

A VARIATIONAL METHOD FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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Introduction. In this paper we present a variational method for functions of bounded boundary rotation and solve certain general extremal problems for these functions. The variational method is based upon a general method of G. M. Goluzin [1], and has previously been used by the present author to solve extremal problems for several classes of univalent functions ([7] and [8]). Once the variation formulas are derived, the details of solving the extremal problems are very similar to those in [7] and [8].

Theorem 1 here has been obtained independently by Professor J. Pfaltzgraff, who used essentially the same methods. I wish to thank Professor Pfaltzgraff for providing me with his unpublished work.

Those functions of bounded boundary rotation which are also *univalent* form a subclass which has recently been studied by M. M. Schiffer and O. Tammi [9]. They use a different (though closely related) variational method. We shall compare their results with ours.

1. Preliminaries. Let V_k denote the class of normalized analytic functions $f(z) = z + a_2z^2 + \dots$ in the open unit disc $D = \{z : |z| < 1\}$ which satisfy the condition $f'(z) \neq 0$ for all z in D , and which map D onto a domain with boundary rotation bounded by $k\pi$.

V. Paatero [6] has shown that $f(z) \in V_k$ iff

$$(1) \quad f'(z) = \exp \left(\int_0^{2\pi} -\log(1 - ze^{-i\theta}) d\psi(\theta) \right)$$

where $\psi(\theta)$ is a real-valued function of bounded variation with

$$\int_0^{2\pi} d\psi(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\psi(\theta)| \leq k.$$

The geometric interpretation is as follows (see [5]): Let $f(z)$ map D onto a domain G . If G is a schlicht domain with a continuously differentiable boundary curve, let $\pi\psi(\theta)$ denote the angle of the tangent vector at the point $f(e^{i\theta})$ to the boundary curve with respect to the positive real axis. The boundary rotation of G is equal to $\pi \int_0^{2\pi} |d\psi(\theta)|$. If G does not have a sufficiently smooth boundary curve,

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the boundary rotation is defined by a limiting process. Finally, if G is a non-schlicht domain without interior branch points, the boundary rotation is similarly defined.

We define S_k to be the subclass of V_k consisting of the univalent functions in V_k .

2. Univalent functions. Clearly, $f(z) \in V_2$ iff $f(z)$ is a normalized univalent function mapping D onto a convex domain. Furthermore, V. Paatero [6] has shown that for $2 \leq k \leq 4$ the classes V_k consist entirely of univalent functions. We strengthen this assertion by showing:

PROPOSITION 1. *For $2 \leq k \leq 4$, the classes V_k consist entirely of close-to-convex functions.*

Proof. The proof of this proposition consists of stating an appropriate definition of close-to-convex functions. We choose the geometric interpretation originally given by Kaplan [3, p. 177]. Let $f(z)$ be analytic in D and continuously differentiable for $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. $f(z)$ is close-to-convex if the angle of the tangent vector to the boundary curve of the image of D under $f(z)$ with respect to the positive real axis either increases as θ increases or else decreases in such a manner that it never drops to a value π radians below a previous value. If $f(e^{i\theta})$ is not sufficiently smooth, we appeal to the suitable limiting process (see also [2, p. 391]).

Now, if $f(z) \in V_k$, $2 \leq k \leq 4$, it clearly satisfies the above conditions and is close-to-convex. Since every close-to-convex function is univalent, see [3], the observation of Paatero is contained here.

3. Variations for V_k . G. M. Goluzin [1] derived variational formulas for classes E_g of analytic functions in D defined by the condition: $f(z) \in E_g$ iff

$$(2) \quad f(z) = \int_0^{2\pi} g(z, \theta) d\psi(\theta)$$

where $g(z, \theta)$ is a given function, analytic in D for $0 \leq \theta \leq 2\pi$, and $\psi(\theta)$ runs through all nondecreasing functions on $0 \leq \theta \leq 2\pi$ subject to the condition $\int_0^{2\pi} d\psi(\theta) = A$.

Goluzin's variations consist of appropriately varying the function $\psi(\theta)$ so as to preserve its monotonicity and total variation.

For a given function $f(z) \in E_g$ having the representation (2) Goluzin derived the following variational formulas:

$$(3) \quad f_*(z) = f(z) + \lambda \int_{\theta_1}^{\theta_2} \frac{\partial g(z, \theta)}{\partial \theta} |\psi(\theta) - c| d\theta$$

where θ_1 and θ_2 are arbitrary but fixed with $0 \leq \theta_1 < \theta_2 \leq 2\pi$, λ varies in the interval $-1 \leq \lambda \leq 1$, and c is real number independent of λ and θ .

$$(4) \quad f_{**}(z) = f(z) + \lambda(g(z, \theta_1) - g(z, \theta_2))$$

where θ_1 and θ_2 with $0 \leq \theta_1 < \theta_2 \leq 2\pi$ are *jump points* for $\psi(\theta)$, and λ varies in the interval $-a < \lambda < a$, where $a > 0$ is sufficiently small.

