MONOTONICITY OF SOLUTIONS OF VOLterra
INTEGRAL EQUATIONS IN BANACH SPACE (1)

BY

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1. Introduction. We shall consider Volterra integral equations

\[ x(t) = x_0 - \int_0^t h(t-\tau)Ax(\tau) \, d\tau \]

where \( x_0 \) and \( x(t) \) belong to a complex Banach space \( X \), \( h(t) \) is a complex-valued function and \( A \) is an operator in \( X \), generally unbounded. We denote by \( B(X) \) the Banach space of bounded linear operators in \( X \), and by \( I \) the identity operator in \( X \). An operator-valued function \( S(t) \), which belongs to \( L^1(0, b; B(X)) \) for any \( b > 0 \), is called a fundamental solution of (1.1) if

\[ S(0) = I - A \int_0^t h(t-\tau)S(\tau) \, d\tau \]

for almost all \( t \). In this definition it is assumed, of course, that the integral on the right-hand side of (1.2) is in the domain of \( A \).

In a recent paper [6], Friedman and Shinbrot have studied the equation (1.1) even in the more general case where \( x_0 \) and \( A \) depend on \( t \) and \( \tau \), respectively. They proved theorems of existence, uniqueness, differentiability and asymptotic behavior of solutions. They also constructed fundamental solutions and derived asymptotic bounds for them. We recall [6] that if \( x_0 \) is in the domain of \( A^\mu \), for some \( \mu > 0 \), then the solution of (1.1) is given by \( S(t)x_0 \).

The purpose of the present paper is to derive monotonicity theorems for solutions of (1.1). We shall generalize some of the monotonicity theorems of Friedman [2] (see also [4]) from the case \( X = \mathbb{R}^1 \) (\( \mathbb{R}^1 \) the one-dimensional Euclidean space) to the case where \( X \) is any Banach space.

In §2 we give some auxiliary results. These results are concerned with Volterra equations in one-dimension (i.e., \( X = \mathbb{R}^1 \)). In particular, we study the behavior of the solutions with respect to a certain parameter.

In §3 we give an integral formula for \( S(t) \) in case \( A \) is a bounded operator. For \( A \) unbounded, we construct a fundamental solution as a limit of fundamental solutions corresponding to the bounded operators \( A(I+A/n)^{-1} \). We prove that the fundamental solution coincides with the fundamental solution of [6, Chapter 1]

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or of [6, Chapter 2] provided the assumptions of [6, Chapter 1] or of [6, Chapter 2], respectively, are satisfied.

In §4 we derive a formula for $S(t)$ in case $A$ is selfadjoint. As a by-product, we obtain monotonicity theorems for $(S(t)x_0, x_0)$.

In §5 we drop the assumption that $A$ is selfadjoint. Instead we assume that the resolvent $(\lambda I - A)^{-1}$ exists for all $\lambda$ except for a sequence $\{\mu_k\}$ of poles, and $0 < \mu_1 < \mu_2 < \cdots, \mu_k \to \infty$ as $k \to \infty$. We obtain a formula for the solution $T(t)x_0$ of (1.1) and then derive monotonicity theorems for $f_0(T(t)x_0)$; here $f_0$ is a bounded linear functional in $X$.

In §6 we give some additional results obtainable with the methods of the previous sections, and some applications to control theory.

2. Auxiliary lemmas. A real-valued function $f(t)$ is said to be completely monotonic in an interval $[a, b)$ if $f \in C^\infty[a, b)$ and for all $n \geq 0$,

$$(-1)^n \frac{d^n f(t)}{dt^n} \geq 0 \quad \text{for all } t \in [a, b).$$

Similarly one defines complete monotonicity in intervals $(a, b), [a, b].$ We recall the following results (see [9]):

**Lemma 2.1.** If $f(t)$ is completely monotonic in an interval $(a, b)$ then $f(t)$ is analytic in $(a, b)$.

**Lemma 2.2.** A function $f(t)$ is completely monotonic in the interval $[0, \infty)$ if and only if

$$f(t) = \int_0^\infty e^{-\lambda t} d\phi(\lambda)$$

where $\phi(\lambda)$ is a bounded nondecreasing function.

**Lemma 2.3.** If $\{f_m(t)\}$ is a sequence of completely monotonic functions in $(a, b)$ and if $f(t)$ is a continuous function in $(a, b)$ such that, for each $t \in (a, b), f_m(t) \to f(t)$ as $m \to \infty$, then $f(t)$ is completely monotonic in $(a, b)$.

Setting $\Delta^n f(t) = f(t + \eta) - f(t),$

$$\Delta^{n+1} f(t) = \Delta^n (\Delta_n f(t)),$$

the assertion of the last lemma is a consequence of the fact (see [9]) that $f(t)$ is completely monotonic if, for any integer $m \geq 1$ and $\eta > 0$,

$$(-1)^m \Delta^n f(t) \geq 0 \quad \text{for } a < t < b - m\eta.$$

In view of Lemma 2.1, if $f(t)$ is a completely monotonic function in $(a, b)$ which does not vanish identically, then $f(t) > 0$ for all $t \in (a, b)$.

We shall need the following result of Miller [8]:

**Lemma 2.4.** If $f(t)$ is a nonzero completely monotonic function in $[0, \infty)$, then $\log f(t)$ is a convex function.
Proof. We have to show that

$$g(t) = f(t)f''(t) - (f'(t))^2 \geq 0.$$  

Using (2.1) we find that

$$g(t) = \int_0^\infty \int_0^\infty \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} \, d\phi(\lambda) \, d\phi(\mu).$$

Next,

$$\int_0^\infty \int_\mu^\infty \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} \, d\phi(\lambda) \, d\phi(\mu) = \int_0^\infty \int_0^\lambda \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} \, d\phi(\lambda) \, d\phi(\mu)$$

$$= \int_0^\infty \int_0^\mu \mu(\mu - \lambda)e^{-(\lambda + \mu)t} \, d\phi(\lambda) \, d\phi(\mu).$$

Therefore

$$g(t) = \int_0^\infty \int_0^\infty (\cdots) + \int_0^\infty \int_\mu^\infty (\cdots) = \int_0^\infty \int_0^\infty (\lambda - \mu)^2e^{-(\lambda + \mu)t} \, d\phi(\lambda) \, d\phi(\mu) \geq 0.$$

Definitions. A function $h(t)$ which belongs to $C(0, \infty)$ and to $L^1(0, 1)$ is said to belong to the class $\mathcal{H}$, if $h(t) \geq 0$, $h(t) \neq 0$, and $h(t)$ is monotone nonincreasing in $(0, \infty)$. If $h \in \mathcal{H}$ and if $\log h(t)$ is a convex function in the interval where $h(t) > 0$, then we say that $h$ belongs to the class $\mathcal{H}'$. Finally, we say that $h \in \mathcal{H}_\alpha$ if $h(t)$ is a nonzero completely monotonic function in $(0, \infty)$ and if $h \in L^1(0, 1)$.

From Lemma 2.4 (applied to $h(t + \epsilon)$, for any $\epsilon > 0$) it follows that if $h \in \mathcal{H}_\alpha$ then $h \in \mathcal{H}'$.

