CHARACTERISTIC CLASSES FOR MODULES OVER GROUPS. I

BY

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Suppose Φ is a group and M is a module over the integral group ring of Φ. Then the homology groups, $H_i(M)$, are also Φ-modules. The usual method of seeing this is to use the standard resolution, S_*, of $\mathbf{Z}$ for M because the summands, S_i, are Φ-modules themselves, and the boundary map is a Φ-homomorphism. However, this complex is a very cumbersome one, and one would like to see if the action of Φ on $H_*(M)$ can be obtained from an arbitrary resolution.

Let D_* be any resolution of $\mathbf{Z}$ for M. The obvious question to ask is whether one can make D_i into a Φ-module in any natural way. One of the results of this paper is to show that this can not be done in general. It is the obstructions to this which give rise to the characteristic classes of the title.

The first section develops the notion of a Φ-*system* for D_* which is an approximation to an action of Φ on D_i. One part of a Φ-system is, for each i and $\sigma \in \Phi$, a $\mathbf{Z}$-homomorphism $A_i(\sigma)\colon D_i \to D_i$ which is a chain map and satisfies an appropriate semilinearity condition. The point is that $A_i(\sigma) \circ A_i(\tau) \neq A_i(\sigma\tau)$ in general. It is easily seen, however, that they are chain homotopic, and it is such a chain homotopy $U_i(\sigma, \tau)$ between $A_i(\sigma) \circ A_i(\tau)$ and $A_i(\sigma\tau)$ which is the other part of a Φ-system and measures the obstruction to the existence of an action of Φ on D_i.

If the complex D_* is reasonable, these obstructions can be described as follows: For each $\sigma, \tau \in \Phi$, $U_i(\sigma, \tau) \in \operatorname{Hom}(D_i, D_{i+1})$ defines an element in

$$\operatorname{Hom}(H_i(M), H_{i+1}(M))$$

which is $H^i(M, H_{i+1}(M))$ if the final (unwritten) coefficients are nice enough. This can be thought of as defining a nonhomogeneous 2-cochain ω^{i+1} for Φ with coefficients in $H^i(M, H_{i+1}(M))$, ω^{i+1} turns out to be a cocycle and its cohomology class

$$v^{i+1}(M) \in H^2(\Phi, H^i(M, H_{i+1}(M)))$$

is what we call the ith *characteristic class* of M. $v^i(M)$ depends only on Φ, M, and the action of Φ on M.

These algebraic characteristic classes satisfy a naturality condition similar to the one satisfied by topological (e.g. Stiefel-Whitney) ones, and if M is $\mathbf{Z}$-free, there is an analogue to the Whitney sum theorem.

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Although the original approach to characteristic classes in topology was via obstructions, another method is to use the spectral sequence of a fibration. The algebraic characteristic classes admit a similar interpretation. In §3 we show that if $E_r^{p,q}$ is the Hochschild-Serre spectral sequence for the split extension of M by Φ, then the map $d_2^{p,q}$ is simply obtained by forming the cup product with the class $v^r(M)$. In [3] we have shown that d_2 for an arbitrary extension can be computed by adding the cohomology class of the extension to $v^p(M)$, and then taking cup products.

A good part of the paper is devoted to computing characteristic classes in particular cases. We exhibit some cases for which these classes are nonzero, including cases in which M is $\boldsymbol{Z}$-free.

Our main general theorems are that if M is $\boldsymbol{Z}$-free, then $2v^i(M)=0$ and if $v^2(M)=0$, then $v^i(M)=0$ for all i.

In later papers we hope to examine the case where M is finite and to investigate higher order characteristic classes.

Part I. The General Case

1. Our starting point is a pair of groups Φ and M and a homomorphism $\phi\colon \Phi \to$ the group of automorphisms of M. For notational convenience, as is customary, we will suppress ϕ, denoting $\phi(\sigma)(m)$ by the simpler $\sigma \cdot m$.

Let R denote the group ring, $Z[M]$, of M. Let (D_*, ∂_*) be a projective resolution of the trivial M-module Z.

$$\longrightarrow D_n \xrightarrow{\partial_n} D_{n-1} \longrightarrow \cdots \longrightarrow D_0 \xrightarrow{\varepsilon} Z \longrightarrow 0$$

Recall that if Γ is an R-module, then $H^n(\mathrm{Hom}_R(D_*, \Gamma))$ is denoted by $H^n(M, R)$.

Definition. A *Φ-system* for (D_*, ∂_*) consists of two sequences of functions as follows:

$$A_n\colon \Phi \to \mathrm{Hom}_Z(D_n, D_n) \quad \text{and} \quad U_n\colon \Phi \times \Phi \to \mathrm{Hom}_Z(D_n, D_{n+1})$$

subject to the conditions listed below:

(i) $\partial_n A_n(\sigma) = A_{n-1}(\sigma)\, \partial_n, \quad n \geqq 1$,

(ii) $\varepsilon A_0(\sigma) = \varepsilon$,

(iii) "$A_n(\sigma)$ is σ-linear", i.e.

$$A_n(\sigma)(r \cdot d) = \sigma(r) \cdot A_n(\sigma)(d) \quad \text{for } d \in D_n \text{ and } r \in R = Z[M],$$

(i′) $\partial_{n+1} U_n(\sigma, \tau) + U_{n+1}(\sigma, \tau)\, \partial_n = A_n(\sigma\tau) - A_n(\sigma) \circ A_n(\tau), \quad n \geqq 1$,

(ii′) $\partial_1 U_0(\sigma, \tau) = A_0(\sigma\tau) - A_0(\sigma) \circ A_0(\tau)$,

(iii′) $U_n(\sigma, \tau)$ is $\sigma\tau$-linear.

Remarks. 1. A_n is *not* a homomorphism, i.e. $A_n(\sigma\tau) \neq A_n(\sigma) + A_n(\tau)$.

2. Intuitively we would like to think of A_n as defining an action of Φ on D_*. This is not what happens in general, i.e. $A_n(\sigma\tau) \neq A_n(\sigma) \circ A_n(\tau)$.

3. The homomorphisms $A_n(\sigma) \in \mathrm{Hom}_Z(D_n, D_n)$ are Z-homomorphisms and not R-homomorphisms. Rather they satisfy the semilinear condition (iii). Similarly for U_n.

EXAMPLE 1. Let (S_*, ∂_*) be the standard resolution, i.e. S_n = the free abelian group generated by the set $M \times \cdots \times M$ ($n+1$ times). Then a Φ-system can be defined by the equations

$$A_n(\sigma)(m_0, \ldots, m_n) = (\sigma(m_0), \ldots, \sigma(m_n)).$$

In this case, of course, $A_n(\sigma\tau) = A_n(\sigma)A_n(\tau)$ so we can take $U_n \equiv 0$. We should remark that in many cases the U_n's are more significant than the A_n's. *One can think of the U_n's as representing the obstruction to finding an action of Φ on D_*.*

Note that if D_n^σ denotes D_n with a new R module structure given by $m \circ x_n = \sigma(m)x_n$, then $A_n(\sigma) \in \mathrm{Hom}_R(D_n, D_n^\sigma)$. This observation reduces the proof of the following proposition to the standard ones.

PROPOSITION 1. *Φ-systems exist for any such (D_*, ∂_*).*

EXAMPLE 2. Let Z_r act on Z_s by $\sigma \cdot t = t^q$ where $q^r = 1$ modulo s. Here σ and t are generators of multiplicative groups Z_r and Z_s of orders r and s respectively.

Let $\Delta = 1 - t \in R = Z[Z_s]$, $N = 1 + t + \cdots + t^{s-1} \in R$, and $\alpha = 1 + t + \cdots + t^{q-1} \in R$. Then it is well known that there is a resolution of Z as follows:

$$\longrightarrow D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\varepsilon} Z \longrightarrow 0$$

where $D_i = R$, $\varepsilon(t) = 1$, $\partial_{2k+1}(1) = \Delta$ and $\partial_{2k}(1) = N$. It is trivial to verify that $A_n(\sigma)$ may be chosen such that

$$A_{2k}(\sigma)(1) = \alpha^k, \qquad A_{2k+1}(\sigma)(1) = \alpha^{k+1}.$$

Define $A_n(\sigma^i) = i$-fold iteration of $A_n(\sigma)$ for $0 \leqq i < r$. It is convenient to let $p = q^{r-1}/s$.

