

ON HOMOTOPY SPHERES BOUNDING HIGHLY CONNECTED MANIFOLDS

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Let $\alpha \in \pi_{n-1}(SO_m)$, $\beta \in \pi_{m-1}(SO_n)$ and consider the manifold $N_{\alpha,\beta}$ obtained by plumbing together the total spaces E_α and E_β of the D^m (respectively D^n) bundle over S^n (respectively S^m) with characteristic classes α (respectively β). In [10] Milnor shows that $\partial N_{\alpha,\beta}$ is almost always a homotopy sphere. In particular, if $m=n$ and $\alpha, \beta \in \text{Im}(\pi_{n-1}(SO_{n-1}) \rightarrow \pi_{n-1}(SO_n))$, then $\partial N_{\alpha,\beta}$ is a homotopy sphere of dimension $(2n-1)$. Let Λ_{2n-1} be the set of all such spheres. It is the object of this note to investigate these spheres.

It is shown in [12] (and [8] by quite different methods) that Λ_{2n-1} is a subgroup of Θ_{2n-1} , the group of homotopy $(2n-1)$ spheres, whose elements represent obstructions to smoothing $(n-1)$ connected PL manifolds of dimension $2n$ where n skeleton can be smoothed. It is of interest, then, to know what the subgroup Λ_{2n-1} is. In this connection we prove

THEOREM 1. *Let $n=4m$ with $m>2$ and $\tau_m = \sigma_{2m}/\text{g.c.d.}[\sigma_m^2, \sigma_{2m}]$ where*

$$\sigma_k = 2^{2k}(2^{2k-1} - 1)B_k a_k j_k / 2k,$$

B_k is the k th Bernoulli number, a_k is 1 or 2 depending on whether k is even or odd, and j_k is the order of the image of the stable J homomorphism, $J: \pi_{4k-1}(SO) \rightarrow \pi_{4k-1}^S$. Then Λ_{2n-1} is a cyclic group of order dividing $j_m^2 \cdot \tau_m$. Furthermore Λ_{8m-1} contains a subgroup of $\Theta_{8m-1}(\partial\pi)$ of order τ_m .

Let P^{8m} be the $(4m-1)$ connected π manifold of index 8 with ∂P a generator of $\Theta_{8m-1}(\partial\pi)$. Let M^{8m} be the closed PL manifold obtained from P by attaching a cone over the boundary. As an application of Theorem 1 we prove

THEOREM 2. *M^{8m} does not have the homotopy type of a closed differentiable manifold.*

Since the manifold M of Theorem 2 satisfies Poincaré duality and the trivial bundle over M is reducible, the Browder-Novikov Theorem on the homotopy type of manifolds [3], [11] gives the

COROLLARY. *There is no reducible vector bundle over M whose dual Pontrjagin classes satisfy the index theorem.*

Received by the editors February 2, 1968 and, in revised form, May 28, 1968.

⁽¹⁾ Partially supported by the National Science Foundation under grant no. NSF GP 6961.

1. **Proofs of Theorems 1 and 2.** Theorems 1 and 2 are both consequences of Lemma 1.5 below. We state the lemmas needed to prove 1.5 first, but defer their proofs to §2. We assume the reader is familiar with the theory in [12].

Let X be an $(n-1)$ connected space satisfying Poincaré duality for the dimension $2n$. Let $\mu_X \in H_{2n}(X)$ be the orientation class.

LEMMA 1.1. *Let ζ^k be a reducible vector bundle over X , $k \gg 2n$. Then there are oriented $(n-1)$ connected smooth handlebodies N, P with P parallelizable and maps f, \bar{f} such that*

- (i) $\partial N = -\partial P$ is a homotopy sphere.
- (ii) $f: N \cup P \rightarrow X$ and $f^! \zeta = \nu$ is the stable normal bundle of $N \cup P$ where $N \cup P$ is the oriented smooth manifold obtained from the disjoint union of N and P by gluing along the boundaries via the identity map.
- (iii) $\bar{f}: \hat{N} \rightarrow X$ is a homotopy equivalence where \hat{N} , the closure of N , is obtained from N by attaching the cone over the boundary.
- (iv) *The diagram*

$$\begin{array}{ccc}
 N \cup P & \xrightarrow{f} & X \\
 q \downarrow & & \uparrow \bar{f} \\
 N \cup P/P \approx \hat{N} & &
 \end{array}$$

is homotopy commutative.