In the following lemma we have collected some results proved in Friedman [2].

**Lemma 2.5.** Consider the integral equation

$$(2.2) \quad x(t) = 1 - \int_0^t h(t - \tau)x(\tau) \, d\tau \quad (0 < t < \infty).$$

(i) If $h \in \mathcal{H}$, then $0 \leq x(t) \leq 1$.

(ii) If $h \in \mathcal{H}'$, then $x(t)$ is monotone nonincreasing.

(iii) If $h \in \mathcal{H}_\alpha$, then $x(t)$ is in $\mathcal{H}_\alpha$.

From [2, Corollary 4, p. 387] we deduce

**Lemma 2.6.** Consider the equation

$$(2.3) \quad x(t) = \int_0^t h(t - \sigma)p(\sigma) \, d\sigma - \lambda \int_0^t h(t - \tau)x(\tau) \, d\tau \quad (0 < t < \infty)$$

where $p(\sigma)$ is a continuous nonnegative function, and $\lambda$ is a positive constant. If $h \in \mathcal{H}'$ then $x(t) \geq 0$.

We shall consider now the equation

$$(2.4) \quad S_\lambda(t) = 1 - \lambda \int_0^t h(t - \tau)S_\lambda(\tau) \, d\tau \quad (0 \leq t < \infty)$$
where $\lambda$ is a complex parameter. By a standard argument one shows that if $h(t)$ is in $C(0, \infty) \cap L^1(0, 1)$ then, for each $\lambda$, there exists a unique solution $S_\lambda(t)$ of (2.4). Furthermore, $S_\lambda(t)$ is continuous in $(t, \lambda)$ $(t \geq 0, \lambda$ complex) and analytic in $\lambda$, for each $t \geq 0$. If $h \in C^n[0, \infty)$ then $\frac{\partial^n S_\lambda(t)}{\partial t^n}$ is continuous in $(t, \lambda)$ (for $t \geq 0, \lambda$ complex) and analytic in $\lambda$, for each $t \geq 0$.

**Lemma 2.7.** If $h \in \mathcal{E}$ then, for $n = 0, 1, 2, \ldots$,

$$(-1)^n (\frac{\partial^n S_\lambda(t)}{\partial \lambda^n}) \geq 0 \text{ if } 0 < \lambda < \infty, 0 < t < \infty. \tag{2.5}$$

**Proof.** The inequality (2.5) for $n = 0$ follows from Lemma 2.5(i). We proceed by induction. We assume that (2.5) holds and prove the same inequality when $n$ is replaced by $n + 1$. Differentiating (2.4) $n + 1$ times with respect to $\lambda$ we get

$$\frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(t) = -(n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_\lambda(\tau) \, d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(\tau) \, d\tau. \tag{2.6}$$

It follows that the function $x(t) = (-1)^{n+1} \frac{\partial^{n+1} S_\lambda(t)}{\partial \lambda^{n+1}}$ satisfies the equation (2.3) with

$$p(\sigma) = (n+1)(-1)^{n+1} \frac{\partial^n S_\lambda(\sigma)}{\partial \lambda^n}.$$ 

By the inductive assumption, $p(\sigma) \geq 0$. Hence we can apply Lemma 2.6 and conclude that $x(t) \geq 0$, i.e., (2.5) holds with $n$ replaced by $n+1$.

**Lemma 2.8.** If $h \in \mathcal{E}'$ then, for $n = 0, 1, 2, \ldots, \lambda > 0$,

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_\lambda(t)}{\lambda} \right] \geq 0 \text{ if } t \neq \lambda. \tag{2.7}$$

**Proof.** We may assume that $h \in C^1[0, \infty)$. Indeed, otherwise we approximate $h(t)$ by a sequence of functions $\{h_m(t)\}$ as follows: $h_m \in \mathcal{E}'$, $h_m(t) \to h(t)$ uniformly on compact subsets of $(0, \infty)$ and

$$\int_0^t |h(t) - h_m(t)| \, dt \to 0.$$

If we know already that the assertion of the lemma holds for the solution $S_{\lambda, m}(t)$ corresponding to $h_m$, then (2.7) is also true since, for any $n \geq 0$,

$$\frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_{\lambda, m}(t)}{\lambda} \right] \to \frac{\partial^n}{\partial \lambda^n} \left[ \frac{S_{\lambda}(t)}{\lambda} \right] \text{ for each } t.$$

Assuming $h$ to be in $C^1[0, \infty)$, it follows that $\partial S_\lambda(t)/\partial t$ exists and satisfies:

$$\frac{\partial S_\lambda(t)}{\partial t} = -\lambda h(t) - \lambda \int_0^t h(t-\tau) \frac{\partial S_\lambda(\tau)}{\partial \tau} \, d\tau. \tag{2.8}$$

The assertion (2.7) is equivalent to the following inequality:

$$(-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial}{\partial t} S_\lambda(t) \right] \geq 0. \tag{2.9}$$
For \( n = 0 \), this inequality follows from Lemma 2.5(ii). We now proceed by induction on \( n \). To pass from \( n \) to \( n + 1 \), we divide both sides of (2.8) by \( \lambda \) and then differentiate both sides \( n + 1 \) times with respect to \( \lambda \). We get

\[
\frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[ \frac{1}{\lambda} \frac{\partial S_{\lambda}(t)}{\partial t} \right] = -(n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial S_{\lambda}(\tau)}{\partial \tau} \right] d\tau
\]

\[
- \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[ \frac{1}{\lambda} \frac{\partial S_{\lambda}(\tau)}{\partial \tau} \right] d\tau.
\]

We can now apply Lemma 2.6 with

\[
\phi^{(k)}(0) = (n+1)(-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial S_{\lambda}(\sigma)}{\partial \sigma} \right].
\]

**Lemma 2.9.** If \( h \in C^\infty \), then, for \( n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots, \lambda > 0 \),

\[
(2.10) \quad (-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[ \frac{1}{\lambda} \frac{\partial^n}{\partial t^n} S_{\lambda}(t) \right] \geq 0 \quad \text{for } t > 0.
\]

**Proof.** Suppose first that \( h(t) \) is in \( C^\infty[0, \infty) \). Then all the derivatives occurring in (2.10) exist for \( t \geq 0 \). We shall establish (2.10) by induction on \( n \). For \( n = 0 \), (2.10) follows from Lemma 2.7. We now assume that (2.10) holds for all \( m \geq 0 \) and \( 0 \leq n \leq k \). We shall prove (2.10) for all \( m \geq 0 \) and \( n = k + 1 \). Differentiating both sides of (2.4) \( k + 1 \) times with respect to \( t \), we get

\[
\frac{\partial^{k+1} S_{\lambda}(t)}{\partial t^{k+1}} = -\lambda h^{(k)}(t) S_{\lambda}(0) - \lambda h^{(k-1)}(t) \frac{\partial S_{\lambda}(0)}{\partial t} - \cdots - \lambda h(t) \frac{\partial^k S_{\lambda}(0)}{\partial t^k}
\]

\[
- \lambda \int_0^t h(t-\tau) \frac{\partial^{k+1} S_{\lambda}(\tau)}{\partial \lambda^{k+1}} d\tau.
\]

Setting

\[
T_{\lambda}(t) = (-1)^{k+1} \frac{1}{\lambda^{k+1}} \frac{\partial^{k+1}}{\partial t^{k+1}} S_{\lambda}(t),
\]

we get

\[
T_{\lambda}(t) = \sum_{t=0}^k (-1)^{k-1} h^{(k-1)}(t) \frac{\partial^t S_{\lambda}(0)}{\partial t^t}
\]

\[
- \lambda \int_0^t h(t-\tau) T_{\lambda}(\tau) d\tau.
\]

We have to prove that, for any \( m \geq 0 \),

\[
(2.12) \quad (-1)^m (\partial^m T_{\lambda}(t)/\partial \lambda_m) \geq 0.
\]

For \( m = 0 \) this follows from the definition of \( T_{\lambda}(t) \) and Lemma 2.5(iii). We now proceed by induction on \( m \).

