MONOTONICITY OF SOLUTIONS OF VOLterra
INTEGRAL EQUATIONS IN BANACH SPACE (1)

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1. Introduction. We shall consider Volterra integral equations

\begin{equation}
    x(t) = x_0 - \int_0^t h(t-\tau)Ax(\tau) \, d\tau
\end{equation}

where \( x_0 \) and \( x(t) \) belong to a complex Banach space \( X \), \( h(t) \) is a complex-valued function and \( A \) is an operator in \( X \), generally unbounded. We denote by \( B(X) \) the Banach space of bounded linear operators in \( X \), and by \( I \) the identity operator in \( X \). An operator-valued function \( S(t) \), which belongs to \( L^1((0, b; B(X)) \) for any \( b > 0 \), is called a fundamental solution of (1.1) if

\begin{equation}
    S(t) = I - A \int_0^t h(t-\tau)S(\tau) \, d\tau
\end{equation}

for almost all \( t \). In this definition it is assumed, of course, that the integral on the right-hand side of (1.2) is in the domain of \( A \).

In a recent paper [6], Friedman and Shinbrot have studied the equation (1.1) even in the more general case where \( x_0 \) and \( A \) depend on \( t \) and \( \tau \), respectively. They proved theorems of existence, uniqueness, differentiability and asymptotic behavior of solutions. They also constructed fundamental solutions and derived asymptotic bounds for them. We recall [6] that if \( x_0 \) is in the domain of \( A^\mu \), for some \( \mu > 0 \), then the solution of (1.1) is given by \( S(t)x_0 \).

The purpose of the present paper is to derive monotonicity theorems for solutions of (1.1). We shall generalize some of the monotonicity theorems of Friedman [2] (see also [4]) from the case \( X = R^1 \) (\( R^1 \) the one-dimensional Euclidean space) to the case where \( X \) is any Banach space.

In §2 we give some auxiliary results. These results are concerned with Volterra equations in one-dimension (i.e., \( X = R^1 \)). In particular, we study the behavior of the solutions with respect to a certain parameter.

In §3 we give an integral formula for \( S(t) \) in case \( A \) is a bounded operator. For \( A \) unbounded, we construct a fundamental solution as a limit of fundamental solutions corresponding to the bounded operators \( A(I + A/n)^{-1} \). We prove that the fundamental solution coincides with the fundamental solution of [6, Chapter 1]

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or of [6, Chapter 2] provided the assumptions of [6, Chapter 1] or of [6, Chapter 2], respectively, are satisfied.

In §4 we derive a formula for \( S(t) \) in case \( A \) is selfadjoint. As a by-product, we obtain monotonicity theorems for \((S(t)x_0, x_0)\).

In §5 we drop the assumption that \( A \) is selfadjoint. Instead we assume that the resolvent \((\lambda I - A)^{-1}\) exists for all \( \lambda \) except for a sequence \( \{\mu_k\} \) of poles, and \( 0 < \mu_1 < \mu_2 < \cdots, \mu_k \to \infty \) as \( k \to \infty \). We obtain a formula for the solution \( T(t)x_0 \) of (1.1) and then derive monotonicity theorems for \( f_\alpha(T(t)x_0) \); here \( f_\alpha \) is a bounded linear functional in \( X \).

In §6 we give some additional results obtainable with the methods of the previous sections, and some applications to control theory.

2. Auxiliary lemmas. A real-valued function \( f(t) \) is said to be completely monotonic in an interval \([a, b)\) or \([a, b]\) and for all \( n \geq 0 \),

\[
(-1)^{n} \frac{d^n f(t)}{dt^n} \geq 0 \quad \text{for all } t \in [a, b).
\]

Similarly one defines complete monotonicity in intervals \((a, b), [a, b]\). We recall the following results (see [9]):

**Lemma 2.1.** If \( f(t) \) is completely monotonic in an interval \((a, b)\) then \( f(t) \) is analytic in \((a, b)\).

**Lemma 2.2.** A function \( f(t) \) is completely monotonic in the interval \([0, \infty)\) if and only if

\[
(2.1) \quad f(t) = \int_{0}^{\infty} e^{-\lambda t} d\phi(\lambda)
\]

where \( \phi(\lambda) \) is a bounded nondecreasing function.

**Lemma 2.3.** If \( \{f_m(t)\} \) is a sequence of completely monotonic functions in \((a, b)\) and if \( f(t) \) is a continuous function in \((a, b)\) such that, for each \( t \in (a, b) \), \( f_m(t) \to f(t) \) as \( m \to \infty \), then \( f(t) \) is completely monotonic in \((a, b)\).

Setting \( \Delta_{m}^{+}f(t)=f(t+\eta)-f(t) \),

\[
\Delta_{m}^{-1}f(t) = \Delta_{m}^{-1}(\Delta_{m}f(t)),
\]

the assertion of the last lemma is a consequence of the fact (see [9]) that \( f(t) \) is completely monotonic if, for any integer \( m \geq 1 \) and \( \eta > 0 \),

\[
(-1)^{m} \Delta_{m}^{+}f(t) \geq 0 \quad \text{for } a < t < b-m\eta.
\]

In view of Lemma 2.1, if \( f(t) \) is a completely monotonic function in \((a, b)\) which does not vanish identically, then \( f(t) > 0 \) for all \( t \in (a, b) \).

We shall need the following result of Miller [8]:

**Lemma 2.4.** If \( f(t) \) is a nonzero completely monotonic function in \([0, \infty)\), then \( \log f(t) \) is a convex function.
Proof. We have to show that
\[ g(t) = f(t)f''(t) - (f'(t))^2 \geq 0. \]

Using (2.1) we find that
\[ g(t) = \int_0^\infty \int_0^\infty \lambda (\lambda - \mu) e^{-\lambda + \mu} d\phi(\lambda) d\phi(\mu). \]

Next,
\[ \int_0^\infty \int_\mu^\infty \lambda (\lambda - \mu) e^{-\lambda + \mu} d\phi(\lambda) d\phi(\mu) = \int_0^\infty \int_0^\lambda \lambda (\lambda - \mu) e^{-\lambda + \mu} d\phi(\lambda) d\phi(\mu) \]
\[ = \int_0^\infty \int_0^\mu \mu (\lambda - \mu) e^{-\lambda + \mu} d\phi(\lambda) d\phi(\mu). \]

Therefore
\[ g(t) = \int_0^\infty \int_0^\mu \cdots + \int_0^\infty \int_\mu^\infty \cdots = \int_0^\infty \int_0^\mu (\lambda - \mu)^2 e^{-\lambda + \mu} d\phi(\lambda) d\phi(\mu) \geq 0. \]

Definitions. A function \( h(t) \) which belongs to \( C(0, \infty) \) and to \( L^1(0, 1) \) is said to belong to the class \( \mathcal{H} \), if \( h(t) \geq 0, h(t) \neq 0, \) and \( h(t) \) is monotone nonincreasing in \( (0, \infty) \). If \( h \in \mathcal{H} \) and if \( \log h(t) \) is a convex function in the interval where \( h(t) > 0 \), then we say that \( h \) belongs to the class \( \mathcal{H}' \). Finally, we say that \( h \in \mathcal{H}_\infty \) if \( h(t) \) is a nonzero completely monotonic function in \( (0, \infty) \) and if \( h \in L^1(0, 1) \).

From Lemma 2.4 (applied to \( h(t + \varepsilon) \), for any \( \varepsilon > 0 \)) it follows that if \( h \in \mathcal{H}_\infty \) then \( h \in \mathcal{H}' \).

