MONOTONICITY OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS IN BANACH SPACE (1)

BY
AVNER FRIEDMAN

1. **Introduction.** We shall consider Volterra integral equations

$$x(t) = x_0 - \int_0^t h(t-\tau)Ax(\tau)\, d\tau \tag{1.1}$$

where x_0 and $x(t)$ belong to a complex Banach space X, $h(t)$ is a complex-valued function and A is an operator in X, generally unbounded. We denote by $B(X)$ the Banach space of bounded linear operators in X, and by I the identity operator in X. An operator-valued function $S(t)$, which belongs to $L^1(0, b; B(X))$ for any $b>0$, is called a *fundamental solution* of (1.1) if

$$S(t) = I - A\int_0^t h(t-\tau)S(\tau)\, d\tau \tag{1.2}$$

for almost all t. In this definition it is assumed, of course, that the integral on the right-hand side of (1.2) is in the domain of A.

In a recent paper [6], Friedman and Shinbrot have studied the equation (1.1) even in the more general case where x_0 and A depend on t and τ, respectively. They proved theorems of existence, uniqueness, differentiability and asymptotic behavior of solutions. They also constructed fundamental solutions and derived asymptotic bounds for them. We recall [6] that if x_0 is in the domain of A^μ, for some $\mu>0$, then the solution of (1.1) is given by $S(t)x_0$.

The purpose of the present paper is to derive monotonicity theorems for solutions of (1.1). We shall generalize some of the monotonicity theorems of Friedman [2] (see also [4]) from the case $X=R^1$ (R^1 the one-dimensional Euclidean space) to the case where X is any Banach space.

In §2 we give some auxiliary results. These results are concerned with Volterra equations in one-dimension (i.e., $X=R^1$). In particular, we study the behavior of the solutions with respect to a certain parameter.

In §3 we give an integral formula for $S(t)$ in case A is a bounded operator. For A unbounded, we construct a fundamental solution as a limit of fundamental solutions corresponding to the bounded operators $A(I+A/n)^{-1}$. We prove that the fundamental solution coincides with the fundamental solution of [6, Chapter 1]

Received by the editors April 9, 1968.

(1) This work was partially supported by the National Science Foundation NSF GP-5558.

or of [6, Chapter 2] provided the assumptions of [6, Chapter 1] or of [6, Chapter 2], respectively, are satisfied.

In §4 we derive a formula for $S(t)$ in case A is selfadjoint. As a by-product, we obtain monotonicity theorems for $(S(t)x_0, x_0)$.

In §5 we drop the assumption that A is selfadjoint. Instead we assume that the resolvent $(\lambda I - A)^{-1}$ exists for all λ except for a sequence $\{\mu_k\}$ of poles, and $0 < \mu_1 < \mu_2 < \cdots$, $\mu_k \to \infty$ as $k \to \infty$. We obtain a formula for the solution $T(t)x_0$ of (1.1) and then derive monotonicity theorems for $f_0(T(t)x_0)$; here f_0 is a bounded linear functional in X.

In §6 we give some additional results obtainable with the methods of the previous sections, and some applications to control theory.

2. **Auxiliary lemmas.** A real-valued function $f(t)$ is said to be *completely monotonic* in an interval $[a, b)$ if $f \in C^\infty[a, b)$ and for all $n \geqq 0$,

$$(-1)^n \, d^n f(t)/dt^n \geqq 0 \quad \text{for all } t \in [a, b).$$

Similarly one defines complete monotonicity in intervals (a, b), $[a, b]$. We recall the following results (see [9]):

Lemma 2.1. *If $f(t)$ is completely monotonic in an interval (a, b) then $f(t)$ is analytic in (a, b).*

Lemma 2.2. *A function $f(t)$ is completely monotonic in the interval $[0, \infty)$ if and only if*

$$f(t) = \int_0^\infty e^{-\lambda t}\, d\phi(\lambda) \tag{2.1}$$

where $\phi(\lambda)$ is a bounded nondecreasing function.

Lemma 2.3. *If $\{f_m(t)\}$ is a sequence of completely monotonic functions in (a, b) and if $f(t)$ is a continuous function in (a, b) such that, for each $t \in (a, b)$, $f_m(t) \to f(t)$ as $m \to \infty$, then $f(t)$ is completely monotonic in (a, b).*

Setting $\Delta_\eta^1 f(t) = f(t+\eta) - f(t)$,

$$\Delta_\eta^{m+1} f(t) = \Delta_\eta^m(\Delta_\eta f(t)),$$

the assertion of the last lemma is a consequence of the fact (see [9]) that $f(t)$ is completely monotonic if, for any integer $m \geqq 1$ and $\eta > 0$,

$$(-1)^m \Delta_\eta^m f(t) \geqq 0 \quad \text{for } a < t < b - m\eta.$$

In view of Lemma 2.1, if $f(t)$ is a completely monotonic function in (a, b) which does not vanish identically, then $f(t) > 0$ for all $t \in (a, b)$.

We shall need the following result of Miller [8]:

Lemma 2.4. *If $f(t)$ is a nonzero completely monotonic function in $[0, \infty)$, then $\log f(t)$ is a convex function.*

Proof. We have to show that

$$g(t) \equiv f(t)f''(t)-(f'(t))^2 \geqq 0.$$

Using (2.1) we find that

$$g(t) = \int_0^\infty \int_0^\infty \lambda(\lambda-\mu)e^{-(\lambda+\mu)t}\, d\phi(\lambda)\, d\phi(\mu).$$

Next,

$$\int_0^\infty \int_\mu^\infty \lambda(\lambda-\mu)e^{-(\lambda+\mu)t}\, d\phi(\lambda)\, d\phi(\mu) = \int_0^\infty \int_0^\lambda \lambda(\lambda-\mu)e^{-(\lambda+\mu)t}\, d\phi(\mu)\, d\phi(\lambda)$$
$$= \int_0^\infty \int_0^\mu \mu(\mu-\lambda)e^{-(\lambda+\mu)t}\, d\phi(\lambda)\, d\phi(\mu).$$

Therefore

$$g(t) = \int_0^\infty \int_0^\mu (\cdots)+\int_0^\infty \int_\mu^\infty (\cdots) = \int_0^\infty \int_0^\mu (\lambda-\mu)^2 e^{-(\lambda+\mu)t}\, d\phi(\lambda)\, d\phi(\mu) \geqq 0.$$

Definitions. A function $h(t)$ which belongs to $C(0, \infty)$ and to $L^1(0, 1)$ is said to belong to the class $\mathscr{H}$, if $h(t) \geqq 0$, $h(t) \not\equiv 0$, and $h(t)$ is monotone nonincreasing in $(0, \infty)$. If $h \in \mathscr{H}$ and if $\log h(t)$ is a convex function in the interval where $h(t)>0$, then we say that h belongs to the class $\mathscr{H}'$. Finally, we say that $h \in \mathscr{H}_\infty$ if $h(t)$ is a nonzero completely monotonic function in $(0, \infty)$ and if $h \in L^1(0, 1)$.

From Lemma 2.4 (applied to $h(t+\varepsilon)$, for any $\varepsilon>0$) it follows that if $h \in \mathscr{H}_\infty$ then $h \in \mathscr{H}'$.

In the following lemma we have collected some results proved in Friedman [2].

