e_{-\alpha}) = 0 = (e_\beta e_\alpha)e_{-\alpha} - \beta(h_\alpha)e_\beta$. By (vi), $e_\beta e_\alpha = 0$ and $A_{\beta,\alpha} = 0$ if and only if $\beta(h_\alpha) = 0 = -\beta(h_\alpha)/\alpha(h_\alpha)$. If $A_{\beta,\alpha} \neq 0$, then $J(e_{\beta+\alpha}, e_\alpha, e_{-\alpha}) = 0 = (e_{\beta+\alpha}e_\alpha)e_{-\alpha} - (\alpha - \beta)(h_\alpha)e_{\beta+\alpha}$. By (vi) $e_{\beta+\alpha}e_\alpha = 0$ and $A_{\beta,\alpha} = -1$ if and only if $\beta(h_\alpha) = \alpha(h_\alpha)$, i.e. $-\beta(h_\alpha)/\alpha(h_\alpha) = -1$.

3. **Classification of the algebras.** Since the remainder of the classification procedure is substantially the same as that of §§4–14 of [5], we shall proceed by merely noting the modifications which must be made in that paper in order to classify the algebras satisfying axioms (i)–(vi) of this paper. Much of the notation

and terminology of [5] will be retained here, and results and proofs which need no modification will not be repeated.

Chief among the modifications to be made is that all statements of the form "The sum of the roots α and β is (or is not) a root" are to be replaced by "$L_\alpha L_\beta \neq 0$" (or "$L_\alpha L_\beta = 0$") respectively.

Similarly, the property of linear dependence is to be replaced by that of sequential dependence, defined as follows:

DEFINITION 3.1. A set S of nonzero roots is *sequentially dependent* if and only if (i) it is linearly dependent, (ii) there exists a sequence of roots $\rho_0, \rho_1, \ldots, \rho_m$ such that $\rho_0 = \rho_m = 0$ and $L_{\rho_{k-1}} L_{\beta_{i_k}} \neq 0$ for $1 \leqq k \leqq m$ where $\beta_{i_k} = \rho_k - \rho_{k-1}$, (iii) for each k, either $\beta_{i_k} \in S$, or $-\beta_{i_k} \in S$, or the difference between the number of occurrences of β_{i_k} and $-\beta_{i_k}$ in $\mathrm{P} = \{\rho_k - \rho_{k-1} \mid 1 \leqq k \leqq m\}$ is a multiple of three, and (iv) there exists $\beta \in S$ such that the difference between the number of occurrences of β and $-\beta$ in P is not a multiple of three.

It follows from Definition 3.1 that if $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is sequentially dependent, then so is the set obtained from S by replacing α_i by $-\alpha_i$ for some i.

The property that a root β is a linear combination of a set T of roots is to be replaced by the stronger property that β is a *sequential linear combination* of T, for which it is also required that $T \cup \beta$ be sequentially dependent. Similarly, any statement that a set T of roots spans a subspace V of H^* should be strengthened to say that T *sequentially spans* V, i.e. T spans V, and every root in V is a sequential linear combination of T. Also, the concept of basis should be replaced by that of *sequential basis*, i.e. a sequentially independent set of roots sequentially spanning the space.

The proof of the very important Lemma 3 of [5] must be considerably expanded, as indicated in the following proof.

LEMMA 3.1. *If α, β, γ are roots where $\gamma \neq 0$ and $L_\alpha L_\beta \neq 0$, then $A_{\alpha+\beta,\gamma} = A_{\alpha,\gamma} + A_{\beta,\gamma}$.*

Proof. Let $\delta = -\alpha - \beta$. Then by (vi) $L_\alpha L_\delta = L_\alpha(L_{-\alpha} L_{-\beta}) \neq 0$, and similarly $L_\beta L_\delta \neq 0$. Since $A_{\rho,-\gamma} = -A_{\rho,\gamma}$ and because of the symmetry among α, β, and δ, if the lemma is false, then there exist α, β, δ such that $L_\alpha L_\beta \neq 0$ and $A_{\alpha,\gamma} + A_{\beta,\gamma} + A_{\delta,\gamma} < 0$ where $A_{\alpha,\gamma} \leqq A_{\beta,\gamma} \leqq A_{\delta,\gamma}$.

If $A_{\delta,\gamma} = 2$, then certainly $A_{\alpha,\gamma} = -2$, $A_{\beta,\gamma} < 0$, and $A_{\alpha+\gamma,\gamma} = 0 = A_{\delta-\gamma,\gamma}$. Then $L_{\alpha+\gamma} L_\beta \neq 0$ since $J(e_{\alpha+\gamma}, e_\beta, e_{-\gamma}) = 0$. Thus the lemma would also be false for the root triple $(\beta, \alpha+\gamma, \delta-\gamma)$. Therefore, without loss of generality we may assume that $A_{\delta,\gamma} < 2$.