In the following lemma we have collected some results proved in Friedman [2].

Lemma 2.5. Consider the integral equation
\[ x(t) = 1 - \int_0^t h(t - \tau) x(\tau) d\tau \quad (0 < t < \infty). \]

(i) If \( h \in \mathcal{H} \), then \( 0 \leq x(t) \leq 1. \)

(ii) If \( h \in \mathcal{H}' \), then \( x(t) \) is monotone nonincreasing.

(iii) If \( h \in \mathcal{H}_\infty \), then \( x(t) \) is in \( \mathcal{H}_\infty \).

From [2, Corollary 4, p. 387] we deduce

Lemma 2.6. Consider the equation
\[ x(t) = \int_0^t h(t - \sigma) p(\sigma) d\sigma - \lambda \int_0^t h(t - \tau) x(\tau) d\tau \quad (0 < t < \infty) \]

where \( p(\sigma) \) is a continuous nonnegative function, and \( \lambda \) is a positive constant. If \( h \in \mathcal{H}' \) then \( x(t) \geq 0 \).

We shall consider now the equation
\[ S_\lambda(t) = 1 - \lambda \int_0^t h(t - \tau) S_\lambda(\tau) d\tau \quad (0 \leq t < \infty) \]
where $\lambda$ is a complex parameter. By a standard argument one shows that if $h(t)$ is in $C(0, \infty) \cap L^1(0, 1)$ then, for each $\lambda$, there exists a unique solution $S_\lambda(t)$ of (2.4). Furthermore, $S_\lambda(t)$ is continuous in $(t, \lambda)$ ($t \geq 0$, $\lambda$ complex) and analytic in $\lambda$, for each $t \geq 0$. If $h \in C^n[0, \infty)$ then $\partial^n S_\lambda(t)/\partial t^n$ is continuous in $(t, \lambda)$ (for $t \geq 0$, $\lambda$ complex) and analytic in $\lambda$, for each $t \geq 0$.

**Lemma 2.7.** If $h \in \mathcal{H}$ then, for $n = 0, 1, 2, \ldots$,

$$ (-1)^n \left( \frac{\partial^n S_\lambda(t)}{\partial \lambda^n} \right) \geq 0 \quad \text{if } 0 < \lambda < \infty, \: 0 < t < \infty. $$

**Proof.** The inequality (2.5) for $n = 0$ follows from Lemma 2.5(i). We proceed by induction. We assume that (2.5) holds and prove the same inequality when $n$ is replaced by $n+1$. Differentiating (2.4) $n+1$ times with respect to $\lambda$ we get

$$ \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(t) = - (n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_\lambda(\tau) \, d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(\tau) \, d\tau. $$

It follows that the function $x(t) = (-1)^{n+1} \frac{\partial^{n+1} S_\lambda(t)}{\partial \lambda^{n+1}}$ satisfies the equation (2.3) with

$$ p(\sigma) = (n+1)(-1)^n \frac{\partial^n S_\lambda(\sigma)}{\partial \lambda^n}. $$

By the inductive assumption, $p(\sigma) \geq 0$. Hence we can apply Lemma 2.6 and conclude that $x(t) \geq 0$, i.e., (2.5) holds with $n$ replaced by $n+1$.

**Lemma 2.8.** If $h \in \mathcal{H}'$ then, for $n = 0, 1, 2, \ldots$, $\lambda > 0$,

$$ (-1)^n \frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_\lambda(t)}{\lambda} \right] \geq 0 \quad \text{if } t < \lambda. $$

**Proof.** We may assume that $h \in C^1[0, \infty)$. Indeed, otherwise we approximate $h(t)$ by a sequence of functions $\{h_m(t)\}$ as follows: $h_m \in \mathcal{H}'$, $h_m(t) \to h(t)$ uniformly on compact subsets of $(0, \infty)$ and

$$ \int_0^t |h(t)-h_m(t)| \, dt \to 0. $$

If we know already that the assertion of the lemma holds for the solution $S_{\lambda,m}(t)$ corresponding to $h_m$, then (2.7) is also true since, for any $n \geq 0$,

$$ \frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_{\lambda,m}(t)}{\lambda} \right] \to \frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_\lambda(t)}{\lambda} \right] \quad \text{for each } t. $$

Assuming $h$ to be in $C^1[0, \infty)$, it follows that $\partial S_\lambda(t)/\partial t$ exists and satisfies:

$$ \frac{\partial S_\lambda(t)}{\partial t} = -\lambda h(t) - \lambda \int_0^t h(t-\tau) \frac{\partial S_\lambda(\tau)}{\partial \tau} \, d\tau. $$

The assertion (2.7) is equivalent to the following inequality:

$$ (-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial}{\partial t} S_\lambda(t) \right] \geq 0. $$
For $n=0$, this inequality follows from Lemma 2.5(ii). We now proceed by induction on $n$. To pass from $n$ to $n+1$, we divide both sides of (2.8) by $\lambda$ and then differentiate both sides $n+1$ times with respect to $\lambda$. We get

$$
\frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[ \frac{1}{\lambda} \frac{\partial S_\lambda(t)}{\partial t} \right] = -(n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial S_\lambda(\tau)}{\partial \tau} \right] d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[ \frac{1}{\lambda} \frac{\partial S_\lambda(\tau)}{\partial \tau} \right] d\tau.
$$

We can now apply Lemma 2.6 with

$$
x(t) = (-1)^{n+2} \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[ \frac{1}{\lambda} \frac{\partial^n S_\lambda(t)}{\partial t^n} \right], \quad p(\sigma) = (n+1)(-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial S_\lambda(\sigma)}{\partial \sigma} \right].
$$

**Lemma 2.9.** If $h \in \mathcal{H}_\omega$ then, for $n=0, 1, 2, \ldots, m=0, 1, 2, \ldots, \lambda > 0$,

$$(2.10) \quad (-1)^m \frac{\partial^m}{\partial \lambda^m} \left[ \frac{1}{\lambda^m} \frac{\partial^m S_\lambda(t)}{\partial t^m} \right] \geq 0 \quad \text{for } t > 0.
$$

**Proof.** Suppose first that $h(t)$ is in $C^\infty[0, \infty)$. Then all the derivatives occurring in (2.10) exist for $t \geq 0$. We shall establish (2.10) by induction on $n$. For $n=0$, (2.10) follows from Lemma 2.7. We now assume that (2.10) holds for all $n^m$ and $m^k$. We shall prove (2.10) for all $m^m$ and $n^k$. Differentiating both sides of (2.4) $k+1$ times with respect to $t$, we get

$$
\frac{\partial^{k+1} S_\lambda(t)}{\partial t^{k+1}} = -\lambda T^{(k)}(t) S_\lambda(0) - \lambda T^{(k-1)}(t) \frac{\partial S_\lambda(0)}{\partial t} - \cdots - \lambda h(t) \frac{\partial^{k} S_\lambda(0)}{\partial t^{k}}
$$

Setting

$$
T_\lambda(t) = (-1)^{k+1} \frac{1}{\lambda^{k+1}} \frac{\partial^{k+1}}{\partial t^{k+1}} S_\lambda(t),
$$

we get

$$
T_\lambda(t) = \sum_{i=0}^k (-1)^{k-i} \frac{h^{(k-i)}(t)}{\lambda^{k-i}} (-1)^i \frac{\partial^i S_\lambda(0)}{\partial t^i}
$$

$$
\lambda \int_0^t h(t-\tau) T_\lambda(\tau) d\tau.
$$

We have to prove that, for any $m^m$,

$$
(2.12) \quad (-1)^m \frac{\partial^m T_\lambda(t)}{\partial \lambda^m} \geq 0.
$$

For $m=0$ this follows from the definition of $T_\lambda(t)$ and Lemma 2.5(iii). We now proceed by induction on $m$.