Lemma 2.5. *Consider the integral equation*

$$x(t) = 1-\int_0^t h(t-\tau)x(\tau)\, d\tau \qquad (0 < t < \infty). \tag{2.2}$$

(i) *If $h \in \mathscr{H}$, then $0 \leqq x(t) \leqq 1$.*
(ii) *If $h \in \mathscr{H}'$, then $x(t)$ is monotone nonincreasing.*
(iii) *If $h \in \mathscr{H}_\infty$, then $x(t)$ is in $\mathscr{H}_\infty$.*

From [2, Corollary 4, p. 387] we deduce

Lemma 2.6. *Consider the equation*

$$x(t) = \int_0^t h(t-\sigma)p(\sigma)\, d\sigma-\lambda \int_0^t h(t-\tau)x(\tau)\, d\tau \qquad (0 < t < \infty) \tag{2.3}$$

where $p(\sigma)$ is a continuous nonnegative function, and λ is a positive constant. If $h \in \mathscr{H}'$ then $x(t) \geqq 0$.

We shall consider now the equation

$$S_\lambda(t) = 1-\lambda \int_0^t h(t-\tau)S_\lambda(\tau)\, d\tau \qquad (0 \leqq t < \infty) \tag{2.4}$$

where λ is a complex parameter. By a standard argument one shows that if $h(t)$ is in $C(0, \infty) \cap L^1(0, 1)$ then, for each λ, there exists a unique solution $S_\lambda(t)$ of (2.4). Furthermore, $S_\lambda(t)$ is continuous in (t, λ) ($t \geqq 0$, λ complex) and analytic in λ, for each $t \geqq 0$. If $h \in C^n[0, \infty)$ then $\partial^n S_\lambda(t)/\partial t^n$ is continuous in (t, λ) (for $t \geqq 0$, λ complex) and analytic in λ, for each $t \geqq 0$.

LEMMA 2.7. *If $h \in \mathscr{H}$ then, for $n=0, 1, 2, \ldots$,*

$$(2.5) \qquad (-1)^n(\partial^n S_\lambda(t)/\partial\lambda^n) \geqq 0 \quad \text{if } 0 < \lambda < \infty, 0 < t < \infty.$$

Proof. The inequality (2.5) for $n=0$ follows from Lemma 2.5(i). We proceed by induction. We assume that (2.5) holds and prove the same inequality when n is replaced by $n+1$. Differentiating (2.4) $n+1$ times with respect to λ we get

$$(2.6) \quad \frac{\partial^{n+1}}{\partial\lambda^{n+1}} S_\lambda(t) = -(n+1)\int_0^t h(t-\tau)\frac{\partial^n}{\partial\lambda^n} S_\lambda(\tau)\,d\tau - \lambda\int_0^t h(t-\tau)\frac{\partial^{n+1}}{\partial\lambda^{n+1}} S_\lambda(\tau)\,d\tau.$$

It follows that the function $x(t) = (-1)^{n+1}\,\partial^{n+1}S_\lambda(t)/\partial\lambda^{n+1}$ satisfies the equation (2.3) with

$$p(\sigma) = (n+1)(-1)^n(\partial^n S_\lambda(\sigma)/\partial\lambda^n).$$

By the inductive assumption, $p(\sigma) \geqq 0$. Hence we can apply Lemma 2.6 and conclude that $x(t) \geqq 0$, i.e., (2.5) holds with n replaced by $n+1$.

LEMMA 2.8. *If $h \in \mathscr{H}'$ then, for $n=0, 1, 2, \ldots, \lambda > 0$,*

$$(2.7) \qquad (-1)^n \frac{\partial^n}{\partial\lambda^n}\left[\frac{S_\lambda(t)}{\lambda}\right] \searrow \text{ if } t \nearrow.$$

Proof. We may assume that $h \in C^1[0, \infty)$. Indeed, otherwise we approximate $h(t)$ by a sequence of functions $\{h_m(t)\}$ as follows: $h_m \in \mathscr{H}'$, $h_m(t) \to h(t)$ uniformly on compact subsets of $(0, \infty)$ and

$$\int_0^1 |h(t) - h_m(t)|\,dt \to 0.$$

If we know already that the assertion of the lemma holds for the solution $S_{\lambda,m}(t)$ corresponding to h_m, then (2.7) is also true since, for any $n \geqq 0$,

$$\frac{\partial^n}{\partial\lambda^n}\left[\frac{S_{\lambda,m}(t)}{\lambda}\right] \to \frac{\partial^n}{\partial\lambda^n}\left[\frac{S_\lambda(t)}{\lambda}\right] \quad \text{for each } t.$$

Assuming h to be in $C^1[0, \infty)$, it follows that $\partial S_\lambda(t)/\partial t$ exists and satisfies:

$$(2.8) \qquad \frac{\partial S_\lambda(t)}{\partial t} = -\lambda h(t) - \lambda\int_0^t h(t-\tau)\frac{\partial S_\lambda(\tau)}{\partial\tau}\,d\tau.$$

The assertion (2.7) is equivalent to the following inequality:

$$(2.9) \qquad (-1)^{n+1}\frac{\partial^n}{\partial\lambda^n}\left[\frac{1}{\lambda}\frac{\partial}{\partial t} S_\lambda(t)\right] \geqq 0.$$

For $n=0$, this inequality follows from Lemma 2.5(ii). We now proceed by induction on n. To pass from n to $n+1$, we divide both sides of (2.8) by λ and then differentiate both sides $n+1$ times with respect to λ. We get

$$\frac{\partial^{n+1}}{\partial\lambda^{n+1}}\left[\frac{1}{\lambda}\frac{\partial S_\lambda(t)}{\partial t}\right] = -(n+1)\int_0^t h(t-\tau)\frac{\partial^n}{\partial\lambda^n}\left[\frac{1}{\lambda}\frac{\partial S_\lambda(\tau)}{\partial\tau}\right]d\tau$$
$$-\lambda\int_0^t h(t-\tau)\frac{\partial^{n+1}}{\partial\lambda^{n+1}}\left[\frac{1}{\lambda}\frac{\partial S_\lambda(\tau)}{\partial\tau}\right]d\tau.$$

We can now apply Lemma 2.6 with

$$x(t) = (-1)^{n+2}\frac{\partial^{n+1}}{\partial\lambda^{n+1}}\left[\frac{1}{\lambda}\frac{\partial S_\lambda(t)}{\partial t}\right], \qquad p(\sigma) = (n+1)(-1)^{n+1}\frac{\partial^n}{\partial\lambda^n}\left[\frac{1}{\lambda}\frac{\partial S_\lambda(\sigma)}{\partial\sigma}\right].$$

Lemma 2.9. *If $h\in\mathscr{H}_\infty$ then, for $n=0, 1, 2, \ldots, m=0, 1, 2, \ldots, \lambda>0$,*

$$(2.10)\qquad (-1)^{m+n}\frac{\partial}{\partial\lambda^m}\left[\frac{1}{\lambda^n}\frac{\partial^n}{\partial t^n}S_\lambda(t)\right] \geqq 0 \quad \textit{for } t > 0.$$

Proof. Suppose first that $h(t)$ is in $C^\infty[0,\infty)$. Then all the derivatives occurring in (2.10) exist for $t\geqq 0$. We shall establish (2.10) by induction on n. For $n=0$, (2.10) follows from Lemma 2.7. We now assume that (2.10) holds for all $m\geqq 0$ and $0\leqq n\leqq k$. We shall prove (2.10) for all $m\geqq 0$ and $n=k+1$. Differentiating both sides of (2.4) $k+1$ times with respect to t, we get

$$\frac{\partial^{k+1}S_\lambda(t)}{\partial t^{k+1}} = -\lambda h^{(k)}(t)S_\lambda(0)-\lambda h^{(k-1)}(t)\frac{\partial S_\lambda(0)}{\partial t}-\cdots-\lambda h(t)\frac{\partial^k S_\lambda(0)}{\partial t^k}$$
$$-\lambda\int_0^t h(t-\tau)\frac{\partial^{k+1}S_\lambda(\tau)}{\partial\tau^{k+1}}d\tau.$$

Setting

$$T_\lambda(t) = (-1)^{k+1}\frac{1}{\lambda^{k+1}}\frac{\partial^{k+1}}{\partial t^{k+1}}S_\lambda(t),$$

we get

$$(2.11)\qquad \begin{aligned} T_\lambda(t) = \sum_{i=0}^{k}&\frac{(-1)^{k-i}h^{(k-i)}(t)}{\lambda^{k-i}}\frac{(-1)^i\,\partial^i S_\lambda(0)/\partial t^i}{\lambda^i}\\ &-\lambda\int_0^t h(t-\tau)T_\lambda(\tau)\,d\tau.\end{aligned}$$

We have to prove that, for any $m\geqq 0$,

$$(2.12)\qquad (-1)^m(\partial^m T_\lambda(t)/\partial\lambda_m) \geqq 0.$$

For $m=0$ this follows from the definition of $T_\lambda(t)$ and Lemma 2.5(iii). We now proceed by induction on m.