Suppose $A_{\delta,\gamma} = 1$. If $A_{\alpha,\gamma} = -2$ and $A_{\beta,\gamma} = 0$, then $A_{\alpha+\gamma,\gamma} = 0$ and $A_{\delta-\gamma,\gamma} = -1$. If $e_\beta e_{-\gamma} = 0$, then $J(e_{\alpha+\gamma}, e_\beta, e_{-\gamma}) = 0$ implies that $L_{\alpha+\gamma} L_\beta \neq 0$. If $e_\beta e_{-\gamma} \neq 0$, then $J(e_\alpha, e_{\beta+\gamma}, e_{-\gamma}) = 0$ implies that $e_\alpha e_{\beta+\gamma} \neq 0$, and by (vi) that $e_{\alpha+\beta+\gamma} e_{-\beta-\gamma} \neq 0$. $J(e_{\beta+\gamma}, e_\alpha, e_\gamma) = 0$ implies that $e_{\alpha+\gamma} e_{\beta+\gamma} = 0$ and $J(e_{-\alpha-\beta-\gamma}, e_{\beta+\gamma}, e_{\alpha+\gamma}) = 0$ implies that $e_{-\alpha-\beta-\gamma} e_{\alpha+\gamma} \neq 0$, i.e. $L_{\alpha+\gamma} L_{\delta-\gamma} \neq 0$. Thus without loss of generality we may assume that $A_{\delta,\gamma} \leqq 0$ unless $A_{\delta,\gamma} = 1$ and $A_{\beta,\gamma} < 0$. If $A_{\delta,\gamma} = 1$, $A_{\beta,\gamma} < 0$, and $A_{\alpha,\gamma} = -1$, we must have $A_{\beta,\gamma}=-1$, but by Lemma 2.7 $\alpha(h_\gamma)/\gamma(h_\gamma)=\beta(h_\gamma)/\gamma(h_\gamma)=1$, and $\delta(h_\gamma)/\gamma(h_\gamma)=-1$, a contradiction. Next, suppose that $A_{\delta,\gamma}=1$, $A_{\beta,\gamma}<0$, and $A_{\alpha,\gamma}=-2$. Then since $J(e_{\alpha+\gamma}, e_\beta, e_{-\gamma})=0$, $(e_{\alpha+\gamma}e_\beta)e_{-\gamma}\neq 0$. Thus $e_{\alpha+\gamma}e_\beta\neq 0$, and so by (vi) $e_{\alpha+\beta+\gamma}e_{-\beta}\neq 0$, and $J(e_{\alpha+\beta+\gamma}, e_\gamma, e_{-\beta})=0$ is contradicted.

Now suppose that $A_{\delta,\gamma}=0$ and $A_{\beta,\gamma}<0$. Then from $J(e_\alpha, e_\beta, e_{-\gamma})=0$ it follows that $(e_\alpha e_\beta)e_{-\gamma}=0$. Thus $e_{-\delta}e_{-\gamma}=0$, and since $A_{\delta,\gamma}=0$, $e_\delta e_{-\gamma}=0$, and $J(e_\beta, e_\delta, e_{-\gamma})=0$ is contradicted.

Suppose next that $A_{\delta,\gamma}<0$. Then $J(e_\alpha, e_\beta, e_{-\gamma})=0$ yields an immediate contradiction.

The only case remaining to be considered is the one in which $A_{\delta,\gamma}=A_{\beta,\gamma}=0$. A change to a more symmetric notation gives $A_{\alpha_1,\gamma}=A_{\alpha_2,\gamma}=0$ and $A_{-\alpha_1-\alpha_2,\gamma}<0$. Thus $e_{\alpha_1+\alpha_2}e_\gamma=0$, $e_{\alpha_1+\alpha_2}e_{-\gamma}\neq 0$, and $e_{\alpha_i}e_\gamma$ and $e_{\alpha_i}e_{-\gamma}$ are either both zero or both nonzero. If $e_{\alpha_1}e_\gamma=e_{\alpha_2}e_\gamma=0$, then $J(e_{\alpha_1}, e_{\alpha_2}, e_{-\gamma})=0$ yields a contradiction; if $e_{\alpha_1}e_\gamma=0$, but $e_{\alpha_2}e_\gamma\neq 0$, then $J(e_{\alpha_1+\alpha_2}, e_{-\alpha_1}, e_\gamma)=0$ yields a contradiction. Therefore assume $e_{\alpha_1}e_\gamma\neq 0$ and $e_{\alpha_2}e_\gamma\neq 0$. Then $e_{\alpha_i}e_{-\gamma}\neq 0$, $e_{\alpha_i\pm\gamma}e_{\alpha_i}=0$, $e_{\pm\alpha_i+\gamma}e_\gamma=0$, $e_{\pm\alpha_i-\gamma}e_\gamma\neq 0$, and $e_{\alpha_i+\gamma}e_{-\alpha_i+\gamma}=0$ for $i=1, 2$. Let (ij) be either permutation of (12). From $J(e_{\alpha_i}, e_{-\alpha_1-\alpha_2}, e_{-\gamma})=0$ it follows that $e_{\alpha_i-\gamma}e_{-\alpha_1-\alpha_2}\neq 0$, and by (vi) $e_{-\alpha_j-\gamma}e_{-\alpha_i+\gamma}\neq 0$ and $e_{-\alpha_j-\gamma}e_{\alpha_1+\alpha_2}\neq 0$. From $J(e_\gamma, e_{\alpha_i+\gamma}, e_{\alpha_j-\gamma})=0$ it follows that $e_{\alpha_j}e_{\alpha_i+\gamma}=0$. $J(e_\gamma, e_{-\alpha_i-\gamma}, e_{-\alpha_j+\gamma})=0$ implies that $e_{-\alpha_i}e_{-\alpha_j+\gamma}\neq 0$. Since $J(e_{\alpha_i+\gamma}, e_{-\alpha_i+\gamma}, e_{-\alpha_j-\gamma})=0$, $\alpha_i-\alpha_j$ must be a root. From $J(e_{\alpha_i-\alpha_j}, e_{\alpha_i}, e_{\alpha_j+\gamma})=0$ it follows that $e_{\alpha_i-\alpha_j}e_{\alpha_i}=0$, and from $J(e_{\alpha_i-\alpha_j}, e_{\alpha_i}, e_{\alpha_j-\gamma})=0$ it follows that $e_{\alpha_1+\alpha_2-\gamma}e_{\alpha_i-\alpha_j}=0$. Thus $e_{-\alpha_i-\gamma}e_{-\alpha_1-\alpha_2+\gamma}=0$ by (vi). $J(e_{\alpha_1+\alpha_2}, e_{-\gamma}, e_{-\alpha_i-\gamma})=0$ implies that $e_{\alpha_1+\alpha_2-\gamma}e_{-\alpha_i-\gamma}=0$, and since $J(e_{\alpha_i+\gamma}, e_{\alpha_1+\alpha_2-\gamma}, e_{-\alpha_1-\alpha_2+\gamma})=0$, $(\alpha_i+\gamma)(h_{\alpha_1+\alpha_2-\gamma})=0$. Therefore