To pass from $m$ to $m+1$, we differentiate (2.11) $m+1$ times with respect to $\lambda$. 


We find
\begin{equation}
\frac{\partial^{m+1} T(x)}{\partial \lambda^{m+1}}(t) = g(t) - (m+1) \int_0^t h(t-\tau) \frac{\partial^m T(x)}{\partial \lambda^m}(\tau) \, d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{m+1} T(x)}{\partial \lambda^{m+1}}(\tau) \, d\tau,
\end{equation}
(2.13)
where
\begin{equation}
g(t) = \sum_{i=0}^k g_i(t),
\end{equation}
(2.14)
g_i(t) = (-1)^{k-i} h^{(k-i)}(\tau) \prod_{s=0}^{m+1} \left( \frac{m+1}{s} \right) \frac{d^s}{d\lambda^s} \frac{1}{\lambda^{n-1}} \cdot \frac{\partial^{m+1-s}}{\partial \lambda^{m+1-s}} \left[ \frac{(-1)^t}{\lambda^t} \frac{d^t}{d\tau^t} S(x)(\tau) \right].
\end{equation}

Set
\begin{equation}
g_{k+1}(t) = -(m+1) \int_0^t h(t-\tau) \frac{\partial^m T(x)}{\partial \lambda^m}(\tau) \, d\tau
\end{equation}
and denote by \( x(t) \) \((0 \leq i \leq k+1)\) the solution of the equation
\begin{equation}
x(t) = (-1)^{m+1} g_i(t) - \lambda \int_0^t h(t-\tau)x_i(\tau) \, d\tau.
\end{equation}
(2.16)
It is clear that
\begin{equation}
(-1)^{m+1} \frac{\partial^{m+1} T(x)}{\partial \lambda^{m+1}}(t) = \sum_{i=0}^{k+1} x_i(t).
\end{equation}
Hence, it suffices to show that \( x_i(t) \geq 0 \) for \( 0 \leq i \leq k+1 \).

The inequality \( x_{k+1}(t) \geq 0 \) follows by applying Lemma 2.6 with
\[ p(\alpha) = (-1)^m (m+1) \frac{\partial^m T(x)}{\partial \lambda^m}; \]
note that by the inductive assumption, \( p(\alpha) \geq 0 \).

From (2.14) and the inductive assumption we easily see that
\begin{equation}
(-1)^{m+1} g_i(t) = \gamma_i(-1)^{k-i} h^{(k-i)}(t) \quad (0 \leq i \leq k)
\end{equation}
where \( \gamma_i = \gamma_i(\lambda) \) is nonnegative. From Theorem 1 and its Corollary 3 in [2] we then have the following: If
\begin{equation}
\frac{h'(a)}{h(a)} \leq \frac{h^{(k-1+i)}(b)}{h^{(k-1-i)}(b)} \quad (0 < a < b < \infty)
\end{equation}
then \( x_i(t) \geq 0 \). Thus, it remains to prove (2.17). We assume here that \( h^{(k-i)}(b) > 0 \) for all \( b > 0 \); if \( h^{(k-i)}(b) = 0 \) for some \( b > 0 \) then \( h^{(k-i)}(t) \equiv 0 \) and \( x_i(t) \equiv 0 \).

Since \( h \in M_{\alpha} \), Lemma 2.4 implies that \( h'(t)/h(t) \not< a \) if \( t \not< a \). Hence (2.17) is a consequence of
\begin{equation}
\frac{h'(b)}{h(b)} \leq \frac{h^{(k-1+i)}(b)}{h^{(k-1-i)}(b)} \quad (0 < b < \infty).
\end{equation}
Now, the function \((-1)^j h^{(j)}(t)\) is completely monotonic. If we apply Lemma 2.4 to this function, we obtain

\[ h^{(j+1)}(b)/h^{(j)}(b) \leq h^{(j+2)}(b)/h^{(j+1)}(b). \]

Applying this inequality for \(j = 0, 1, \ldots, k - i - 1\), we get (2.18).

We have proved Lemma 2.9 assuming that \(h(t)\) is in \(C^\infty[0, \infty)\). Consider now the general case, where we merely assume that \(h \in \mathcal{H}_\omega\). Then we can apply the previous result to the solution \(S_{\lambda, \epsilon}(t)\) of (2.4) with \(h(t)\) replaced by \(h(t + \epsilon), \epsilon > 0\). Since, for each \(t > 0, m \geq 0\),

\[ \partial^m S_{\lambda, \epsilon}(t)/\partial \lambda^m \to \partial^m S_\lambda(t)/\partial \lambda^m, \]

the assertion of the lemma for \(S_\lambda(t)\) follows upon applying Lemma 2.3.

The last lemma of this section is the following:

**Lemma 2.10.** Let \(h \in \mathcal{H}\). Then, for \(n = 0, 1, 2, \ldots\),

\[ (-1)^n (\partial^n S_\lambda(t)/\partial \lambda^n) \leq n!/\lambda^n \quad (0 < \lambda < \infty, 0 < t < \infty). \]

**Proof.** From Lemma 2.7 and (2.6) we obtain

\[ (-1)^{n+1} \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(\tau) \, d\tau \leq (-1)^n \frac{n+1}{\lambda} \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_\lambda(\tau) \, d\tau. \]

Applying this relation successively, we find that

\[ (-1)^m \int_0^t h(t-\tau) \frac{\partial^m}{\partial \lambda^m} S_\lambda(\tau) \, d\tau \leq \frac{m!}{\lambda^{m+1}} \quad (m = 0, 1, 2, \ldots). \]

Hence from (2.6), with \(n + 1 = m\), we get

\[ (-1)^n \frac{\partial^n}{\partial \lambda^n} S_\lambda(t) \leq (-1)^{n-1} m \int_0^t h(t-\tau) \frac{\partial^{n-1}}{\partial \lambda^{n-1}} S_\lambda(\tau) \, d\tau \leq \frac{m!}{\lambda^n} \]

for \(m = 1, 2, \ldots\).

### 3. Integral formula for \(S(t)\)

**Theorem 3.1.** Let \(X\) be a Banach space. We denote by \(\sigma(A)\) the spectrum of an operator \(A\).

**Theorem.** Let \(A\) be a bounded operator and let \(\Gamma\) be any continuously differentiable closed Jordan curve containing \(\sigma(A)\) in its interior. Let \(h(t)\) be any function in \(C(0, \infty) \cap L^1(0, 1)\). Then the operator-valued function

\[ S(t) = \frac{1}{2\pi i} \int_{\Gamma} (A - \lambda)^{-1} S_\lambda(t) \, d\lambda \]

is the unique fundamental solution of (1.1).

The orientation of \(\Gamma\), in (3.1), is taken counterclockwise.