To pass from \( m \) to \( m + 1 \), we differentiate (2.11) \( m + 1 \) times with respect to \( \lambda \).
We find
\[
\frac{\partial^{m+1} T_\lambda(t)}{\partial \lambda^{m+1}} = g(t) - (m+1) \int_0^t h(t-\tau) \frac{\partial^m T_\lambda(\tau)}{\partial \lambda^m} d\tau
\]
(2.13)
\[-\lambda \int_0^t h(t-\tau) \frac{\partial^{m+1} T_\lambda(\tau)}{\partial \lambda^{m+1}} d\tau,
\]
where
\[
g(t) = \sum_{i=0}^k g_i(t),
\]
(2.14)
\[g_i(t) = (-1)^{k-i} h^{k-i}(t) \sum_{s=0}^{m+1} \binom{m+1}{s} \frac{1}{\lambda^{k-s}} \cdot \frac{\partial^{m+1-s}}{\partial \lambda^{m+1-s}} \left[ \frac{(-1)^i \partial^i}{\partial t^i} S_\lambda(0) \right].
\]

Set
\[
g_{k+1}(t) = -(m+1) \int_0^t h(t-\tau) \frac{\partial^m T_\lambda(\tau)}{\partial \lambda^m} d\tau
\]
(2.15)
and denote by \(x_i(0) (0 \leq i \leq k+1)\) the solution of the equation
\[
x_i(t) = (-1)^{m+1} g_i(t) - \lambda \int_0^t h(t-\tau)x_i(\tau) d\tau.
\]
(2.16)
It is clear that
\[
(-1)^{m+1} \frac{\partial^{m+1} T_\lambda(t)}{\partial \lambda^{m+1}} = \sum_{i=0}^{k+1} x_i(t).
\]

Hence, it suffices to show that \(x_i(0) \geq 0\) for \(0 \leq i \leq k+1\).

The inequality \(x_{k+1}(0) \geq 0\) follows by applying Lemma 2.6 with
\[
p(\sigma) = (-1)^m (m+1) \frac{\partial^m T_\lambda(\sigma)}{\partial \lambda^m};
\]

note that by the inductive assumption, \(p(\sigma) \geq 0\).

From (2.14) and the inductive assumption we easily see that
\[
(-1)^{m+1} g_i(t) = \gamma_i(-1)^{k-i} h^{k-i}(t) \quad (0 \leq i \leq k)
\]
where \(\gamma_i = \gamma_i(\lambda)\) is nonnegative. From Theorem 1 and its Corollary 3 in [2] we then have the following: If
\[
h'(a) \leq \frac{h'(b)}{h(b)} \quad (0 < a < b < \infty)
\]
(2.17)
then \(x_i(t) \geq 0\). Thus, it remains to prove (2.17). We assume here that \(h^{k-i}(b) > 0\) for all \(b > 0\); if \(h^{k-i}(b) = 0\) for some \(b > 0\) then \(h^{k-i}(t) \equiv 0\) and \(x_i(t) \equiv 0\).

Since \(h \in \mathcal{H}_\infty\), Lemma 2.4 implies that \(h'(t)/h(t) \not\geq f\). Hence (2.17) is a consequence of
\[
h'(b) \leq \frac{h'(b)}{h(b)} \quad (0 < b < \infty).
\]
(2.18)
Now, the function \((-1)^j h^{(j)}(t)\) is completely monotonic. If we apply Lemma 2.4 to this function, we obtain
\[
h^{(j+1)}(b)/h^{(j)}(b) \leq h^{(j+2)}(b)/h^{(j+1)}(b).
\]
Applying this inequality for \(j=0, 1, \ldots, k-i-1\), we get (2.18).

We have proved Lemma 2.9 assuming that \(h(t)\) is in \(C^\infty[0, \infty)\). Consider now the general case, where we merely assume that \(h \in \mathcal{H}_\infty\). Then we can apply the previous result to the solution \(S_{h, e}(t)\) of (2.4) with \(h(t)\) replaced by \(h(t+\varepsilon), \varepsilon > 0\). Since, for each \(t > 0, m \geq 0\),
\[
\partial^m S_{h, e}(t)/\partial \lambda^m \to \partial^m S_h(t)/\partial \lambda^m,
\]
the assertion of the lemma for \(S_h(t)\) follows upon applying Lemma 2.3.

The last lemma of this section is the following:

**Lemma 2.10.** Let \(h \in \mathcal{H}\). Then, for \(n=0, 1, 2, \ldots\),
\[
(2.19) \quad (-1)^n (\partial^n S_h(t)/\partial \lambda^n) \leq n!/\lambda^n \quad (0 < \lambda < \infty, 0 < t < \infty).
\]

**Proof.** From Lemma 2.7 and (2.6) we obtain
\[
(2.20) \quad (-1)^{n+1} \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_h(\tau) \, d\tau \leq (-1)^n \frac{n+1}{\lambda} \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_h(\tau) \, d\tau.
\]
Applying this relation successively, we find that
\[
(2.21) \quad (-1)^m \int_0^t h(t-\tau) \frac{\partial^m}{\partial \lambda^m} S_h(\tau) \, d\tau \leq \frac{m!}{\lambda^{m+1}} \quad (m = 0, 1, 2, \ldots).
\]
Hence from (2.6), with \(n+1 = m\), we get
\[
(2.22) \quad (-1)^m \frac{\partial^m}{\partial \lambda^m} S_h(t) \leq (-1)^{m-1} m \int_0^t h(t-\tau) \frac{\partial^{m-1}}{\partial \lambda^{m-1}} S_h(\tau) \, d\tau \leq \frac{m!}{\lambda^m}
\]
for \(m=1, 2, \ldots\).

3. **Integral formula for \(S(t)\).** Let \(X\) be a Banach space. We denote by \(\sigma(A)\) the spectrum of an operator \(A\).

**Theorem 3.1.** Let \(A\) be a bounded operator and let \(\Gamma\) be any continuously differentiable closed Jordan curve containing \(\sigma(A)\) in its interior. Let \(h(t)\) be any function in \(C(0, \infty) \cap L^1(0, 1)\). Then the operator-valued function
\[
(3.1) \quad S(t) \equiv \frac{1}{2\pi i} \int_\Gamma (A-I)^{-1} S_h(t) \, d\lambda
\]
is the unique fundamental solution of (1.1).