$$(\alpha_1+\alpha_2-\gamma)(h_{\alpha_1+\alpha_2-\gamma}) = (\alpha_1+\gamma)(h_{\alpha_1+\alpha_2-\gamma})+(\alpha_2+\gamma)(h_{\alpha_1+\alpha_2-\gamma}) = 0,$$

contradicting Lemma 2.1. Thus $A_{\alpha,\gamma}+A_{\beta,\gamma}=A_{\alpha+\beta,\gamma}$ if $L_\alpha L_\beta\neq 0$.

An alternate proof is necessary for Lemma 4 of [5]. We give one based on Lemma 3.1.

LEMMA 3.2. *If $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is sequentially dependent, then $|A_{\alpha_i,\alpha_j}|=0$.*

Proof. By Lemma 3.1 with the sequence $\rho_0, \rho_1, \dots, \rho_m$ establishing the sequential dependence of S is associated a linear dependence relation among the rows of (A_{α_i,α_j}). Hence $|A_{\alpha_i,\alpha_j}|=0=|\bar{A}_{\alpha_i,\alpha_j}|$.

The fact that for characteristic 3, 2α is a root whenever α is a root invalidates a statement in the last paragraph of the proof of Theorem 3 of [5]. The following lemma is a substitute for that portion of the proof.

LEMMA 3.3. *Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a set S of roots such that $\beta_u=\sum_{i=1}^n \alpha_{k_i}$ for $\alpha_{k_i}\in S$, $(\cdots((L_{\alpha_{k_1}}L_{\alpha_{k_2}})L_{\alpha_{k_3}})\cdots)L_{\alpha_{k_n}}\neq 0$, and $A_{\beta_u,\alpha_j}=0$ for all $\alpha_j\in S$. Then $\beta_u=0$.*

Proof. By Lemma 2.6, if $\beta_u\neq 0$, $A_{\alpha_j,\beta_u}=0$ for all $\alpha_j\in S$. Thus $A_{\beta_u,\beta_u}=0$ by Lemma 3.1, a contradiction since $A_{\beta,\beta}=2$ for $\beta\neq 0$.

In connection with the proof of Lemma 7 of [5], the following lemma should be noted.

LEMMA 3.4. *If* $\{\beta, \alpha_1, \ldots, \alpha_r\}$ *is sequentially independent, then so is* $\{\beta-\alpha_q, \alpha_1, \ldots, \alpha_r\}$ *where* $L_\beta L_{-\alpha_q} \neq 0$.

Proof. Let $\beta' = \beta - \alpha_q$, and suppose the lemma to be false. Then there is a sequence $0=\rho_0, \rho_1, \ldots, \rho_n = 0$ satisfying the conditions given in Definition 3.1 for sequential dependence of $\{\beta', \alpha_1, \ldots, \alpha_r\}$ and such that for some k, $L_{\rho_k} L_{\pm\beta'} \neq 0$. Without loss of generality, assume $L_{\rho_k} L_{\beta'} \neq 0$. Then since $J(e_{\rho_k}, e_\beta, e_{-\alpha_q}) = 0$ and $(e_\beta e_{-\alpha_q}) e_{\rho_k} \neq 0$, either $(e_{\rho_k} e_{-\alpha_q}) e_\beta \neq 0$ or $(e_{\rho_k} e_\beta) e_{-\alpha_q} \neq 0$. Thus it is possible to insert another root between ρ_k and ρ_{k+1} for every k such that $\rho_{k+1} - \rho_k = \pm\beta'$ in such a way as to form a sequence satisfying the conditions given in Definition 3.1 for sequential dependence of $\{\beta, \alpha_1, \ldots, \alpha_r\}$. This contradiction establishes the lemma.

The following lemma corresponds to Lemma 8 of [5].

LEMMA 3.5. *Let* $S = \{\alpha_1, \ldots, \alpha_r\}$ *be a semisimple system of roots which is the union of simple systems each of which is maximal in the complete set of roots of L with respect to H. Let* $T = \{\beta_1, \ldots, \beta_t\}$ *be a simple system of roots with respect to H. Then either*

(a) $S \cup T$ *is sequentially independent and* $A_{\beta_j, \alpha_i} = 0$ *for all* β_j, α_i; *or*

(b) *every* β_j *is a sequential linear combination of S, and for every* β_j *there is an* α_i *such that* $A_{\beta_j, \alpha_i} \neq 0$.