**Proof.** The uniqueness of the fundamental solution follows by standard arguments. It remains to verify (1.2). We have
\[ T \equiv I - A \int_0^t h(t - \tau)S(\tau) \, d\tau \]
\[ = I - \int_0^t h(t - \tau) \left\{ \frac{1}{2\pi i} \int \lambda (\lambda I - A)^{-1} S_{\lambda}(\tau) \, d\lambda \right\} \, d\tau. \]

Changing the order of integration and using (2.4), we get
\[ T = I + \frac{1}{2\pi i} \int \lambda \left[ (\lambda I - A)^{-1} - \frac{I}{\lambda} \right] [S_{\lambda}(t) - 1] \, d\lambda \]
\[ = I + S(t) - \frac{1}{2\pi i} \int \lambda (\lambda I - A)^{-1} \, d\lambda + \frac{I}{2\pi i} \int \frac{1 - S_{\lambda}(t)}{\lambda} \, d\lambda. \]

By Cauchy's theorem we easily find that
\[ \frac{1}{2\pi i} \int \lambda (\lambda I - A)^{-1} \, d\lambda = I. \]

Also,
\[ \int \frac{1 - S_{\lambda}(t)}{\lambda} \, d\lambda = \int \left\{ \int_0^t h(t - \tau)S_{\lambda}(\tau) \, d\tau \right\} \, d\lambda \]
\[ = \int_0^t h(t - \tau) \left\{ \int \lambda S_{\lambda}(\tau) \, d\lambda \right\} \, d\tau = 0. \]

We obtain that \( T = S(t). \) This proves (1.2).

Theorem 3.1 can easily be extended to more general integral equations. For example, we shall construct a solution of

\[ (3.2) \quad S(t, s) = I - \int_s^t h(t - \tau, \tau)AS(\tau, s) \, d\tau. \]

Denote by \( S_{\lambda}(t, s) \) the solution of

\[ (3.3) \quad S_{\lambda}(t, s) = 1 - \lambda \int_s^t h(t - \tau, \tau)S_{\lambda}(\tau, s) \, d\tau. \]

Then we have

**Theorem 3.1'.** Let \( A, \Gamma \) be as in Theorem 3.1 and let \( h(t, \tau) \) be a continuous function for \( t \geq 0, \tau \geq 0. \) Then the unique solution of (3.2) is given by

\[ (3.4) \quad S(t, \tau) = \frac{1}{2\pi i} \int \Gamma \left[ \lambda (\lambda I - A)^{-1} \right] S_{\lambda}(t, \tau) \, d\lambda. \]

The proof is similar to the proof of Theorem 3.1.

We next consider the case where \( A \) is not necessarily a bounded operator.

**Definition.** A linear operator \( A \) in \( X \) is said to belong to the class \( \mathfrak{A} \) if

(i) \( A \) is closed and densely defined;
(ii) \( \sigma(A) \subseteq \{ \lambda; |\arg \lambda| \leq \pi/2 - \varepsilon, \Re \lambda \geq \lambda_0 \} \) for some \( \varepsilon > 0, \lambda_0 > 0; \)
(iii) \( \|(\lambda I - A)^{-1}\| \leq c/|\lambda| \) if \( |\arg \lambda| > \pi/2 - \varepsilon. \)
Definition. A complex-valued function \( h(t) \) is said to belong to the class \( \mathcal{W} \) if

(i) \( h(0) > 0 \);

(ii) \( h \in C^1[0, \infty) \) and \( h(t) \) is absolutely continuous;

(iii) for any \( b > 0 \), \( h(t) \) is in \( L^p(0, b) \) for some \( p > 1 \).

In [6, Chapter 1] it was proved that if \( A \in \mathfrak{A} \) and \( h \in \mathcal{W} \), then there exists a fundamental solution \( W(t) \) of (1.1), in a sense different than (1.2). Thus, \( W(t) \) satisfies the equation

\[
W(t) = I - \int_0^t h(t-\tau) A W(\tau) \, d\tau
\]

in the following sense

\[
\tilde{W}(t) = e^{-tA} + \int_0^t e^{-(t-\tau)A} F(W; \tau) \, d\tau
\]

where \( e^{-tA} \) is the analytic semigroup of \( -A \), and

\[
\tilde{W}(t) = W(t) + \int_0^t \frac{h(0)}{h(0)} W(t) + \frac{1}{h(0)} \int_0^t h(t-\tau) W(\tau) \, d\tau,
\]

\[
F(W; \tau) = \frac{h(0)}{h(0)} W(t) + \frac{1}{h(0)} \int_0^t h(t-\tau) W(\tau) \, d\tau.
\]

Denoting by \( D(A^\mu) \) the domain of \( A^\mu \) (see [7] for the definition of \( A^\mu \)), we have the following result: If \( x_0 \in D(A^\mu) \) for some \( \mu > 0 \), then \( W(t)x_0 \) is the unique solution of (1.1). (The solutions of (1.1) are assumed to be such that \( \|Ax(\tau)\| \) is integrable in every bounded interval \( (0, b) \).)

We introduce the operators

\[
A_n = (I + A/n)^{-1}.
\]

One easily verifies that \( \|A_n\| \leq Cn \) and

\[
(\lambda I - A_n)^{-1} = -\frac{1}{n-\lambda} I + \frac{n^2}{(n-\lambda)^2} \left[ \frac{n\lambda}{n+\lambda} I - A \right]^{-1}.
\]

Denote by \( \Gamma_n \) a continuously differentiable closed Jordan curve which contains \( \sigma(A_n) \), and set

\[
S_n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda I - A_n)^{-1} S_n(t) \, d\lambda.
\]

**Theorem 3.2.** Let \( h \in \mathcal{W} \), \( A \in \mathfrak{A} \). Then, for any \( x_0 \in X \),

\[
\lim_{n \to \infty} S_n(t)x_0 = W(t)x_0
\]

uniformly with respect to \( t \) in bounded intervals \( [0, b) \).
Proof. Suppose first that \( x_0 \in \mathcal{D}(A) \). Set

\[
\begin{align*}
    u_n(t) &= S^n(t)x_0, \\
    u(t) &= W(t)x_0,
\end{align*}
\]

\[
\begin{align*}
    \tilde{u}_n(t) &= u_n(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u_n(\tau) \, d\tau, \\
    \tilde{u}(t) &= u(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u(\tau) \, d\tau.
\end{align*}
\]

To prove (3.12) it suffices to show that

\[
\lim_{n \to \infty} \tilde{u}_n(t) = \tilde{u}(t)
\]
uniformly in \( t \) in bounded intervals \([0, b)\).

From [6] we have

\[
\begin{align*}
    \tilde{u}(t) &= e^{-tA}x_0 + \int_0^t e^{-(t-\tau)A} \left[ \frac{\hat{h}(0)}{h(0)} u(\tau) + \frac{1}{h(0)} \int_0^\tau \hat{h}(\tau-s)u(s) \, ds \right] \, d\tau.
\end{align*}
\]

Similarly,

\[
\begin{align*}
    \tilde{u}_n(t) &= e^{-tA}x_0 + \int_0^t e^{-(t-\tau)A} \left[ \frac{\hat{h}(0)}{h(0)} u_n(\tau) + \frac{1}{h(0)} \int_0^\tau \hat{h}(\tau-s)u_n(s) \, ds \right] \, d\tau.
\end{align*}
\]

From the definition of \( e^{-tA} \) (see [7]) as an integral of the form

\[
\frac{1}{2\pi i} \int_C e^{\lambda(t+\lambda I)}^{-1} \, d\lambda
\]
and from the relation

\[
\|[A_n - A]^{-1}y_0\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{(for any} \ y_0 \in X)
\]
we find that, for any continuous function \( v(s) \),

\[
\lim_{n \to \infty} \|[e^{-tA} - e^{-tA_n}]A^{-1}v(s)\| = 0
\]
uniformly with respect to \( t, s \) in bounded sets of \([0, \infty)\).