The orientation of \(\Gamma\), in (3.1), is taken counterclockwise.

**Proof.** The uniqueness of the fundamental solution follows by standard arguments. It remains to verify (1.2). We have
\[ T \equiv I - A \int_0^t h(t - \tau) S(\tau) \, d\tau \]
\[ = I - \int_0^t h(t - \tau) \left\{ \frac{1}{2\pi i} \int_{\Gamma} \lambda (\lambda I - A)^{-1} S_\lambda(\tau) \, d\lambda \right\} \, d\tau. \]

Changing the order of integration and using (2.4), we get
\[ T = I + \frac{1}{2\pi i} \int_{\Gamma} \left[ (\lambda I - A)^{-1} - \frac{I}{\lambda} \right] \left[ S_\lambda(t) - 1 \right] \, d\lambda \]
\[ = I + S(t) - \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} \, d\lambda + \frac{I}{2\pi i} \int_{\Gamma} \frac{1 - S_\lambda(t)}{\lambda} \, d\lambda. \]

By Cauchy’s theorem we easily find that
\[ \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} \, d\lambda = I. \]

Also,
\[ \int_{\Gamma} \frac{1 - S_\lambda(t)}{\lambda} \, d\lambda = \int_{\Gamma} \left\{ \int_0^t h(t - \tau) S_\lambda(\tau) \, d\tau \right\} \, d\lambda \]
\[ = \int_0^t h(t - \tau) \left\{ \int_{\Gamma} S_\lambda(\tau) \, d\lambda \right\} \, d\tau = 0. \]

We obtain that \( T = S(t) \). This proves (1.2).

Theorem 3.1 can easily be extended to more general integral equations. For example, we shall construct a solution of
\[ (3.2) \quad S(t, s) = I - \int_s^t h(t - \tau, \tau) A S(\tau, s) \, d\tau. \]

Denote by \( S_\lambda(t, s) \) the solution of
\[ (3.3) \quad S_\lambda(t, s) = 1 - \lambda \int_s^t h(t - \tau, \tau) S_\lambda(\tau, s) \, d\tau. \]

Then we have

**Theorem 3.1'.** Let \( A, \Gamma \) be as in Theorem 3.1 and let \( h(t, \tau) \) be a continuous function for \( t \geq 0, \tau \geq 0 \). Then the unique solution of (3.2) is given by
\[ (3.4) \quad S(t, \tau) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_\lambda(t, \tau) \, d\lambda. \]

The proof is similar to the proof of Theorem 3.1.

We next consider the case where \( A \) is not necessarily a bounded operator.

**Definition.** A linear operator \( A \) in \( X \) is said to belong to the class \( \mathfrak{A} \) if
(i) \( A \) is closed and densely defined;
(ii) \( \sigma(A) \subseteq \{ \lambda; |\arg \lambda| \leq \pi/2 - \varepsilon, \Re \lambda \geq \lambda_0 \} \) for some \( \varepsilon > 0, \lambda_0 > 0 \);
(iii) \( \| (\lambda I - A)^{-1} \| \leq c/|\lambda| \) if \( |\arg \lambda| > \pi/2 - \varepsilon \).
Definition. A complex-valued function \( h(t) \) is said to belong to the class \( \mathcal{X} \) if

\( \text{(i)} \ h(0) > 0; \)

\( \text{(ii)} \ h \in C^1[0, \infty) \) and \( h(t) \) is absolutely continuous;

\( \text{(iii)} \) for any \( b > 0 \), \( \tilde{h}(t) \) is in \( L^p(0, b) \) for some \( p > 1 \).

In [6, Chapter 1] it was proved that if \( A \in \mathcal{A} \) and \( h \in \mathcal{X} \), then there exists a fundamental solution \( W(t) \) of (1.1), in a sense different than (1.2). Thus, \( W(t) \) satisfies the equation

\[ W(t) = I - \int_0^t h(t - \tau) A W(\tau) \, d\tau \]

in the following sense

\[ \tilde{W}(t) = e^{-tA} + \int_0^t e^{-(t-\tau)A} F(W; \tau) \, d\tau \]

where \( e^{-tA} \) is the analytic semigroup of \( -A \), and

\[ W(t) = W(t) + \int_0^t \frac{h(t - \tau)}{h(0)} W(\tau) \, d\tau, \]

\[ F(W; \tau) = \frac{h(0)}{h(0)} W(t) + \frac{1}{h(0)} \int_0^t h(t - \tau) W(\tau) \, d\tau. \]

Denoting by \( D(A^\mu) \) the domain of \( A^\mu \) (see [7] for the definition of \( A^\mu \)), we have the following result: If \( x_0 \in D(A^\mu) \) for some \( \mu > 0 \), then \( W(t)x_0 \) is the unique solution of (1.1). (The solutions of (1.1) are assumed to be such that \( \|Ax(\tau)\| \) is integrable in every bounded interval \((0, b)\).)

We introduce the operators

\[ A_n = A(I + A/n)^{-1}. \]

One easily verifies that \( \|A_n\| \leq Cn \) and

\[ (\lambda I - A_n)^{-1} = \frac{1}{\lambda - n} I + \frac{n^2}{(\lambda - n)^2} \left[ \frac{n}{\lambda} I - A \right]^{-1}. \]

Denote by \( \Gamma_n \) a continuously differentiable closed Jordan curve which contains \( \sigma(A_n) \), and set

\[ S^n(t) = \int_{\Gamma_n} (\lambda I - A_n)^{-1} S_n(t) \, d\lambda. \]

Theorem 3.2. Let \( h \in \mathcal{X}, A \in \mathcal{A} \). Then, for any \( x_0 \in X \),

\[ \lim_{n \to \infty} S^n(t)x_0 = W(t)x_0 \]

uniformly with respect to \( t \) in bounded intervals \([0, b)\).
Proof. Suppose first that $x_0 \in D(A)$. Set

$$u_n(t) = S^n(t)x_0, \quad u(t) = W(t)x_0,$$

$$\bar{u}_n(t) = u_n(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u_n(\tau) \, d\tau,$$

$$\bar{u}(t) = u(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u(\tau) \, d\tau.$$

To prove (3.12) it suffices to show that

$$\lim_{n \to \infty} \bar{u}_n(t) = \bar{u}(t)$$

uniformly in $t$ in bounded intervals $[0, b)$.

From [6] we have

$$\bar{u}(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} \left[ \frac{h(0)}{h(0)} u(s) + \frac{1}{h(0)} \int_0^s h(s-s')u(s') \, ds' \right] ds.$$

Similarly,

$$\bar{u}_n(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} \left[ \frac{h(0)}{h(0)} u_n(s) + \frac{1}{h(0)} \int_0^s h(s-s')u_n(s') \, ds' \right] ds.$$

From the definition of $e^{-tA}$ (see [7]) as an integral of the form

$$\frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda I + A)^{-1} \, d\lambda$$

and from the relation

$$\| [A_n - A] A^{-1} y_0 \| \to 0 \quad \text{as} \quad n \to \infty \quad \text{(for any} \quad y_0 \in X)$$

we find that, for any continuous function $v(s)$,

$$\lim_{n \to \infty} \| e^{-tA} - e^{-tA_n} A^{-1} v(s) \| = 0$$

uniformly with respect to $t, s$ in bounded sets of $[0, \infty)$.

