A RADON-NIKODYM THEOREM FOR STONE ALGEBRA VALUED MEASURES

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Introduction. Let $C(S)$ be the ring of continuous real valued functions on a compact Hausdorff space S. Stone [5] shows that each bounded subset of $C(S)$ has a least upper bound (in $C(S)$) if and only if the closure of each open subset of S is open; in this event we call $C(S)$ a Stone algebra. Throughout this paper $C(S)$ is a Stone algebra.

It is convenient to adjoin an object $+\infty$, not in $C(S)$, and extend the partial ordering of $C(S)$ to $C(S) \cup \{+\infty\}$ in the obvious way. When $\{a_n : n=1, 2, \ldots\}$ is an unbounded set in $C(S)$ we define $\bigvee_{n=1}^{\infty} a_n$ to be $+\infty$.

Following [6] we define a $C(S)$-valued measure on a measurable space $(X, \mathscr{B})$ to be a map $m: \mathscr{B} \to C(S) \cup \{+\infty\}$ such that $mE \geqq 0$ for each E and, if $\{E_n\}$ $n=1, 2, \ldots$ is a pairwise disjoint sequence of sets in $\mathscr{B}$ then

$$m \bigcup_{n=1}^{\infty} E_n = \bigvee_{n=1}^{\infty} \sum_{r=1}^{n} mE_r.$$

In the special case where $C(S)$ is identifiable with the self-adjoint part of a commutative von Neumann algebra the latter condition is equivalent to

$$mE = \lim \sum_{r=1}^{n} mE_r,$$

where the limit is taken in the weak operator topology. But Floyd [1] gives an example of a Stone algebra such that there is no Hausdorff vector topology in which each bounded monotone increasing sequence converges to its least upper bound. This pathology causes some difficulty.

In [6] a theory of Stone algebra valued measures and integrals was constructed and in [7] this was applied to give a spectral theorem for normal operators on a Kaplansky-Hilbert module. It is not necessary to have read these papers to understand the work here although we shall use some of the results obtained in [6].

The goal of this paper is an analogue for Stone algebra valued measures of the classical Radon-Nikodym Theorem. In a later publication I intend to discuss applications of the results of this paper to averaging operators and to Boolean algebras.

The following analogue of the Radon-Nikodym Theorem might be conjectured.

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Let l and m be finite $C(S)$-valued measures on a measurable space $(X, \mathscr{B})$ and suppose that $mE=0$ implies $lE=0$. Then there exists a $\mathscr{B}$-measurable function $f\colon X \to \boldsymbol{R}$ such that $lE=\int_E f\,dm$ for each $E \in \mathscr{B}$. It is easy to show that this is false. Let $X=S=\{a, b\}$, a two-point space, then $C(S)$ can be identified with the space of all ordered pairs of real numbers. Let $\mathscr{B}$ be the collection of all subsets of X. Define measures l and m by

$$l\{a\} = m\{b\} = (1, 0) \quad \text{and} \quad l\{b\} = m\{a\} = (0, 1).$$

Assume f exists and take $E=\{a\}$. Then

$$(1, 0) = l\{a\} = \int_{\{a\}} f\,dm = f(a)(0, 1)$$

so that $(1, 0)$ is a scalar multiple of $(0, 1)$ which is impossible.

In §4 a generalization of the Radon-Nikodym Theorem is obtained. In the example above the measures are "twisted" relative to each other. We restrict ourselves to *modular measures* and are then able to define absolute continuity in such a way as to exclude "twisting".

Modular measures are defined in §2. When a $C(S)$-valued measure is modular its L^p-spaces form modules over $C(S)$ and integration with respect to the measure is a module homomorphism of L^1 into $C(S)$.

When $C(S)$ satisfies the countable chain condition and the measure is modular then L^2 is a Kaplansky-Hilbert module. This gives new examples of Kaplansky-Hilbert modules. More important for our purpose is that it enables us to use Kaplansky's powerful results.

It is now of interest to know when a Stone algebra satisfies the countable chain condition. If S is the Boolean space of a complete Boolean algebra satisfying the countable chain condition then $C(S)$ also satisfies the countable chain condition. It is well known that when μ is a finite real valued measure on a space $(X, \mathscr{B})$ then $L^\infty(X, \mathscr{B}, \mu)$ satisfies the countable chain condition. We show that, when m is a finite $C(S)$-valued measure, if $C(S)$ satisfies the countable chain condition then so also does $L^\infty(X, \mathscr{B}, m)$ and thus $L^\infty(X, \mathscr{B}, m)$ is a Stone algebra.