To apply these variational formulas to V_k , we express the function $\psi(\theta)$ appearing in (1) as the difference of two increasing functions, $\psi(\theta) = \alpha(\theta) - \beta(\theta)$. Since

$$\int_0^{2\pi} d\psi(\theta) = \int_0^{2\pi} d\alpha(\theta) - \int_0^{2\pi} d\beta(\theta) = 2$$

and

$$\int_0^{2\pi} |d\psi(\theta)| = \int_0^{2\pi} d\alpha(\theta) + \int_0^{2\pi} d\beta(\theta) \leq k,$$

we conclude that

$$(5) \quad \int_0^{2\pi} d\alpha(\theta) \leq \frac{k}{2} + 1$$

and

$$(6) \quad \int_0^{2\pi} d\beta(\theta) \leq \frac{k}{2} - 1.$$

The representation (1) for V_k can now be rewritten as

$$(7) \quad f'(z) = \exp \left(\int_0^{2\pi} \log(1 - ze^{-i\theta}) d\beta(\theta) - \int_0^{2\pi} \log(1 - ze^{-i\theta}) d\alpha(\theta) \right)$$

where $\alpha(\theta)$ and $\beta(\theta)$ are nondecreasing functions on $[0, 2\pi]$ satisfying (5) and (6) respectively.

(7) can be written as

$$f'(z) = \exp(H(z) - G(z))$$

where

$$H(z) = \int_0^{2\pi} \log(1 - ze^{-i\theta}) d\beta(\theta)$$

and

$$G(z) = \int_0^{2\pi} \log(1 - ze^{-i\theta}) d\alpha(\theta).$$

As $\alpha(\theta)$ and $\beta(\theta)$ range over all nondecreasing functions satisfying (5) and (6) respectively, we obtain two classes of the form E_g described above. We apply the Goluzin formulas to these classes.

Varying $H(z)$ we obtain for V_k ,

$$\begin{aligned} f'_*(z) &= \exp(H_*(z) - G(z)) = \exp \left(H(z) - G(z) + \lambda \int_{\theta_1}^{\theta_2} \frac{ize^{-i\theta}}{1 - ze^{-i\theta}} |\beta(\theta) - c| d\theta \right) \\ &= f'(z) \cdot \exp \left(\lambda \int_{\theta_1}^{\theta_2} \frac{ize^{-i\theta}}{1 - ze^{-i\theta}} |\beta(\theta) - c| d\theta \right). \end{aligned}$$

Expanding the exponential up to first order terms in λ we have,

$$(8) \quad f'_*(z) = f'(z) + \lambda \int_{\theta_1}^{\theta_2} f'(z) \frac{ize^{-i\theta}}{1 - ze^{-i\theta}} |\beta(\theta) - c| d\theta + O(\lambda^2).$$

Here, $O(\lambda^2)$ is an analytic function of z , and the error is uniform in each disc $|z| \leq r < 1$.

Varying $G(z)$ we obtain the same formula with $\beta(\theta)$ replaced by $\alpha(\theta)$.

If θ_1 and θ_2 are jump points for $\alpha(\theta)$ or $\beta(\theta)$ we have the following variational formula for V_k :

$$(9) \quad f'_{**}(z) = f'(z) + \lambda f'(z)(\log(1 - ze^{-i\theta_1}) - \log(1 - ze^{-i\theta_2})) + O(\lambda^2).$$

It should be remarked at this point that the formulas (8) and (9) are identical in form to the variational formulas derived in [8] for convex functions of order α . It is for this reason that, as mentioned in the introduction, the details to the solution of extremal problems for V_k will be very similar to those in [8].

4. A general extremal problem. The solution of extremal problems for V_k is equivalent to finding the functions $\alpha(\theta)$ and $\beta(\theta)$ which appear in the representation (7) for the extremal function. We begin with a general extremal problem for V_k .