Subtracting (3.14) from (3.15) and using (3.17) with \( v(t) = Ax_0 \) and with \( v(t) = Au(t) \), we get

\[
\|\tilde{u}(t) - \tilde{u}_n(t)\| \leq C(t) \int_0^t \|\tilde{u}(\tau) - \tilde{u}_n(\tau)\| \, d\tau + \varepsilon_n(t)
\]
where \( C(t) \) is bounded in bounded intervals \([0, b)\), and \( \varepsilon_n(t) \to 0 \), as \( n \to \infty \), uniformly in \( t \) in bounded intervals \([0, b)\). The last inequality gives (3.13).

Having proved (3.12) for \( x_0 \in \mathcal{D}(A) \), we next notice that, in any bounded interval \( 0 \leq t \leq b \), \( \|S^n(t)\| \leq C \) where \( C \) is a constant independent of \( n, t \). In fact, since (3.6)-(3.8) hold for \( A = A_n, \ W = S^n \), the latter bound follows from the estimates on \( W \) obtained in [6, Chapter 1]. It follows that (3.12) holds for all \( x_0 \in X \), uniformly in \( t \) in bounded intervals.
Remark. A result similar to Theorem 3.2 holds also with respect to the more general integral equation (3.2).

Definition. A complex-valued function $h(t)$ is said to belong to the class $\mathscr{H}'$ if

(i) $h(0) > 0$ and $h(t)$ is absolutely continuous in $[0, \infty)$;

(ii) $h(t) \in L^1(0, \infty)$.

If $h \in \mathscr{H}'$ then we can introduce the function

$$g(s) = h(0) + h^*(s) \quad \text{for } \Re s \geq 0,$$

where $f^*(s)$ indicates the Laplace transform of $f(t)$. Then $g(s) = sh^*(s)$ if $\Re s > 0$.

It follows that $h^*(s)$ can be defined by continuity for $\Re s \geq 0, s \neq 0$. If $g(0) \neq 0$, then we let $h^*(0) = \infty$, and introduce the set

$$\Delta = \{ -1/h^*(s); \Re s \geq 0 \}.$$

As proved in [6, Chapter 2], if $h \in \mathscr{H}'$, $A \in \mathfrak{A}$, and if

$$g(s) \neq 0 \quad \text{for all } s \text{ with } \Re s \geq 0,$$

then there exists a fundamental solution $S(t)$ of (1.1) in the sense defined in §1 (cf. (1.2)), and it belongs to $L^p(0, \infty; B(X))$ for any $p \geq 2$.

Analogously to Theorem 3.2, we have

**Theorem 3.3.** Let $h \in \mathscr{H}'$, $A \in \mathfrak{A}$, and let (3.20), (3.21) hold. Then for any $p \geq 2$, and for any $x_0 \in X$,

$$\lim_{n \to \infty} \int_0^\infty \| S_n(t)x_0 - S(t)x_0 \|_p \, dt = 0.$$

**Proof.** In [6, Chapter 2] it was proved that

$$S(t) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} S_\lambda(t) \, d\lambda$$

for an appropriate curve $C$ lying in the resolvent set $\rho(A)$ of $A$, where $S_\lambda(t)$ is the inverse Laplace transform of the function

$$\frac{1}{s + \lambda g(s)}.$$

One can easily verify that if $S_\lambda(t)$ is the solution of (2.4) then its Laplace transform coincides with the function (3.24). Hence, by the uniqueness of the inverse Laplace transform we conclude that the function $S_\lambda(t)$ occurring in (3.23) coincides with the solution of (2.4).

Using the definition of $S^n(t)$ in (3.11) and Cauchy's theorem, we have:

$$S^n(t) = \frac{1}{2\pi i} \int_C (\lambda I - A_n)^{-1} S_\lambda(t) \, dt.$$
Noting that
\[ \|(\lambda I - A_n)^{-1}x_0 - (\lambda I - A)^{-1}x_0\| \leq \|(\lambda I - A_n)^{-1}\| \|(A_n - A)(\lambda I - A)^{-1}x_0\| \leq \varepsilon_n/|\lambda|, \]
where \( \varepsilon_n \to 0 \) if \( n \to \infty \), we obtain from (3.23), (3.25):
\[ \|S^n(t)x_0 - S(t)x_0\| \leq c\varepsilon_n \int_C |S_\lambda(t)| \frac{|d\lambda|}{|\lambda|}. \]
Since, by [6],
\[ \left\{ \int_0^\infty |S_\lambda(t)|^p \, dt \right\}^{1/p} \leq \frac{c}{|\lambda|^{1/p}} \text{ if } p \geq 2, \]
we obtain
\[ \left\{ \int_0^\infty \|S^n(t)x_0 - S(t)x_0\|^p \, dt \right\}^{1/p} \leq c\varepsilon_n \int_C \frac{|d\lambda|}{|\lambda|^{1+1/p}} \to 0 \]
as \( n \to \infty \). This proves (3.22).

4. Monotonicity for \( A \) selfadjoint. Let \( X \) be a Hilbert space and let \( A \) be a selfadjoint operator in \( X \). We say that \( A \) is strictly positive if the number
\[ \delta_A = \inf_{x \neq 0} \frac{(Ax, x)}{(x, x)} \]
is positive. The main result of the present section is the following:

**Theorem 4.1.** Let \( X \) be a Hilbert space and let \( A \) be a strictly positive selfadjoint operator, with the spectral decomposition of the identity \( \{E_\lambda\} \). If \( h \in \mathcal{H} \) then the operator \( S(t) \) given by
\[ (4.1) \quad S(t)x_0 = \int_{\delta_A}^\infty S_\mu(t) \, dE_\mu x_0 \quad (x_0 \in X) \]
is a fundamental solution of (1.1).

Note that (4.1) is formally obtained from (3.1) and the formula
\[ (\lambda I - A)^{-1} = \int_{\delta_A}^\infty \frac{dE_\mu}{\lambda - \mu} \]
using the Cauchy formula.

**Proof.** Since \( 0 \leq S_\mu(t) \leq 1 \), the integral in (4.1) exists and \( \|S(t)x_0\| \leq \|x_0\| \). \( S(t)x_0 \) is clearly continuous in \( t \). Next, by Fubini’s theorem,
\[ \int_0^t h(t - \tau)S(\tau)x_0 \, d\tau = \int_{\delta_A}^\infty \left\{ \int_0^t h(t - \tau)S_\mu(\tau) \, d\tau \right\} dE_\mu x_0. \]
Using (2.4) we find the expression on the right is equal to
\[ \int_{\delta_A}^\infty \left\{ \frac{1}{\mu} - \frac{S_\mu(t)}{\mu} \right\} dE_\mu x_0 = A^{-1}x_0 - A^{-1}S(t)x_0, \]
where (4.1) has been used. We have thus proved that

$$\int_0^t h(t-\tau)S(\tau)x_0 \, d\tau$$

lies in $D(A)$ and that if we apply $A$ to this integral we obtain $x_0 - S(t)x_0$. This completes the proof of (1.2).

