EXTREMAL QUASICONFORMAL MAPPINGS WITH
PRESCRIBED BOUNDARY VALUES

BY

RICHARD S. HAMILTON

1. Let \( R \) be a Riemann surface whose universal covering space is conformally equivalent to the unit disk. We can regard \( R \) as the interior of a Riemann surface with boundary \( R^* \) whose boundary is as large as possible (see §3). A quasiconformal map \( f \) of \( R \) onto another Riemann surface \( S \) has a unique continuous extension \( f^* \) mapping \( R^* \) onto \( S^* \). Two quasiconformal maps \( f \) and \( g \) of \( R \) onto \( S \) are homotopic modulo the boundary if \( f^* = g^* \) on \( R^* - R \) and there exists a homotopy between \( f^* \) and \( g^* \) which is constant on \( R^* - R \). If \( R^* - R \) is empty then \( f \) is just homotopic to \( g \).

Let \( K \) be the complex cotangent bundle of \( R \). Then \( \beta(f) \), the Beltrami differential of \( f \), is an element of \( L^\infty(\bar{KK}^{-1}) \), the Banach space of all essentially bounded sections of the bundle \( \bar{KK}^{-1} \). (Locally \( \nu \in L^\infty(\bar{KK}^{-1}) \) is given by \( \nu_a \frac{dz_a}{dz_a} \).) \( L^\infty(\bar{KK}^{-1}) \) is the dual of \( L^1(K^2) \), the Banach space of integrable sections of the bundle \( K^2 \). (Locally \( \varphi \in L^1(K^2) \) is given by \( \varphi_a \frac{dz_a}{dz_a} \).) Let \( A(R) \) denote the closed subspace of \( L^1(K^2) \) of all integrable analytic quadratic differentials on \( R \). By \( \|\beta(f)\|_{A(R)} \) we denote the norm of the restriction of \( \beta(f) \), regarded as a linear functional on \( L^1(K^2) \), to the closed subspace \( A(R) \).

The quasiconformal map \( f: R \to S \) is extremal if \( \|\beta(f)\|_{A(R)} \leq \|\beta(g)\| \) for all maps \( g \) homotopic to \( f \) modulo the boundary. The main result is

**Theorem 1.** If \( f: R \to S \) is extremal then \( \|\beta(f)\|_{A(R)} = \|\beta(f)\| \).

**Corollary 1.** If \( R \) is a compact surface minus a finite number of points, then \( A(R) \) is finite dimensional and consists of all meromorphic quadratic differentials on the compact surface with poles of order at most one at the deleted points. Furthermore there exists a nonzero \( \theta \in A(R) \) and a constant \( c \) with \( \beta(f) = c\theta/|\theta| \).

**Remark.** This result is classical and due to Teichmüller.

**Proof.** Since \( A(R) \) is finite dimensional, by Theorem 1 there exists a nonzero \( \theta \in A(R) \) with \( \int_R \beta(f) \theta = \int_R \|\beta(f)\| \|\theta| \). This can happen only if \( \beta(f) \theta = \|\beta(f)\| \|\theta| \), and therefore \( \beta(f) = \|\beta(f)\| \theta/|\theta| \).

**Corollary 2.** If \( f: R \to S \) is extremal then \( \|\beta(f)\|_{R - K} = \|\beta(f)\| \) for every compact proper subset \( K \) of \( R \).

**Proof.** By Theorem 1 we can find a sequence \( \theta_n \in A(R) \) with \( \|\theta_n\| = 1 \) and \( \int_R \beta(f) \theta_n \to \|\beta(f)\| \). Since the value of an analytic function at a point is the average

---

Received by the editors June 9, 1967 and, in revised form, March 3, 1968.
of its values over a disk centered at that point, the $\theta_n$ are uniformly bounded on every compact subset. Using Cauchy's formula their derivatives are uniformly bounded also, and by passing to a subsequence we may assume that the $\theta_n$ converge uniformly on compact subsets to some $\theta \in A(R)$. Suppose $\| \beta(f) \| R-K < \| \beta(f) \|$. Since

$$\left| \int_R \beta(f) \theta_n \right| \leq \| \beta(f) \| \int_K |\theta_n| + \| \beta(f) \| R-K \int_{R-K} |\theta_n|$$

we must have $\int_{R-K} |\theta_n| \to 0$, and the $\theta_n$ converge to $\theta$ in $L^1$ norm. We then have $\| \theta \|=1$ and $\beta(f)(\theta)=\| \beta(f) \|$, and we can repeat the previous argument to show $\beta(f)=\beta(f)\| \theta \|$. This gives a contradiction.