LEMMA. *Let $U_n \in \mathrm{Hom}_R(D_n, D_{n+1})$ be defined by*

$$U_{2k} = 0$$

$$U_{2k+1}(1) = \frac{(q^r)^{k+1} - 1}{s} \in R.$$

Then $\partial_{n+1}U_n + U_{n-1}\partial_n = A_n(\sigma)^r -$ identity ($n \geqq 1$) and $\partial_1 U_0 = 0$.

Proof. This reduces to establishing

$$\frac{(q^r)^k - 1}{s} N = (A_{2k}(\sigma) - \mathrm{id})(1).$$

Now $A_{2k}(\sigma)^2(1) = A_{2k}(\sigma)(\alpha^k) = A_{2k}(\sigma)(\alpha^k \cdot 1) = \sigma(\alpha^k)A_{2k}(\sigma)(1) = \sigma(\alpha^k)a^k$, etc. Thus $(A_{2k}(\sigma)^r - \mathrm{id})(1) = \{\sigma^{r-1}(\alpha) \cdots \sigma(\alpha)\alpha\}^k - 1$; thus we must show that

$$\beta^k - 1 = ((q^r)^k - 1 \cdot N)/s,$$

where $\beta=\sigma^{r-1}(\alpha)\cdots\sigma(a)\cdot\alpha\in R$. But $\partial_1\beta=\partial_1 A_1(\sigma)^r(1)=A_0(\sigma)^r\,\partial_1(1)=\partial_1(1)$, i.e. $\Delta\beta=\Delta$. So $\Delta\beta^k=\Delta$. Thus $\Delta(\beta^k-1)=0$, or $\beta^k-1\in\operatorname{Ker}\partial_1=\operatorname{Im}\partial_2=\{nN\mid n\in Z\}$. But then $\beta^k-1=n_kN$ for some integer n_k. Applying the ring homomorphism $\varepsilon\colon R\to Z$ we conclude that

$$\varepsilon(\beta)^k-\varepsilon(1)=n_k\in(N)\quad\text{or}\quad(q^r)^k-1=n_k\cdot s.$$

This completes the proof of the lemma.

It is easy now to see how to choose $U_n(\sigma^i,\sigma^j)$. In fact we may choose

$$\text{(1)}\qquad \begin{aligned} U_n(\sigma^i,\sigma^j) &= 0 && \text{if } i+j<r\\ &= A_n(\sigma^{i+j})\circ U_n(1) && \text{if } i+j\geqq r.\end{aligned}$$

Returning to the general theory we suppose that Γ is both an M-module and a Φ-module such that

$$\sigma(m\cdot\gamma)=\sigma(m)\cdot\sigma(\gamma)\qquad\forall\sigma\in\Phi,\ m\in M\text{ and }\gamma\in\Gamma.$$

Then we can extend A_* to $\operatorname{Hom}_R(D_*,\Gamma)$ by $(A_n(\sigma)\cdot f)(x_n)=\sigma(f(A_n(\sigma^{-1})x_n))$. The following is readily verified.

PROPOSITION 2. *The above map induces an action of* Φ *on* $H^n(M;\Gamma)$. *This action is independent of the* Φ*-system and of the particular resolution* (D_*,∂_*).

Of course there is entirely analogously an action of Φ on $H_n(M,\Gamma)$ (i.e. on the homology of $D_*\otimes_R\Gamma$). We will denote by $\sigma_*(\chi)$ this action of σ on $\chi\in H_n(M,\Gamma)$.

This action is usually defined via the complex (S_*,∂_*) of Example 1 (see e.g. [4]).

2. **Characteristic classes.** Let k be a principal ideal domain on which M acts trivially and assume further that $H_n(M,k)$ is k-projective. (It seems likely that this assumption is unduly restrictive but it is convenient.) Thus from the universal coefficient sequence we know that $H^n(M,A)\simeq\operatorname{Hom}_k(H_n(M,k),A)$ where A is a k-module on which M acts trivially.

We remark that these assumptions are automatically satisfied in two important cases:

(a) k is a field,

(b) M is a finitely generated free abelian group and $k=Z$.

We will omit specific references to k in much of what follows.

Let $f^n\in\operatorname{Hom}_R(D_n,H_n(M,k))$ be a cocycle representing the cohomology class corresponding to the identity map in $\operatorname{Hom}_k(H_n(M,k),H_n(M,k))$. It is clear that $A_n(\sigma)\cdot f^n$ represents the same cohomology class, so for each $\sigma\in\Phi$, there is some $F_\sigma^{n-1}\in\operatorname{Hom}_R(D_{n-1},H_n(M))$, such that $A_n(\sigma)\cdot f^n-f^n=F_\sigma^{n-1}\partial_n$. Define $u^n(\sigma,\tau)\in\operatorname{Hom}_R(D_{n-1},H_n(M))$ by

$$\text{(2)}\qquad u^n(\sigma,\tau)=A_{n-1}(\sigma)\cdot F_\tau^{n-1}-F_{\sigma\tau}^{n-1}+F_\sigma^{n-1}+(\sigma,\tau)_*[f^nU_{n-1}(\tau^{-1},\sigma^{-1})].$$

LEMMA. *$u^n(\sigma, \tau)$ is a cocycle and hence defines an element $\omega^n(\sigma, \tau)$, of $H^{n-1}(M, H_n(M, k))$.*

Proof.

$$\begin{aligned}\delta^{n-1}u^n(\sigma, \tau) &= u^n(\sigma, \tau)\,\partial_n\\ &= \sigma_*\cdot F_\tau^{n-1}A_{n-1}(\sigma^{-1})\,\partial_n + F_{\sigma\tau}^{n-1}\,\partial_n + F_\sigma^{n-1}\circ\partial_n\\ &\quad + (\sigma\tau)_*f^n(-\partial_n U_n(\tau^{-1}, \sigma^{-1}) - A_n(\tau^{-1})A_n(\sigma^{-1}) + A_n(\tau^{-1}\sigma^{-1}))\\ &= \sigma_*(A_n(\tau)\cdot f^n - f^n)A_n(\sigma^{-1}) - F_{\sigma\tau}^{n-1}\,\partial_n + F_\sigma^{n-1}\,\partial_n\\ &\quad + \{-(\sigma\tau)_*f^nA_n(\tau^{-1})A_n(\sigma^{-1}) + A_n(\sigma\tau)\cdot f^n\}\\ &= \sigma_*\tau_*f^nA_n(\tau^{-1})A_n(\sigma^{-1}) - A_n(\sigma)\cdot f^n - \{A_n(\sigma\tau)\cdot f^n - f^n\}\\ &\quad + \{A_n(\sigma)\cdot f^n - f^n\} + \{-(\sigma\tau)_*f^nA_n(\tau^{-1})A_n(\sigma^{-1}) + A_n(\sigma\tau)\cdot f^n\}\\ &= 0.\end{aligned}$$

THEOREM 1. *The function $\omega^n\colon \Phi\times\Phi \to H^{n-1}(M, H_n(M, k))$ is a Φ-cocycle and represents therefore an element $v^n \in H^2(\Phi, H^{n-1}(M, H_n(M, k)))$.*

Proof. $(\delta\omega^n)(\sigma, \tau, \rho) = A_n(\sigma)\sigma\cdot\omega^n(\tau, \rho) - \omega^n(\sigma\tau, \rho) + \omega^n(\sigma, \tau\rho) - \omega^n(\sigma, \tau)$. We work of course with a representative cocycle. Such a one is obtained by replacing ω by u.