For the Radon-Nikodym Theorem to hold and for L^2-spaces to be Kaplansky-Hilbert modules it is not essential that the modular measures take their values in a Stone algebra satisfying the countable chain condition but this is a simple and convenient condition for applications.

1. **L^p-spaces.** Throughout this paper, unless we state otherwise, X is a nonempty set and $\mathscr{B}$ is a σ-algebra of subsets of X; $m\colon \mathscr{B} \to C(S)$ is a positive $C(S)$-valued measure which is σ-additive in the sense defined in the introduction; $\mathscr{L}^1(X, m)$ is the set of real valued m-integrable functions on X; for $1 \leqq p < \infty$, $\mathscr{L}^p(X, m)$ is the set of $\mathscr{B}$-measurable functions f such that $|f|^p \in \mathscr{L}^1(X, m)$; $\mathscr{L}^\infty(X, m)$ is the set of $\mathscr{B}$-measurable functions such that for each $f \in \mathscr{L}^\infty(X, m)$ there can be found $F \in \mathscr{B}$ with $mF=0$ and f bounded on $X-F$.

We call $(X, \mathscr{B}, m, C(S))$ a *complete* $C(S)$-measure space if $mE=0$ and $N\subset E$ implies $N\in\mathscr{B}$.

LEMMA 1.1. *Let $N\in\mathscr{N}$ iff $N\subset E$ for some $E\in\mathscr{B}$ such that $mE=0$. Let $\mathscr{B}^*$ be the collection of all sets of the form $B\cup N$ where $B\in\mathscr{B}$ and $N\in\mathscr{N}$. Then $\mathscr{B}^*$ is a σ-algebra of subsets of X and there is a unique measure $m^*: \mathscr{B}^*\to C(S)\cup\{+\infty\}$ such that $m^*B=mB$ for each $B\in\mathscr{B}$. Moreover $(X, \mathscr{B}^*, m^*, C(S))$ is a complete $C(S)$-measure space and if f^* is $\mathscr{B}^*$-measurable there can be found a $\mathscr{B}$-measurable function f such that $m^*\{x\in X : f^*(x)\neq f(x)\}=0$.*

The first part of the lemma is straightforward. Let f^* be a positive $\mathscr{B}^*$-measurable function and $\{f_n^*\}$, $n=1, 2, \dots$, a monotone increasing sequence of $\mathscr{B}^*$-simple functions such that $f^*(x)=\sup f_n^*(x)$ for each x. For each n we can find a $\mathscr{B}$-simple function f_n such that $f_n\leqq f_n^*$ and $m^*\{x : f_n(x)<f_n^*(x)\}=0$. Let $f=\sup f_n$ then $f\leqq f^*$ and

$$\begin{aligned}\{x : f^*(x) \neq f(x)\} &= \bigcup_{n=1}^{\infty}\{x : f(x) < f_n^*(x)\}\\ &\subset \bigcup_{n=1}^{\infty}\{x : f_n(x) < f_n^*(x)\}.\end{aligned}$$

So $m^*\{x : f(x)\neq f^*(x)\}=0$ and the result follows.

LEMMA 1.2. *For $1\leqq p\leqq\infty$, $\mathscr{L}^p(X, m)$ is a vector lattice.*

This is a trivial consequence of the definition of $\mathscr{L}^p(X, m)$ and the inequality, valid for $1\leqq p<\infty$,

$$|f+g|^p \leqq (|f|+|g|)^p \leqq 2^p(|f|^p+|g|^p).$$

When f and g are $\mathscr{B}$-measurable functions f is said to be *equivalent* to g, $f\sim g$, if $m\{x : f(x)\neq g(x)\}=0$. It is easy to verify that this does define an equivalence relation; that $f\sim g$ iff $f-g\sim 0$; and that the collection of functions equivalent to 0 is a subspace of $\mathscr{L}^p(X, m)$. We define $L^p(X, m)$ to be the quotient of $\mathscr{L}^p(X, m)$ by this equivalence relation, then $L^p(X, m)$ is a vector lattice with respect to the induced vector operations and partial ordering. Although $\mathscr{L}^p(X, m)$ may be properly contained in $\mathscr{L}^p(X, m^*)$ it follows from Lemma 1.1 that $L^p(X, m)=L^p(X, m^*)$.