**Remark.** Given any quasiconformal map $f: R \to S$, it follows from the usual compactness properties of quasiconformal mappings that there exists at least one extremal map homotopic to $f$ modulo the boundary. Strebel [7] has shown that when $R$ is the unit disk such an extremal need not have the form $\beta(f)=\omega(\theta)/\theta$ and need not be unique.

2. The proof of Theorem 1 will be modeled upon the following general result.

**Theorem 2.** Let $B$ be a Banach space and $B^*$ its dual. Let $M$ be a $C^1$ submanifold of $B^*$. Suppose that the dual norm in $B^*$ assumes its minimum, or maximum, on $M$ at a point $x$ in $M$, and that there exists a closed subspace $A$ of $B$ such that the tangent space to $M$ at $x$ is the subspace of $B^*$ orthogonal to $A$. Then $\| x \| A = \| x \|$.

**Proof.** Suppose $\| x \| A < \| x \|$. By the Hahn-Banach extension theorem we can find a linear functional $y$ in $B^*$ with $y \| A=x \| A$ and $\| y \|=\| x \| A < \| x \|$. Since $y-x$ vanishes on $A$, we can construct a $C^1$ path $\alpha: (-\varepsilon, \varepsilon) \to M$ with $\alpha(0)=x$ and $D\alpha(0)(1)=y-x$.

Then for sufficiently small positive $t$

$$\| \alpha(t)-\alpha(0) - D\alpha(0)(t) \| / t < \| x \| - \| y \|$$

since $\| x \| - \| y \| > 0$. But then

$$\| \alpha(t) \| < \| \alpha(0) + D\alpha(0)(t) \| + (\| x \| - \| y \|)t$$

$$\leq \| x + t(y-x) \| + (\| x \| - \| y \|)t$$

$$\leq (1-t)\| x \| + t\| y \| + t\| x \| - t\| y \| = \| x \|$$

and the norm does not assume its minimum on $M$ at $x$. Similarly, for sufficiently small negative $t$, $\| \alpha(t) \| > \| x \|$ and the norm does not assume its maximum on $M$ at $x$ either.

In the case where $B$ is a Hilbert space this condition yields a familiar result.

**Corollary 3.** If $B$ is a Hilbert space, then $x$ is orthogonal to the tangent space to $M$ at $x$.

**Proof.** Since $\| x \| A = \| x \|$, $x \in A$. 


3. We can represent $R$ as the quotient of the unit disk $D$ under the Fuchsian group $\Gamma$. Write the boundary of $D$ as the disjoint union of the closed set $\Lambda(\Gamma)$ of limits of fixed points of $\Gamma$ and the relatively open set $\Phi(\Gamma)/\Gamma$ of points of discontinuity of $\Gamma$. Then $R^* = D \cup \Phi(\Gamma)/\Gamma$ is a Riemann surface with boundary whose interior is $R$, and any such surface is (conformally equivalent to) an open subset of $R^*$. Thus the boundary of $R^*$ is as large as possible.

If $f: R \to S$ is a quasiconformal map, we may represent $S$ also as the quotient of the disk by another Fuchsian group $\Delta$, and the map $f$ is covered by a quasiconformal map $F$ of the unit disk to itself which has a continuous extension $F^*$ on the closed disk covering a continuous extension $f^*: R^* \to S^*$ of $f$.

**Theorem 3.** Two maps $f, g: R \to S$ are homotopic modulo the boundary if and only if they can be covered by maps $F^*$ and $G^*$ of the disk to itself which agree on the boundary of the disk.

**Proof.** First suppose $f$ and $g$ are homotopic modulo the boundary, and let $h^*(t): R^* \to S^*$ be the homotopy with $h^*(0) = f^*$, $h^*(1) = g^*$ and $h^*$ constant on $R^* - R$. Then by the Covering Homotopy Theorem we can cover $h^*(t)$ with a homotopy $H^*(t): D \to D$ which is constant on $\Phi(\Gamma)$.