After expanding and cancelling many pairs of terms we obtain the expression

$$\begin{aligned}&\sigma\tau F_\rho^{n-1}(A_{n-1}(\tau^{-1})A_{n-1}(\sigma^{-1}) - A_{n-1}(\tau^{-1}\sigma^{-1}))\\ &\qquad + \sigma\tau\rho f^n(U_{n-1}(\rho^{-1}, \tau^{-1})A_{n-1}(\sigma^{-1}) - U_{n-1}(\rho^{-1}, \tau^{-1}\sigma^{-1})\\ &\qquad\qquad + U_{n-1}(\rho^{-1}\tau^{-1}, \sigma^{-1}) - \sigma\tau f^nU_{n-1}(\tau^{-1}, \sigma^{-1})).\end{aligned}$$

Using property (i′) of Φ-systems the first term becomes

$$\begin{aligned}&-\sigma\tau F_\rho^{n-1}(\partial_n U_{n-1}(\tau^{-1}, \sigma^{-1}) + U_{n-2}(\tau^{-1}, \sigma^{-1})\,\partial_{n-1})\\ &\qquad = -\sigma\tau\{\rho f^nA_n(\rho^{-1}) - f^n\}U_{n-1}(\tau^{-1}, \sigma^{-1}) + \text{a coboundary}.\end{aligned}$$

Thus except for a coboundary the whole expression becomes, except for sign,

$$\begin{aligned}&+\sigma\tau\rho f^n\{A_n(\rho^{-1})U_{n-1}(\tau^{-1}, \sigma^{-1}) - U_{n-1}(\rho^{-1}, \tau^{-1})A_{n-1}(\sigma^{-1})\\ &\qquad + U_{n-1}(\rho^{-1}, \tau^{-1}\sigma^{-1}) - U_{n-1}(\rho^{-1}\tau^{-1}, \sigma^{-1})\}.\end{aligned}$$

If we rewrite this as $\sigma\tau\rho\omega^{n-1}(\sigma, \tau\rho)$, a direct computation shows that

$$\partial_n, \omega^{n-1} + \omega^{n-2}\,\partial_{n-1} = 0 \quad\text{and}\quad \partial_1\omega^0 = 0.$$

Since D_* is a projective resolution it follows that for some $(\sigma\tau\rho)^{-1}$-linear functions $B_n \in \operatorname{Hom}(D_n, D_{n+2})$ we have $\partial_{n+2}B_n + B_{n-1}\,\partial_n = \omega^n$ and $\partial_2 B_0 = \omega^0$. Taking account of the fact that f^n is a cocycle, this shows that the whole expression is a coboundary and thus $(\delta\omega^n)(\sigma, \tau, \rho) = 0 \in H^{n-1}(M, H_n(M, k))$ which was to be proved.

THEOREM 2. *The cohomology class of ω^n depends only on the homomorphism $\phi\colon \Phi \to \operatorname{Aut}(M)$. More particularly, it is independent of the choice of f^n, F_σ^{n-1}, the Φ-system and the particular resolution D_*.*

We omit the proof since it is even more tedious than the previous ones, and no more difficult. We suggest however that the effect of the various choices be considered one-by-one.

DEFINITION. The cohomology class of ω^n in $H^2(\Phi, H^{n-1}(M, H_n(M, k)))$ is called the nth *characteristic* class. It is denoted $v^n(M)$ or sometimes simply v^n.

Naturality. Let $h\colon \Phi' \to \Phi$ be any homomorphism. Then Φ' acts on M via h and the action of Φ on M. For notational convenience we denote by M', M with this action of Φ'. Thus $v^n(M') \in H^2(\Phi', H^{n-1}(M', H_n(M', k)))$ are defined. It is trivial to see that h induces a homomorphism.

$$h^*\colon H^2(\Phi, H^{n-1}(M, H_n(M))) \to H^2(\Phi', H^{n-1}(M', H_n(M')))$$

and to prove the following proposition.

THEOREM 3 (NATURALITY). $h^*(v^n(M)) = v^n(M')$.

EXAMPLE 2 (*Continued*). We choose $k = Z_s$. Thus $D_n \otimes_R Z_s = R \otimes_R Z_s \simeq Z_s$ and $\partial_n \otimes 1 = 0$. Thus $H_n(Z_s, Z_s) \cong D_n \otimes_R Z_s \cong Z_s$ for $n \geqq 1$. It is clear that $\gamma_n = 1 \otimes 1 \in Z_n \otimes Z_s$ is a generator and that $\sigma_*(\gamma_{2k}) = A_{2k}(\sigma)(1) \otimes 1 = \alpha^k \otimes 1 = \alpha^{k-1} \otimes q = \cdots = 1 \otimes q^k = q^k 1 \otimes 1 = q^k \gamma_{2k}$. Similarly $\sigma_*(\gamma_{2k+1}) = q^{k+1}\gamma_{2k+1}$. Now $f^n \in \operatorname{Hom}_R(D_n, H_n)$ is characterized by $f^n(1) = \gamma_n$. By definition $A_{2k}(\sigma^{-1}) \cdot f^{2k} = \sigma_*^{-1} \circ f^{2k} \circ A_{2k}(\sigma)$ and so its value on 1 is $\sigma_*^{-1} f^{2k}(\alpha^k \cdot 1) = \sigma_*^{-1}(q^k \cdot \gamma_{2k}) = \gamma_{2k}$; i.e. $A_{2k}(\sigma^{-1}) \cdot f^{2k} = f^{2k}$. Similarly $A_n(\sigma^i) \cdot f^n = f^n$ so $F_\sigma^{n-1} = 0$ is legitimate. Thus $u^n(\sigma^i, \sigma^j) = -\sigma_*^{i+j} f^n \circ U_{n-1}(\sigma^{-i}, \sigma^{-j})$. Thus we conclude that $v^{2k+1} = 0$ (since $U_{2k} = 0$). To compute further however we must determine the Φ-module structure of $H^{n-1}(M, H_n(M)) = \operatorname{Hom}(H_{n-1}, H_n)$ in this case. It is easy from the above to see that for even n, σ acts trivially while for odd n, σ_* is multiplication by q. The next task is to describe $H^2(\Phi, H^{k-1}(M, H_n(M)))$. Since Φ is cyclic in this case we may make use of the fact that $H^2(\Phi, A) = A^\Phi/\Sigma \cdot A$, i.e. the quotient of the fixed elements by the image of Σ where $\Sigma = 1 + \sigma + \sigma^2 + \cdots + \sigma^{r-1}$. If $\omega \Phi \times \Phi \to A$ is a two-cocycle representing an element $\{\omega\} \in H^2(\Phi, A)$ then it is readily verified that $\sum_{i=0}^{r-1} \omega(\sigma^i, \sigma) \in A^\Phi$ and represents the corresponding class in $A^\Phi/\Sigma A$. Our attention is therefore focused on

$$\sigma_*^r f^{2k} U_{2k-1} = f^{2k} U_{2k-1} = \frac{q^{rk}-1}{s} \cdot f^{2k}.$$

Putting everything together

$$v^{2k} \in H^2(\Phi, H^{2k-1}(M, H_{2k}(M))) = Z_s/r \cdot Z_s$$

and corresponds to $-(q^{rk}-1)/s$. Now it is easy to see that if $(q^r-1)/s = p$ then $(q^{rk}-1)/s = k \cdot p \bmod s$.

Summarizing. If Z_r acts on Z_s by $\sigma(t) = t^q$ where $q^r - 1 = ps$ then $v^{2k+1} = 0$ and $v^{2k} \in h^2(Z_r, H^{2k-1}(Z_s, H_{2k}(Z_s))) \simeq Z_s/r \cdot Z_s$ corresponds to $-k \cdot p$.

In the special case $r = 2$, $s = 8$, $q = 5$ we conclude that $v^{4m+2} \neq 0$ for any m while all others are zero.

We remark that this shows that one *cannot* find a Φ-system A'_n for the *complex* of the example satisfying $A'_n(\sigma\tau)=A'_n(\sigma)\circ A'_n(\tau)$. For then $U'_{n-1}=0$ would be permissible and then $v^n=0$ would be a consequence.

3. **The second differential.** Let $G=$ the semidirect product of M and Φ. Thus M is a normal subgroup of G with quotient Φ. Indeed we can represent each element, g, of G uniquely as a pair $(m,\sigma)\in M\times\Phi$. The multiplication in G is then

$$(m_1,\sigma_1)(m_2,\sigma_2) = (m_1+\sigma_1(m_2), \sigma_1\sigma_2).$$

Hochschild and Serre have shown in [4], that if B is some appropriate coefficient group, there is a spectral sequence

$$E_r^{p,q}(B) \Rightarrow H^n(G; B),$$

where $E_2^{p,q}(B)=H^p(\Phi, H^q(M,B))$.

In [2] and [3], the present authors found an interpretation of the second differential in this spectral sequence (actually the emphasis there is the comparison of this spectral sequence with those of *other* extensions of Φ by M). We wish to recall that proposition and to relate the characteristic classes with the differential.