To pass from m to $m+1$, we differentiate (2.11) $m+1$ times with respect to λ.

We find

$$\begin{aligned}\frac{\partial^{m+1}T_\lambda(t)}{\partial\lambda^{m+1}} &= g(t)-(m+1)\int_0^t h(t-\tau)\frac{\partial^m T_\lambda(\tau)}{\partial\lambda^m}\,d\tau \\ &\quad -\lambda\int_0^t h(t-\tau)\frac{\partial^{m+1}T_\lambda(\tau)}{\partial\lambda^{m+1}}\,d\tau,\end{aligned} \tag{2.13}$$

where

$$\begin{aligned} g(t) &= \sum_{i=0}^{k} g_i(t), \\ g_i(t) &= (-1)^{k-i}h^{(k-i)}(t)\sum_{s=0}^{m+1}\binom{m+1}{s}\frac{d^s}{d\lambda^s}\frac{1}{\lambda^{k-i}}\cdot\frac{\partial^{m+1-s}}{\partial\lambda^{m+1-s}}\left[\frac{(-1)^i}{\lambda^i}\frac{\partial^i}{\partial t^i}S_\lambda(0)\right].\end{aligned} \tag{2.14}$$

Set

$$g_{k+1}(t) = -(m+1)\int_0^t h(t-\tau)\frac{\partial^m T_\lambda(\tau)}{\partial\lambda^m}\,d\tau \tag{2.15}$$

and denote by $x_i(t)$ $(0\leqq i\leqq k+1)$ the solution of the equation

$$x_i(t) = (-1)^{m+1}g_i(t)-\lambda\int_0^t h(t-\tau)x_i(\tau)\,d\tau. \tag{2.16}$$

It is clear that

$$(-1)^{m+1}\frac{\partial^{m+1}T_\lambda(t)}{\partial\lambda^{m+1}} = \sum_{i=0}^{k+1} x_i(t).$$

Hence, it suffices to show that $x_i(t)\geqq 0$ for $0\leqq i\leqq k+1$.

The inequality $x_{k+1}(t)\geqq 0$ follows by applying Lemma 2.6 with

$$p(\sigma) = (-1)^m(m+1)\,\partial^m T_\lambda(\sigma)/\partial\lambda^m;$$

note that by the inductive assumption, $p(\sigma)\geqq 0$.

From (2.14) and the inductive assumption we easily see that

$$(-1)^{m+1}g_i(t) = \gamma_i(-1)^{k-i}h^{(k-i)}(t) \qquad (0\leqq i\leqq k)$$

where $\gamma_i=\gamma_i(\lambda)$ is nonnegative. From Theorem 1 and its Corollary 3 in [2] we then have the following: If

$$\frac{h'(a)}{h(a)} \leqq \frac{h^{(k-i+1)}(b)}{h^{(k-i)}(b)} \qquad (0<a<b<\infty) \tag{2.17}$$

then $x_i(t)\geqq 0$. Thus, it remains to prove (2.17). We assume here that $h^{(k-i)}(b)>0$ for all $b>0$; if $h^{(k-i)}(b)=0$ for some $b>0$ then $h^{(k-i)}(t)\equiv 0$ and $x_i(t)\equiv 0$.

Since $h\in\mathscr{H}_\infty$, Lemma 2.4 implies that $h'(t)/h(t)\nearrow$ if $t\nearrow$. Hence (2.17) is a consequence of

$$\frac{h'(b)}{h(b)} \leqq \frac{h^{(k-i+1)}(b)}{h^{(k-i)}(b)} \qquad (0<b<\infty). \tag{2.18}$$

Now, the function $(-1)^j h^{(j)}(t)$ is completely monotonic. If we apply Lemma 2.4 to this function, we obtain

$$h^{(j+1)}(b)/h^{(j)}(b) \leqq h^{(j+2)}(b)/h^{(j+1)}(b).$$

Applying this inequality for $j=0, 1, \ldots, k-i-1$, we get (2.18).

We have proved Lemma 2.9 assuming that $h(t)$ is in $C^\infty[0, \infty)$. Consider now the general case, where we merely assume that $h \in \mathscr{H}_\infty$. Then we can apply the previous result to the solution $S_{\lambda,\varepsilon}(t)$ of (2.4) with $h(t)$ replaced by $h(t+\varepsilon)$, $\varepsilon > 0$. Since, for each $t > 0$, $m \geqq 0$,

$$\partial^m S_{\lambda,\varepsilon}(t)/\partial\lambda^m \to \partial^m S_\lambda(t)/\partial\lambda^m,$$

the assertion of the lemma for $S_\lambda(t)$ follows upon applying Lemma 2.3.

The last lemma of this section is the following:

Lemma 2.10. *Let $h \in \mathscr{H}$. Then, for $n=0, 1, 2, \ldots$,*

$$(-1)^n(\partial^n S_\lambda(t)/\partial\lambda^n) \leqq n!/\lambda^n \qquad (0 < \lambda < \infty, 0 < t < \infty). \tag{2.19}$$

Proof. From Lemma 2.7 and (2.6) we obtain

$$(-1)^{n+1}\int_0^t h(t-\tau)\frac{\partial^{n+1}}{\partial\lambda^{n+1}}S_\lambda(\tau)\,d\tau \leqq (-1)^n\frac{n+1}{\lambda}\int_0^t h(t-\tau)\frac{\partial^n}{\partial\lambda^n}S_\lambda(\tau)\,d\tau.$$

Applying this relation successively, we find that

$$(-1)^m\int_0^t h(t-\tau)\frac{\partial^m}{\partial\lambda^m}S_\lambda(\tau)\,d\tau \leqq \frac{m!}{\lambda^{m+1}} \qquad (m = 0, 1, 2, \ldots).$$

Hence from (2.6), with $n+1=m$, we get

$$(-1)^m\frac{\partial^m}{\partial\lambda^m}S_\lambda(t) \leqq (-1)^{m-1}m\int_0^t h(t-\tau)\frac{\partial^{m-1}}{\partial\lambda^{m-1}}S_\lambda(\tau)\,d\tau \leqq \frac{m!}{\lambda^m}$$

for $m=1, 2, \ldots$.

3. **Integral formula for $S(t)$.** Let X be a Banach space. We denote by $\sigma(A)$ the spectrum of an operator A.

Theorem 3.1. *Let A be a bounded operator and let Γ be any continuously differentiable closed Jordan curve containing $\sigma(A)$ in its interior. Let $h(t)$ be any function in $C(0, \infty) \cap L^1(0, 1)$. Then the operator-valued function*

$$S(t) \equiv \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1}S_\lambda(t)\,d\lambda \tag{3.1}$$

is the unique fundamental solution of (1.1).