Proof. The proof is given in §2.

If n is even, $HJ \oplus S: \pi_{n-1}(SO_n) \rightarrow Z \oplus \pi_{n-1}(SO)$ is a monomorphism where H is the Hopf invariant, J the J homomorphism, and S the suspension. Hence if $(H; \langle, \rangle; \alpha)$ is an n -space, α is completely determined by $HJ\alpha$ and $S\alpha$. By recalling that $HJ\alpha(x) = \langle x, x \rangle$ and $S\alpha$ is a homomorphism, we will consider an n -space (n even) to be a triple $(H; \langle, \rangle; \beta)$ where H is a free abelian group, \langle, \rangle is a non-singular bilinear form on H ; and $\beta: H \rightarrow \pi_{n-1}(SO)$ is a homomorphism.

If ζ^k is any stable bundle over X , we may define a map

$$\beta_\zeta: H_n(X) \rightarrow \pi_{n-1}(SO)$$

by observing that every element $x \in H_n(X)$ is spherical. Thus there is a map $f_x: S^n \rightarrow X$ representing x . Let $\beta_\zeta(x)$ be the characteristic class of $f_x^! \zeta$. Clearly β_ζ is a homomorphism.

LEMMA 1.2. *Let n be even. Let X, ζ^k , and N be as in 1.1. Then N is the handlebody corresponding to the n -space $(H_n(X); \langle, \rangle; -\beta_\zeta)$. Thus N is determined completely by X and ζ .*

Proof. See §2.

Let $n=4m$ and λ, μ be integers. Then $N_{\lambda, \mu}$ will denote the handlebody obtained by plumbing together the total spaces of the D^n bundles over S^n with characteristic

classes $\lambda', \mu' \in \pi_{n-1}(SO_n)$ satisfying $(HJ \oplus S)(\lambda') = (0, \lambda)$ and $(HJ \oplus S)(\mu') = (0, \mu)$. Since $HJ(\lambda') = 0$, $\lambda' \in \text{Im}(\pi_{n-1}(SO_{n-1}) \rightarrow \pi_{n-1}(SO))$ and $\partial N_{\lambda, \mu}$ is a homotopy sphere (see [10]). We note for future reference that the n space corresponding to $N_{\lambda, \mu}$ has $H = Z + Z$ with generators e_1 and e_2 . Relative to this basis, \langle , \rangle has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally $S\alpha(e_1) = \lambda$, $S\alpha(e_2) = \mu$.

The order of the image of $J: \pi_{4m-1}(SO) \rightarrow \pi_{4m-1}^S$ will be denoted by j_m .

LEMMA 1.3. *Let $n = 4m$ and N_i^{2n} , $i = 1, 2$, be $(n - 1)$ connected handlebodies with boundaries homotopy spheres. If the closures \hat{N}_i of N_i have the same homotopy type, then there is an integer λ such that $\partial N_1 = \partial N_2 \# \partial N_{\lambda, j_m}$.*

Proof. The proof is given in §2.

LEMMA 1.4. $\lambda \partial N_{1, \mu} = \mu \partial N_{\lambda, 1}$ and $\partial N_{\lambda, \mu} = \lambda \mu \cdot \partial N_{1, 1}$.

Proof. See §2.