Proof. Without loss of generality suppose that there is an integer k, $0 \leqq k \leqq t$, such that $A_{\beta_j, \alpha_i} = 0$ for all α_i if $j \leqq k$ and $A_{\beta_j, \alpha_i} \neq 0$ for some α_i if $j > k$. From Lemmas 2.6 and 3.1 it follows that $A_{\beta_j, \beta_{j'}} = 0$ for $j' \leqq k < j$. Since T is simple, it is indecomposable, and so either $k=0$ or $k=t$. If $k=0$, (b) holds. Therefore suppose $k=t$ and that (a) does not hold, i.e., $A_{\beta_j, \alpha_i} = 0$ for all β_j, α_i and $S \cup T$ is sequentially dependent. Letting $\alpha_i = \gamma_i$ and $\beta_j = \gamma_{j+r}$, we have $|A_{\gamma_i, \gamma_j}| = |A_{\alpha_i, \alpha_j}| \cdot |A_{\beta_i, \beta_j}|$ since $A_{\alpha_i, \beta_j} = A_{\beta_j, \alpha_i} = 0$. By Lemma 6 of [5], $|A_{\alpha_i, \alpha_j}| \cdot |A_{\beta_i, \beta_j}| > 0$, contradicting Lemma 3.2.

We are now able to establish the existence of a sequential basis.

COROLLARY 3.1. *There exists a semisimple system S which is a sequential basis for* H^*.

Proof. Let S' be a semisimple system of roots which is the union of simple systems each of which is maximal in the complete set of roots of L with respect to H. If S' is not a sequential basis for H^*, there exist roots which are not sequential linear combinations of S'. From these we choose a maximal simple system T. By Lemma 3.5, $S' \cup T$ is sequentially independent, and $A_{\beta,\alpha} = 0$ for $\alpha \in S'$, $\beta \in T$. Thus semisimplicity of $S' \cup T$ will follow if it can be shown that $L_\alpha L_\beta = 0$ whenever $\alpha \in S'$ and $\beta \in T$. But this follows from Lemma 7 of [5]. Thus $S' \cup T$ is semisimple and T is maximal in the complete set of roots. Hence S' may be replaced by $S' \cup T$ and the above procedure repeated, and since there are only finitely many roots, sufficient repetition of this same procedure yields a sequential basis.

An alternate proof must be supplied for one of the statements in the proof of Theorem 4 of [5]:

LEMMA 3.6. *Let S_i, $i=1,\ldots,s$ be disjoint maximal simple systems such that $S=S_1 \cup S_2 \cup \cdots \cup S_s$ is a basis for H^*. Let T_i be the set of all nonzero roots which are sequential linear combinations of S_i. If $\alpha \in T_i$ and $\beta \in T_j$ where $j\neq i$, then $L_\alpha L_\beta=0$.*

Proof. If $\alpha \in T_i$ and $\beta \in T_j$ where $i\neq j$, then by Lemma 7 of [5] and Lemmas 2.6 and 3.1, $A_{\beta,\alpha}=0$, while if $\beta \in T_i$, then by Lemma 3.5 there exists $\alpha_i \in S_i$ such that $A_{\beta,\alpha_i}\neq 0$. Thus $T_i \cap T_j$ is empty. Since $A_{\beta,\alpha}=0$ for $\alpha \in T_i$, $\beta \in T_j$, it follows that if $L_\beta L_\alpha \neq 0$, then $L_\beta L_{-\alpha}\neq 0$, and $L_{\beta+\alpha}L_\alpha=0$. Thus $\{-\alpha, \alpha+\beta\}$ is simple, and by Lemma 3.5, $\alpha+\beta \in T_i$. Similarly $\alpha+\beta\in T_j$. Thus $T_i \cap T_j$ is nonempty, a contradiction.

The remarks made in [5] concerning types G_2 and G_3 may be omitted in the case of characteristic 3 because of Lemma 2.3, but the special remarks concerning type A_n when $p \mid n+1$ must also be made about type E_6. It should be noted that, in contrast with the situation elsewhere, the linear dependence relation of Lemma 24 of [5] is not to be associated with a sequential dependence relation.

The contradictions achieved in the proof of Lemma 15 of [5] must be put in the product-of-root-spaces form instead of the sum-of-roots form, i.e.: in case 1, $(\cdots(L_{\alpha_2}L_{\alpha_3})\cdots)L_{\alpha_{r-1}})L_{\alpha_r})L_{\alpha_r})L_{\alpha_1}=L_{\alpha_r}$, but $L_{\alpha_r}L_{-\alpha_1}=0$, contradicting (vi); in case 2, $(\cdots(L_{\alpha_1}L_{\alpha_2})\cdots)L_{\alpha_{r-1}}=L_{\alpha_r}$, but $L_{\alpha_r}L_{-\alpha_{r-1}}=0$, contradicting (vi); in case i, $(\cdots(L_{\alpha_1}L_{\alpha_2})L_{\alpha_2})\cdots)L_{\alpha_{r-1}})L_{\alpha_{r-1}})L_{\alpha_r}=L_{\alpha_{r-1}}$, but $L_{\alpha_{r-1}}L_{-\alpha_r}=0$, contradicting (vi); and in case I

$$(\cdots(L_{\alpha_5}L_{\alpha_4})L_{\alpha_3})L_{\alpha_3})L_{\alpha_2})L_{\alpha_2})L_{\alpha_4})L_{\alpha_3})L_{\alpha_3})L_{\alpha_3})L_{\alpha_1} = L_{\alpha_3},$$

but $L_{\alpha_3}L_{-\alpha_1}=0$, contradicting (vi).