The orientation of Γ, in (3.1), is taken counterclockwise.

Proof. The uniqueness of the fundamental solution follows by standard arguments. It remains to verify (1.2). We have

$$T \equiv I - A\int_0^t h(t-\tau)S(\tau)\,d\tau$$
$$= I - \int_0^t h(t-\tau)\left\{\frac{1}{2\pi i}\int_\Gamma A(\lambda I - A)^{-1}S_\lambda(\tau)\,d\lambda\right\}d\tau.$$

Changing the order of integration and using (2.4), we get

$$T = I + \frac{1}{2\pi i}\int_\Gamma \left[(\lambda I - A)^{-1} - \frac{I}{\lambda}\right][S_\lambda(t) - 1]\,d\lambda$$
$$= I + S(t) - \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1}\,d\lambda + \frac{I}{2\pi i}\int_\Gamma \frac{1 - S_\lambda(t)}{\lambda}\,d\lambda.$$

By Cauchy's theorem we easily find that

$$\frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1}\,d\lambda = I.$$

Also,

$$\int_\Gamma \frac{1 - S_\lambda(t)}{\lambda}\,d\lambda = \int_\Gamma \left\{\int_0^t h(t-\tau)S_\lambda(\tau)\,d\tau\right\}d\lambda$$
$$= \int_0^t h(t-\tau)\left\{\int_\Gamma S_\lambda(\tau)\,d\lambda\right\}d\tau = 0.$$

We obtain that $T = S(t)$. This proves (1.2).

Theorem 3.1 can easily be extended to more general integral equations. For example, we shall construct a solution of

$$S(t, s) = I - \int_s^t h(t-\tau, \tau)AS(\tau, s)\,d\tau. \tag{3.2}$$

Denote by $S_\lambda(t, s)$ the solution of

$$S_\lambda(t, s) = 1 - \lambda\int_s^t h(t-\tau, \tau)S_\lambda(\tau, s)\,d\tau. \tag{3.3}$$

Then we have

THEOREM 3.1′. *Let A, Γ be as in Theorem* 3.1 *and let $h(t, \tau)$ be a continuous function for $t \geqq 0$, $\tau \geqq 0$. Then the unique solution of* (3.2) *is given by*

$$S(t, \tau) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1}S_\lambda(t, \tau)\,d\lambda. \tag{3.4}$$

The proof is similar to the proof of Theorem 3.1.

We next consider the case where A is not necessarily a bounded operator.

DEFINITION. A linear operator A in X is said to belong to the class $\mathfrak{A}$ if

(i) A is closed and densely defined;

(ii) $\sigma(A) \subset \{\lambda;\ |\arg\lambda| \leqq \pi/2 - \varepsilon,\ \mathrm{Re}\,\lambda \geqq \lambda_0\}$ for some $\varepsilon > 0$, $\lambda_0 > 0$;

(iii) $\|(\lambda I - A)^{-1}\| \leqq c/|\lambda|$ if $|\arg\lambda| > \pi/2 - \varepsilon$.

DEFINITION. A complex-valued function $h(t)$ is said to belong to the class $\mathscr{K}$ if

(i) $h(0)>0$;

(ii) $h \in C^1[0, \infty)$ and $\dot{h}(t)$ is absolutely continuous;

(iii) for any $b>0$, $\ddot{h}(t)$ is in $L^p(0, b)$ for some $p>1$.

In [6, Chapter 1] it was proved that if $A \in \mathfrak{A}$ and $h \in \mathscr{K}$, then there exists a fundamental solution $W(t)$ of (1.1), in a sense different than (1.2). Thus, $W(t)$ satisfies the equation

$$W(t) = I-\int_0^t h(t-\tau)AW(\tau)\,d\tau \tag{3.5}$$

in the following sense

$$\tilde{W}(t) = e^{-tA}+\int_0^t e^{-(t-\tau)A}F(W;\tau)\,d\tau \tag{3.6}$$

where e^{-tA} is the analytic semigroup of $-A$, and

$$\tilde{W}(t) = W(t)+\frac{1}{h(0)}\int_0^t \dot{h}(t-\tau)W(\tau)\,d\tau, \tag{3.7}$$

$$F(W;\tau) = \frac{\dot{h}(0)}{h(0)}\,W(t)+\frac{1}{h(0)}\int_0^t \ddot{h}(t-\tau)W(\tau)\,d\tau. \tag{3.8}$$

Denoting by $D(A^\mu)$ the domain of A^μ (see [7] for the definition of A^μ), we have the following result: If $x_0 \in D(A^\mu)$ for some $\mu>0$, then $W(t)x_0$ is the unique solution of (1.1). (The solutions of (1.1) are assumed to be such that $\|Ax(\tau)\|$ is integrable in every bounded interval $(0, b)$.)

We introduce the operators

$$A_n = A(I+A/n)^{-1}. \tag{3.9}$$

One easily verifies that $\|A_n\| \leqq Cn$ and

$$(\lambda I-A_n)^{-1} = -\frac{1}{n-\lambda}\,I+\frac{n^2}{(n-\lambda)^2}\left[\frac{n\lambda}{n+\lambda}\,I-A\right]^{-1}. \tag{3.10}$$

Denote by Γ_n a continuously differentiable closed Jordan curve which contains $\sigma(A_n)$, and set

$$S^n(t) = \frac{1}{2\pi i}\int_{\Gamma_n} (\lambda I-A_n)^{-1}S_\lambda(t)\,d\lambda. \tag{3.11}$$

THEOREM 3.2. *Let $h \in \mathscr{K}$, $A \in \mathfrak{A}$. Then, for any $x_0 \in X$,*

$$\lim_{n\to\infty} S^n(t)x_0 = W(t)x_0 \tag{3.12}$$

uniformly with respect to t in bounded intervals $[0, b)$.

Proof. Suppose first that $x_0 \in D(A)$. Set

$$u_n(t) = S^n(t)x_0, \qquad u(t) = W(t)x_0,$$

$$\tilde{u}_n(t) = u_n(t)+\frac{1}{h(0)}\int_0^t h(t-\tau)u_n(\tau)\,d\tau,$$

$$\tilde{u}(t) = u(t)+\frac{1}{h(0)}\int_0^t h(t-\tau)u(\tau)\,d\tau.$$

To prove (3.12) it suffices to show that

$$\lim_{n\to\infty} \tilde{u}_n(t) = \tilde{u}(t) \tag{3.13}$$

uniformly in t in bounded intervals $[0, b)$.

From [6] we have

$$\tilde{u}(t) = e^{-tA}x_0+\int_0^t e^{-(t-\tau)A}\left[\frac{\dot{h}(0)}{h(0)}u(\tau)+\frac{1}{h(0)}\int_0^\tau \ddot{h}(\tau-s)u(s)\,ds\right]d\tau. \tag{3.14}$$

Similarly,

$$\tilde{u}_n(t) = e^{-tA_n}x_0+\int_0^t e^{-(t-\tau)A_n}\left[\frac{\dot{h}(0)}{h(0)}u_n(\tau)+\frac{1}{h(0)}\int_0^\tau \ddot{h}(\tau-s)u_n(s)\,ds\right]d\tau. \tag{3.15}$$

From the definition of e^{-tA} (see [7]) as an integral of the form

$$\frac{1}{2\pi i}\int_C e^{\lambda t}(\lambda I+A)^{-1}\,d\lambda$$

and from the relation

$$\|[A_n-A]A^{-1}y_0\| \to 0 \quad \text{as } n\to\infty \text{ (for any } y_0 \in X) \tag{3.16}$$

we find that, for any continuous function $v(s)$,

$$\lim_{n\to\infty} \|[e^{-tA}-e^{-tA_n}]A^{-1}v(s)\| = 0 \tag{3.17}$$

uniformly with respect to t, s in bounded sets of $[0, \infty)$.