Theorems 1 and 2 are both consequences of the following key:

LEMMA 1.5. *Let $n = 4m$ ($m > 2$). Then $\partial N_{j_m, j_m}$ is an element of order τ_m .*

Proof. Let ζ_r be the l -plane bundle over S^n , $l \gg 4m$, with characteristic class rj_m , $r = 0, \pm 1, \pm 2, \dots$. Then ζ_r is fiber homotopically trivial for any r . Let $X = S_1^n \times S_2^n$ and $\pi_i: X \rightarrow S_i^n$, $i = 1, 2$, be the projections. Then $\xi = \pi_1^!(\zeta_{-1}) \oplus \pi_2^!(\zeta_{-1})$ is a fiber homotopically trivial bundle over X . Thus ξ has the same fiber homotopy type as the stable normal bundle of X . Hence ξ is reducible. By 1.1 and 1.2, then, there is a unique handlebody N with $\partial N \in \Theta_{2n-1}(\partial\pi)$ and a homotopy equivalence

$$\bar{f}: N \cup C(\partial N) \rightarrow X$$

such that $\bar{f}^!\xi|N$ is the stable normal bundle of N . In fact N is the handlebody corresponding to the n -space $(H_n(X), \langle , \rangle, -\beta_\xi)$. Let $e_i \in H_n(X)$ correspond to the inclusion $S_i^n \rightarrow S_1^n \times S_2^n$, $i = 1, 2$. Then e_1, e_2 are generators of $H_n(X) = Z \oplus Z$ satisfying $e_i e_j = 1 - \delta_{ij}$ and $-\beta_\xi(e_1) = j_m$, $-\beta_\xi(e_2) = tj_m$. Since this is exactly the n -space corresponding to N_{j_m, tj_m} it follows that $\partial N_{j_m, tj_m} \in \Theta_{2n-1}(\partial\pi)$ for all t .

In order to compute the order of $\partial N_{j_m, tj_m}$ in $\Theta_{2n-1}(\partial\pi)$, we compute the index $\tau(P)$ of the π -manifold P of 1.1. But, in the notation of 1.1, since $f^!\xi$ is the stable normal bundle of $P \cup N$,

$$\begin{aligned} \tau(P) &= \tau(P) + \tau(N) = \tau(P \cup N) = \langle L_{2m}(\bar{p}_1(f^!\xi), \dots, \bar{p}_{2m}(f^!\xi)), \mu_{P \cup N} \rangle \\ &= \langle L_{2m}(\bar{p}_1(\xi), \dots, \bar{p}_{2m}(\xi)), \mu_X \rangle. \end{aligned}$$

(The last equality follows since $f_*\mu_{P \cup N} = \mu_X$.)

Simple computations using the Whitney sum formula show that

$$\bar{p}_m(\xi) = -[p_m(\zeta_{-1}) \times 1 + 1 \times p_m(\zeta_{-1})], \quad \bar{p}_{2m}(\xi) = p_m(\zeta_{-1}) \times p_m(\zeta_{-1}).$$

Thus $2\bar{p}_{2m}(\xi) = (\bar{p}_m(\xi))^2$. In the case that the only nonzero Pontrjagin classes are in dimensions m and $2m$, $L_{2m} = s_{2m}p_{2m} + \frac{1}{2}(s_m^2 - s_{2m})p_m^2$ where $s_i = 2^{2i}(2^{2i-1} - 1)B_i/(2i)!$. Thus $L_{2m}(\bar{p}_m(\xi), \bar{p}_{2m}(\xi)) = s_m^2 p_{2m}(\xi) = s_m^2 p_m(\xi_{-1}) \times p_m(\xi_{-t})$. By a formula due to Kervaire $p_m(\xi_r) = a_m(2m-1)!rj_m e$ for an appropriate generator $e \in H^n(S^n)$. Since we may orient X in any desired fashion we may suppose that $\mu_X = e \times e$. Substituting for s_m , $p_m(\xi_{-1})$, and $p_m(\xi_{-t})$ and combining the formulae above then gives

$$\tau(P) = \sigma_m^2 t.$$

By [7, Theorem 7.5], $\partial N_{j_m, t j_m} = \partial P = S^{8m-1}$ if and only if $\tau(P) \equiv 0 \pmod{\sigma_{2m}}$. Thus $\partial N_{j_m, t j_m} = S^{8m-1}$ if and only if $\sigma_{2m} | \sigma_m^2 t$, i.e. if and only if $\sigma_{2m}/\text{g.c.d.}[\sigma_m^2, \sigma_{2m}]$ divides t . Since $\partial N_{j_m, t j_m} = t \partial N_{j_m, j_m}$ (cf. 1.4) the lemma follows.