A new proof must be substituted for a small portion of the proof of Lemma 25 of [5] as indicated by the following lemma.

LEMMA 3.7. *If $L(\zeta)>1$ and $L_{\zeta-j\alpha_i}L_{-\alpha_i}\neq 0$ for $0\leqq j\leqq k-1$, but $L_{\zeta-k\alpha_i}L_{-\alpha_i}=0$, then $L(\zeta-k\alpha_i)>0$.*

Proof. If $L(\zeta-k\alpha_i)\leqq 0$, then $\zeta=j\alpha_i$ for some integer j, but $L(j\alpha_i)\leqq 1$ for any integer j, thus contradicting the initial hypothesis.

Thus we have determined the root systems of the algebras satisfying (i)–(vi) and found them to be "classical," i.e. there exists a one-to-one addition-preserving mapping of such a root system onto the root system of one of Chevalley's [2] analogues of the complex semisimple Lie algebras.

The isomorphism theorem (Theorem 9 of [5]) is valid for those root systems for which (xiv) of [5] is valid. We shall show that (xiv) of [5] holds if L is not of type C_2.

LEMMA 3.8. *If L satisfies* (i)–(vi) *and is not of type C_2, then $\overline{A}_{\beta,\alpha}=-\beta(h_\alpha)/\alpha(h_\alpha)$ for all roots α, β where $\alpha\neq 0$.*

Proof. If $A_{\beta,\alpha}=\pm 1$, or if $A_{\beta,\alpha}=0$ and $L_\beta L_{\pm\alpha}=0$, the result is immediate from Lemma 2.7. The remaining cases, $A_{\beta,\alpha}=\pm 2$ and $A_{\beta,\alpha}=0$ where $L_\beta L_{\pm\alpha}\neq 0$, occur together, i.e. $L_\beta L_{\pm\alpha}\neq 0$ and $A_{\beta-\alpha,\alpha}=-2$ and $A_{\beta+\alpha,\alpha}=2$ are equivalent. Therefore, since $(\beta\pm\alpha)(h_\alpha)/\alpha(h_\alpha)=\beta(h_\alpha)/\alpha(h_\alpha)\pm 1$, we need only establish that $\beta(h_\alpha)=0$ whenever $A_{\beta,\alpha}=0$ and $L_\beta L_{\pm\alpha}\neq 0$. Since algebras of type C_2 are excluded by the hypothesis of the lemma, only types B_n, C_n ($n\geqq 3$) and F_4 must be considered. As in Chapter 4 of [3], we denote the nonzero elements of a root system of type B_n by $\{\pm\omega_i, \pm\omega_i\pm\omega_j\}$ where $1\leqq i<j\leqq n$. The only pairs (α,β) of these roots satisfying $A_{\beta,\alpha}=0$ and $L_\beta L_{\pm\alpha}\neq 0$ are $(\pm\omega_i, \pm\omega_j)$ for $i\neq j$. Since $J(e_{\omega_i}, e_{-\omega_i}, e_{\omega_j\pm\omega_k})=0$, we have $(\omega_j\pm\omega_k)(h_{\omega_i})=0$ for $i\neq j$, k, and so $\omega_j(h_{\omega_i})=0$ for $i\neq j$. Again, as in Chapter 4 of [3], we denote the roots in a system of type F_4 by

$$\{\pm\omega_i, \ \pm\omega_i\pm\omega_j, \ \tfrac{1}{2}(\pm\omega_1\pm\omega_2\pm\omega_3\pm\omega_4)\}$$

where $1\leqq i<j\leqq 4$. The only pairs (α,β) satisfying $A_{\beta,\alpha}=0$ and $L_\beta L_{\pm\alpha}\neq 0$ are $(\pm\omega_i, \pm\omega_j)$ for $i\neq j$ and $(\frac{1}{2}\sum_{i=1}^4 \varepsilon_i'\omega_i, \frac{1}{2}\sum_{i=1}^4 \varepsilon_i\omega_i)$ where $\varepsilon_i=\pm 1$ and $\varepsilon_i'=\varepsilon_i$ for two values of i and $\varepsilon_i'=-\varepsilon_i$ for the other two values of i. That $\omega_j(h_{\omega_i})=0$ for $i\neq j$ follows for F_4 as it did for B_n. For the second case, without loss of generality, let $\alpha=\frac{1}{2}\sum_{i=1}^4(-1)^i\varepsilon_i\omega_i$. Since $J(e_{\varepsilon_1\omega_1+\varepsilon_2\omega_2}, e_\alpha, e_{-\alpha})=0$, it follows that