First fix an integer n. The universal coefficient theorem gives

$$H^n(M,k) \cong \operatorname{Hom}_k(H_n(M,k),k).$$

Thus $H^n(M,k)$ and $H_n(M,k)$ are paired to k. This induces a cup-product pairing of spectral sequences as follows:

$$\bar{E}_r^{p,q}\otimes\tilde{E}_r^{p'q,q'}\to E_r^{p+p',q+q'}$$

where

$$\bar{E}_r^{p,q} = E_r^{p,q}(H^N(M,k))$$
$$\tilde{E}_r^{p,q} = E_r^{p,q}(H_N(M,k))$$

and

$$E_r^{p,q} = E_r^{p,q}(k).$$

In particular $\tilde{E}_2^{0,N}\simeq H^0(\Phi, \operatorname{Hom}_k(H_N(M,k), H_N(M,k)))$. This group contains an element f^N, which corresponds to the identity in $\operatorname{Hom}_k(H_N(M,k), H_N(M,k))$. Note also that $\bar{E}_2^{p,0}=H^p(\Phi, H^0(M, H^N(M,k)))$ is isomorphic to $H^p(\Phi, H^N(M,k)) = E_2^{p,N}$. Calling this isomorphism θ, Proposition 2.2 of [3] can be stated as follows.

Proposition 2. *Let* $\chi\in E_2^{p,N}$ *then*

$$d_2(\chi) = (-1)^p\theta(\chi)\cup d_2(f^N). \tag{3}$$

We should remark that this is not at all deep; it is confusing due to the superabundance of notation. The significance of it is that it "reduces" the computation of d_2 to that of $d_2(f^N)\in H^2(\Phi, H^{N-1}(M, H_N(M,k)))$. The following proposition can be interpreted as giving a computational hold on $d_2(f^N)$.

THEOREM 4. $v^N(M)=d_2(f^N)$.

Proof. The proof of Theorem 6.1 of [3] shows that, using the standard complex (S_*, ∂_*), $d_2(f^N)$ is represented by the cocycle

$$Y^n(\sigma, \tau) = A_{n-1}(\sigma)\cdot F_\tau^{n-1}-F_{\sigma\tau}^{n-1}+F_\sigma^{n-1} \in \operatorname{Hom}_R(S_{N-1}, H_N(M)).$$

For this complex $U_k=0$ is permissible. Thus $\{Y^N\}=\{u^N\}$.

We remark that our interest in the classes v^N stems from this theorem. One would hope that it would give some information about $H^n(G, k)$ (which it does). However, the proposition can be turned around to give us information about v^N. The proposition below is such a situation.

PROPOSITION 3. $v^1(M)=0$.

Proof. Since $d_2\colon E_2^{0,1}\to E_2^{2,0}$ is such that $v^1(M)=d_2(f^1)$ we see that $v^1\neq 0$ implies $E_3^{2,0}\neq E_2^{2,0}$. But $E_\infty^{2,0}(B)=E_2^{2,0}(B)/d_2(E_2^{0,1}(B))$. Also $E_2^{N,0}(B)=H^n(\Phi_0, H^0(M,B))$ $\cong H^n(\Phi, B)$. Under this identification indeed

$$E_\infty^{n,0}(B) \cong \text{Image}\,(H^n(\Phi, B)\xrightarrow{\pi^*} H^n(G, B))$$

where π is the canonical projection from G to $\Phi=G/M$. Since there is a homomorphism $\rho\colon \Phi\to G$ such that $\pi\rho=1_\Phi$, it follows that π^* is a monomorphism and hence that $E_\infty^{n,0}(B)=E_2^{n,0}(B)$. Thus we conclude that $d_2E_2^{0,1}(B)\to E_2^{2,0}(B)$ is zero. For appropriate choice of B this gives $d_2=0$.

PART II. THE FREE ABELIAN CASE

1. We assume throughout Part II that M is a free abelian group. We will specialize the preceding by choosing $k=Z$ although that is not strictly necessary for much of what we do.

Our first task is to describe a particular resolution (D_*, ∂_*). Let $\{m_i\}$ $i\in J$ be an indexed free basis for M. Let $N=$a free $(R=Z[M])$ module on symbols $\{X_i\}$ $i\in J$. It is well known that there is an acyclic resolution of Z via R-free modules as follows:

(4a) $D_n=\Lambda_R^n N$, $n\geqq 0$.

(4b) $\varepsilon\colon D_0\equiv R\to Z$ satisfies $\varepsilon(m)=1$ $\forall m\in M$.

(4c) $\partial_1\colon D_1\to D_0$ satisfies $\partial_1\chi_i=m_i-1\in R$ $\forall i\in J$.

(4d) $\partial_{m+n}(X\wedge Y)=\partial_n X\wedge Y+(-1)^n X\wedge\partial_m Y$ for $X\in D_n$ and $Y\in D_m$.

Clearly $\partial\otimes\text{id}=0$ and so $H_n(M, Z)\cong D_n\otimes_R Z\cong\Lambda^n(D_1\otimes_R Z)\cong\Lambda^n M$. The multiplication in D_* corresponds to Pontrjagin multiplication in $H_*(M)$. We will exploit this multiplicative structure below.

THEOREM 5. $2v^2(M)=0$ *for free abelian groups* M.

Proof. Consider $f\in\operatorname{Hom}_R(S_2, H_2(M))$ where $f(1, m_1, m_2)=m_1\wedge m_2\in\Lambda^2 M$ $\cong H_2(M)$. The lemma below shows that f and $-2f^2$ are cohomologous, i.e., there is an $h \in \operatorname{Hom}_R(S_1, H_2)$ such that $f = -2f^2 + \delta h$. According to equation (2), in this case, v^2 is given up to sign by $\sigma \cdot F_\tau^1 - F_{\sigma\tau}^1 + F_\sigma^1$ where $\delta F_\sigma^1 = \sigma \cdot f^2 - f^2$. Since $\sigma \cdot f = f$ we conclude that $0 = -2\delta F_\sigma^1 + \delta(\sigma \cdot h - h)$. Thus we may define $g_\sigma \in H(M, H_2(M))$ by $g_\sigma =$ the cohomology class of $-2F_\sigma^1 + \sigma \cdot h - h$. Now it is clear that

$$(\delta g)(\sigma, \tau) = \sigma \cdot g\tau - g\sigma\tau + g\sigma = -2\{\sigma \cdot F_\tau^1 - F_{\sigma\tau}^1 + F_\sigma^1\}.$$

Thus $2v^2(M) = 0$.

LEMMA. *f represents the class in $H^2(M, H_2(M)) \simeq \operatorname{Hom}(H_2(M), H_2(M))$ corresponding to twice the identity map.*

Proof. We construct part of a chain transformation $\phi: D_* \to S_*$. Choose $\phi_0(1) = (1)$, $\phi_1(X_i) = +(1, m_i) \in S_1$, and

$$\phi_2(X_{i_1} \wedge X_{i_2}) = +(1, m_{i_1}, m_{i_2}) - (m_{i_1}, m_{i_2}, m_{i_1}, m_{i_2})$$

for $i_1 < i_2$ relative to some ordering of J. It is readily verified that $\partial_2\phi_2 = \phi_1\partial_2$, $\partial_1\phi_1 = \phi_0\partial_1$, and $\varepsilon\phi_0 = \varepsilon$. It is also trivial that f is in fact a cocycle. Thus the cohomology class is determined by $f\phi_2\varepsilon : \operatorname{Hom}_R(D_2, H_2(M))$. Now $(f\phi_2)(X_{i_1} \wedge X_{i_2}) = +m_{i_1} \wedge m_{i_2} - (m_{i_1}^{-1} m_{i_2}) \wedge m_{i_2} = +2m_{i_1} \wedge m_{i_2}$, then $+f\phi_2$ represents twice the identity.

We will see later that this implies $2v^n(m) = 0$ for all n.

For the moment we record the following trivial facts.