For each $\gamma \in \Gamma$ there exists a $\delta \in \Delta$ with $H^*(0)\gamma = \delta H^*(0)$. Fix $x$ in the interior of $D$ and consider the two curves in $D$ given by $H^*(t)\gamma(x)$ and $\delta H^*(t)x$. Since both are liftings of the same curve in $S$ and both have the same initial point, they must agree. Thus $H^*(t)\gamma = \delta H^*(t)$ for all $t$.

But then $H^*(t)$ must be constant on the fixed points of $\Gamma$, and hence on the whole boundary of the disk. Therefore $F^* = H^*(0)$ and $G^* = H^*(1)$ cover $f$ and $g$ and agree on the boundary of the disk.

Conversely suppose $F^*$ and $G^*$ agree on the boundary of the disk. Define $H^*(t)(z)$ to be the point which divides the noneuclidean line segment between $F^*(z)$ and $G^*(z)$ in the ratio $t:(1-t)$. Then $H^*(t)$ covers a homotopy

$$h^*(t): R^* \to S^*$$

between $f^*$ and $g^*$ constant on $R^* - R$.

Let $f_n: R \to S$ be a sequence of maps homotopic to $f$ modulo the boundary, with $\|\beta(f_n)\| \leq k < 1$. Cover $f$ and $f_n$ with maps $F^*$ and $F^*_n$ of the closed disk to itself which all agree on the boundary, and with $\|\beta(F_n)\| \leq k < 1$. Some subsequence of the $F^*_n$ will converge uniformly to a quasiconformal map $G^*$ (see Ahlfors [1]) which agrees with $F^*$ on the boundary of the disk, and which covers a quasiconformal map $g: R \to S$ homotopic to $f$ modulo the boundary. Moreover $\|\beta(g)\| \leq \lim inf \|\beta(f_n)\|$ so we may choose the $f_n$ to make $g$ extremal.

4. In order to apply Theorem 2, I need the following result, which occurs (implicitly) in Bers [3].
Theorem 4. Let \( N \) be the set of all Beltrami differentials of quasiconformal maps of \( R \) onto itself homotopic to the identity map modulo the boundary. Then \( N \) is an analytic submanifold of \( L^\infty(\overline{KK}^{-1}) \) in a neighborhood of zero whose tangent space at zero is the subspace orthogonal to \( A(R) \).

For completeness I shall outline the proof. Let \( D \) be the unit disk and \( D' \) its complement in the sphere. Define

\[
P_\mu(z) = (n!/2\pi i) \int_D \mu(\zeta)(\zeta - z)^{-n-1} \, d\zeta \wedge d\bar{\zeta}.
\]

For \( \mu \in L^\infty(D) \), \( P_\mu \) is analytic in \( D' \) with a zero at infinity of order at least \( n+1 \), and \( d/dz P_\mu = P_{\mu+1} \) in \( D' \). Moreover for \( \mu \in L^p(D) \), any \( p > 2 \), \( P_\mu \) is Hölder-continuous in the entire sphere and has generalized derivatives \( \partial/\partial z P_\mu = \mu \) and \( \partial/\partial z P_{\mu+1} = P_{\mu+1} \). In \( D \), \( P_1 \) is a singular integral and by the Calderón-Zygmund inequality it is a bounded linear operator of \( L^p(D) \) into itself, whose norm, by the Riesz convexity theorem, approaches 1 as \( p \) approaches 2. We can then prove (see Ahlfors [2, p. 97]) that

\[
w(\mu)(z) = z + P_0(I-P_1)^{-1}\mu(z)
\]

is a quasiconformal map of the sphere to itself with Beltrami differential \( \mu \) on \( D \) and analytic on \( D' \).