THEOREM 1. *Let $\zeta \neq 0$ be a given point in D , and let $F(X_1, X_2, \dots, X_{n+1})$ be analytic in a neighborhood of each point $F(f'(\zeta), f''(\zeta), \dots, f^{(n)}(\zeta), \zeta), f(z) \in V_k$. Then the functional*

$$J(f') = \operatorname{Re} F(f'(\zeta), f''(\zeta), \dots, f^{(n)}(\zeta), \zeta)$$

attains its maximum (or minimum) in V_k only for a function of the form

$$(10) \quad f'(z) = \prod_{j=1}^M (1 - \varepsilon_j z)^{\beta_j} / \prod_{j=1}^N (1 - e_j z)^{\alpha_j}$$

where $M < n, N \leq n, |\varepsilon_j| = |e_j| = 1$ and $\sum_{j=1}^M \beta_j \leq k/2 - 1$ and $\sum_{j=1}^N \alpha_j \leq k/2 + 1$.

Proof. The compactness of V_k assures the existence of a function $f(z) \in V_k$ which maximizes (or minimizes) $J(f')$. Let $f(z)$ with

$$f'(z) = \exp \int_0^{2\pi} \log(1 - ze^{-i\theta})(d\beta(\theta) - d\alpha(\theta))$$

be such a function. For the functions $f'_*(z)$ defined by (8) we have

$$(11) \quad J(f'_*) = J(f') + \lambda \int_{\theta_1}^{\theta_2} \operatorname{Re} \sum_{j=1}^n d_j [i\zeta f'(\zeta)(e^{i\theta} - \zeta)^{-1}]^{(j-1)} |\beta(\theta) - c| d\theta + O(\lambda^2)$$

where $d_j = \partial F(X_1, \dots, X_{n+1}) / \partial X_j$ evaluated at the extreme point

$$(f'(\zeta), \dots, f^{(n)}(\zeta), \zeta).$$

(Here, $f^{(0)}(z) = f(z)$.)

The extremal property of $f(z)$ implies that

$$(12) \quad \int_{\theta_1}^{\theta_2} Q(\theta) |\beta(\theta) - c| d\theta = 0$$

where

$$Q(\theta) = \operatorname{Re} \sum_{j=1}^n d_j [i\zeta f'(\zeta)(e^{i\theta} - \zeta)^{-1}]^{(j-1)}.$$

V_k is a rotation invariant family, and a theorem of Kirwan [4] assures us that at least one $d_j \neq 0$.

If the continuous function $Q(\theta)$ does not vanish in the interval (θ_1, θ_2) , (12) implies that $\beta(\theta) = c$ throughout that interval. Now, $Q(\theta) = 0$ is equivalent to an equation of degree at most $2n$ in $e^{i\theta}$. We therefore conclude that $\psi(\theta)$ is a step function with at most $2n$ jump points and furthermore, these jumps can only occur at the zeros of $Q(\theta)$. Let these points be $\theta_1, \dots, \theta_{2n}$, with $\theta_1 < \theta_2 < \dots < \theta_{2n}$.

We now construct the functions $f'_{**}(z)$ defined by (9). For these functions we have

$$(13) \quad \begin{aligned} J(f'_{**}) &= J(f') \\ &+ \lambda \operatorname{Re} \sum_{j=1}^n d_j [f'(\zeta)(\log(1 - \zeta e^{-i\theta_1}) - \log(1 - \zeta e^{-i\theta_2}))]^{(j-1)} + O(\lambda^2). \end{aligned}$$

The extremal property of $f'(z)$ implies that the coefficient of λ in (13) must vanish. Thus, the function

$$R(\theta) = \operatorname{Re} \sum_{j=1}^n d_j [f'(\zeta) \log(1 - \zeta e^{-i\theta})]^{(j-1)}$$

has the same value at θ_1 and θ_2 . The Mean Value Theorem implies that $R'(\theta)$ must vanish at some point θ' , with $\theta_1 < \theta' < \theta_2$. But $R'(\theta) = Q(\theta)$ and this is a contradiction.

We conclude that $\beta(\theta)$ could not have jump points at each of the $2n$ roots of $Q(\theta)$, but at most at every other root. Thus, $\beta(\theta)$ is a step function with at most n jump points.

A similar procedure with $\beta(\theta)$ replaced by $\alpha(\theta)$ in (8) and (9) yields the conclusion that $\alpha(\theta)$ is a step function having at most n jump points. Evaluating (7) with these $\alpha(\theta)$ and $\beta(\theta)$ we arrive at (10) and the theorem is proved.

For the class V'_k of functions with boundary rotation equal to $k\pi$, the conclusion of Theorem 1 is strengthened to $\sum \alpha_j = k/2 + 1$ and $\sum \beta_j = k/2 - 1$. Furthermore, as we shall see in the next section, for certain specific extremal problems in V_k we also conclude $\sum \alpha_j = k/2 + 1$ and $\sum \beta_j = k/2 - 1$.