PROPOSITION 3. *Let A_*, U_* be any Φ-system for (D_*, ∂_*). Then we may choose $F_\sigma = 0$ for all $\sigma \in I$ and consequently*

$$\omega^n(\sigma, \tau) = +\{\sigma_* \tau_* f^n U_{n-1}(\tau^{-1}, \sigma^{-1})\}$$

is a representative for $v^n(M)$. Since $f^n: \Lambda_R^n D_1 \to H_n(M)$ is R-linear and M acts trivially on $H_n(M)$, f^n actually factors through the canonical projection

$$\Lambda_R^n D_1 \to \Lambda_R^n D_1 \otimes_R Z = \Lambda^n(D_1 \otimes_R Z) = \Lambda^n M = H_n(M).$$

So does $A_n(\sigma) \cdot f^n$ and in fact they both correspond after the canonical projection to the identity map. Hence they are equal, and so $F_\sigma = 0$ is appropriate.

PROPOSITION 4. *There is a Φ-system $\{A_n\}$ for the above complex satisfying*

(4e) *$A_0(\sigma)m = \sigma(m)$, $m \in M \subset R = D_0$ and $\sigma \in I$, and*

(4f) *$A_{n+m}(\sigma)(Y_n \wedge Y_m) = A_n(\sigma) Y_n \wedge A_m(\sigma) Y_m$ for $Y_n \in D_n$ and $Y_m \in D_m$.*

Proof. Choose A_0 as prescribed by (4e). Choose $A_1(\sigma)$ so that A is σ-linear and such that $\partial_1 A_1(\sigma) = A_0(\sigma)\,\partial_1$. Then define $A_m(\sigma)(X_1 \wedge \cdots \wedge X_m) = A_1(\sigma)X_1 \wedge \cdots \wedge A_1(\sigma)X_m$.

Slightly less trivial is the following: Choose an ordering of the index set J.

PROPOSITION. *There is a collection of $U_n(\sigma, \tau)$ satisfying*

(4g) *$U_0(\sigma, \tau) = 0$ and*

(4h) *$U_n(\sigma, \tau)(X_{i_1} \wedge \cdots \wedge X_{i_n}) = U_{n-1}(\sigma, \tau)(X_{i_1} \wedge \cdots \wedge X_{i_{n-1}}) \wedge A_1(\sigma)A_1(\tau)X_{i_n}$ $+(-1)^{n-1}A_{n-1}(\sigma, \tau)(X_{i_1} \wedge \cdots \wedge X_{i_{n-1}}) \wedge U_1(\sigma, \tau)X_{i_n}$ for $i_1 < i_2 < \cdots < i_n$.*

Proof. Choose U_0 as specified and let $U_1(\sigma, \tau)$ be any $\sigma\tau$-linear function satisfying $\partial_2 U_1(\sigma, \tau) = A_1(\sigma, \tau) - A_1(\sigma)\cdot A_1(\tau)$. Define U_n for $n \geqq 2$ by the formula above. We now inductively verify that $\partial_{n+1} U_n(\sigma, \tau) + U_{n-1}(\sigma, \tau)\,\partial_n = A_n(\sigma, \tau) - A_n(\sigma) A_n(\tau)$.

For convenience we omit most of the complication in the subscripts.

$$
\begin{aligned}
&(\partial_{n+1} U_n + U_{n-1}\,\partial_n)(X_1 \wedge \cdots \wedge X_n) \\
&\quad = \partial_{n+1}((-1)^{n-1} A(\sigma, \tau)(X_1 \wedge \cdots \wedge X_{n-1}) \wedge U_1(X_n) \\
&\qquad\qquad + U_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) A_1(\sigma) A_1(\tau)(X_n)) \\
&\qquad + U_{n-1}(\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) X_n + (-1)^{n-1}(X_1 \wedge \cdots \wedge X_{n-1})\,\partial_1 X_n) \\
&\quad = (-1)^{n-1}\,\partial_{n+1}(A(\sigma, \tau)(X_1 \wedge \cdots \wedge X_{n-1}) \wedge U_1(X_n)) \\
&\qquad + \partial_n U_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) A_1(\sigma)(A_1(\tau) X_n) \\
&\qquad + (-1)^n U_{n-1}(X_1 \wedge \cdots \wedge X_{n-1})(A_0(\sigma) A_0(\tau)\,\partial_1 X_n) \\
&\qquad + (-1)^{n-2} A_{n-2}(\sigma, \tau)\,\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) U_1 X_n \\
&\qquad + [U_{n-2}\,\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1})] \\
&\qquad \cdot (A_1(\sigma)(A_1(\tau) X_n) + (-1)^{n-1} U_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) A_0(\sigma\tau) \partial_1 X_n).
\end{aligned}
$$

Here we have made use of the fact that the $\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) \wedge X_n$ is an alternating sum of R-basis elements for which our formula defining U_{n-1} is defined and of the fact U_{n-1} is $\sigma\tau$-linear.

Thus the expression becomes:

$$
\begin{aligned}
&(-1)^{n-1}\,\partial_{n-1}(A(\sigma, \tau)(X_1 \wedge \cdots \wedge X_{n-1}) \wedge U_1 X_n) \\
&\qquad + (\partial U + U\partial)(X_1 \wedge \cdots \wedge X_{n-1}) \wedge A_1(\sigma)(A_1(\tau) X_n) \\
&\qquad + (-1)^{n-2} A_{n-2}(\sigma, \tau)(\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1})) U_1(X_n) \\
&\quad = (-1)^{n-1} A(\sigma, \tau)\,\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) \wedge U_1 X_n + A_{n-1}(\sigma\tau)(X_1 \wedge X_{n-1}) \\
&\qquad\qquad \cdot \partial_2 U_1 X_n \\
&\qquad + \{A_{n-1}(\sigma, \tau) - A_{n-1}(\sigma) A_{n-1}(\tau)\}(X_1 \wedge \cdots \wedge X_{n-1}) \wedge A_1(\sigma)(A_1(\tau) X_n) \\
&\qquad + (-1)^{n-2} A_{n-2}(\sigma, \tau)\,\partial_{n-1}(X_1 \wedge \cdots \wedge X_{n-1}) \wedge U_1 X_n \\
&\quad = A_{n-1}(\sigma, \tau)(X_1 \wedge \cdots \wedge X_{n-1})(A_1(\sigma, \tau) X_n - A_1(\sigma)(A_1(\tau) X_n)) \\
&\qquad + [A_{n-1}(\sigma, \tau) - A_{n-1}(\sigma) A_{n-1}(\tau)](X_1 \wedge \cdots \wedge X_{n-1}) A_1(\sigma)(A_1(\tau) X_n) \\
&\quad = A_n(\sigma, \tau)(X_1 \wedge \cdots \wedge X_n) - A_{n-1}(\sigma)(A_{n-1}(\tau)(X_1 \wedge \cdots \wedge X_{n-1})) \\
&\qquad\qquad \cdot A_1(\sigma)(A_1(\tau) X_n) \\
&\quad = A_n(\sigma, \tau)(X_1 \wedge \cdots \wedge X_n) - A_n(\sigma)(A_{n-1}(\tau)(X_1 \wedge \cdots \wedge X_{n-1}) \wedge A_1(\tau) X_n) \\
&\quad = \{A_n(\sigma, \tau) - A_n(\sigma) \circ A_n(\tau)\}(X_1 \wedge \cdots \wedge X_n).
\end{aligned}
$$

We remark that the above proposition depends crucially on the ordering of the index set. I.e., the formulae defining U_n do not hold for an arbitrary product $Y_1 \wedge \cdots \wedge Y_n$ where $Y_i \in D_1$.

We use the following algebraic fact.

PROPOSITION 5. *There are natural maps*

$$J_n\colon \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) \to \operatorname{Hom}(\Lambda^{n-1} M, \Lambda^n M)$$

satisfying

$$J_n(F)(x_1 \wedge \cdots \wedge x_{n-1}) = \sum (-1)^{i+1} x_1 \wedge \cdots \wedge x_{i-1} \wedge F(x_1) \wedge x_{i+1} \wedge \cdots \wedge x_{n-1}.$$

Proof. It is clear that a similar map

$$J_n(\tilde{F})\colon \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) \to \operatorname{Hom}(\otimes^{n-1} M, \Lambda^n M)$$

exists. We need only verify that $J_n(\tilde{F})$ annihilates elements of the form

$$x_1 \otimes \cdots \otimes x_i \otimes x \otimes x \otimes y_1 \otimes \cdots \otimes y_k$$

where $i+k+2=n-1$.