We now prove Theorems 1 and 2.

Proof of Theorem 1. It follows directly from [12, Theorem 2 (1)] that Λ_{2n-1} is a cyclic group generated by $\partial N_{1,1}$. An argument similar to 1.4 however shows that $\partial N_{j_m, j_m} = j_m^2 \partial N_{1,1}$. Thus Theorem 1 follows from 1.5.

Proof of Theorem 2. Suppose that M has the homotopy type of a closed differentiable manifold M' . Then M' is the closure of a $(4m-1)$ connected handlebody N' with $\partial N' = S^{2n-1}$ and by 1.3 $\partial P = S^{2n-1} \# \partial N_{\lambda, j_m} = \partial N_{\lambda, j_m}$. Since ∂P generates $\Theta_{8m-1}(\partial\pi)$, the theorem follows by proving that $\partial N_{\lambda, j_m}$ cannot be a generator of $\Theta_{8m-1}(\partial\pi)$.

To see the last statement we recall that since $\partial N_{j_m, j_m} = j_m \partial N_{1, j_m}$ by 1.4 and the order of $\partial N_{j_m, j_m} = \tau_m$, the order of $\partial N_{1, j_m}$ divides $j_m \tau_m$. Thus the power of 2 in the order of $\partial N_{1, j_m}$ is less than or equal to the power of 2 in $j_m \tau_m$. But an easy computation shows that τ_m is odd and it is well known that the power of 2 in j_m is less than the power of 2 in $16m$, at least for $m \geq 3$. Thus the power of 2 in the order of $\partial N_{1, j_m}$ is less than that in $16m$. By Kervaire-Milnor [7], the order of $\Theta_{8m-1}(\partial\pi)$ is $\sigma_{2m}/8$. An easy computation now shows that the power of 2 in $16m$ is less than that in $\sigma_{2m}/8$. Thus $\partial N_{1, j_m}$ does not generate $\Theta_{8m-1}(\partial\pi)$ and $\partial N_{\lambda, j_m} = \lambda \partial N_{1, j_m}$ cannot generate $\Theta_{8m-1}(\partial\pi)$.

The author would like to thank the referee for suggesting the argument above which greatly simplifies the original proof.

2. Proof of Lemmas 1.1-1.4. In this section we prove the lemmas used in the proofs of Theorems 1 and 2. The proofs rely heavily on the classification theory of [12] and we shall assume the reader's familiarity with the theory and notation of that paper. In particular we shall use n -spaces and the computation of the Grothendieck group of n -spaces freely.

Proof of 1.1. Let $g: S^{2n+k} \rightarrow T(\zeta)$ be such that the image of $[g]$ under the Hurewicz homomorphism is the generator $\phi(\mu_X) \in H_{2n+k}(T(\zeta))$ where ϕ is the Thom isomorphism. By the usual methods of transverse regularity and surgery we may assume $M = g^{-1}(X)$ is $(n-1)$ connected. We shall decompose M in the desired fashion by showing that the n -space $(H_n(M); \langle \ , \ \rangle; \alpha)$ corresponding to M may be decomposed as the direct sum of n -spaces.