Remembering that \( R = D/\Gamma \), let \( L^\infty(D, \Gamma) = L^\infty(\overline{KK}^{-1}) \) be the Banach subspace of \( \mu \in L^\infty(D) \) with \( \mu = (\mu \circ \gamma) \arg^{-2}\gamma' \) for all \( \gamma \in \Gamma \). Also let \( B(D', \Gamma) \) be the Banach space of all analytic functions \( \varphi \) in \( D' \) with a zero of order at least 4 at infinity, which satisfy \( \varphi = (\varphi \circ \gamma)(\gamma')^2 \) and whose norm sup \( (z\bar{z}-1)^2|\varphi(z)| \) is finite. This is just the norm of the quadratic differential \( \varphi(z) \, dz^2 \) in the Poincaré metric on \( D' \). Let \([f]\) denote the Schwarzian derivative of \( f \). Since \( \mu \) is invariant under \( \Gamma \), for each \( \gamma \in \Gamma \) the map \( w(\mu) \circ \gamma \circ w(\mu)^{-1} \) is an analytic homeomorphism of the sphere and hence itself a Möbius transformation \( \delta \). Then \( w(\mu) \circ \gamma = \delta \circ w(\mu) \), and

\[
([w(\mu)] \circ \gamma)(\gamma')^2 = [w(\mu)].
\]

Moreover by a theorem of Nehari [6] on schlicht mappings sup \( (z\bar{z}-1)^2|[w(\mu)]| \leq 6 \). Hence \( \Lambda(\mu) = [w(\mu)] \) belongs to \( B(D', \Gamma) \) and \( \|\Lambda(\mu)\| \leq 6 \).

Lemma 1. \( \Lambda : L^\infty(D, \Gamma) \to B(D', \Gamma) \) is a complex analytic map and \( D\Lambda(0) = P_3 \).

Proof. Fix a point \( z \in D \). Since \( (I - \mu P_1)^{-1}\mu \) is a uniformly convergent power series in \( \mu \), \( w(\mu)(z) \), \( d/dz w(\mu)(z) \), \ldots, \( d^n/dz^n w(\mu)(z) \) are all analytic functions of \( \mu \). Therefore so is \( \Lambda(\mu)(z) \). Let \( \gamma \) be the circle of radius 1 in the \( t \)-plane. By the Cauchy integral formula, for \( \|\nu\| < 1 - \|\mu\| \),

\[
D\Lambda(\mu)(\nu)(z) = (1/2\pi i) \int_\gamma \Lambda(\mu + t\nu)(z)/t^2 \, dt.
\]
Since $|\Lambda(\mu + tv)(z)| \leq 6(z\bar{z} - 1)^{-2}$, $D\Lambda(\mu)$ is a bounded complex linear map of $L^\infty(D, \Gamma)$ into $B(D', \Gamma)$. Also for $|c| \leq 1/2$

$$|\Lambda(\mu + cv)(z) - \Lambda(\mu)(z) - D\Lambda(\mu)(cv)(z)|$$

$$\leq \frac{1}{2\pi} \int_0^\pi |\Lambda(\mu + tv)(z)| \sqrt{(t - c)^{-1} - t^{-1} - ct^{-2}} \, dt$$

$$\leq 12c^2(z\bar{z} - 1)^{-2}.$$ 

Therefore $\|\Lambda(\mu + cv) - \Lambda(\mu) - D\Lambda(\mu)(cv)\| \leq 12c^2$ so $\Lambda$ is in fact differentiable with derivative $D\Lambda(\mu)$. Since $D\Lambda(\mu)$ is complex-linear, $\Lambda$ is analytic. By evaluating at $z$ again we may compute $Dw(0)(\mu)(z) = P_\mu(\mu)$ and $D\Lambda(0)(\mu)(z) = P_\mu(\mu)$. Therefore $D\Lambda(0) = P_\mu$.

Define a continuous linear map $S : B(D', \Gamma) \to L^\infty(D, \Gamma)$ by

$$S_\varphi(z) = c(1 - z\bar{z})^2 (z - \bar{z})^4.$$ 

Using a reproducing formula for analytic functions (see Bers [3, Lecture 3, p. 6]), $D\Lambda(0) \circ S$ is the identity for an appropriate choice of the constant $c$. This proves that $D\Lambda(0)$ maps $L^\infty(D, \Gamma)$ onto $B(D', \Gamma)$ and its kernel is a closed split subspace. By the inverse function theorem $\Lambda^{-1}(0)$ is an analytic submanifold in a neighborhood of zero.