Subtracting (3.14) from (3.15) and using (3.17) with $v(t)=Ax_0$ and with $v(t)=Au(t)$, we get

$$\|\tilde{u}(t)-\tilde{u}_n(t)\| \leqq C(t)\int_0^t \|\tilde{u}(\tau)-\tilde{u}_n(\tau)\|\,d\tau+\varepsilon_n(t)$$

where $C(t)$ is bounded in bounded intervals $[0, b)$, and $\varepsilon_n(t) \to 0$, as $n\to\infty$, uniformly in t in bounded intervals $[0, b)$. The last inequality gives (3.13).

Having proved (3.12) for $x_0 \in D(A)$, we next notice that, in any bounded interval $0 \leqq t \leqq b$, $\|S^n(t)\| \leqq C$ where C is a constant independent of n, t. In fact, since (3.6)–(3.8) hold for $A=A_n$, $W=S^n$, the latter bound follows from the estimates on W obtained in [6, Chapter 1]. It follows that (3.12) holds for all $x_0 \in X$, uniformly in t in bounded intervals.

REMARK. A result similar to Theorem 3.2 holds also with respect to the more general integral equation (3.2).

DEFINITION. A complex-valued function $h(t)$ is said to belong to the class $\mathscr{K}'$ if

(i) $h(0)>0$ and $h(t)$ is absolutely continuous in $[0, \infty)$;

(ii) $\dot{h}(t) \in L^1(0, \infty)$.

If $h \in \mathscr{K}'$ then we can introduce the function

$$g(s) = h(0)+\dot{h}^\wedge(s) \quad \text{for } \operatorname{Re} s \geqq 0, \tag{3.18}$$

where $f^\wedge(s)$ indicates the Laplace transform of $f(t)$. Then $g(s)=sh^\wedge(s)$ if $\operatorname{Re} s>0$. It follows that $h^\wedge(s)$ can be defined by continuity for $\operatorname{Re} s \geqq 0$, $s \neq 0$. If $g(0)\neq 0$, then we let $h^\wedge(0)=\infty$, and introduce the set

$$\Delta \equiv \{-1/h^\wedge(s);\ \operatorname{Re} s \geqq 0\}. \tag{3.19}$$

As proved in [6, Chapter 2], if $h \in \mathscr{K}'$, $A \in \mathfrak{A}$, and if

$$g(s) \neq 0 \quad \text{for all } s \text{ with } \operatorname{Re} s \geqq 0, \tag{3.20}$$

$$\Delta \subset \rho(A), \tag{3.21}$$

then there exists a fundamental solution $S(t)$ of (1.1) in the sense defined in §1 (cf. (1.2)), and it belongs to $L^p(0, \infty; B(X))$ for any $p \geqq 2$.

Analogously to Theorem 3.2, we have

THEOREM 3.3. *Let $h \in \mathscr{K}'$, $A \in \mathfrak{A}$, and let* (3.20), (3.21) *hold. Then for any $p \geqq 2$, and for any $x_0 \in X$,*

$$\lim_{n\to\infty} \int_0^\infty \|S^n(t)x_0 - S(t)x_0\|^p \, dt = 0. \tag{3.22}$$

Proof. In [6, Chapter 2] it was proved that

$$S(t) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} S_\lambda(t) \, d\lambda \tag{3.23}$$

for an appropriate curve C lying in the resolvent set $\rho(A)$ of A, where $S_\lambda(t)$ is the inverse Laplace transform of the function

$$1/(s+\lambda g(s)). \tag{3.24}$$

One can easily verify that if $S_\lambda(t)$ is the solution of (2.4) then its Laplace transform coincides with the function (3.24). Hence, by the uniqueness of the inverse Laplace transform we conclude that the function $S_\lambda(t)$ occurring in (3.23) coincides with the solution of (2.4).

Using the definition of $S^n(t)$ in (3.11) and Cauchy's theorem, we have:

$$S^n(t) = \frac{1}{2\pi i} \int_C (\lambda I - A_n)^{-1} S_\lambda(t) \, dt. \tag{3.25}$$

Noting that

$$\|(\lambda I-A_n)^{-1}x_0-(\lambda I-A)^{-1}x_0\| \leqq \|(\lambda I-A_n)^{-1}\| \, \|(A_n-A)(\lambda I-A)^{-1}x_0\| \leqq \varepsilon_n/|\lambda|$$

where $\varepsilon_n \to 0$ if $n \to \infty$, we obtain from (3.23), (3.25):

$$\|S^n(t)x_0-S(t)x_0\| \leqq c\varepsilon_n \int_C |S_\lambda(t)| \frac{|d\lambda|}{|\lambda|}.$$

Since, by [6],

$$\left\{\int_0^\infty |S_\lambda(t)|^p \, dt\right\}^{1/p} \leqq \frac{c}{|\lambda|^{1/p}} \quad \text{if } p \geqq 2,$$

we obtain

$$\left\{\int_0^\infty \|S^n(t)x_0-S(t)x_0\|^p \, dt\right\}^{1/p} \leqq c\varepsilon_n \int_C \frac{|d\lambda|}{|\lambda|^{1+1/p}} \to 0$$

as $n \to \infty$. This proves (3.22).

4. **Monotonicity for A selfadjoint.** Let X be a Hilbert space and let A be a selfadjoint operator in X. We say that A is *strictly positive* if the number

$$\delta_A = \inf_{x \neq 0} \frac{(Ax, x)}{(x, x)}$$

is positive. The main result of the present section is the following:

THEOREM 4.1. *Let X be a Hilbert space and let A be a strictly positive selfadjoint operator, with the spectral decomposition of the identity $\{E_\lambda\}$. If $h \in \mathscr{H}$ then the operator $S(t)$ given by*

$$S(t)x_0 = \int_{\delta_A}^\infty S_\mu(t) \, dE_\mu x_0 \qquad (x_0 \in X) \tag{4.1}$$

is a fundamental solution of (1.1).

Note that (4.1) is formally obtained from (3.1) and the formula

$$(\lambda I-A)^{-1} = \int_{\delta_A}^\infty \frac{dE_\mu}{\lambda-\mu},$$

using the Cauchy formula.

Proof. Since $0 \leqq S_\mu(t) \leqq 1$, the integral in (4.1) exists and $\|S(t)x_0\| \leqq \|x_0\|$. $S(t)x_0$ is clearly continuous in t. Next, by Fubini's theorem,

$$\int_0^t h(t-\tau)S(\tau)x_0 \, d\tau = \int_{\delta_A}^\infty \left\{\int_0^t h(t-\tau)S_\mu(\tau) \, d\tau\right\} dE_\mu x_0.$$

Using (2.4) we find the expression on the right is equal to

$$\int_{\delta_A}^\infty \left\{\frac{1}{\mu}-\frac{S_\mu(t)}{\mu}\right\} dE_\mu x_0 = A^{-1}x_0-A^{-1}S(t)x_0,$$

where (4.1) has been used. We have thus proved that

$$\int_0^t h(t-\tau)S(\tau)x_0 \, d\tau$$

lies in $D(A)$ and that if we apply A to this integral we obtain $x_0 - S(t)x_0$. This completes the proof of (1.2).