From Theorem 4.1 and Lemma 2.5(i), (ii) we obtain

**Corollary 1.** For any $x_0 \in X$,

$$0 \leq (S(t)x_0, x_0) \leq \|x_0\|^2 \quad (0 < t < \infty).$$

**Corollary 2.** If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,

$$\text{if } t \not\in \mathcal{H} \text{ if } t \not\in \mathcal{H} \quad (0 < t < \infty).$$

If $S_\mu(t) \not\in \mathcal{H}$ when $t = \mu > 0$, then we obtain from (2.4) the bound

$$S_\mu(t) \leq \left[1 + \mu \int_0^t h(\tau) \, d\tau\right]^{-1}.$$

We conclude

**Corollary 3.** If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,

$$\text{if } t \not\in \mathcal{H} \text{ if } t \not\in \mathcal{H} \quad (0 < t < \infty).$$

We next have

**Corollary 4.** If $h \in \mathcal{H}_e$ then, for any $x_0 \in X$,

$$(-1)^n \frac{d^n}{dt^n} (S(t)x_0, x_0) \geq 0 \quad (n = 0, 1, 2, \ldots; 0 < t < \infty).$$

**Proof.** By Lemma 2.5(iii), the functions

$$T_n(t) = \int_{\delta_k}^{m+\delta_k} S_\mu(t) \, d(E_\mu x_0, x_0) \quad (m = 1, 2, \ldots)$$

are completely monotone in $(0, \infty)$. Since, for each $t > 0$, $T_n(t) \to (S(t)x_0, x_0)$ as $m \to \infty$, the assertion of the corollary follows from Lemma 2.3.

**Remark.** Theorem 4.1 extends, with the same proof, to the case of the integral equation (3.2).

5. Monotonicity for general $A$.

**Definition.** A closed linear operator $A$ with a dense domain is said to belong to the class $\mathcal{R}'$ if it satisfies the following properties:

(i) $(\lambda - A)^{-1}$ exists for all complex $\lambda$, except for a sequence $\{\mu_k\}$ (which may be finite) of positive and increasing numbers with no finite limit.

(ii) At each $\mu_k$, $(\lambda - A)^{-1}$ has a pole, i.e.,

$$\lambda (\lambda - A)^{-1} = \sum_{j=1}^{m_k} \frac{B_{k,j}(\lambda)}{\lambda - \mu_k} + B_{k,0}(\lambda)$$

where $(\lambda - A)^{-1}$ has been used. We have thus proved that

$$\int_0^t h(t-\tau)S(\tau)x_0 \, d\tau$$

lies in $D(A)$ and that if we apply $A$ to this integral we obtain $x_0 - S(t)x_0$. This completes the proof of (1.2).

From Theorem 4.1 and Lemma 2.5(i), (ii) we obtain

**Corollary 1.** For any $x_0 \in X$,

$$0 \leq (S(t)x_0, x_0) \leq \|x_0\|^2 \quad (0 < t < \infty).$$

**Corollary 2.** If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,

$$\text{if } t \not\in \mathcal{H} \text{ if } t \not\in \mathcal{H} \quad (0 < t < \infty).$$

If $S_\mu(t) \not\in \mathcal{H}$ when $t = \mu > 0$, then we obtain from (2.4) the bound

$$S_\mu(t) \leq \left[1 + \mu \int_0^t h(\tau) \, d\tau\right]^{-1}.$$

We conclude

**Corollary 3.** If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,

$$\text{if } t \not\in \mathcal{H} \text{ if } t \not\in \mathcal{H} \quad (0 < t < \infty).$$

We next have

**Corollary 4.** If $h \in \mathcal{H}_e$ then, for any $x_0 \in X$,

$$(-1)^n \frac{d^n}{dt^n} (S(t)x_0, x_0) \geq 0 \quad (n = 0, 1, 2, \ldots; 0 < t < \infty).$$

**Proof.** By Lemma 2.5(iii), the functions

$$T_n(t) = \int_{\delta_k}^{m+\delta_k} S_\mu(t) \, d(E_\mu x_0, x_0) \quad (m = 1, 2, \ldots)$$

are completely monotone in $(0, \infty)$. Since, for each $t > 0$, $T_n(t) \to (S(t)x_0, x_0)$ as $m \to \infty$, the assertion of the corollary follows from Lemma 2.3.

**Remark.** Theorem 4.1 extends, with the same proof, to the case of the integral equation (3.2).
where $B_{k,l}$ are bounded operators and $B_{k,0}(\lambda)$ is an analytic function (with values in $B(X)$) in a neighborhood of $\lambda = \mu_k$.

From (3.1), (5.1) and the residue theorem, we formally obtain the formula

\[ S(t)x_0 = \sum_{k=1}^{+\infty} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j}x_0. \]

(We consider here the case where $\{\mu_k\}$ is an infinite sequence; the modifications for the case of a finite sequence are trivial.)

To show that $S(t)$ is a fundamental solution (under certain assumptions), we introduce the operators $T_p(t)$ defined by

\[ T_p(t) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j}, \]

and set

\[ \Delta_p(t) = -T_p(t) + I - A \int_0^t h(t-\tau)T_p(\tau)\,d\tau. \]

Applying $\lambda I - A = (\lambda - \mu_k)I + (\mu_k I - A)$ to both sides of (5.1), we obtain the relations:

\[ AB_{k,m_k} - \mu_k B_{k,m_k} = 0, \]

\[ AB_{k,j} - \mu_k B_{k,j} = B_{k,j+1} \quad (1 \leq j \leq m_k-1). \]

Using these relations and (2.6), (2.4), we get

\[ \Delta_p(t) = -\sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} - I - A \int_0^t h(t-\tau) \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{\mu_k}(t)}{\partial \mu^{j-1}} \mu_k B_{k,j}\,d\tau \]

\[ = I - \sum_{k=1}^{p} B_{k,1}. \]

**Definition.** We denote by $X_A$ the set of all elements $x_0$ of $X$ for which

\[ \sum_{k=1}^{+\infty} \sum_{j=1}^{m_k} \frac{1}{\mu_k^{j-1}} \|B_{k,j}x_0\| < \infty, \]

\[ \lim_{p \to \infty} \sum_{k=1}^{p} B_{k,1}x_0 = x_0. \]

**Theorem 5.1.** Let $h \in \mathcal{H}$, $A \in \mathcal{A}'$, $x_0 \in X_A$. Then the limit

\[ T(t)x_0 \equiv \lim_{p \to \infty} T_p(t)x_0 \]
exists uniformly with respect to \( t, 0 \leq t < \infty \), and

\[
T(t)x_0 = x_0 - A \int_0^t h(t-\tau)T(\tau)x_0 \, d\tau.
\]

**Proof.** The uniform convergence of the sequence \( \{T_p(t)x_0\} \) follows from Lemma 2.10 and the assumption (5.7). From (5.6) we have

\[
T_p(t)x_0 = x_0 - A \int_0^t h(t-\tau)T_p(\tau)x_0 \, d\tau - \Delta_p(t)x_0,
\]

where

\[
\Delta_p(t)x_0 = x_0 - \sum_{k=1}^p B_{k-1}x_0.
\]

By (5.8), \( \Delta_p(t)x_0 \to 0 \) as \( p \to \infty \). Hence, taking \( p \to \infty \) in (5.11) and using the assumption that \( A \) is a closed operator, we conclude that the integral

\[
\int_0^t h(t-\tau)T(\tau)x_0 \, d\tau
\]

belongs to the domain of \( A \) and that (5.10) holds.

**Corollary 1.** If, in addition to the assumptions of Theorem 5.1, we assume that \( x_0 \in D(A) \) and \( Ax_0 \in X_A \), then

\[
T(t)x_0 = x_0 - \int_0^t h(t-\tau)AT(\tau)x_0 \, d\tau
\]

and \( T(t)x_0 \) is continuous for \( t \geq 0 \).