**Lemma 2.** $\Lambda^{-1}(0) = N$.

**Proof.** First suppose $\mu \in \Lambda^{-1}(0)$. Then the Schwarzian derivative of $w(\mu)$ is zero on $D'$, so $w(\mu)$ agrees on $D'$ with a Möbius transformation $A$. Let $w = A^{-1} \circ w(\mu)$. Then $w$ is $\mu$-quasiconformal on $D$ and the identity on $D'$. Since $\mu \in L^\infty(D, G)$, $w$ covers a $\mu$-quasiconformal map of $R$ onto itself homotopic to the identity modulo the boundary.

Conversely any such map can be lifted to a quasiconformal map $w$ of $D$ onto itself which leaves the boundary fixed. Let $\mu = \beta(w)$, and extend $w$ to be the identity in $D'$. Then $w(\mu)w^{-1}$ is an analytic one-to-one map of the sphere onto itself and therefore is a Möbius transformation whose Schwarzian derivative in $D'$ is $\Lambda(\mu) = 0$.

Finally

$$D\Lambda(0)(\mu) = P_\mu = \frac{6/2\pi i}{D} \int_D \frac{\mu(\zeta)}{(\zeta - z)^4} \, d\zeta \wedge d\bar{\zeta}$$

and the functions $(\zeta - z)^{-4}$ are dense in $A(D)$, the integrable analytic functions in $D$. Moreover we know (see Earle [4]) that each element of $A(R)$ is a Poincaré series of an element of $A(D)$. Therefore $T_0 = \text{Ker } D\Lambda(0)$ is the subspace of $L^\infty(KK^{-1})$ orthogonal to $A(R)$. 


5. The composition of two quasiconformal maps is again quasiconformal. If
\( \mu = \beta(f) \) and \( \nu = \beta(g) \) then
\[
\frac{\partial(g \circ f)}{\partial z} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{w}} \frac{\partial f}{\partial z} \\
= \frac{\partial g}{\partial w} \frac{\partial f}{\partial z} (1 + \mu \nu \arg^{-2} \frac{\partial f}{\partial z}) \\
\frac{\partial(g \circ f)}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{w}} \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{w}} \frac{\partial f}{\partial \overline{z}} \\
= \frac{\partial g}{\partial \overline{w}} \frac{\partial f}{\partial \overline{z}} (\mu + \nu \arg^{-2} \frac{\partial f}{\partial \overline{z}}).
\]
Here \( f^\# = \nu \arg^{-2} \frac{\partial f}{\partial z} \) is the pull-back of \( \nu \frac{d\overline{w}}{dw} \) as a tensor. We then have
\[
\beta(g \circ f) = (\mu + f^\# \nu) (1 + \bar{\mu} f^\# \bar{\nu}).
\]
We may regard this as a calculation in local coordinates for Riemann surfaces.
Let \( R, S, \) and \( T \) be Riemann surfaces, \( f: R \to S \) and \( g: S \to T \) quasiconformal maps, and \( K \) and \( J \) the complex cotangent bundles on \( R \) and \( S \). The tensor pull-back defines a linear isometric isomorphism \( f^\#: L^\infty(J^{-1}) \to L^\infty(K^{-1}) \). It is an isometry because \( |f^\# \nu_a| = |\nu_b| \circ f \beta_a \), and an isomorphism because \( (f^\#)^{-1} = (f^{-1})^\# \). Then if \( \mu = \beta(f) \) and \( \nu = \beta(g) \) we have as before
\[
\beta(g \circ f) = (\mu + f^\# \nu) (1 + \bar{\mu} f^\# \bar{\nu}).
\]
This suggests that we define a map of the Beltrami differentials on \( S \) to the Beltrami differentials on \( R \) by
\[
C(f)(\nu) = (\mu + f^\# \nu) (1 + \bar{\mu} f^\# \bar{\nu}).
\]
The Beltrami differentials on \( R \) are the points in the unit ball in \( L^\infty(K^{-1}) \), which I denote by \( B(R) \). The map \( C(f): B(S) \to B(R) \) is analytic. Indeed \( (1 + \bar{\mu} f^\# \bar{\nu})^{-1} \) admits a uniformly convergent power series since \( \|f^\# \nu\| < 1 \). Moreover \( \beta(g \circ f) = C(f) \beta(g) \), and since every Beltrami differential is the Beltrami differential of some quasiconformal map, it follows that \( C(g \circ f) = C(g) \circ C(f) \). Therefore, \( C(f)^{-1} = C(f^{-1}) \) and \( C(f) \) is bi-analytic.
Let \( d \) be the Poincaré metric in the disk \( D \), given by
\[
d(z, w) = \frac{1}{2} \log \frac{1 + r}{1 - r} \quad \text{where} \quad r = |(z - w)/(1 - \overline{z}w)|.
\]
If \( \nu \) and \( \pi \) are two Beltrami differentials and \( \alpha \) and \( \beta \) are two coordinate charts then since
\[
\nu_\alpha = \nu_\beta \arg^2 \left( \frac{dz_\beta}{dz_\alpha} \right) \quad \text{and} \quad \pi_\beta = \pi_\alpha \arg^2 \left( \frac{dz_\alpha}{dz_\beta} \right)
\]
the number \( d(\nu(x), \pi(x)) = d(\nu_\alpha(x), \pi_\alpha(x)) \) is invariantly defined for almost all \( x \), Define a metric on \( B(R) \), the Beltrami differentials on \( R \), by
\[
\gamma(\nu, \pi) = \text{ess sup} \ d(\nu(x), \pi(x))
\]
where the essential supremum is taken over almost all \( x \) in \( R \). This metric is natural in the sense that it makes each map \( C(f): B(S) \to B(R) \) an isometry (if we define a metric \( \tau \) on \( B(S) \) in the same way). To establish this result it is necessary only to observe that in terms of a local coordinate chart the map \( C(f) \) is induced by a Möbius transformation from the unit disk in each fibre of \( L^\infty(J^{-1}) \) to the unit
disk in the corresponding fibre of $L^\infty(\bar{KK}^{-1})$, and the Poincaré metric is invariant under Möbius transformations. This metric is shown to be induced by a Finsler structure on $B(R)$ by Earle and Eells in [5].