From Theorem 4.1 and Lemma 2.5(i), (ii) we obtain

COROLLARY 1. *For any* $x_0 \in X$,

$$0 \leqq (S(t)x_0, x_0) \leqq \|x_0\|^2 \qquad (0 < t < \infty). \tag{4.2}$$

COROLLARY 2. *If* $h \in \mathscr{H}'$ *then, for any* $x_0 \in X$,

$$(S(t)x_0, x_0) \searrow \textit{ if } t \nearrow \qquad (0 < t < \infty). \tag{4.3}$$

If $S_\mu(t) \searrow$ when $t \nearrow$ $(\mu > 0)$, then we obtain from (2.4) the bound

$$S_\mu(t) \leqq \left[1 + \mu \int_0^t h(\tau)\, d\tau\right]^{-1}.$$

We conclude

COROLLARY 3. *If* $h \in \mathscr{H}'$ *then, for any* $x_0 \in X$,

$$(S(t)x_0, x_0) \leqq \int_{\delta_A}^{\infty} \left[1 + \mu \int_0^t h(\tau)\, d\tau\right]^{-1} d(E_\mu x_0, x_0). \tag{4.4}$$

We next have

COROLLARY 4. *If* $h \in \mathscr{H}_\infty$ *then, for any* $x_0 \in X$,

$$(-1)^n \frac{d^n}{dt^n}(S(t)x_0, x_0) \geqq 0 \qquad (n = 0, 1, 2, \ldots; 0 < t < \infty). \tag{4.5}$$

Proof. By Lemma 2.5(iii), the functions

$$T_m(t) = \int_{\delta_A}^{m+\delta_A} S_\mu(t)\, d(E_\mu x_0, x_0) \qquad (m = 1, 2, \ldots)$$

are completely monotonic in $(0, \infty)$. Since, for each $t > 0$, $T_m(t) \to (S(t)x_0, x_0)$ as $m \to \infty$, the assertion of the corollary follows from Lemma 2.3.

REMARK. Theorem 4.1 extends, with the same proof, to the case of the integral equation (3.2).

5. **Monotonicity for general** A.

DEFINITION. A closed linear operator A with a dense domain is said to belong to the class $\mathfrak{A}'$ if it satisfies the following properties:

(i) $(\lambda I - A)^{-1}$ exists for all complex λ, except for a sequence $\{\mu_k\}$ (which may be finite) of positive and increasing numbers with no finite limit.

(ii) At each μ_k, $(\lambda I - A)^{-1}$ has a pole, i.e.,

$$(\lambda I - A)^{-1} = \sum_{j=1}^{m_k} \frac{B_{k,j}}{(\lambda - \mu_k)^j} + B_{k,0}(\lambda) \tag{5.1}$$

where $B_{k,j}$ are bounded operators and $B_{k,0}(\lambda)$ is an analytic function (with values in $B(X)$) in a neighborhood of $\lambda=\mu_k$.

From (3.1), (5.1) and the residue theorem, we formally obtain the formula

$$S(t)x_0 = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\frac{1}{(j-1)!}\frac{\partial^{j-1}S_{\mu_k}(t)}{\partial\mu^{j-1}}B_{k,j}x_0. \tag{5.2}$$

(We consider here the case where $\{\mu_k\}$ is an infinite sequence; the modifications for the case of a finite sequence are trivial.)

To show that $S(t)$ is a fundamental solution (under certain assumptions), we introduce the operators $T_p(t)$ defined by

$$T_p(t) = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{1}{(j-1)!}\frac{\partial^{j-1}S_{\mu_k}(t)}{\partial\mu^{j-1}}B_{k,j}, \tag{5.3}$$

and set

$$\Delta_p(t) = -T_p(t)+I-A\int_0^t h(t-\tau)T_p(\tau)\,d\tau. \tag{5.4}$$

Applying $\lambda I-A=(\lambda-\mu_k)I+(\mu_k I-A)$ to both sides of (5.1), we obtain the relations:

$$\begin{aligned} AB_{k,m_k}-\mu_k B_{k,m_k} &= 0,\\ AB_{k,j}-\mu_k B_{k,j} &= B_{k,j+1} \qquad (1\leqq j\leqq m_k-1). \end{aligned} \tag{5.5}$$

Using these relations and (2.6), (2.4), we get

$$\begin{aligned} \Delta_p(t) &= -\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{1}{(j-1)!}\frac{\partial^{j-1}S_{\mu_k}(t)}{\partial\mu^{j-1}}B_{k,j}+I\\ &\quad -\int_0^t h(t-\tau)\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{1}{(j-1)!}\frac{\partial^{j-1}S_{\mu_k}(t)}{\partial\mu^{j-1}}\mu_k B_{k,j}\,d\tau\\ &\quad -\int_0^t h(t-\tau)\sum_{k=1}^{p}\sum_{j=1}^{m_k-1}\frac{1}{(j-1)!}\frac{\partial^{j-1}S_{\mu_k}(t)}{\partial\mu^{j-1}}B_{k,j+1}\,d\tau\\ &= I-\sum_{k=1}^{p}B_{k,1}. \end{aligned} \tag{5.6}$$

DEFINITION. We denote by X_A the set of all elements x_0 of X for which

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\frac{\|B_{k,j}x_0\|}{\mu_k^{j-1}} < \infty, \tag{5.7}$$

$$\lim_{p\to\infty}\sum_{k=1}^{p}B_{k,1}x_0 = x_0. \tag{5.8}$$

THEOREM 5.1. *Let $h\in\mathscr{H}$, $A\in\mathfrak{A}'$, $x_0\in X_A$. Then the limit*

$$T(t)x_0 \equiv \lim_{p\to\infty} T_p(t)x_0 \tag{5.9}$$

exists uniformly with respect to t, $0 \leqq t < \infty$, *and*

$$T(t)x_0 = x_0 - A\int_0^t h(t-\tau)T(\tau)x_0\,d\tau. \tag{5.10}$$

Proof. The uniform convergence of the sequence $\{T_p(t)x_0\}$ follows from Lemma 2.10 and the assumption (5.7). From (5.6) we have

$$T_p(t)x_0 = x_0 - A\int_0^t h(t-\tau)T_p(\tau)x_0\,d\tau - \Delta_p(t)x_0, \tag{5.11}$$

where

$$\Delta_p(t)x_0 = x_0 - \sum_{k=1}^{p} B_{k,1}x_0.$$

By (5.8), $\Delta_p(t)x_0 \to 0$ as $p \to \infty$. Hence, taking $p \to \infty$ in (5.11) and using the assumption that A is a closed operator, we conclude that the integral

$$\int_0^t h(t-\tau)T(\tau)x_0\,d\tau$$

belongs to the domain of A and that (5.10) holds.

COROLLARY 1. *If, in addition to the assumptions of Theorem* 5.1, *we assume that* $x_0 \in D(A)$ *and* $Ax_0 \in X_A$, *then*

$$T(t)x_0 = x_0 - \int_0^t h(t-\tau)AT(\tau)x_0\,d\tau \tag{5.12}$$

and $T(t)x_0$ *is continuous for* $t \geqq 0$.

Proof. We have

$$T_p(t)x_0 = x_0 - \int_0^t h(t-\tau)AT_p(\tau)x_0\,d\tau - \Delta_p(t)x_0, \tag{5.13}$$

and $AT_p(\tau)x_0 = T_p(\tau)(Ax_0)$. Since $Ax_0 \in X_A$, $T_p(\tau)(Ax_0) \to T(\tau)(Ax_0)$ as $p \to \infty$, uniformly with respect to τ. It follows that $T(\tau)x_0$ is in $D(A)$, and $AT(\tau)x_0 = T(\tau)(Ax_0)$. Now take $p \to \infty$ in (5.13).