Let $f=g|M$ and $\mu_M \in H_{2n}(M)$ be the orientation class satisfying $f_*\mu_M = \mu_X$. Let

$$H' = \ker (f_*: H_n(M) \rightarrow H_n(X)),$$

$$H'' = \text{im}((\cap \mu_m)f^*: H^n(X) \rightarrow H^n(M)).$$

Then it is well known that $H_n(M) \approx H' \oplus H''$ and that this splitting extends to the intersection pairing; that is $\langle x, y \rangle = 0$ for any $x \in H', y \in H''$. Thus if $z = x + y$ is an element of $H_n(M)$ with $x \in H', y \in H'', \alpha(z) = \alpha(x + y) = \alpha(x) + \alpha(y) + \langle x, y \rangle \partial \iota_n = \alpha(x) + \alpha(y)$. Hence $(H_n(M); \langle, \rangle; \alpha) = (H'; \langle, \rangle'; \alpha') \oplus (H''; \langle, \rangle''; \alpha'')$ where $\langle, \rangle' = \langle, \rangle|H'; \alpha' = \alpha|H'; \langle, \rangle'' = \langle, \rangle|H''; \alpha'' = \alpha|H''$. Since \langle, \rangle is nonsingular, so are \langle, \rangle' and \langle, \rangle'' and the n -space for M is decomposed as stated.

Let P and N be the handlebodies corresponding to

$$(H'; \langle, \rangle'; \alpha') \quad \text{and} \quad (H''; \langle, \rangle''; \alpha'')$$

respectively. It follows immediately from the construction that ∂P and ∂N are homotopy spheres and that $M^{2n} - \text{int}(D^{2n})$ is diffeomorphic to the boundary connected sum of P and N . Thus $S^{2n-1} = \partial P \# \partial N, \partial P = -\partial N$ and M is almost diffeomorphic to $P \cup N$. In the sequel, we identify M with $P \cup N$ under this almost diffeomorphism. Since almost diffeomorphic manifolds have the same stable normal bundle, $f^!\zeta$ is the stable normal bundle of $P \cup N$.

The handlebody P is a π -manifold for if $h: S^n \rightarrow M$ is an embedding representing any element $x \in H'$, since $f_*x = 0, f^!\zeta|h(S^n) = \nu_M|S^n$ is trivial where ν_M denotes the stable normal bundle of M . Thus the normal bundle $\nu(h(S^n), M)$ of $h(S^n)$ in M is stably trivial since $\alpha(x)$ is the characteristic of $\nu(h(S^n), M)$; this means that $S\alpha(x) = 0$ for any $x \in H'$. Hence P is a π -manifold.

It follows that $f|P$ is null homotopic and that there is a map $\bar{f}: \hat{N} \rightarrow X$ such that the diagram in (iv) is homotopy commutative.

To see (iii) note that since $f_*: H_n(N \cup P) \rightarrow H_n(X)$ is onto, so is \bar{f}_* . On the other hand, since $H_n(P) = H' = \ker f_* = \ker \bar{f}_*q_*: H_n(N \cup P) \rightarrow H_n(X)$ and $H_n(P) = \ker q_*: H_n(N \cup P) \rightarrow H_n(N \cup P, P) \approx H_n(N \cup P/P), \bar{f}_*$ is a monomorphism in dimension n . Finally since f and q have degree 1, so does \bar{f} . Thus $\bar{f}_*: H_*(\hat{N}) \rightarrow H_*(X)$ is an isomorphism and (iii) follows.

Proof of 1.2. It is easy to see that the map \bar{f} of 1.1 induces an isomorphism of the n -space corresponding to N with $(H_n(X); \langle, \rangle; -\beta_c)$.

Proof of 1.3. The proof is divided into two cases.

Case 1. The index $\tau(\hat{N}_1) = 0 = \tau(\hat{N}_2)$. In this case since the quadratic form $\langle x, x \rangle, x \in H_n(N_1) = H_n(\hat{N}_1)$, assumes only even values and is unimodular, there is a symplectic basis $b_1, \dots, b_s, c_1, \dots, c_s$ for $H_n(\hat{N}_1)$. Thus $b_i b_j = 0 = c_i c_j$ and $b_i c_j = \delta_{ij}$.