For the rest of this paper, unless we specify otherwise, $\mathscr{L}^p(X)$ is $\mathscr{L}^p(X, m^*)$ and $L^p(X)$ is $L^p(X, m)=L^p(X, m^*)$.

When f is a $\mathscr{B}^*$-measurable function we define $[f]$ to be the set of all $\mathscr{B}^*$-measurable functions equivalent to f. This notation will sometimes be misused by writing f for $[f]$ and, when $[f]=\xi$, by writing ξ for f.

As in classical measure theory the *essential supremum* of the modulus of a function is defined by

$$\operatorname{ess\,sup}|f| = \inf\{\lambda\in \boldsymbol{R}^+ : m\{x : |f(x)| > \lambda\} = 0\}.$$

A measurable function on X is in $\mathscr{L}^\infty(X, m)$ if and only if its modulus has a finite essential supremum. The essential supremum of the modulus of a function defines a seminorm on $\mathscr{L}^\infty(X, m)$ which induces a norm on $L^\infty(X, m)$.

LEMMA 1.3. *When $L^\infty(X)$ is given the essential supremum norm and the natural multiplication it is a Banach algebra and is isometrically isomorphic to the algebra of all real valued continuous functions on some compact Hausdorff space.*

Let $B^\infty(X)$ be the algebra of bounded real valued $\mathscr{B}$-measurable functions on X equipped with the supremum norm and let $\mathscr{M}$ be the subspace of $B^\infty(X)$ consisting of functions vanishing outside a set of zero m-measure. Then $B^\infty(X)$ is the self-adjoint part of a commutative C^*-algebra and so isometrically isomorphic to $C(K)$ for some compact Hausdorff K. Further $\mathscr{M}$ is a closed ideal and $L^\infty(X)$ can be identified in a natural way with $B^\infty(X)/\mathscr{M}$. The result follows by verifying that this identification induces the same norm on $L^\infty(X)$ as the essential supremum norm.

LEMMA 1.4. *Let $f \in \mathscr{L}^p(X, m)$ and $g \in \mathscr{L}^q(X, m)$ where $1 < p < \infty$ and $1/p + 1/q = 1$ then $fg \in \mathscr{L}^1(X, m)$ and $|fg| \leqq p^{-1}|f|^p + q^{-1}|g|^q$.*

When $0 < \lambda < 1$ and α and β are positive real numbers the inequality $\alpha^\lambda \beta^{1-\lambda} \leqq \lambda\alpha + (1-\lambda)\beta$ is well known. Put $\alpha = |f(x)|^p$, $\beta = |g(x)|^q$ and $\lambda = 1/p$ then the required inequality follows. Thus fg is a measurable function dominated by a function in $\mathscr{L}^1(X, m)$ so fg is in $\mathscr{L}^1(X, m)$.

We observe that if $f \in \mathscr{L}^\infty(X)$ and $g \in \mathscr{L}^p(X)$ then $fg \in \mathscr{L}^p(X)$. For any measurable f and g we define $[f][g]$ to be $[fg]$.

LEMMA 1.5. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. If $f \in \mathscr{L}^p(X, m)$ and $g \in \mathscr{L}^q(X, m)$ then*

$$\int_X |fg|\, dm \leqq \left(\int_X |f|^p\, dm\right)^{1/p} \left(\int_X |g|^q\, dm\right)^{1/q}.$$

Let Y be a subset of X with $mY < +\infty$ and let a and b be positive elements of $L^\infty(Y, m)$. By Lemma 1.3 we may identify $L^\infty(Y, m)$ with $C(E)$ for some compact Hausdorff space E. Misusing our notation, we define $T: C(E) \to C(S)$ by $Th = \int_Y h\, dm$.