$$(\varepsilon_1\omega_1+\varepsilon_2\omega_2)(h_\alpha) = 0.$$

Similarly $(\varepsilon_3\omega_3+\varepsilon_4\omega_4)(h_\alpha)=0$. Therefore $\beta(h_\alpha)=0$. For type C_n, $n>2$, with nonzero roots $\{\pm 2\omega_i, \ \pm\omega_i\pm\omega_j\}$ where $1\leqq i<j\leqq n$, it is sufficient to show that $(\omega_i-\omega_j)(h_{\omega_i+\omega_j})=0$. Let k be distinct from i and j, and choose $e_{\omega_i+\omega_j}$, $e_{-\omega_j-\omega_k}$, and $e_{\omega_i-\omega_k}$ in such a way that $e_{\omega_i+\omega_j}e_{-\omega_j-\omega_k}=e_{\omega_i-\omega_k}$. Then, since $J(e_{\omega_i+\omega_j}, e_{-\omega_i-\omega_j}, e_{-\omega_j-\omega_k})=0$ implies that $e_{\omega_i-\omega_k}e_{-\omega_i-\omega_j}=(\omega_j+\omega_k)(h_{\omega_i+\omega_j})e_{-\omega_j-\omega_k}$, it follows from $J(e_{\omega_i+\omega_j}, e_{-\omega_i-\omega_j}, e_{\omega_i-\omega_k})=0$ that $(\omega_j+\omega_k)(h_{\omega_i+\omega_j})=-(\omega_i+\omega_j)(h_{\omega_i+\omega_j})$. Similarly $(\omega_i+\omega_k)(h_{\omega_i+\omega_j})=-(\omega_i+\omega_j)(h_{\omega_i+\omega_j})$. Thus $(\omega_i-\omega_j)(h_{\omega_i+\omega_j})=0$.

With the modifications noted in this section, the results of [5] are valid for characteristic three and can be summarized as in the following theorem.

THEOREM 3.1. *Let L be a finite-dimensional Lie algebra over a field K of characteristic* 3 *satisfying axioms* (i)–(vi). *Then L is a direct sum of simple algebras satisfying the same axioms. For each such simple algebra, there is a fundamental system S of roots which is a sequential basis for H^*, where S is of type A_n* ($n\geqq 1$), *B_n* ($n\geqq 3$), *C_n* ($n\geqq 2$), *D_n* ($n\geqq 4$), *E_6, E_7, E_8, or F_4*, (*whose diagrams appear in Figure* 1 *of* [5]). *If L is simple and not of type C_2, then L is uniquely determined up to isomorphism by K and the diagram of S.*

It is readily verified that the algebras defined by Chevalley [2] of the types listed in Theorem 3.1 satisfy axioms (i)–(vi). Thus by the uniqueness part of this theorem, the algebras satisfying (i)–(vi) and not of type C_2 are completely classified. A Lie algebra of type C_2 (with root systems $\{\pm\omega_i, \pm\omega_i\pm\omega_2\}$ for $i = 1, 2$) which satisfies (i)–(vi), but has degenerate Killing form, can be constructed provided only that $\omega_1(h_{\omega_1})\omega_1(h_{\omega_2})=\omega_2(h_{\omega_1})\omega_2(h_{\omega_2})$, $(\omega_1\pm\omega_2)(h_{\omega_i})\neq 0$, and $\omega_1(h_{\omega_1})\omega_2(h_{\omega_2})\neq 0$. It is not necessary that h_{ω_1} and h_{ω_2} be linearly independent, and so a sequential basis need not be a basis. A check of the algebras of the other types reveals that their sequential bases are linearly independent except for types A_n for $n\equiv 2 \pmod 3$ and E_6, in which cases $\sum_{j=1}^n j\alpha_j=0$, and $\alpha_1,\dots,\alpha_{n-1}$ is a basis.

4. **Algebras with nondegenerate Killing form.** In this section we shall assume that L is a Lie algebra with nondegenerate Killing form (,) over an algebraically closed field K of characteristic 3, and that H is a Cartan subalgebra for L. It will be shown that L satisfies axioms (i)–(vi).

The fact that L satisfies axioms (i)–(iv) is readily established from the first three sections of [6] and from the observation that because of the definition of the Killing form, L can have no nonzero abelian ideals. It also follows from [6] that $(\operatorname{ad} e_\alpha)^3=0$ for any element $e_\alpha \in L_\alpha$, the root space for the nonzero root α, that $\alpha(h_\alpha)\neq 0$ where $0\neq h_\alpha \in L_\alpha L_{-\alpha}$, that $e_\alpha e_{-\alpha}\neq 0$ if and only if $(e_\alpha, e_{-\alpha})\neq 0$ and that L_α and $L_{-\alpha}$ are dual spaces relative to the Killing form.

In order to prove that (v) and (vi) are satisfied, it is helpful to first establish that each root space L_α is one-dimensional if α is nonzero.

LEMMA 4.1. *If α is a nonzero root of L, then $L_1=L_{-\alpha}+L_\alpha L_{-\alpha}+L_\alpha$ is a simple Lie algebra.*

Proof. Every nonzero element of L_1 can be expressed in the form $h+x_\alpha+y_{-\alpha}$ where $h\in L_\alpha L_{-\alpha}$, $x_\alpha \in L_\alpha$, $y_{-\alpha}\in L_{-\alpha}$. Let I be a nonzero ideal of L_1, and let $l=h+x_\alpha+y_{-\alpha}$ be a nonzero element of I with the least possible number of nonzero components h, x_α, $y_{-\alpha}$. From the duality of L_α and $L_{-\alpha}$ and from Corollary 3.2 of [6], it follows that we may choose l so that $h\neq 0$. Then if $l\notin L_\alpha L_{-\alpha}$, it follows that, since $\alpha(h_\alpha)\neq 0$, hl is a nonzero element of I with fewer nonzero components than l, a contradiction. Therefore $L_\alpha L_{-\alpha}\in I$, and since $\alpha(h_\alpha)\neq 0$, we conclude that $I=L_1$. Thus L_1 is simple.