It is clear since $x \wedge x=0$ that all terms except perhaps two are separately zero and since $F(x) \in \Lambda^2 M$ we have $x \wedge F(x)=F(x) \wedge x$.

Via canonical isomorphisms we may view J_n as a homomorphism of Φ-modules

$$J_n\colon H^1(M, H_2(M)) \to H^{n-1}(M, H_n(M)).$$

THEOREM 6. $(J_{n+1})_*(v^2(M))=v^{n+1}(M)$.

Proof. Using the Φ-system as above and expanding on $-$ we see that

$$-u^{n+1}(\sigma, \tau)(X_{i_1} \wedge \cdots \wedge X_{i_n})$$
$$= \sum (-1)^{j+1} X_{i_1} \wedge \cdots \wedge X_{i_j} \wedge (f^2 U_1(\sigma, \tau) X_{i_{j+1}}) \wedge \cdots \wedge X_n.$$

Untangling the definitions gives the theorem.

We have the remarkable corollaries.

COROLLARY. $v^2(M)=0$ *iff* $v^n(M)=0$ *for all n.*

COROLLARY. $2v^n(M)=0$.

COROLLARY. *If* Φ *is a finite group with an odd number of elements, then* $v^n(M)=0$ *for all n.*

This last corollary gives a much more satisfactory explanation of the facts alluded to in [2] concerning the integral representations of $\boldsymbol{Z}_p$.

We remark, lest the reader become too optimistic, that $v^n(M)\neq 0$ in general. An example is given below.

Together with the results of [3], these corollaries give a fair hold on the 2nd differential in the Hochschild-Serre spectral sequence for any extension of a free abelian group.

2. **The direct sum theorem.** We suppose, in this section, that a Φ-module $M=M' \oplus M''$. We wish to express $v^i(M)$ in terms of $v^j(M')$ and $v^n(M'')$. In view of Theorem 6 of the preceding section the critical case is $v^2(M)$. A general formula follows from that special case. It turns out to be surprisingly complicated.

Recalling the canonical isomorphisms

$$\Lambda^n(M' \oplus M'') \cong \bigoplus_{i+j=n} \Lambda^i(M') \otimes \Lambda^j(M'')$$

we see that $H^1(M', H_2(M')) \cong \mathrm{Hom}\,(\Lambda^1 M', \Lambda^2 M')$ is a direct summand of $H^1(M, H_2(M'))$, as is $H^1(M, H_2(M))$. In an obvious sense then we have the following theorem.

THEOREM 7. $v^2(M' \oplus M'') = v^2(M') + v^2(M'')$.

Proof. Let $R' = Z[M']$ and $R'' = Z[M'']$; then $R = Z[M] \cong R'' \otimes_Z R''$. In fact $D_* = D'_* \otimes D''_*$—for we may choose a basis for $M' \oplus M''$ that is the union of a basis for M' and a basis for M''. It follows that we may find $A_i(\sigma)$ and $U_i(\sigma, \tau)$ for D_* in terms of the corresponding homomorphisms in D'_* and D''_*. The details harbor no surprises and lead directly to the formula given.

For the convenience of the reader we formulate precisely the more general formula. For this purpose we introduce the following notation:

$$C'_j\colon \mathrm{Hom}\,(\Lambda^i M', \Lambda^{i+1} M') \to \mathrm{Hom}\,(\Lambda^{i+j} M, \Lambda^{i+j+1} M)$$

$$\begin{aligned} C'_j(F)(x \otimes y) &= F(x) \otimes y \quad \text{if } x \in \Lambda^i M' \text{ and } y \in \Lambda^j M'' \\ &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

Of course we are making use of the canonical isomorphisms above.

Similarly we have

$$C''_j\colon \mathrm{Hom}\,(\Lambda^i M'', \Lambda^{i+1} M'') \to \mathrm{Hom}\,(\Lambda^{i+j} M, \Lambda^{i+j+1} M)$$

$$\begin{aligned} C''_j(F)(x \otimes y) &= (-1)^j x \otimes F(y) \quad \text{if } x \in \Lambda^j M' \text{ and } y \in \Lambda^i M'' \\ &= 0 \qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

With this notation then we have the following formula.

THEOREM 8. $v^{n+1}(M' \oplus M'') = \sum_{i+j=n} C'_{j*}(v^{i+1}(M')) + C''_{i*}(v^{j+1}(M''))$.

The proof is immediate from Theorem 6 and the previous theorem.

EXAMPLE 3. $\Phi = Z_2 + Z_2$, $M =$ the group ring of Φ modulo its invariant elements, i.e.

$$0 \longrightarrow Z \xrightarrow{\ j\ } Z\,[\Phi] \xrightarrow{\ p\ } M \longrightarrow 0$$

is an exact sequence of Φ-modules with $j(1) = \sum_{\sigma \in \Phi} \sigma$.

Let ε_1 and ε_2 be generators of Φ and define $m_i = \rho(\varepsilon_i)$ $i = 1, 2$ and $m_3 = \rho(\varepsilon_1 \varepsilon_2)$. It is easy to see that relative to the basis (m_1, m_2, m_3) ε_1 and ε_2 induce the transformations described by the matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Let $t_i = m_i - 1 \in R = Z[M]$. It is easy to see that

$$\begin{aligned} \varepsilon_1(t_1) &= -\{m_1^{-1}m_3^{-1}t_1 + m_1^{-1}m_2^{-1}m_3^{-1}t_2 + m_3^{-1}t_3\} \\ \varepsilon_1(t_2) &= t_3 \\ \varepsilon_1(t_3) &= t_2 \end{aligned}$$

and hence that one may define $A_1(\varepsilon_1)$ by

$$\begin{aligned} A_1(\varepsilon_1)\colon X_1 &\mapsto -\{m_1^{-1}m_3^{-1}X_1 + m_1^{-1}m_2^{-1}m_3^{-1}X_2 + m_3^{-1}X_3\} \\ X_2 &\mapsto X_3 \\ X_3 &\mapsto X_2 \end{aligned}$$

Similarly one can show that the following is a permissible choice for $A_1(\varepsilon_2)$.

$$\begin{aligned} A_1(\varepsilon_2)\colon X_1 &\mapsto X_3 \\ X_2 &\mapsto -\{m_1^{-1}m_2^{-1}m_3^{-1}X_1 + m_2^{-1}m_3^{-1}X_2 + m_3^{-1}X_3\} \\ X_3 &\mapsto X_1 \end{aligned}$$

A direct computation shows that both $A_1(\varepsilon_1) \circ A_1(\varepsilon_1)$ and $A_1(\varepsilon_2) \circ A_1(\varepsilon_2)$ = the identity map: $D_1 \to D_1$. (Recall that $A_1(\varepsilon_i)$ is ε_i-linear!)

We may choose $A_1(\varepsilon_1\varepsilon_2) = A_1(\varepsilon_1) \circ A_1(\varepsilon_2)$ and $A_1(\mathrm{id})$ = identity. This completes the computation of A_* in view of Proposition 4. It is now, of course, possible to compute all of $U_i(\sigma, \tau)$, but not all of them need be computed. Formula (7) gives an explicit nonhomogeneous two-dimensional Φ-cocycle with coefficients in $H^1(M, H_2(M))$ which represents $v^2(M)$. In view of the special choice of Φ it is possible to describe a two-dimensional cocycle much more efficiently.