We shall need the following estimate comparing the metric $\tau$ with the $L^\infty$ norm.

**Theorem 5.** Let $v$ and $\pi$ be Beltrami differentials. Then

$$\tau(v, \pi) - \tau(v, 0) \leq \|v - \pi\| + o(\|\pi\|)$$

where $o(t)/t \to 0$ as $t \to 0$.

**Proof.** The Poincaré metric is very close to the Euclidean metric at the origin, so that

$$d(z, w) - d(z, 0) \leq |z - w| - |z| + o(|w|)$$

where $o(t)/t \to 0$ as $t \to 0$. To prove this, regard $d(z, w)$ and $|z - w|$ as two functions of $w$ for fixed $z \neq 0$ and evaluate the derivatives $\partial / \partial w$ at $w = 0$. In both cases a laborious calculation yields $-\bar{z}/2|z|$. Since $d(z, w)$ and $|z - w|$ are real the derivatives $\partial / \partial \bar{w}$ are obtained by conjugation. Hence both partial derivatives agree at $w = 0$ and are continuous, so the estimate follows from the Mean Value Theorem. On the other hand, if $z = 0$ we may regard

$$d(0, w) = (1/2) \log (1 + |w|)/(1 - |w|)$$

as a function of $|w|$, and taking its derivative at $|w| = 0$ we again obtain the required estimate.

By replacing $o(t)$ by sup $\{o(u) \mid 0 \leq u \leq t\}$ we may assume that the error estimate $o(t)$ is monotone nondecreasing. Then for the metric $\tau$ we have

$$\tau(v, \pi) = \text{ess sup} d(v(x), \pi(x))$$

$$\leq \text{ess sup} \{d(v(x), 0) + |v(x) - \pi(x)| - |v(x)| + o(|\pi(x)|)\}$$

$$\leq \text{ess sup} \{d(v(x), 0) - |v(x)|\} + \text{ess sup} |v(x) - \pi(x)| + \text{ess sup} o(|\pi(x)|).$$

But $d(z, 0) - |z|$ is a monotonic increasing function of $|z|$, since

$$d/dr(1/2) \log (1 + r)/(1 - r) = 1/(1 - r^2).$$

Therefore

$$\tau(v, \pi) \leq \tau(v, 0) - \|v\| + \|v - \pi\| + o(\|\pi\|)$$

which proves the theorem.