Subtracting (3.14) from (3.15) and using (3.17) with $v(t) = Ax_0$ and with $v(t) = Au(t)$, we get

$$\| \bar{u}(t) - \bar{u}_n(t) \| \leq C(t) \int_0^t \| \bar{u}(\tau) - \bar{u}_n(\tau) \| \, d\tau + \varepsilon_n(t)$$

where $C(t)$ is bounded in bounded intervals $[0, b)$, and $\varepsilon_n(t) \to 0$, as $n \to \infty$, uniformly in $t$ in bounded intervals $[0, b)$. The last inequality gives (3.13).

Having proved (3.12) for $x_0 \in D(A)$, we next notice that, in any bounded interval $0 \leq t \leq b$, $\| S^n(t) \| \leq C$ where $C$ is a constant independent of $n, t$. In fact, since

(3.6)–(3.8) hold for $A = A_n$, $W = S^n$, the latter bound follows from the estimates on $W$ obtained in [6, Chapter 1]. It follows that (3.12) holds for all $x_0 \in X$, uniformly in $t$ in bounded intervals.
Remark. A result similar to Theorem 3.2 holds also with respect to the more general integral equation (3.2).

Definition. A complex-valued function $h(t)$ is said to belong to the class $\mathscr{F}$ if

(i) $h(0) > 0$ and $h(t)$ is absolutely continuous in $[0, \infty)$;

(ii) $h(t) \in L^1(0, \infty)$.

If $h \in \mathscr{F}$ then we can introduce the function

$$g(s) = h(0) + h^\prime(s) \text{ for Re } s \geq 0,$$

where $f^\prime(s)$ indicates the Laplace transform of $f(t)$. Then $g(s) = sh^\prime(s)$ if Re $s > 0$.

It follows that $h^\prime(s)$ can be defined by continuity for Re $s \geq 0$, $s \neq 0$. If $g(0) \neq 0$, then we let $h^\prime(0) = \infty$, and introduce the set

$$\Delta \equiv \{-1/h^\prime(s); \text{ Re } s \geq 0\}.$$

As proved in [6, Chapter 2], if $h \in \mathscr{F}$, $A \in \mathfrak{A}$, and if

$$g(s) \neq 0 \text{ for all } s \text{ with Re } s \geq 0,$$

then there exists a fundamental solution $S(t)$ of (1.1) in the sense defined in §1 (cf. (1.2)), and it belongs to $L^p(0, \infty; B(X))$ for any $p \geq 2$.

Analogously to Theorem 3.2, we have

**Theorem 3.3.** Let $h \in \mathscr{F}$, $A \in \mathfrak{A}$, and let (3.20), (3.21) hold. Then for any $p \geq 2$, and for any $x_0 \in X$,

$$\lim_{n \to \infty} \int_0^\infty \|S^n(t)x_0 - S(t)x_0\|^p dt = 0.$$

**Proof.** In [6, Chapter 2] it was proved that

$$S(t) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1}S_\lambda(t) d\lambda$$

for an appropriate curve $C$ lying in the resolvent set $\rho(A)$ of $A$, where $S_\lambda(t)$ is the inverse Laplace transform of the function

$$1/(s + \lambda g(s)).$$

One can easily verify that if $S_\lambda(t)$ is the solution of (2.4) then its Laplace transform coincides with the function (3.24). Hence, by the uniqueness of the inverse Laplace transform we conclude that the function $S_\lambda(t)$ occurring in (3.23) coincides with the solution of (2.4).

Using the definition of $S^n(t)$ in (3.11) and Cauchy's theorem, we have:

$$S^n(t) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1}S_\lambda(t) dt.$$
Noting that
\[
\| (\lambda I - A_n)^{-1} x_0 - (\lambda I - A)^{-1} x_0 \| \leq \| (\lambda I - A_n)^{-1} \| \| (A_n - A)(\lambda I - A)^{-1} x_0 \| \leq \varepsilon_n |\lambda|
\]
where \( \varepsilon_n \to 0 \) if \( n \to \infty \), we obtain from (3.23), (3.25):
\[
\bigg\| S(t) x_0 - S(t) x_0 \bigg\| \leq c \varepsilon_n \int_C |S(\lambda(t))| \frac{d\lambda}{|\lambda|}
\]
Since, by [6],
\[
\left\{ \int_0^\infty |S(\lambda(t))|^p \, dt \right\}^{1/p} \leq \frac{c}{|\lambda|^{1/p}} \quad \text{if } p \geq 2,
\]
we obtain
\[
\left\{ \int_0^\infty \bigg\| S(t) x_0 - S(t) x_0 \bigg\|^p \, dt \right\}^{1/p} \leq c \varepsilon_n \int_C \frac{d\lambda}{|\lambda|^{1+1/p}} \to 0
\]
as \( n \to \infty \). This proves (3.22).

4. Monotonicity for \( A \) selfadjoint. Let \( X \) be a Hilbert space and let \( A \) be a self-adjoint operator in \( X \). We say that \( A \) is strictly positive if the number
\[
\delta_A = \inf_{x \neq 0} \frac{(Ax, x)}{(x, x)}
\]
is positive. The main result of the present section is the following:

**Theorem 4.1.** Let \( X \) be a Hilbert space and let \( A \) be a strictly positive selfadjoint operator, with the spectral decomposition of the identity \( \{E_\lambda\} \). If \( h \in H \) then the operator \( S(t) \) given by

\[
S(t) x_0 = \int_{\delta_A}^\infty S_\lambda(t) \, dE_\mu x_0 \quad (x_0 \in X)
\]
is a fundamental solution of (1.1).

Note that (4.1) is formally obtained from (3.1) and the formula
\[
(\lambda I - A)^{-1} = \int_{\delta_A}^\infty \frac{dE_\mu}{\lambda - \mu}
\]
using the Cauchy formula.

**Proof.** Since \( 0 \leq S_\lambda(t) \leq 1 \), the integral in (4.1) exists and \( \| S(t) x_0 \| \leq \| x_0 \| \). \( S(t) x_0 \) is clearly continuous in \( t \). Next, by Fubini’s theorem,
\[
\int_0^t h(t - \tau) S(\tau) x_0 \, d\tau = \int_{\delta_A}^\infty \left\{ \int_0^t h(t - \tau) S_\lambda(\tau) \, d\tau \right\} \, dE_\mu x_0.
\]
Using (2.4) we find the expression on the right is equal to
\[
\int_{\delta_A}^\infty \left\{ \frac{1}{\mu} - \frac{S_\lambda(t)}{\mu} \right\} \, dE_\mu x_0 = A^{-1} x_0 - A^{-1} S(t) x_0,
\]
where (4.1) has been used. We have thus proved that

$$\int_0^t \! h(t-\tau)S(\tau)x_0 \, d\tau$$

lies in $D(A)$ and that if we apply $A$ to this integral we obtain $x_0 - S(t)x_0$. This completes the proof of (1.2).