By a result of Kaplansky [4], since there exists $h\in L_\alpha L_{-\alpha}$, viz. $h_\alpha/\alpha(h_\alpha)$, such that $\alpha(h)=1$, L_1 is either three-dimensional or seven-dimensional, and L_α and $L_{-\alpha}$ are either one-dimensional or three-dimensional.

LEMMA 4.2. *All the root spaces L_α for nonzero roots α are one-dimensional.*

Proof. It is sufficient to show that none of these root spaces is three-dimensional. Therefore, suppose that for some nonzero root β the dimension n_β of L_β is three. Let L_1 be the seven-dimensional algebra generated by L_β. The adjoint representation for L induces representations ρ_α of L_1 on each subspace of L of the form $L_{\alpha-\beta}+L_\alpha+L_{\alpha+\beta}$. Since (,) is nondegenerate, and $(s, t)=0$ for $s\in L_\sigma$, $t\in L_\tau$ if $\sigma+\tau\neq 0$, and since $\beta(h_\beta)\neq 0$, it follows that for any nonzero element $x\in L_1$ there exists $y\in L_1$ such that $(x, y)\neq 0$. Furthermore, since (x, y) is the sum of trace forms for representations ρ_α, at least one such form is nonzero and thus nondegenerate because of the simplicity of L_1. Because L_β and $L_{-\beta}$ are three-dimensional, the trace form for ρ_0 is degenerate, and so the trace form for ρ_α must be nondegenerate for some nonzero root $\alpha\neq\pm\beta$. Thus there exist linear transformations $H\in\rho_\alpha(L_\beta L_{-\beta})$; E_i, $i=1, 2, 3, \in\rho_\alpha(L_{-\beta})$; and F_i, $i=1, 2, 3, \in\rho_\alpha(L_\beta)$ such that $[E_i, H]=-E_i$, $[F_i, H]=F_i$, $[F_i, E_i]=H$, $[F_i, E_j]=0$ for $i\neq j$, and $[F_i, F_j]=E_k$, $[E_i, E_j]=-F_k$, where $(i\,j\,k)$ is a cyclic permutation of $(1\ 2\ 3)$. Clearly H is diagonal with diagonal elements

$\alpha(h)$, $(\alpha+\beta)(h)$, $(\alpha-\beta)(h)$ where $h=\rho_\alpha^{-1}(H)$ is a nonzero multiple of h_β. Since $H=[F_i, E_i]$, its trace must be zero, and since the trace form is nondegenerate, the trace of H^2 must be nonzero. These conditions on the trace can only be satisfied if at least two of the root spaces L_α, $L_{\alpha+\beta}$, $L_{\alpha-\beta}$ are one-dimensional. L_1, being seven-dimensional, cannot be faithfully represented on a two-dimensional space. If $L_{\alpha-\beta}+L_\alpha+L_{\alpha+\beta}$ is three-dimensional, H must be either of the form

$$\operatorname{diag}(\lambda, \lambda+1, \lambda-1) \quad \text{or} \quad \operatorname{diag}(\lambda, \lambda-1, \lambda+1)$$

in order to have weight spaces corresponding to the weights ± 1, as required; however the product of these three-dimensional weight spaces would be two-dimensional, a contradiction. Therefore, without loss of generality, we may assume that $n_\alpha=3$ and $n_{\alpha+\beta}=n_{\alpha-\beta}=1$. Since the trace of H is zero, we have $\alpha(h)=0$. Since F_1, F_2, and F_3 generate $\rho_\alpha(L_1)$, and since $L_{\alpha+\beta}H\neq 0$, there exists F_i such that $L_{\alpha+\beta}F_i\neq 0$. Since $n_{\alpha-\beta}=1$, there exists a two-dimensional subspace of $\rho_\alpha(L_\beta)$ which annihilates $L_{\alpha+\beta}$. Without loss of generality, from the construction of L_1 in [4] it follows that F_1, F_2, and F_3 can be chosen so that $L_{\alpha+\beta}F_1=L_{\alpha+\beta}F_2=0$. Let $x_{\alpha+\beta}$ be a basis for $L_{\alpha+\beta}$, and let $x_{\alpha+\beta}F_3=x_{\alpha-\beta}$. Since E_1, E_2, and E_3 generate $\rho_\alpha(L_1)$, and since $L_{\alpha+\beta}H\neq 0$, there exists E_i such that $x_{\alpha+\beta}E_i\neq 0$. Since $0=(x_{\alpha+\beta}E_i)H=x_{\alpha+\beta}(HE_i-E_i)$, it follows that $x_{\alpha+\beta}H=x_{\alpha+\beta}$, i.e. $\beta(h)=1$, and so $x_{\alpha-\beta}H=-x_{\alpha-\beta}$. Since $x_{\alpha+\beta}=x_{\alpha+\beta}H=x_{\alpha+\beta}(F_1E_1-E_1F_1)=-(x_{\alpha+\beta}E_1)F_1$, we have $x_{\alpha+\beta}E_1\neq 0$. Let $x_{\alpha+\beta}E_1=x_\alpha$. Then $x_\alpha F_1=-x_{\alpha+\beta}$. $x_\alpha=x_{\alpha+\beta}E_1=x_{\alpha+\beta}(F_2F_3-F_3F_2)=-x_{\alpha-\beta}F_2$. Thus $0=x_{\alpha-\beta}(F_2E_1-E_1F_2)=-x_\alpha E_1$, and $0=x_\alpha H=x_\alpha(F_1E_1-E_1F_1)=-x_{\alpha+\beta}E_1=-x_\alpha\neq 0$. Thus the assumption of the existence of a three-dimensional root space leads to a contradiction, and the lemma is established.