Let $f: \hat{N}_1 \rightarrow \hat{N}_2$ be a homotopy equivalence. Then $f_*: H_*(\hat{N}_1) \rightarrow H_*(\hat{N}_2)$ preserves intersections and the map $J\alpha$; see [12, Lemma 8]. Then $b'_i = f_*b_i, c'_i = f_*c_i, i = 1, \dots, s$, is a symplectic basis for $H_n(\hat{N}_2)$ and $J\alpha_1(b_i) = J\alpha_2(b'_i)$. Thus $\alpha_1(b_i) - \alpha_2(b'_i) \in \ker (J: \pi_{n-1}(SO_n) \rightarrow \pi_{2n-1}(S^n))$ and since

$$\begin{array}{ccc}
 \pi_{n-1}(SO_n) & \xrightarrow{S} & \pi_{n-1}(SO) \\
 \downarrow J & & \downarrow J \\
 \pi_{2n-1}(S^n) & \xrightarrow{E} & \pi_{n-1}^S
 \end{array}$$

commutes up to sign, $JS(\alpha_1(b_i) - \alpha_2(b'_i)) = 0$. Hence, since $n = 4m$, $S\alpha_1(b_i) = S\alpha_2(b'_i) + \beta_i j_m$ for some integer β_i where we have identified j_m with a generator of the kernel of the stable J homomorphism. Similarly $S\alpha_1(c_i) = S\alpha_2(c'_i) + \gamma_i j_m$ for some integer γ_i .

Following Wall, let $\chi_k \in H_n(\hat{N}_k)$, $k = 1, 2$, be such that $\langle \chi_k, x \rangle = S\alpha(x)$ for any $x \in H_n(\hat{N}_k)$. Since $b_1, \dots, b_s, c_1, \dots, c_s$ is a symplectic basis $\chi_1 = \sum_{i=1}^s S\alpha_1(c_i)b_i + \sum_{i=1}^s S\alpha_1(b_i)c_i$. Similarly $\chi_2 = \sum_{i=1}^s S\alpha_2(c'_i)b'_i + \sum_{i=1}^s S\alpha_2(b'_i)c'_i$. Let

$$\lambda = \sum_{i=1}^s (\gamma_i S\alpha_2(b'_i) + \beta_i S\alpha_2(c'_i) + \beta_i \gamma_i j_m)$$

and consider $N'_1 = N_1 \natural (S^n \times S^n - \hat{D}^{2n})$ and $N'_2 = N_2 \natural N_{\lambda, j_m}$ where \natural denotes boundary connected sum. We shall show that N'_1 and N'_2 determine the same element in the Grothendieck group of n -spaces.

By [12, Theorem 2, (1)], the element of \mathcal{G}_n corresponding to any $(n - 1)$ connected handlebody N^{2n} is determined completely by the rank r of $H_n(M)$, the index τ of the intersection pairing, and the element $\chi^2 = \langle \chi, \chi \rangle$. In the case at hand we have easily $\text{rank } H_n(N'_1) = \text{rank } H_n(N'_2)$ and $\tau(N'_1) = \tau(N'_2)$. Finally, since $\chi(N'_1) = \chi(N_1) + 0$ and $\chi(N'_2) = \chi(N_2) + \chi(N_{\lambda, j_m})$ we have

$$\begin{aligned}
 \chi(N'_1)^2 &= 2 \sum_{i=1}^s S\alpha_1(b_i)S\alpha_1(c_i) \\
 &= 2 \sum_{i=1}^s (S\alpha_2(b'_i) + \beta_i j_m)(S\alpha_2(c'_i) + \gamma_i j_m) \\
 &= 2 \sum_{i=1}^s S\alpha_2(b'_i)S\alpha_2(c'_i) + 2j_m \lambda \\
 &= \chi(N_2)^2 + \chi(N_{\lambda, j_m})^2 \\
 &= \chi(N'_2)^2
 \end{aligned}$$