6. Let $N$ be as before the set of all Beltrami differentials of quasiconformal maps of $R$ onto itself homotopic to the identity modulo the boundary.

**Theorem 6.** Let $f: R \to S$ be extremal and $\mu = \beta(f)$. Then $\tau(\mu, 0) \leq \tau(\mu, \pi)$ for all $\pi \in N$.

**Proof.** Suppose $\pi \in N$ and $\tau(\mu, \pi) < \tau(\mu, 0)$. We know that $\pi = \beta(g)$ for some quasiconformal map $g: R \to R$ homotopic to the identity modulo the boundary.
If \( H: R \times [0, 1] \to R \) is a homotopy between \( g \) and the identity fixing the boundary then \( f \circ g^{-1} \circ H \) is a homotopy between \( f \) and the map \( k = f \circ g^{-1} \) which leaves the boundary fixed. Since \( k \circ g = f \), \( C(g)\beta(k) = \beta(f) \). Let \( \mu = \beta(f) \) and \( \lambda = \beta(g) \). Since \( C(g) \) is an isometry and \( C(g)0 = \beta(g) = \pi \),
\[
\tau(\lambda, 0) = \tau(C(g)\lambda, C(g)0) = \tau(\mu, \pi) < \tau(\mu, 0).
\]

But \( \tau(\mu, 0) \) is a monotone increasing function of \( \|\mu\| \) since \( d(x, 0) \) is a monotone increasing function of \( z \). Therefore \( \|\lambda\| < \|\mu\| \). Since \( \lambda = \beta(k) \) and \( k = \) homotopic to \( f \) modulo the boundary, \( f \) is not extremal.

It is now easy to complete the proof of Theorem 1 by imitating the proof of Theorem 2, using the estimate in Theorem 5. Suppose that \( f: R \to S \) is quasiconformal with Beltrami differential \( \mu = \beta(f) \) but that \( \|\mu\| A(R) \leq \|\mu\| \). By the Hahn-Banach Theorem we can find \( v \in L^\infty(\overline{K}K^{-1}) \) with \( \|v\| A(R) = \mu \| A(R) \) and \( \|v\| = \|\mu\| A(R) \leq \|\mu\| \). Then \( \mu - v \in A(R)^+ \) and \( A(R)^+ \) is the tangent space at \( 0 \) to the analytic submanifold \( N \). Consequently we can find a \( C^1 \) path \( \alpha: (-\epsilon, \epsilon) \to N \) with \( \alpha(0) = 0 \) and \( Da(0)(l) = \mu - v \). Then, restricting our attention to positive \( t \),
\[
\|\alpha(t) - \alpha(0) - Da(0)(t)\| / t = \|\alpha(t) - t\mu + t v\| / t \to 0 \quad \text{as} \quad t \to 0.
\]

This makes \( \|\alpha(t)\| \leq Kt \) for some constant \( K \) and all sufficiently small \( t \), so \( \alpha(\|\alpha(t)\|) / t \to 0 \) as \( t \to 0 \) as well.

Now by the estimate of Theorem 5
\[
\tau(\mu, \alpha(t)) - \tau(\mu, 0) \leq \|\mu - \alpha(t)\| - \|\mu\| + o(\|\alpha(t)\|).
\]

Also \( \|\mu - \alpha(t)\| \leq (1 - t)\|\mu\| + t\|v\| + \|\alpha(t) - t\mu + tv\| \). Combining these inequalities with the estimates above we see that
\[
\tau(\mu, \alpha(t)) - \tau(\mu, 0) \leq \beta(t) - \tau(\|\mu\| - \|v\|)
\]
where \( \beta(t) / t \to 0 \) as \( t \to 0 \). Since \( \|\mu\| - \|v\| > 0 \) we must have \( \tau(\mu, \alpha(t)) < \tau(\mu, 0) \) for all sufficiently small positive \( t \). Then since \( \alpha(t) \in N \) it follows from Theorem 6 that the map \( f: R \to S \) is not extremal.

**References**


**Cornell University, Ithaca, New York**