LEMMA 4.3. *L satisfies* (v), *i.e. for any root β and any nonzero root α there exists an integer m such that $L_{\beta+m\alpha}L_\alpha=0$.*

Proof. This lemma is an immediate consequence of Lemma 4.2 and the fact that $(\operatorname{ad} e_\alpha)^3=0$ where e_α is a nonzero element of L_α.

LEMMA 4.4. *L satisfies* (vi), *i.e. if $(L_\beta L_\alpha)L_{-\alpha}=0$, then $L_\beta L_\alpha=0$.*

Proof. Assume that L does not satisfy (vi). Let bases e_ϕ for L_ϕ be chosen in such a way that $e_\phi e_{-\phi}=h_\phi$. Then there exist nonzero roots β and α such that $e_\beta e_\alpha\neq 0$, and $e_{\beta+\alpha}e_{-\alpha}=0$. This is already a contradiction if $\beta=\pm\alpha$. Therefore assume $\beta\neq\pm\alpha$. Since $(e_{\beta+\alpha}e_{-\alpha}, e_{-\beta})=0$, also $e_{\beta+\alpha}e_{-\beta}=0$ and $e_{-\alpha}e_{-\beta}=0$. From $(e_\beta e_\alpha, e_{-\beta-\alpha})\neq 0$ it follows that $e_\alpha e_{-\beta-\alpha}\neq 0$, and $e_\beta e_{-\beta-\alpha}\neq 0$. $J(e_\beta, e_\alpha, e_{-\alpha})=0=-\beta(h_\alpha)e_\beta+(e_{-\alpha}e_\beta)e_\alpha$ and $J(e_{\beta+\alpha}, e_\alpha, e_{-\alpha})=0=(e_{\beta+\alpha}, e_\alpha)e_{-\alpha}-(\beta+\alpha)(h_\alpha)e_{\beta+\alpha}$. Since $\beta(h_\alpha)$ and $(\beta+\alpha)(h_\alpha)$ are not both zero, $L_{\beta-\alpha}\neq 0$, and either $e_{\beta+\alpha}e_\alpha\neq 0$ or $e_{\beta-\alpha}e_\alpha\neq 0$. Since $(\operatorname{ad} e_\alpha)^3=0$ and $e_\beta e_\alpha\neq 0$, only one of these two possibilities can occur. Therefore, either $\beta(h_\alpha)=0$, $e_{\beta+\alpha}e_\alpha\neq 0$, and $e_{\beta-\alpha}e_\alpha=0$, or $(\beta+\alpha)(h_\alpha)=0$, $e_{\beta-\alpha}e_\alpha\neq 0$, and $e_{\beta+\alpha}e_\alpha=0$. Since $\beta(h_\alpha)$ and $\alpha(h_\beta)$ are zero if and only if $(h_\alpha, h_\beta)=0$, it follows that if $\beta(h_\alpha)\neq 0$, then $\alpha(h_\beta)\neq 0$, and just as $(\beta+\alpha)(h_\alpha)=0$, so also $(\beta+\alpha)(h_\beta)=0$. From $J(e_\beta, e_\alpha, e_{-\beta-\alpha})=0$ it follows that $h_{\beta+\alpha}$ is a linear combination of h_β and h_α.

Thus if $\beta(h_\alpha)\neq 0$, then $(\beta+\alpha)(h_{\beta+\alpha})=0$, a contradiction. Therefore $\beta(h_\alpha)=\alpha(h_\beta)=0$, $e_{\beta+\alpha}e_\alpha\neq 0$, $e_{\beta+\alpha}e_\beta\neq 0$, $e_{\beta-\alpha}e_\alpha=0$, and $e_{\beta-\alpha}e_\beta=0$. Thus, since $(\beta+\alpha)(h_\beta)\neq 0$, $J(e_{-\beta-\alpha}, e_\beta, e_{-\beta})=0$ implies that $e_{-\beta}e_{-\beta-\alpha}\neq 0$. Also $J(e_{\beta-\alpha}, e_\alpha, e_{-\alpha})=0$ implies that $e_{-\alpha}e_{\beta-\alpha}\neq 0$. Thus $(e_{-\beta-\alpha}, e_{\beta-\alpha}e_{-\alpha})\neq 0$, and $e_{-\beta-\alpha}e_{\beta-\alpha}\neq 0$. Thus $e_\alpha(\operatorname{ad} e_{-\beta-\alpha})^3$ $\neq 0$, a contradiction, and the lemma is established.

We have now shown that the Lie algebras with nondegenerate Killing form over algebraically closed fields of characteristic three satisfy axioms (i)–(vi), and a complete classification follows upon determination of the Killing forms for the algebras of Theorem 3.1. The result is:

THEOREM 4.1. *Let L be a Lie algebra with nondegenerate Killing form over an algebraically closed field of characteristic three. Then L is a direct sum of one or more algebras of types* A_n, $n\not\equiv 2 \pmod 3$; B_n, $n\not\equiv 2 \pmod 3$; C_n, $n\not\equiv 2 \pmod 3$; *or* D_n, $n\not\equiv 1 \pmod 3$.

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UNIVERSITY OF COLORADO,
BOULDER, COLORADO