From Theorem 4.1 and Lemma 2.5(i), (ii) we obtain

**Corollary 1.** For any $x_0 \in X$,

(4.2) \[ 0 \leq (S(t)x_0, x_0) \leq \|x_0\|^2 \quad (0 < t < \infty). \]

**Corollary 2.** If $h \in \mathcal{H}''$ then, for any $x_0 \in X$,

(4.3) \[ (S(t)x_0, x_0) \leq 0 \quad (0 < t < \infty). \]

If $S_n(t) \nrightarrow$ when $t \nrightarrow (\mu > 0)$, then we obtain from (2.4) the bound

$$S_n(t) \leq \left[ 1 + \mu \int_0^t h(\tau) \, d\tau \right]^{-1}.$$

We conclude

**Corollary 3.** If $h \in \mathcal{H}''$ then, for any $x_0 \in X$,

(4.4) \[ (S(t)x_0, x_0) \leq \int_0^\infty \left[ 1 + \mu \int_0^t h(\tau) \, d\tau \right]^{-1} d(E_n x_0, x_0). \]

We next have

**Corollary 4.** If $h \in \mathcal{H}_n$ then, for any $x_0 \in X$,

(4.5) \[ (-1)^n \frac{d^n}{dt^n} (S(t)x_0, x_0) \geq 0 \quad (n = 0, 1, 2, \ldots ; 0 < t < \infty). \]

**Proof.** By Lemma 2.5(iii), the functions

$$T_m(t) = \int_{\delta_4}^{\delta_4 + \delta_4} S_n(t) \, d(E_n x_0, x_0) \quad (m = 1, 2, \ldots)$$

are completely monotonic in $(0, \infty)$. Since, for each $t > 0$, $T_m(t) \rightarrow (S(t)x_0, x_0)$ as $m \rightarrow \infty$, the assertion of the corollary follows from Lemma 2.3.

**Remark.** Theorem 4.1 extends, with the same proof, to the case of the integral equation (3.2).

### 5. Monotonicity for General $A$

**Definition.** A closed linear operator $A$ with a dense domain is said to belong to the class $\mathcal{R}$ if it satisfies the following properties:

(i) $(\lambda I - A)^{-1}$ exists for all complex $\lambda$, except for a sequence $\{\mu_k\}$ (which may be finite) of positive and increasing numbers with no finite limit.

(ii) At each $\mu_k$, $(\lambda I - A)^{-1}$ has a pole, i.e.,

(5.1) \[ (\lambda I - A)^{-1} = \sum_{j=1}^{m_k} \frac{B_{k,j}}{(\lambda - \mu_k)^j} + B_{k,0}(\lambda) \]
where $B_{k,j}$ are bounded operators and $B_{k,0}(\lambda)$ is an analytic function (with values in $B(X)$) in a neighborhood of $\lambda = \mu_k$.

From (3.1), (5.1) and the residue theorem, we formally obtain the formula

$$S(t)x_0 = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{jk}(t)}{\partial \mu^{j-1}} B_{k,j}x_0. \tag{5.2}$$

(We consider here the case where $\{\mu_k\}$ is an infinite sequence; the modifications for the case of a finite sequence are trivial.)

To show that $S(t)$ is a fundamental solution (under certain assumptions), we introduce the operators $T_p(t)$ defined by

$$T_p(t) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{jk}(t)}{\partial \mu^{j-1}} B_{k,j}, \tag{5.3}$$

and set

$$\Delta_p(t) = -T_p(t) + I - A \int_0^t h(t-\tau)T_p(\tau) \, d\tau. \tag{5.4}$$

Applying $\lambda I - A = (\lambda - \mu_k)I + (\mu_kI - A)$ to both sides of (5.1), we obtain the relations:

$$AB_{k,m_k} - \mu_k B_{k,m_k} = 0, \quad AB_{k,j} - \mu_k B_{k,j} = B_{k,j+1} \quad (1 \leq j \leq m_k-1). \tag{5.5}$$

Using these relations and (2.6), (2.4), we get

$$\Delta_p(t) = -\sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{jk}(t)}{\partial \mu^{j-1}} B_{k,j} + I$$

$$- \int_0^t h(t-\tau) \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{jk}(t)}{\partial \mu^{j-1}} \mu_k B_{k,j} \, d\tau$$

$$- \int_0^t h(t-\tau) \sum_{k=1}^{p} \sum_{j=1}^{m_k-1} \frac{1}{(j-1)!} \frac{\partial^{j-1}S_{jk}(t)}{\partial \mu^{j-1}} B_{k,j+1} \, d\tau \tag{5.6}$$

$$= I - \sum_{k=1}^{p} B_{k,1}.$$  

DEFINITION. We denote by $X_A$ the set of all elements $x_0$ of $X$ for which

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left\| B_{k,j}x_0 \right\| \mu_k^{-1} < \infty, \tag{5.7}$$

$$\lim_{p \to \infty} \sum_{k=1}^{p} B_{k,1}x_0 = x_0. \tag{5.8}$$

THEOREM 5.1. Let $h \in \mathcal{H}$, $A \in \mathcal{A}'$, $x_0 \in X_A$. Then the limit

$$T(t)x_0 = \lim_{p \to \infty} T_p(t)x_0 \tag{5.9}$$
exists uniformly with respect to $t$, $0 \leq t < \infty$, and

$$T(t)x_0 = x_0 - A \int_0^t h(t-\tau)T(\tau)x_0 \, d\tau. \quad (5.10)$$

**Proof.** The uniform convergence of the sequence $\{T_p(t)x_0\}$ follows from Lemma 2.10 and the assumption (5.7). From (5.6) we have

$$T_p(t)x_0 = x_0 - A \int_0^t h(t-\tau)T_p(\tau)x_0 \, d\tau - \Delta_p(t)x_0, \quad (5.11)$$

where

$$\Delta_p(t)x_0 = x_0 - \sum_{k=1}^{p} B_{k,1}x_0.$$ 

By (5.8), $\Delta_p(t)x_0 \to 0$ as $p \to \infty$. Hence, taking $p \to \infty$ in (5.11) and using the assumption that $A$ is a closed operator, we conclude that the integral

$$\int_0^t h(t-\tau)T(\tau)x_0 \, d\tau$$

belongs to the domain of $A$ and that (5.10) holds.

**Corollary 1.** If, in addition to the assumptions of Theorem 5.1, we assume that $x_0 \in D(A)$ and $Ax_0 \in X_A$, then

$$T(t)x_0 = x_0 - \int_0^t h(t-\tau)AT(\tau)x_0 \, d\tau \quad (5.12)$$

and $T(t)x_0$ is continuous for $t \geq 0$.

**Proof.** We have

$$T_p(t)x_0 = x_0 - \int_0^t h(t-\tau)AT_p(\tau)x_0 \, d\tau - \Delta_p(t)x_0, \quad (5.13)$$

and $AT_p(\tau)x_0 = T_p(\tau)(Ax_0)$. Since $Ax_0 \in X_A$, $T_p(\tau)(Ax_0) \to T(\tau)(Ax_0)$ as $p \to \infty$, uniformly with respect to $\tau$. It follows that $T(\tau)x_0$ is in $D(A)$, and $AT(\tau)x_0 = T(\tau)(Ax_0)$. Now take $p \to \infty$ in (5.13).