**Proof.** We have

\[
T_p(t)x_0 = x_0 - \int_0^t h(t-\tau)AT_p(\tau)x_0 \, d\tau - \Delta_p(t)x_0,
\]

and \( AT_p(\tau)x_0 = T_p(\tau)(Ax_0) \). Since \( Ax_0 \in X_A \), \( T_p(\tau)(Ax_0) \to T(\tau)(Ax_0) \) as \( p \to \infty \), uniformly with respect to \( t \). It follows that \( T(\tau)x_0 \) is in \( D(A) \), and \( AT(\tau)x_0 = T(\tau)(Ax_0) \). Now take \( p \to \infty \) in (5.13).

**Corollary 2.** Let \( h \in \mathcal{H} \cap \mathcal{X} \), \( A \in \mathcal{A} \cap \mathcal{A}' \), \( x_0 \in X_A \), \( Ax_0 \in X_A \). Then

\[
T(t)x_0 = W(t)x_0
\]

where \( W(t) \) is the fundamental solution of (1.1) occurring in Theorem 3.2.

Indeed, both sides of (5.14) are solutions of (1.1). By the uniqueness assertion of [6, Theorem 1], they must coincide.

**Corollary 3.** Let \( h \in \mathcal{H} \cap \mathcal{X}' \), \( A \in \mathcal{A} \cap \mathcal{A}' \), \( x_0 \in X_A \), and assume also that \( \bar{h}(t) \in L^1(0, \infty) \), and that (3.20), (3.21) hold. Then

\[
T(t)x_0 = S(t)x_0
\]

where \( S(t) \) is the fundamental solution of (1.1) occurring in Theorem 3.3.
This follows from (5.10) and the uniqueness assertion of [6, Theorem 10]. We recall [6] that if \( h \in \mathcal{H} \) then the condition (3.20) is equivalent to the condition

\[
(5.16) \quad h(\infty) > 0.
\]

We shall now study monotonicity of the scalar function \( f_0(T(t)x_0) \), where \( f_0 \) is any bounded linear functional in \( X \).

**Theorem 5.2.** Let \( h \in \mathcal{H}, \quad A \in \mathcal{A}, \quad x_0 \in X_A, \quad f_0 \in X^* \). If

\[
(5.17) \quad (-1)^{j-1} f_0(B_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m, \quad 1 \leq k < \infty)
\]

then \( f_0(T(t)x_0) \geq 0 \) for all \( t \geq 0 \).

**Proof.** From (5.3) and Lemma 2.7 we immediately have that \( f_0(T_p(t)x_0) \geq 0 \). Now take \( p \to \infty \).

**Theorem 5.3.** Let \( h \in \mathcal{H}, \quad A \in \mathcal{A}, \quad x_0 \in X_A, \quad f_0 \in X^* \). If (5.17) holds and, in addition,

\[
(5.18) \quad (-1)^{j-1} f_0(AB_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m, \quad 1 \leq k < \infty)
\]

then \( f_0(T(t)x_0) \geq 0 \) if \( t \neq 0 \).

Note that \( AB_{k,j}x_0 \) is well defined for any \( x_0 \in X \).

Before proving this theorem, we state and prove the following theorem.

**Theorem 5.4.** Let \( h \in \mathcal{H}, \quad A \in \mathcal{A}, \quad x_0 \in X_A, \quad f_0 \in X^* \). Set

\[
(5.19) \quad B_{k,j}^n = (-1)^{m_k - j - 1} \sum_{i=0}^{k} \binom{n}{i} \mu_k^{n-i} B_{k,m_k-j-i}.
\]

If

\[
(5.20) \quad f_0(B_{k,j}^nx_0) \geq 0 \quad \text{for} \quad 0 \leq j \leq m_k, \quad 1 \leq k < \infty, \quad 0 \leq n < \infty,
\]

then

\[
(5.21) \quad (-1)^n \frac{d^n}{dt^n} f_0(T(t)x_0) \geq 0 \quad \text{for} \quad 0 \leq n < \infty, \quad 0 < t < \infty.
\]

**Proof.** Let \( \Gamma \) be a circle about \( \mu_k \) such that all the points \( \mu_j \) with \( j \neq k \) lie outside \( \Gamma \). By the residue theorem,

\[
(5.22) \quad \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_{\lambda}(t) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n (\lambda I - A)^{-1} \left( \frac{S_{\lambda}(t)}{\lambda^n} \right) \, d\lambda = J(t).
\]

Set

\[
Q_n(t) = (-1)^n \lambda^{-n} \frac{\partial^n S_{\lambda}(t)}{\partial t^n}
\]
and write
\[ \lambda^n = \sum_{i=0}^{n} \binom{n}{i} \mu_k^{n-i} (\lambda - \mu_k)^i. \]

From the residue theorem we then get
\[
(-1)^n \frac{d^n g(t)}{dt^n} = \frac{1}{2\pi i} \int_{\Gamma} \left[ \sum_{i=0}^{n} \binom{n}{i} \mu_k^{n-i} (\lambda - \mu_k)^i \right] \left[ \sum_{j=1}^{mk} \frac{B_{k,j}}{(\lambda - \mu_k)^j} \right] Q_n(t) \, d\lambda
\]
\[
= \sum_{j=1}^{mk} B_{k,j} \sum_{i=0}^{\frac{n}{j}} \binom{n}{i} \mu_k^{n-i} \frac{1}{(j-i-1)!} \frac{\partial^{j-i-1} Q_{\mu_k}^n(t)}{\partial \mu_k^{j-i-1}}
\]
\[
= \sum_{q=0}^{m-1} \left\{ B_{k,m_k-q} \binom{n}{q} \mu_k^{n-q} + B_{k,m_k-2q+1} \binom{n}{n} \mu_k^{n-2q+1} + \cdots + B_{k,m_k} \binom{n}{n} \mu_k^{n-q} \right\}
\]
\[
\times \frac{1}{(m_k - q - 1)!} \frac{\partial^{m_k - q - 1} Q_{\mu_k}^n(t)}{\partial \mu_k^{m_k - q - 1}}
\]

where
\[ \binom{n}{i} = 0 \text{ if } i > n. \]

Thus, by (5.19) the last sum is equal to
\[
\sum_{q=0}^{m_k-1} (-1)^{m_k - q - 1} \frac{1}{(m_k - q - 1)!} \frac{\partial^{m_k - q - 1} Q_{\mu_k}^n(t)}{\partial \mu_k^{m_k - q - 1}} B_{k,q}^n.
\]

Hence, recalling (5.22) and (5.3), we have
\[
(-1)^n \frac{d^n T_p(t)x_0}{dt^n} = \sum_{k=1}^{m_k} \sum_{q=0}^{m_k-1} (-1)^{m_k - q - 1} \frac{1}{(m_k - q - 1)!} \frac{\partial^{m_k - q - 1} Q_{\mu_k}^n(t)}{\partial \mu_k^{m_k - q - 1}} B_{k,q}^n x_0.
\]

Using Lemma 2.9 and the assumption (5.20), we conclude that
\[
(-1)^n \frac{d^n f_0(T_p(t)x_0)}{dt^n} \leq 0 \text{ for } n = 0, 1, 2, \ldots; t > 0.
\]

Since \( T_p(t)x_0 \to T(t)x_0 \) as \( p \to \infty \), the assertion of the theorem follows from Lemma 2.3.