Letting $\Omega = Z[\Phi]$ we may construct a Ω-free acyclic resolution, T_* of Z as follows:

$$\cdots \xrightarrow{\partial_3} \Omega\alpha_{0,2} \oplus \Omega\alpha_{1,1} \oplus \Omega\alpha_{2,0} \xrightarrow{\partial_2} \Omega\alpha_{0,1} \oplus \Omega\alpha_{1,0} \xrightarrow{\partial_1} \Omega\alpha_{0,0} \xrightarrow{\varepsilon} Z \longrightarrow 0.$$

Here $\alpha_{i,j}$ is a free generator of T_{i+j}. The maps in question are

$$\varepsilon(\alpha_{0,0}) = 1,$$

$$\begin{aligned} \partial_1\colon \alpha_{1,0} &\mapsto (\varepsilon_1 - 1)\alpha_{0,0}, \\ \alpha_{0,1} &\mapsto (\varepsilon_2 - 1)\alpha_{0,0}, \end{aligned}$$

$$\begin{aligned} \partial_2\colon \alpha_{2,0} &\mapsto (\varepsilon_1 + 1)\alpha_{1,0}, \\ \alpha_{1,1} &\mapsto (\varepsilon_1 - 1)\alpha_{0,1} - (\varepsilon_2 - 1)\alpha_{1,0}, \\ \alpha_{0,2} &\mapsto (\varepsilon_2 + 1)\alpha_{0,1}. \end{aligned}$$

A chain map ψ_* from this resolution to the standard nonhomogeneous resolution (see [1] for terminology) is given by

$$\psi_0: \alpha_{0,0} \mapsto [\ \ ],$$
$$\psi_1: \alpha_{1,0} \mapsto [\varepsilon_1],$$
$$\alpha_{0,1} \mapsto [\varepsilon_2],$$
$$\psi_2: \alpha_{2,0} \mapsto [\varepsilon_1, \varepsilon_1]+[1, 1],$$
$$\alpha_{1,1} \mapsto [\varepsilon_1, \varepsilon_2]-[\varepsilon_2, \varepsilon_1],$$
$$\alpha_{0,2} \mapsto [\varepsilon_2, \varepsilon_2]+[1, 1].$$

It follows that if h is any nonhomogeneous 2-cocycle with coefficients in the Φ-module A, then $h \circ \psi_2$ represents the same cohomology class. Then formula (7) shows that $v^2(M) \in H^2(\Phi, H^1(M, H_2(M))$ is determined by ω^2 where

$$\omega^2 \circ \psi_2(\alpha_{2,0}) = -\{\varepsilon_1^{-1}\varepsilon_1^{-1}f_2U_1(\varepsilon_1, \varepsilon_1)+1\cdot 1\cdot f_2U_1(1, 1)\} \in \mathrm{Hom}\,(H_1, H_2),$$
$$\omega^2 \circ \psi_2(\alpha_{1,1}) = -\{\varepsilon_1^{-1}\varepsilon_2^{-1}f_2U_1(\varepsilon_1, \varepsilon_2)-\varepsilon_2^{-1}\varepsilon_1^{-1}f_2U_1(\varepsilon_2, \varepsilon_1)\},$$

and

$$\omega^2 \circ \psi_2(\alpha_{0,2}) = -\{\varepsilon_2^{-1}\varepsilon_2^{-1}f_2U_1(\varepsilon_2, \varepsilon_2)+1\cdot 1\cdot f_2U_1(1, 1)\}.$$

In view of the remarks above that $A_1(\varepsilon_i)^2=\mathrm{id}$ we may choose $U_1(\varepsilon_i, \varepsilon_i)=0$. As usual we may choose $U_1(1, 1)=0$. Since $A_1(\varepsilon_1\varepsilon_2)=A_1(\varepsilon_1)A_1(\varepsilon_2)$ by definition we may choose $U_1(\varepsilon_1, \varepsilon_2)=0$. Thus our cocycle takes the form

$$\alpha_{2,0} \mapsto 0,$$
$$\alpha_{1,1} \mapsto \{\varepsilon_2^{-1}\varepsilon_1^{-1}f_2U_1(\varepsilon_2, \varepsilon_1)\},$$
$$\alpha_{0,2} \mapsto 0.$$

We must now calculate $U_1(\varepsilon_2, \varepsilon_1)$. We have

$$\begin{aligned}\partial_2U_1(\varepsilon_2, \varepsilon_1) &= A_1(\varepsilon_2\varepsilon_1)-A_1(\varepsilon_2) \circ A_1(\varepsilon_1)\\ &= A_1(\varepsilon_1) \circ A_1(\varepsilon_2)-A_1(\varepsilon_2) \circ A_1(\varepsilon_1).\end{aligned}$$

A direct calculation shows that this right-hand side is the homomorphism

$$X_1 \mapsto (m_1^{-1}-m_1^{-1}m_3^{-1})X_1+(1-m_3^{-1})X_2+(m_1^{-1}m_3^{-1}-m_2m_3^{-1})X_3,$$
$$X_2 \mapsto (m_3^{-1}-1)X_1+(m_2^{-1}m_3^{-1}-m_2^{-1})X_2+(m_1m_3^{-1}-m_2^{-1}m_3^{-1})X_3,$$
$$X_3 \mapsto (m_1^{-1}m_2^{-1}m_3^{-1}-m_1^{-1}m_3^{-1})X_1+(m_2^{-1}m_3^{-1}-m_1^{-1}m_2^{-1}m_3^{-1})X_2+0,$$

and thus one can choose $U_1(\varepsilon_2, \varepsilon_1)$ to be

$$\begin{aligned}U_1(\varepsilon_2, \varepsilon_1)\colon X_1 &\mapsto 0 \quad -m_1^{-1}m_3^{-1}X_1 \wedge X_3 \quad +m_3^{-1}X_2 \wedge X_3,\\ X_2 &\mapsto 0 \quad -m_3^{-1}X_1 \wedge X_3 \quad -m_2^{-1}m_3^{-1}X_2 \wedge X_3,\\ X_3 &\mapsto -m_1^{-1}m_2^{-1}m_3^{-1}X_1 \wedge X_2+0+0.\end{aligned}$$

Consequently if we identify $H_i(M)$ with $\Lambda^i M$

$$\begin{aligned}\phi = \varepsilon_2^{-1}\varepsilon_1^{-1}f_2U_1(\varepsilon_2, \varepsilon_1)\colon m_1 &\mapsto 0 \quad +m_1 \wedge m_3+m_2 \wedge m_3,\\ m_2 &\mapsto 0 \quad -m_1 \wedge m_3-m_2 \wedge m_3,\\ m_3 &\mapsto -m_1 \wedge m_3.\end{aligned}$$

Summarizing. $v^2(M) \in H^2(\Phi; \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M))$ is given by the cocycle sending $\alpha_{2,0}$ and $\alpha_{0,2}$ to zero but sending $\alpha_{1,1}$ to the homomorphism, ϕ, just described.

We now contend that $v^2(M)\neq 0$. For suppose g is a 1-cochain whose coboundary is the above cocycle. Then if

$$y_{0,1} = g(\alpha_{0,1}) \in \operatorname{Hom}(M, \Lambda^2(M)), \quad \text{and} \quad y_{1,0} = g(\alpha_{1,0}),$$

then we would have

(1) $0=(1+\varepsilon_1)y_{0,1}$,

(2) $\phi=(\varepsilon_1-1)y_{0,1}-(\varepsilon_2-1)y_{1,0}$, and

(3) $0=(1+\varepsilon_2)y_{1,0}$.

We now make use of the structure of M as a module over the subgroups generated by ε_i. Equation (1) implies that $y_{0,1}$ defines an element in $H^1(Z_2^{(1)}, \operatorname{Hom}(M, \Lambda^2 M))$ where $Z_2^{(1)}=$ the subgroup of Φ generated by ε_1. A similar statement holds for $y_{1,0}$. A lemma proved below shows $H^1(Z_2^{(i)}, \operatorname{Hom}(M, \Lambda^2 M))=0$. Thus we conclude the existence of $x_{0,1}$ and $x_{1,0} \in \operatorname{Hom}(M, \Lambda^2 M)$ such that $y_{1,0}=(\varepsilon_1-1)x_{1,0}$ and $y_{0,1}=(\varepsilon_2-1)x_{0,1}$. Thus equation (2) implies the existence of ξ such that

$$\phi = (\varepsilon_1-1)(\varepsilon_2-1)\cdot\xi \in \operatorname{Hom}(M, \Lambda^2 M).$$

We now demonstrate that such an equation is impossible. Note that $M=\Lambda^1 M$ and $\Lambda^2 M$ are paired to $\Lambda^3 M$ which is easily seen to be a trivial Φ-module of rank 1 with $m_1 \wedge m_2 \wedge m_3$ as generator.

Note that

$$\begin{aligned}\phi(m_2) \wedge m_3-\phi(m_1) \wedge m_2 &= (-m_1 \wedge m_3-m_2 \wedge m_3) \wedge m_3\\ &\quad -(m_1 \wedge m_3+m_2 \wedge m_3) \wedge m_2 = 0+m_1 \wedge m_2 \wedge m_3\end{aligned}$$

is a generator. On the other hand

$$\begin{aligned}\{(\varepsilon_1-1)(\varepsilon_2-1)\xi\}(m_2) \wedge m_3-\{(\varepsilon_1-1)(\varepsilon_2-1)\xi\}(m_1) \wedge m_2\\ = -2\xi(m_1+m_3) \wedge (m_2+m_3)\end{aligned}$$

(detail is easy though tedious) which cannot be a generator.