COROLLARY 2. *Let* $h \in \mathscr{H} \cap \mathscr{K}$, $A \in \mathfrak{A} \cap \mathfrak{A}'$, $x_0 \in X_A$, $Ax_0 \in X_A$. *Then*

$$T(t)x_0 = W(t)x_0 \tag{5.14}$$

where $W(t)$ *is the fundamental solution of* (1.1) *occurring in Theorem* 3.2.

Indeed, both sides of (5.14) are solutions of (1.1). By the uniqueness assertion of [6, Theorem 1], they must coincide.

COROLLARY 3. *Let* $h \in \mathscr{H} \cap \mathscr{K}'$, $A \in \mathfrak{A} \cap \mathfrak{A}'$, $x_0 \in X_A$, *and assume also that* $t\dot{h}(t) \in L^1(0, \infty)$, *and that* (3.20), (3.21) *hold. Then*

$$T(t)x_0 = S(t)x_0 \tag{5.15}$$

where $S(t)$ *is the fundamental solution of* (1.1) *occurring in Theorem* 3.3.

This follows from (5.10) and the uniqueness assertion of [6, Theorem 10].

We recall [6] that if $h \in \mathscr{H}$ then the condition (3.20) is equivalent to the condition

$$h(\infty) > 0. \tag{5.16}$$

We shall now study monotonicity of the scalar function $f_0(T(t)x_0)$, where f_0 is any bounded linear functional in X.

THEOREM 5.2. *Let $h \in \mathscr{H}$, $A \in \mathfrak{A}'$, $x_0 \in X_A$, $f_0 \in X^*$. If*

$$(-1)^{j-1} f_0(B_{k,j}x_0) \geqq 0 \qquad (1 \leqq j \leqq m_k,\ 1 \leqq k < \infty) \tag{5.17}$$

then $f_0(T(t)x_0) \geqq 0$ for all $t \geqq 0$.

Proof. From (5.3) and Lemma 2.7 we immediately have that $f_0(T_p(t)x_0) \geqq 0$. Now take $p \to \infty$.

THEOREM 5.3. *Let $h \in \mathscr{H}'$, $A \in \mathfrak{A}'$, $x_0 \in X_A$, $f_0 \in X^*$. If (5.17) holds and, in addition,*

$$(-1)^{j-1} f_0(AB_{k,j}x_0) \geqq 0 \qquad (1 \leqq j \leqq m_k,\ 1 \leqq k < \infty) \tag{5.18}$$

then $f_0(T(t)x_0) \searrow$ if $t \nearrow$.

Note that $AB_{k,j}x_0$ is well defined for any $x_0 \in X$.

Before proving this theorem, we state and prove the following theorem.

THEOREM 5.4. *Let $h \in \mathscr{H}_\infty$, $A \in \mathfrak{A}'$, $x_0 \in X_A$, $f_0 \in X^*$. Set*

$$\tilde{B}^n_{k,j} = (-1)^{m_k - j - 1} \sum_{i=0}^{j} \binom{n}{i} \mu_k^{n-i} B_{k,m_k-j-i}. \tag{5.19}$$

If

$$f_0(\tilde{B}^n_{k,j}x_0) \geqq 0 \quad \text{for } 0 \leqq j \leqq m_k,\ 1 \leqq k < \infty,\ 0 \leqq n < \infty, \tag{5.20}$$

then

$$(-1)^n \frac{d^n}{dt^n} f_0(T(t)x_0) \geqq 0 \quad \text{for } 0 \leqq n < \infty,\ 0 < t < \infty. \tag{5.21}$$

Proof. Let Γ be a circle about μ_k such that all the points μ_j with $j \neq k$ lie outside Γ. By the residue theorem,

$$\begin{aligned} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} &= \frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} S_\lambda(t)\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma \lambda^n (\lambda I - A)^{-1} \left(\frac{S_\lambda(t)}{\lambda^n} \right) d\lambda \equiv J(t). \end{aligned} \tag{5.22}$$

Set

$$Q^n_\lambda(t) = (-1)^n \lambda^{-n} \frac{\partial^n S_\lambda(t)}{\partial t^n}$$

and write

$$\lambda^n = \sum_{i=0}^{n} \binom{n}{i} \mu_k^{n-i}(\lambda-\mu_k)^i.$$

From the residue theorem we then get

$$\begin{aligned}(-1)^n \frac{d^n J(t)}{dt^n} &= \frac{1}{2\pi i}\int_\Gamma \left[\sum_{i=0}^{n}\binom{n}{i}\mu_k^{n-i}(\lambda-\mu_k)^i\right]\left[\sum_{j=1}^{m_k}\frac{B_{k,j}}{(\lambda-\mu_k)^j}\right] Q_\lambda^n(t)\,d\lambda \\ &= \sum_{j=1}^{m_k} B_{k,j}\sum_{i=0}^{j-1}\binom{n}{i}\mu_k^{n-i}\frac{1}{(j-i-1)!}\frac{\partial^{j-i-1}Q_{\mu_k}^n(t)}{\partial\mu^{j-i-1}} \\ &= \sum_{q=0}^{m_k-1}\left\{B_{k,m_k-q}\binom{n}{0}\mu_k^n + B_{k,m_k-q+1}\binom{n}{1}\mu_k^{n-1} + \cdots + B_{k,m_k}\binom{n}{q}\mu_k^{n-q}\right\} \\ &\qquad\qquad \times \frac{1}{(m_k-q-1)!}\frac{\partial^{m_k-q-1}Q_{\mu_k}^n(t)}{\partial\mu^{m_k-q-1}}\end{aligned}$$

where

$$\binom{n}{i} = 0 \quad \text{if } i > n.$$

Thus, by (5.19) the last sum is equal to

$$\sum_{q=0}^{m_k-1}(-1)^{m_k-q-1}\frac{1}{(m_k-q-1)!}\frac{\partial^{m_k-q-1}Q_{\mu_k}^n(t)}{\partial\mu^{m_k-q-1}}\tilde{B}_{k,q}^n.$$

Hence, recalling (5.22) and (5.3), we have

$$(5.23)\quad (-1)^n\frac{d^n T_p(t)x_0}{dt^n} = \sum_{k=1}^{p}\sum_{q=0}^{m_k-1}(-1)^{m_k-q-1}\frac{1}{(m_k-q-1)!}\frac{\partial^{m_k-q-1}Q_{\mu_k}^n(t)}{\partial\mu^{m_k-q-1}}\tilde{B}_{k,q}^n x_0.$$

Using Lemma 2.9 and the assumption (5.20), we conclude that

$$(-1)^n\frac{d^n f_0(T_p(t)x_0)}{dt^n} \geqq 0 \quad \text{for } n = 0, 1, 2, \ldots; t > 0.$$

Since $T_p(t)x_0 \to T(t)x_0$ as $p \to \infty$, the assertion of the theorem follows from Lemma 2.3.

Using the notation Δ_η^m (following Lemma 2.3), we can state

COROLLARY. *If* (5.20) *is assumed to hold only for* $0 \leqq n \leqq n_0$, *then*

$$(5.24)\qquad (-1)^n\Delta_\eta^n f_0(T(t)x_0) \geqq 0 \quad \textit{for } 0 \leqq n \leqq n_0, 0 \leqq t < t+\eta < \infty.$$

Indeed, the proof of Theorem 5.4 shows that (5.24) holds with $T(t)x_0$ replaced by $T_p(t)x_0$. Since $T_p(t)x_0 \to T(t)x_0$ as $p \to \infty$, (5.22) follows.