Using the notation \( \Delta^0_n \) (following Lemma 2.3), we can state

**COROLLARY.** If (5.20) is assumed to hold only for \( 0 \leq n \leq n_0 \), then
\[
(-1)^n \Delta^0_n f_0(T(t)x_0) \leq 0 \text{ for } 0 \leq n \leq n_0, 0 \leq t < t + \eta < \infty.
\]

Indeed, the proof of Theorem 5.4 shows that (5.24) holds with \( T(t)x_0 \) replaced by \( T_p(t)x_0 \). Since \( T_p(t)x_0 \to T(t)x_0 \) as \( p \to \infty \), (5.22) follows.

**REMARK.** Let the assumptions of Corollary 2 to Theorem 5.1 hold and let \( h \in C^{\eta}[0, \infty) \). Then, by [6] and (5.14), \( T(t)x_0 \) has \( n_0 \) continuous derivatives in \([0, \infty)\). Hence, (5.24) implies that
\[
(-1)^n (\partial^n f_0(T(t)x_0)/\partial t^n) \leq 0 \text{ for } 0 \leq n \leq n_0, 0 \leq t < \infty.
\]
Proof of Theorem 5.3. We shall use the formula

\[ T_p(t)x_0 = \sum_{k=1}^{p} \sum_{q=0}^{m_k-1} (-1)^{m_k-q} \left[ \frac{\partial^{m_k-q-1}}{\partial \mu^{m_k-q-1}} \left( \frac{S_\mu(t)}{\mu} \right) \right]_{\mu=\mu_k} B_{k,q}x_0 \]

which one obtains by the same method that was used before to derive (5.23). Since

\[ B_{k,q} = (-1)^{m_k-q-1}(\mu_k B_{k,m_k-q} + B_{k,m_k-q+1}) \quad (1 \leq q \leq m_k-1), \]

(5.5) shows that the inequalities (5.18) imply the inequalities (5.20) for \( n = 1 \). Hence, (5.26) gives

\[ f_\sigma(T_p(t)x_0) \text{ if } t \neq 0. \]

Since \( T_p(t)x_0 \to T(t)x_0 \) as \( p \to \infty \), the proof is complete.

Remark. If \( X \) is a finite-dimensional Banach space, then any linear operator \( A \) whose eigenvalues are positive numbers is in \( \mathfrak{A}' \). Furthermore, the series in (5.2) now consists of a finite number of terms. Hence (5.7) holds. (5.8) is also valid; in fact, it easily follows using the residue theorem. Thus \( X_\mathfrak{A} = X \).

6. Additional results. In the previous two sections we have derived theorems which involved the functions \( S_\lambda(t) \) for \( \lambda > 0 \). A crucial step in the derivation of these theorems was the behavior of the function \( S_\lambda(t) \) for positive values of the parameter \( \lambda \). Since analogous results on the behavior of \( S_\lambda(t) \) for \( \lambda \) complex are not available in the literature, we cannot extend, at present, the results of \( \S \S 4, 5 \) to operators \( A \) with \( \sigma(A) \) which is not contained in the real interval \( 0 < \lambda < \infty \).

However, for some special functions \( h(t) \), the behavior of \( S_\lambda(t) \), for complex \( \lambda \), is known with sufficient precision. We give here one example where \( h(t) = t^{-\alpha} \) for some \( 0 < \alpha < 1 \). Then \( S_\lambda(t) = E_\beta(-\gamma t^\beta) \) where \( \beta = 1 - \alpha \), \( \gamma = \Gamma(\beta) \) and where \( E_\beta(z) \) is the Mittag-Leffler function

\[ E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)}. \]

From a well-known asymptotic formula for \( E_\beta(z) \) (see [1, p. 207]) we find that

\[ |S_\lambda(t) - \gamma_0 t^\beta| \leq \frac{c}{|\lambda|^{\frac{1}{2}}} \text{ if } |\arg \lambda t^\beta| < \frac{1+\alpha}{2} \pi, \]

provided \( |\lambda t^\beta| \geq c_0 > 0 \); here \( \gamma_0 = (\Gamma(\alpha)\Gamma(1-\alpha))^{-1} \).

Let us assume that the resolvent set \( \rho(A) \) of \( A \) contains the sector \( |\arg \lambda| > (1+\alpha)\pi/2 \) and that \( \|(\lambda I - A)^{-1}\| \leq c/(1 + |\lambda|) \) for \( \lambda \) in this sector. We define \( S(t) \) by (3.23) and choose \( C \) in \( \rho(A) \) such that (6.1) holds for \( \lambda \in C \). It follows that \( S(t) \) is a bounded operator for each \( t > 0 \). Furthermore, it varies continuously in \( t \). One can also continue \( S(t) \) analytically into a sector \( |\arg \lambda| < \delta \) for some \( \delta > 0 \).

Since \( h(0) = \infty \), the results of [6] do not cover the present case of \( h(t) = t^{-\alpha} \). There arises the question in what sense is \( S(t) \) a fundamental solution.
So far we have only considered solutions of equations of the form (1.1) where \(x_0\) is independent of \(t\). But some of the results extend without difficulty to the equations

\[
x(t) = k(t)x_0 - \int_0^t h(t-\tau)Ax(\tau)\,d\tau,
\]

where \(k(t)\) is a scalar function.

The solution is given by (see [6]):

\[
x(t) = k(0)S(t)x_0 + \int_0^t k(\tau)S(t-\tau)x_0\,d\tau
\]

where \(k(\tau) = dk(\tau)/d\tau\). Hence, if \(X\) is a Hilbert space,

\[
(x(t), x_0) = k(0)(S(t)x_0, x_0) + \int_0^t k(\tau)(S(t-\tau)x_0, x_0)\,d\tau.
\]

This relation combined with the results of §§4, 5 yields monotonicity properties for \((x(t), x_0)\). For example, if \(k(0) \geq 0, k(\tau) \geq 0\), then \((x(t), x_0) \geq 0\).

APPLICATIONS. If \(h \in \mathcal{H}_o\) then we have proved several theorems to the effect that \((S(t)x_0, x_0)\) is completely monotonic in \(t\). Since \((S(0)x_0, x_0) = (x_0, x_0) \neq 0\), we conclude that \((S(t)x_0, x_0) > 0\) for all \(t > 0\). In particular, \(S(t)x_0 \neq 0\) for all \(t > 0\). Thus the solutions of (1.1) have the “weak backward uniqueness” property as defined in [5]. This fact is important in the study of optimal-control for trajectories \(x(t)\) given by

\[
x(t) = u(t) + \int_0^t h(t-\tau)Ax(\tau)\,d\tau
\]

where \(u(t)\) is the control function. It enables us to prove uniqueness of time-optimal controls (see [3], [5, p. 42]).

If \(A\) is selfadjoint and if \(h \in \mathcal{H}'\) and \(h\) is strictly decreasing, then we can again assert that \((S(t)x_0, x_0) > 0\) for all \(x_0 \neq 0, t > 0\). Indeed, otherwise we get, from (4.1), \(S_\mu(t_0) = 0\) for some \(\mu > 0, t_0 > 0\). But then, by Lemma 2.5, \(S_\mu(t) = 0\) if \(t > t_0\). Using (2.4) we then see that \(S_\mu(t) < S_\mu(t_0)\) if \(t > t_0\); a contradiction.

If \(h \in \mathcal{H}'\), then we have proved several theorems to the effect that \((S(t)x_0, x_0) \not< 0\) if \(t \not> \). This can be used to answer some questions of controllability; for instance, to show that a point \(x_0\) can be “steered,” by a suitable control, to any given neighborhood of \(0\) (cf. [3]).

REFERENCES


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