**Corollary 2.** Let $h \in \mathcal{L} \cap \mathcal{L}', A \in \mathcal{A} \cap \mathcal{A}'$, $x_0 \in X_A$, $Ax_0 \in X_A$. Then

$$T(t)x_0 = W(t)x_0 \quad (5.14)$$

where $W(t)$ is the fundamental solution of (1.1) occurring in Theorem 3.2.

Indeed, both sides of (5.14) are solutions of (1.1). By the uniqueness assertion of [6, Theorem 1], they must coincide.

**Corollary 3.** Let $h \in \mathcal{H} \cap \mathcal{H}'$, $A \in \mathcal{A} \cap \mathcal{A}'$, $x_0 \in X_A$, and assume also that $ih(t) \in L^1(0, \infty)$, and that (3.20), (3.21) hold. Then

$$T(t)x_0 = S(t)x_0 \quad (5.15)$$

where $S(t)$ is the fundamental solution of (1.1) occurring in Theorem 3.3.
This follows from (5.10) and the uniqueness assertion of [6, Theorem 10].
We recall [6] that if \( h \in \mathcal{H} \) then the condition (3.20) is equivalent to the condition
\[
(5.16) \quad h(\infty) > 0.
\]

We shall now study monotonicity of the scalar function \( f_0(T(t)x_0) \), where \( f_0 \) is any bounded linear functional in \( X \).

**Theorem 5.2.** Let \( h \in \mathcal{H}, A \in \mathcal{A}, x_0 \in X_A, f_0 \in X^* \). If
\[
(5.17) \quad (-1)^j f_0(B_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m_k, 1 \leq k < \infty)
\]
then \( f_0(T(t)x_0) \geq 0 \) for all \( t \geq 0 \).

**Proof.** From (5.3) and Lemma 2.7 we immediately have that \( f_0(T_p(t)x_0) \geq 0 \).
Now take \( p \rightarrow \infty \).

**Theorem 5.3.** Let \( h \in \mathcal{H}', A \in \mathcal{A}', x_0 \in X_A, f_0 \in X^* \). If (5.17) holds and, in addition,
\[
(5.18) \quad (-1)^j f_0(AB_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m_k, 1 \leq k < \infty)
\]
then \( f_0(T(t)x_0) \perp t \) if \( t \neq . \)

Note that \( AB_{k,j}x_0 \) is well defined for any \( x_0 \in X \).

Before proving this theorem, we state and prove the following theorem.

**Theorem 5.4.** Let \( h \in \mathcal{H}_o, A \in \mathcal{A}', x_0 \in X_A, f_0 \in X^* \). Set
\[
(5.19) \quad \overline{B}_{k,j} = (-1)^{m_k-j-1} \sum_{i=0}^{k} i^{n-i} \mu_{k-i} B_{k,j-i}n_i.
\]

If
\[
(5.20) \quad f_0(\overline{B}_{k,j}x_0) \geq 0 \quad for \ 0 \leq j \leq m_k, 1 \leq k < \infty, 0 \leq n < \infty,
\]
then
\[
(5.21) \quad (-1)^n \frac{d^n}{dt^n} f_0(T(t)x_0) \geq 0 \quad for \ 0 \leq n < \infty, 0 < t < \infty.
\]

**Proof.** Let \( \Gamma \) be a circle about \( \mu_k \) such that all the points \( \mu_j \) with \( j \neq k \) lie outside \( \Gamma \). By the residue theorem,
\[
(5.22) \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_{\lambda}(t) \, d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} \lambda^n (\lambda I - A)^{-1} \left( \frac{S_{\lambda}(t)}{\lambda^n} \right) \, d\lambda \equiv J(t).
\]

Set
\[
Q^n(t) = (-1)^n \lambda^{-n} \frac{\partial^n S_{\lambda}(t)}{\partial t^n}
\]

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and write

\[ \lambda^n = \sum_{i=0}^{n} \binom{n}{i} \mu_k^{n-i}(\lambda - \mu_k)^i. \]

From the residue theorem we then get

\[
(-1)^n \frac{d^n\mathcal{J}(t)}{dt^n} = \frac{1}{2\pi i} \int \left[ \sum_{i=0}^{n} \binom{n}{i} \mu_k^{n-i}(\lambda - \mu_k)^i \right] \left[ \sum_{j=1}^{m_k} \frac{B_{k,j}}{(\lambda - \mu_k)} \right] Q_n(t) \, d\lambda
\]

\[
= \sum_{j=1}^{m_k} B_{k,j} \sum_{i=0}^{j-1} \binom{n}{i} \mu_k^{n-i} \frac{1}{(j-i-1)!} \frac{\partial^{j-i-1}Q_{\mu_k}(t)}{\partial \mu_k^{j-i-1}}
\]

\[
= \sum_{q=0}^{m_k-1} \left\{ B_{k,m_k-q} \binom{n}{0} \mu_k^n + B_{k,m_k-q+1} \binom{n}{1} \mu_k^{n-1} + \cdots + B_{k,m_k} \binom{n}{q} \mu_k^{n-q} \right\}
\]

\[
\times \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1}Q_{\mu_k}(t)}{\partial \mu_k^{m_k-q-1}}
\]

where

\[
\binom{n}{i} = 0 \quad \text{if } i > n.
\]

Thus, by (5.19) the last sum is equal to

\[
\sum_{q=0}^{m_k-1} (-1)^{m_k-q-1} \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1}Q_{\mu_k}(t)}{\partial \mu_k^{m_k-q-1}} B_{k,q}.
\]

Hence, recalling (5.22) and (5.3), we have

\[
(-1)^n \frac{d^nT_p(t)x_0}{dt^n} = \sum_{k=1}^{p} \sum_{q=0}^{m_k-1} (-1)^{m_k-q-1} \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1}Q_{\mu_k}(t)}{\partial \mu_k^{m_k-q-1}} B_{k,q} x_0.
\]

Using Lemma 2.9 and the assumption (5.20), we conclude that

\[
(-1)^n \frac{d^n\mathcal{J}(t)x_0}{dt^n} \geq 0 \quad \text{for } n = 0, 1, 2, \ldots; t > 0.
\]

Since \( T_p(t)x_0 \rightarrow T(t)x_0 \) as \( p \rightarrow \infty \), the assertion of the theorem follows from Lemma 2.3.

Using the notation \( \Delta^n_\alpha \) (following Lemma 2.3), we can state

**Corollary.** If (5.20) is assumed to hold only for \( 0 \leq n \leq n_0 \), then

\[
(-1)^n \frac{d^n\mathcal{J}(t)x_0}{dt^n} \geq 0 \quad \text{for } 0 \leq n \leq n_0, 0 \leq t < t + \eta < \infty.
\]

Indeed, the proof of Theorem 5.4 shows that (5.24) holds with \( T(t)x_0 \) replaced by \( T_p(t)x_0 \). Since \( T_p(t)x_0 \rightarrow T(t)x_0 \) as \( p \rightarrow \infty \), (5.22) follows.