LEMMA. $H^1(Z_2^{(i)}, \operatorname{Hom}(M, \Lambda^2 M))=0$.

Proof. As we have already remarked, $\Lambda^3 M\simeq Z$ is a trivial Φ-module and hence $\Lambda^2 M\simeq \operatorname{Hom}_Z(M, Z)\equiv M^*$. We have the exact sequence

$$(*) \qquad 0 \to Z \to Z[\Phi] \to M \to 0.$$

Thus

$$(*') \qquad 0 \leftarrow M^* \leftarrow \operatorname{Hom}(Z[\Phi], M^*) \leftarrow \operatorname{Hom}(M, M^*) \leftarrow 0.$$

According to [1, p. 199], the middle module is $Z[\Phi]$-free and hence $H^j(Z_2^{(i)}, \mathrm{Hom}(M, M^*)) \simeq H^{j-1}(Z_2^{(i)}, M^*)$ for $j>1$; and from

$$(*'') \qquad 0 \to M^* = Z[\Phi]^* \to Z^* \to 0$$

we conclude that $H^{j-1}(Z_2^{(i)}, M^*) \simeq H^{j-1}(Z_2^{(i)}, Z^*) =$ for $j-1>1$. But it is well known that $H^j(Z_2^{(i)}, Z)$ is zero for odd i and $H^j(Z_2, \quad)$ is periodic of order 2. This completes the proof that $v^2(M) \neq 0$.

The methods of the previous lemma can be used to determine the group in which v^2 lies, i.e. $H^2(\Phi, H'(M, H_2(M)))$. Since $(*)$, $(*')$, $(*'')$ are exact sequences of Φ-modules, we get

$$\begin{aligned} H^2(\Phi, H^1(M, H_2(M))) &\simeq H^2(\Phi, \mathrm{Hom}(M, \Lambda^2 M)) \\ &\simeq H^2(\Phi, \mathrm{Hom}(M, M^*)) \simeq H^1(\Phi, M^*). \end{aligned}$$

But $\boldsymbol{Z}[\Phi]^* \simeq \boldsymbol{Z}[\Phi]$, so $(*'')$ shows that $M^* \simeq I$, the augmentation ideal of $\boldsymbol{Z}[\Phi]$. In any case we have from $(*'')$

$$\cdots \longrightarrow H^0(\Phi, \boldsymbol{Z}[\Phi]) \xrightarrow{\varepsilon_*} H^0(\Phi, \boldsymbol{Z}) \longrightarrow H^1(\Phi, M^*) \longrightarrow 0$$

and $H^0(\Phi, \boldsymbol{Z}[\Phi]) =$ ideal generated by $\sum = \sum_{\sigma \in \Phi}$ in $\boldsymbol{Z}[\Phi]$ and $H^0(\Phi, \boldsymbol{Z}) = \boldsymbol{Z}$, so the question is, what is $\varepsilon(\sum)$? But ε is just the usual augmentation map, so $\varepsilon(\sum) = 4$ and $H^1(\Phi, M^*) = \boldsymbol{Z}_4$, so we have proved

THEOREM 9. *Let Φ be $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ & M the quotient of $\boldsymbol{Z}[\Phi]$ by the ideal generated by $\sum = \sum_{\sigma \in \Phi} \sigma$. Then*

$$H^2(\Phi, H^1(M, H_2(M))) = \boldsymbol{Z}_4$$

and $v^2(M)$ is the element of order 2 in $\boldsymbol{Z}_4$.

Note that $v^i(M) = 0$ for $i>2$, since $v^i(M) = 0$ for $i>3$ since M has rank 3 and v^3 happens to lie in a 0 group, i.e.

$$\begin{aligned} H^2(\Phi, H^2(M, H_3(M))) &= H^2(\Phi, H^2(M)) = H^2(\Phi, \Lambda^2 M) \\ &= H^2(\Phi, M^*) = H^1(\Phi, \boldsymbol{Z}) = [H^0(\boldsymbol{Z}_2, \boldsymbol{Z}) \otimes H^1(\boldsymbol{Z}_2, \boldsymbol{Z})] \\ &\quad \otimes [H^1(\boldsymbol{Z}_2, \boldsymbol{Z}) \otimes H^0(\boldsymbol{Z}_2, \boldsymbol{Z})] = 0. \end{aligned}$$

REMARK. The above is the only example we know of a nonzero characteristic class of a $\boldsymbol{Z}$-free module.

EXAMPLE 2*. In this example we briefly consider the dual of

$$M, M^* = \mathrm{Hom}(M, \boldsymbol{Z}) = I,$$

the augmentation ideal of $\boldsymbol{Z}[\Phi]$. Although M and I are closely related, we prove

PROPOSITION A. $H^2(\Phi, H^1(I, H_2(I))) = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

PROPOSITION B. $v^i(I) = 0$ *for all i.*

For Proposition A, we merely list the string of isomorphisms:

$$\begin{aligned} H^2(\Phi, H^1(I, H_2(I))) &= H^2(\Phi, \mathrm{Hom}(I, \Lambda_2 I)) = H^2(\Phi, \mathrm{Hom}(I, M)) \\ &= H^3(\Phi, M) = H^4(\Phi, \boldsymbol{Z}) = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2. \end{aligned}$$

For B, we list a Φ-system for the associated Koszul complex which has the property that $A_1(\lambda)\circ A_1(\eta)=A_1(\eta)\circ A_1(\lambda)$ and $[A_1(\sigma)]^2=[A_1(\tau)]^2=[A_1(\sigma\tau)]^2=$ the identity. Hence we get $U\equiv 0$, so by Theorem 5, $U_*\equiv 0$. Let i_1, i_2, i_3 be the generators of I, and

$$\begin{aligned} \sigma\cdot i_1 &= i_1^{-1} & \tau\cdot i_1 &= i_3 i_2^{-1} \\ \sigma\cdot i_2 &= i_3 i_1^{-1} & \tau\cdot i_2 &= i_2^2 \\ \sigma\cdot i_3 &= i_2 i_1^{-1} & \tau\cdot i_3 &= i_1 i_2^{-1} \\ [A_1(\sigma)](X_1) &= -i_1^{-1}X_1 & [A_1(\tau)](X_1) &= -i_3 i_2^{-1}X_2+X_3 \\ [A_1(\sigma)](X_2) &= -i_3 i_1^{-1}X_1+X_3 & [A_1(\tau)](X_2) &= -i_2^{-1}X_2 \\ [A_1(\sigma)](X_3) &= -i_2 i_1^{-1}X_1+X_2 & [A(\tau)](X_3) &= -i_1 i_2^{-1}X_2+X_1. \end{aligned}$$

3. **Future developments.** First, we would like to say that it now seems likely that the main results of [3] can be reformulated more generally so as to omit all restrictions on the final coefficients Γ and to make more obvious the applications to other categories, e.g. Lie algebras, associative algebras, etc. If this turns out to be the case, the general theory of this paper will then apply to this less restrictive situation.

As for the category of modules over groups, in a later paper we hope to investigate the case in which the module is finite in a similar manner that the $\boldsymbol{Z}$-free case was examined in the second half of this paper. Eventually there should be a unified treatment which would presumably utilize a complex defined by Tate.

Finally, the theorems of §1 show that $v^2(M)$ is really the object of primary concern (at least in the $\boldsymbol{Z}$-free case). We would like to find classes

$$v^{(i)}\in H^i(\Phi, H^{i-1}(M, H_i(M)))$$

with $v^{(2)}=v^2$ and with $v^{(i)}$ determining the differential d_i in the spectral sequence for the split extension $\Phi\cdot M$. Note that if one takes final coefficients $\Gamma=\boldsymbol{Z}_3$, say, then we know that $d_2=0$, and it is easy to see that d_3 on the third row is given by a cup product with an element

$$v^{(3)}\in H^3(\Phi, H^2(M, H_3(M, \boldsymbol{Z}_3))).$$

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