5. Distortion and rotation theorems. We begin with the known distortion theorem for V_k [5].

COROLLARY 1. *Let $f(z) \in V_k$. Then,*

$$\frac{(1 - |z|)^{k/2 - 1}}{(1 + |z|)^{k/2 + 1}} \leq |f'(z)| \leq \frac{(1 + |z|)^{k/2 - 1}}{(1 - |z|)^{k/2 + 1}}.$$

These bounds are sharp and are attained by a function of the form (10) with $M = N = 1$.

Proof. Set $J(f') = |f'(\zeta)|$ in Theorem 1.

COROLLARY 2. Let $f(z) \in V_k$. Then for all $z \in D$,

$$|\arg f'(z)| \leq k \sin^{-1} |z|$$

where $\sin^{-1} 0 = 0$.

This bound is sharp and is attained by a function of the form (10) with $M = N = 1$.

Proof. Set $J(f') = \operatorname{Re} \pm i \log f'(\zeta)$ in Theorem 1, and recall that

$$|\arg(1 - re^{i\theta})| \leq \sin^{-1} r.$$

It is interesting to observe that this is the same sharp bound on $|\arg f'(z)|$ obtained by Schiffer and Tammi [9, p. 143], for S_k .

If the functional $J(f)$ has the form $J(f) = \operatorname{Re} F(\log f'(\zeta))$ we can strengthen the result of Theorem 1. In this case, the extremal function must be of the form

$$f'(z) = (1 - \varepsilon z)^{k/2 - 1} / (1 - ez)^{k/2 + 1}$$

with $|\varepsilon| = |e| = 1$.

6. A general coefficient theorem.

THEOREM 2. Let $F(X_2, \dots, X_n)$ be analytic in C^{n-1} . Then the functional

$$C(f) = \operatorname{Re} F(a_2, \dots, a_n)$$

where $f(z) = z + a_2 z^2 + \dots$, attains its maximum (or minimum) in V_k only for a function of the form

$$(14) \quad f'(z) = \prod_{j=1}^N (1 - \varepsilon_j z)^{\beta_j} / \prod_{j=1}^M (1 - e_j z)^{\alpha_j}$$

where $N \leq n - 1$, $M \leq n - 1$, $|\varepsilon_j| = |e_j| = 1$, and $\sum_{j=1}^N \beta_j \leq k/2 - 1$ and $\sum_{j=1}^M \alpha_j \leq k/2 + 1$.

Proof. The proof is quite similar to the proof of Theorem 1 and we merely sketch it.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be extremal for the problem under consideration with

$$f'(z) = \exp \int_0^{2\pi} \log(1 - ze^{-i\theta})(d\beta(\theta) - d\alpha(\theta)).$$

For the functions $f'_*(z)$ given by (8) we have

$$f'_*(z) = z + \sum_{n=2}^{\infty} \left(a_n + \lambda \int_{\theta_1}^{\theta_2} \delta a_n |\beta(\theta) - c| d\theta \right) z^n + O(\lambda^2)$$

with

$$\delta a_n = \frac{i}{n} \sum_{j=0}^{n-2} (j+1) a_{j+1} \exp(-i(n-j-1)\theta), \quad (a_1 = 1).$$

Computing $C(f_*)$ and exploiting the extremal property of $f(z)$ we arrive at the condition

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \sum_{j=2}^n d_j \delta a_j |\beta(\theta) - c| d\theta = 0.$$

The continuous function $Q(\theta) = \operatorname{Re} \sum_{j=2}^n d_j \delta a_j$ has at most $2n - 2$ roots with respect to θ in $[0, 2\pi]$. Thus, $\beta(\theta)$ is a step function with at most $2n - 2$ jump points. Now, constructing $C(f_{**})$ we can prove, using the same reasoning as in the proof of Theorem 1, that $\beta(\theta)$ and $\alpha(\theta)$ are step functions having at most $n - 1$ jump points, and the theorem is proved.

Setting $C(f) = |a_2|$, we conclude the known result (see e.g., [5]) that the function maximizing $|a_2|$ over V_k is a function of the form (14) with $N = M = 1$, $\beta_1 = k/2 - 1$, $\alpha_1 = k/2 + 1$.

For the problem of maximizing $|a_n|$ over S_k , Schiffer and Tammi [9, p. 137], have found the extremal function to be a mapping of D onto a polygon with N corners and $N \leq 2(n - 2)$. This agrees with the conclusion of Theorem 2. The expression for the extremal function in Theorem 2, (14), gives a bit more information about the structure of the extremal function.

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