REMARK. Let the assumptions of Corollary 2 to Theorem 5.1 hold and let $h \in C^{n_0}[0, \infty)$. Then, by [6] and (5.14), $T(t)x_0$ has n_0 continuous derivatives in $[0, \infty)$. Hence, (5.24) implies that

$$(5.25)\qquad (-1)^n(\partial^n f_0(T(t)x_0)/\partial t^n) \geqq 0 \quad \text{for } 0 \leqq n \leqq n_0, 0 \leqq t < \infty.$$

Proof of Theorem 5.3. We shall use the formula

$$(5.26)\qquad T_p(t)x_0 = \sum_{k=1}^{p}\sum_{q=0}^{m_k-1}(-1)^{m_k-q-1}\left[\frac{\partial^{m_k-q-1}}{\partial\mu^{m_k-q-1}}\left(\frac{S_\mu(t)}{\mu}\right)\right]_{\mu=\mu_k}\tilde{B}^1_{k,q}x_0$$

which one obtains by the same method that was used before to derive (5.23). Since

$$\tilde{B}^1_{k,q} = (-1)^{m_k-q-1}(\mu_k B_{k,m_k-q}+B_{k,m_k-q+1})\qquad (1 \leqq q \leqq m_k-1),$$

(5.5) shows that the inequalities (5.18) imply the inequalities (5.20) for $n=1$. Hence, (5.26) gives

$$f_0(T_p(t)x_0)\searrow \text{ if } t\nearrow.$$

Since $T_p(t)x_0 \to T(t)x_0$ as $p\to\infty$, the proof is complete.

REMARK. If X is a finite-dimensional Banach space, then any linear operator A whose eigenvalues are positive numbers is in $\mathfrak{A}'$. Furthermore, the series in (5.2) now consists of a finite number of terms. Hence (5.7) holds. (5.8) is also valid; in fact, it easily follows using the residue theorem. Thus $X_A=X$.

6. **Additional results.** In the previous two sections we have derived theorems which involved the functions $S_\lambda(t)$ for $\lambda>0$. A crucial step in the derivation of these theorems was the behavior of the function $S_\lambda(t)$ for positive values of the parameter λ. Since analogous results on the behavior of $S_\lambda(t)$ for λ complex are not available in the literature, we cannot extend, at present, the results of §§4, 5 to operators A with $\sigma(A)$ which is not contained in the real interval $0<\lambda<\infty$.

However, for some special functions $h(t)$, the behavior of $S_\lambda(t)$, for complex λ, is known with sufficient precision. We give here one example where $h(t)=t^{-\alpha}$ for some $0<\alpha<1$. Then $S_\lambda(t)=E_\beta(-\gamma\lambda t^\beta)$ where $\beta=1-\alpha$, $\gamma=\Gamma(\beta)$ and where $E_\beta(z)$ is the Mittag-Leffler function

$$E_\beta(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\beta+1)}.$$

From a well-known asymptotic formula for $E_\beta(z)$ (see [1, p. 207]) we find that

$$(6.1)\qquad \left|S_\lambda(t)-\frac{\gamma_0}{\lambda t^\beta}\right| \leqq \frac{c}{|\lambda|^2t^{2\beta}}\quad \text{if } |\arg\lambda t^\beta| < \frac{1+\alpha}{2}\pi,$$

provided $|\lambda t^\beta|\geqq c_0>0$; here $\gamma_0=(\Gamma(\alpha)\Gamma(1-\alpha))^{-1}$.

Let us assume that the resolvent set $\rho(A)$ of A contains the sector $|\arg\lambda|>(1+\alpha)\pi/2$ and that $\|(\lambda I-A)^{-1}\|\leqq c/(1+|\lambda|)$ for λ in this sector. We define $S(t)$ by (3.23) and choose C in $\rho(A)$ such that (6.1) holds for $\lambda\in C$. It follows that $S(t)$ is a bounded operator for each $t>0$. Furthermore, it varies continuously in t. One can also continue $S(t)$ analytically into a sector $|\arg\lambda|<\delta$ for some $\delta>0$.

Since $h(0)=\infty$, the results of [6] do not cover the present case of $h(t)=t^{-\alpha}$. There arises the question in what sense is $S(t)$ a fundamental solution.

So far we have only considered solutions of equations of the form (1.1) where x_0 is independent of t. But some of the results extend without difficulty to the equations

$$x(t) = k(t)x_0 - \int_0^t h(t-\tau)Ax(\tau)\,d\tau, \tag{6.2}$$

where $k(t)$ is a scalar function.

The solution is given by (see [6]):

$$x(t) = k(0)S(t)x_0 + \int_0^t \dot{k}(\tau)S(t-\tau)x_0\,d\tau \tag{6.3}$$

where $\dot{k}(\tau) = dk(\tau)/d\tau$. Hence, if X is a Hilbert space,

$$(x(t), x_0) = k(0)(S(t)x_0, x_0) + \int_0^t \dot{k}(\tau)(S(t-\tau)x_0, x_0)\,d\tau. \tag{6.4}$$

This relation combined with the results of §§4, 5 yields monotonicity properties for $(x(t), x_0)$. For example, if $k(0) \geqq 0$, $\dot{k}(\tau) \geqq 0$, then $(x(t), x_0) \geqq 0$.

Applications. If $h \in \mathscr{H}_\infty$ then we have proved several theorems to the effect that $(S(t)x_0, x_0)$ is completely monotonic in t. Since $(S(0)x_0, x_0) = (x_0, x_0) \neq 0$, we conclude that $(S(t)x_0, x_0) > 0$ for all $t > 0$. In particular, $S(t)x_0 \neq 0$ for all $t > 0$. Thus the solutions of (1.1) have the "weak backward uniqueness" property as defined in [5]. This fact is important in the study of optimal-control for trajectories $x(t)$ given by

$$x(t) = u(t) + \int_0^t h(t-\tau)Ax(\tau)\,d\tau \tag{6.5}$$

where $u(t)$ is the control function. It enables us to prove uniqueness of time-optimal controls (see [3], [5, p. 42]).

If A is selfadjoint and if $h \in \mathscr{H}'$ and h is strictly decreasing, then we can again assert that $(S(t)x_0, x_0) > 0$ for all $x_0 \neq 0$, $t > 0$. Indeed, otherwise we get, from (4.1), $S_\mu(t_0) = 0$ for some $\mu > 0$, $t_0 > 0$. But then, by Lemma 2.5, $S_\mu(t) = 0$ if $t > t_0$. Using (2.4) we then see that $S_\mu(t) < S_\mu(t_0)$ if $t > t_0$; a contradiction.

If $h \in \mathscr{H}'$, then we have proved several theorems to the effect that $(S(t)x_0, x_0) \searrow$ if $t \nearrow$. This can be used to answer some questions of controllability; for instance, to show that a point x_0 can be "steered," by a suitable control, to any given neighborhood of 0 (cf. [3]).

References

1. A. Erdélyi et al., "Higher transcendental functions," in *Bateman manuscript project*, Vol. III, McGraw-Hill, New York, 1955.

2. A. Friedman, *On integral equations of Volterra type*, J. Analyse Math. **11** (1963), 381–413.

3. ———, *Optimal control for hereditary processes*, Arch. Rational Mech. Anal. **15** (1964), 396–416.

4. A. Friedman, *Periodic behavior of solutions of Volterra integral equations*, J. Analyse Math. **15** (1965), 287–303.

5. ———, *Optimal control in Banach spaces*, J. Math. Anal. Appl. **19** (1967), 35–55.

6. A. Friedman and M. Shinbrot, *Volterra integral equations in Banach space*, Trans. Amer. Math. Soc. **126** (1967), 131–179.

7. T. Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246–274.

8. R. K. Miller, *On Volterra integral equations with non-negative integrable resolvents*, J. Math. Anal. Appl. **22** (1968), 319–340.

9. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946.

Northwestern University,
Evanston, Illinois