**Remark.** Let the assumptions of Corollary 2 to Theorem 5.1 hold and let \( h \in C^{n_0}[0, \infty) \). Then, by [6] and (5.14), \( T(t)x_0 \) has \( n_0 \) continuous derivatives in \([0, \infty)\). Hence, (5.24) implies that

\[
(-1)^n (\partial^n h(T(t)x_0)/\partial t^n) \geq 0 \quad \text{for } 0 \leq n \leq n_0, 0 \leq t < \infty.
\]
Proof of Theorem 5.3. We shall use the formula

\[(5.26) \quad T_p(t)x_0 = \sum_{\lambda=1}^{p} \sum_{q=1}^{m_k-1} (-1)^{m_k-q-1} \left[ \frac{\partial^{m_k-q-1}}{\partial \mu^{m_k-q-1}} \left( \frac{S_\mu(t)}{\mu} \right) \right]_{\mu=\lambda_k} B_{k,q} x_0 \]

which one obtains by the same method that was used before to derive (5.23). Since

\[B_{k,q} = (-1)^{m_k-q-1}(\mu_k B_{k,m_k-q} + B_{k,m_k-q+1}) \quad (1 \leq q \leq m_k - 1),\]

(5.5) shows that the inequalities (5.18) imply the inequalities (5.20) for \(n = 1\). Hence, (5.26) gives

\[f_\lambda(T_p(t)x_0) = \lambda t, \quad \text{if } t \neq 0.\]

Since \(T_p(t)x_0 \to T(t)x_0\) as \(p \to \infty\), the proof is complete.

Remark. If \(X\) is a finite-dimensional Banach space, then any linear operator \(A\) whose eigenvalues are positive numbers is in \(\mathcal{B}\). Furthermore, the series in (5.2) now consists of a finite number of terms. Hence (5.7) holds. (5.8) is also valid; in fact, it easily follows using the residue theorem. Thus \(X_A = X\).

6. Additional results. In the previous two sections we have derived theorems which involved the functions \(S_\lambda(t)\) for \(\lambda > 0\). A crucial step in the derivation of these theorems was the behavior of the function \(S_\lambda(t)\) for positive values of the parameter \(\lambda\). Since analogous results on the behavior of \(S_\lambda(t)\) for \(\lambda\) complex are not available in the literature, we cannot extend, at present, the results of §§4, 5 to operators \(A\) with \(\sigma(A)\) which is not contained in the real interval \(0 < \lambda < \infty\).

However, for some special functions \(h(t)\), the behavior of \(S_\lambda(t)\), for complex \(\lambda\), is known with sufficient precision. We give here one example where \(h(t) = t^{-\alpha}\) for some \(0 < \alpha < 1\). Then \(S_\lambda(t) = E_\beta(-\gamma t^\alpha)\) where \(\beta = 1 - \alpha\), \(\gamma = \Gamma(\beta)\) and where \(E_\beta(z)\) is the Mittag-Leffler function

\[E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)}.\]

From a well-known asymptotic formula for \(E_\beta(z)\) (see [1, p. 207]) we find that

\[(6.1) \quad \left| S_\lambda(t) - \frac{\gamma_0}{\lambda t^\beta} \right| \leq \frac{C}{|\lambda|^{q_2/2\beta}} \quad \text{if } |\arg \lambda t^\beta| < \frac{1+\alpha}{2} \pi,\]

provided \(|\lambda t^\beta| \geq \varepsilon_0 > 0\); here \(\gamma_0 = (\Gamma(\alpha)\Gamma(1-\alpha))^{-1}\).

Let us assume that the resolvent set \(\rho(A)\) of \(A\) contains the sector \(|\arg \lambda| > (1+\alpha)\pi/2\) and that \(||(\lambda I - A)^{-1}|| \leq c/(1 + |\lambda|)\) for \(\lambda\) in this sector. We define \(S(t)\) by (3.23) and choose \(C\) in \(\rho(A)\) such that (6.1) holds for \(\lambda \in C\). It follows that \(S(t)\) is a bounded operator for each \(t > 0\). Furthermore, it varies continuously in \(t\). One can also continue \(S(t)\) analytically into a sector \(|\arg \lambda| < \delta\) for some \(\delta > 0\).

Since \(h(0) = \infty\), the results of [6] do not cover the present case of \(h(t) = t^{-\alpha}\). There arises the question in what sense is \(S(t)\) a fundamental solution.
So far we have only considered solutions of equations of the form (1.1) where $x_0$ is independent of $t$. But some of the results extend without difficulty to the equations

\[(6.2) \quad x(t) = k(t)x_0 - \int_0^t h(t-\tau)Ax(\tau)\,d\tau,\]

where $k(t)$ is a scalar function.

The solution is given by (see [6]):

\[(6.3) \quad x(t) = \mathcal{S}(0)5(0)x_0 + \int_0^t k(\tau)\mathcal{S}(t-\tau)x_0\,d\tau\]

where $k(\tau) = dk(\tau)/d\tau$. Hence, if $X$ is a Hilbert space,

\[(6.4) \quad (x(t), x_0) = k(0)(\mathcal{S}(t)x_0, x_0) + \int_0^t k(\tau)(\mathcal{S}(t-\tau)x_0, x_0)\,d\tau.\]

This relation combined with the results of §§4, 5 yields monotonicity properties for $(x(t), x_0)$. For example, if $k(0) \geq 0$, $k(\tau) \geq 0$, then $(x(t), x_0) \geq 0.$

**Applications.** If $h \in \mathcal{H}_\sigma$ then we have proved several theorems to the effect that $(\mathcal{S}(t)x_0, x_0)$ is completely monotonic in $t$. Since $(\mathcal{S}(0)x_0, x_0) = (x_0, x_0) \neq 0$, we conclude that $(\mathcal{S}(t)x_0, x_0) > 0$ for all $t > 0$. In particular, $\mathcal{S}(t)x_0 \neq 0$ for all $t > 0$. Thus the solutions of (1.1) have the “weak backward uniqueness” property as defined in [5]. This fact is important in the study of optimal-control for trajectories $x(t)$ given by

\[(6.5) \quad x(t) = u(t) + \int_0^t h(t-\tau)Ax(\tau)\,d\tau\]

where $u(t)$ is the control function. It enables us to prove uniqueness of time-optimal controls (see [3], [5, p. 42]).

If $A$ is selfadjoint and if $h \in \mathcal{H}'$ and $h$ is strictly decreasing, then we can again assert that $(\mathcal{S}(t)x_0, x_0) > 0$ for all $x_0 \neq 0$, $t > 0$. Indeed, otherwise we get, from (4.1), $S_u(t_0) = 0$ for some $\mu > 0$, $t_0 > 0$. But then, by Lemma 2.5, $S_u(t) = 0$ if $t > t_0$. Using (2.4) we then see that $S_u(t) < S_u(t_0)$ if $t > t_0$; a contradiction.

If $h \in \mathcal{H}'$, then we have proved several theorems to the effect that $(\mathcal{S}(t)x_0, x_0) \prec 0$ if $t > t$. This can be used to answer some questions of controllability; for instance, to show that a point $x_0$ can be “steered,” by a suitable control, to any given neighborhood of 0 (cf. [3]).

**References**


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