since $\langle x, y \rangle = 0$ for any elements $x \in H_n(N_2) \subset H_n(N'_2)$ and $y \in H_n(N_{\lambda, j_m}) \subset H_n(N'_2)$. Thus N'_1 and N'_2 determine the same element of \mathcal{G}_n . Since the map $N \rightarrow \partial N$ induces a homomorphism $\partial: \mathcal{G}_n \rightarrow \mathcal{O}_{2n-1}$, it follows that $\partial N'_1 = \partial N'_2$. Hence $\partial N_1 = \partial N_1 \# S^{2n-1} = \partial N'_1 = \partial N'_2 = \partial N_2 \# \partial N_{\lambda, j_m}$ and Case 1 of the proposition follows.

Case 2. General case. Since \hat{N}_1 has the homotopy type of \hat{N}_2 , the index $\tau(\hat{N}_1) = \tau(\hat{N}_2)$. But also 8 divides $\tau(\hat{N}_k)$, $k = 1, 2$. Therefore there is a π -manifold P with $\partial P \neq \emptyset$ such that $\tau(P) = \tau(\hat{N}_k)$. Consider the manifolds M_k obtained by closing $N_k \natural (-P)$, $k = 1, 2$, where $-P$ denotes P with the opposite orientation. Since \hat{N}_1 and \hat{N}_2 have the same homotopy type, a simple argument shows that M_1 and M_2 have the same homotopy type. But also $\tau(M_k) = \tau(\hat{N}_k) - \tau(P) = 0$, $k = 1, 2$. By Case 1, therefore, $\partial N_1 \# (-\partial P) = \partial(N_1 \natural -P) = \partial(N_2 \natural -P) \# \partial N_{\lambda, j_m} = \partial N_2 \# -\partial P \# \partial N_{\lambda, j_m}$ for some λ . The lemma follows.

Proof of 1.4. Under the identification $\mathcal{G}_{4m} = Z \oplus Z \oplus Z$ of [12, Theorem 2, (1)], $N_{\lambda, \mu}$ corresponds to $(1, 0, \lambda\mu)$. Since $\partial: \mathcal{G}_{4m} \rightarrow \Theta_{4m-1}$ depends only on the last two coordinates, the lemma follows.

BIBLIOGRAPHY

1. J. F. Adams, *On the groups $J(X)$* . IV, *Topology* **5** (1966), 21–71.
2. R. Bott, *A note on the Samelson product in the classical groups*, *Comment. Math. Helv.* **34** (1960), 249–256.
3. W. Browder, *Homotopy type of differentiable manifolds*, *Colloquium on Algebraic Topology*, Aarhus, 1962, pp. 42–46.
4. A. Haefliger, *Plongements différentiables de variétés dans variétés*, *Comment. Math. Helv.* **36** (1961), 47–82.
5. W. C. Hsiang, J. Levine and R. H. Szczarba, *On the normal bundle of a homotopy sphere embedded in Euclidean space*, *Topology* **3** (1965), 173–181.
6. M. Kervaire and J. W. Milnor, *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, *Proc. Internat. Congr. Math.*, Edinburgh, 1958.
7. ———, *Groups of homotopy spheres*, *Ann. of Math.* **77** (1963), 504–537.
8. A. Kosinski, *On the inertia group of π -manifolds*, *Amer. J. Math.* **89** (1967), 227–248.
9. M. Mahowald, *On the order of the image of J* , *Topology* **6** (1967), 371–378.
10. J. W. Milnor, *Some differentiable structures on spheres*, *Amer. J. Math.* **81** (1959), 962–972.
11. S. P. Novikov, *Diffeomorphisms of simply connected manifolds*, *Dokl. Akad. Nauk SSSR* **143** (1962), 1046–1049 = *Soviet Math. Dokl.* **3** (1962), 540–543.
12. C. T. C. Wall, *Classification of $(n-1)$ connected $2n$ manifolds*, *Ann. of Math.* **75** (1962), 163–189.

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