For each $s \in S$ the map $h \to (Th)(s)$ is a positive linear functional on $C(E)$. Let μ_s be the unique (real valued) Baire measure on E such that

$$\int_E h\, d\mu_s = (Th)(s) \quad \text{for each } h \in C(E).$$

By the classical Hölder inequality,

$$\int_E ab\, d\mu_s \leqq \left(\int_E a^p\, d\mu_s\right)^{1/p} \left(\int_E b^q\, d\mu_s\right)^{1/q}.$$

So

$$(Tab)(s) \leqq (Ta^p)^{1/p}(s)(Tb^q)^{1/q}(s).$$

Thus

$$\int_Y ab\, dm \leqq \left(\int_Y a^p\, dm\right)^{1/p}\left(\int_Y b^q\right)^{1/q}.$$

Let $f \in \mathscr{L}^p(X, m)$ and $g \in \mathscr{L}^q(X, m)$. Then we can find monotone increasing sequences of positive simple functions $\{f_n\}$, $n=1, 2, \dots$ and $\{g_n\}$, $n=1, 2, \dots$ such that $f_n \to |f|$ and $g_n \to |g|$. For each n, f_n and g_n vanish outside a set of finite m-measure and are bounded functions. So

$$\begin{aligned}\int_X f_n g_n\, dm &\leqq \left(\int_X f_n^p\, dm\right)^{1/p}\left(\int_X g_n^q\, dm\right)^{1/q}\\ &\leqq \left(\int_X |f|^p\, dm\right)^{1/p}\left(\int_X |g|^q\, dm\right)^{1/q}.\end{aligned}$$

By Proposition 3.3 of [6]

$$\bigvee_{n=1}^{\infty} \int_X f_n g_n\, dm = \int_X |fg|\, dm.$$

This completes the proof.

We introduce the notation $\phi([f]) = \int_X f\, dm^*$ for $f \in \mathscr{L}^1(X, m^*)$.

Lemma 1.6. *If $1 \leqq p < \infty$ and f, g are in $\mathscr{L}^p(X)$, then*

$$\phi(|f+g|^p)^{1/p} \leqq \phi(|f|^p)^{1/p} + \phi(|g|^p)^{1/p}.$$

The inequality is trivial when $p=1$. Suppose $1<p$ and let $q=p/(p-1)$ so that $1/p+1/q=1$. Since $|f+g|^{p-1}$ is in $\mathscr{L}^q(X)$ we have by Lemma 1.5

$$\begin{aligned}\phi(|f+g|^{p-1}|f|) &\leqq \phi(|f+g|^{(p-1)q})^{1/q}\phi(|f|^p)^{1/p}\\ &= \phi(|f+g|^p)^{1-1/p}\phi(|f|^p)^{1/p}.\end{aligned}$$

But

$$\phi(|f+g|^p) \leqq \phi(|f+g|^{p-1}|f|) + \phi(|f+g|^{p-1}|g|).$$

So

$$\phi(|f+g|^p) \leqq \phi(|f+g|^p)^{1-1/p}\{\phi(|f|^p)^{1/p} + \phi(|g|^p)^{1/p}\}$$

and the result follows.

It is now clear that for $1 \leqq p < \infty$ we can define a seminorm $\|\ \|_p$ on $\mathscr{L}^p(X)$ by defining $\|f\|_p$ to be the norm of $\phi(|f|^p)^{1/p}$ as an element of $C(S)$. Then $\|\ \|_p$ induces a norm on $L^p(X)$ which we will also denote by $\|\ \|_p$.

Order limits and order sums were defined in [6] as follows. If $\{a_r\}$, $r=1, 2, \dots$, is a $C(S)$-valued sequence and

$$\bigvee_{n=1}^{\infty} \bigwedge_{r=n}^{\infty} a_r = \bigwedge_{n=1}^{\infty} \bigvee_{r=n}^{\infty} a_r$$

then this common value is LIM a_r. Further $\sum_1^{\frown} a_r$ is the order limit of the partial sums $\{\sum_1^n a_r\}$, $n=1, 2, \ldots$.

PROPOSITION 1.1. *Let $f_n \in \mathscr{L}^p(X)$ for $n=1, 2, \ldots$ where $1 \leqq p < \infty$ and suppose $\sum_1^{\frown} \phi(|f_n|^p)^{1/p} < \infty$. Then there is an $f \in \mathscr{L}^p(X)$ such that*

$$\operatorname*{LIM}_n \phi\left(\left|f - \sum_1^n f_r\right|^p\right)^{1/p} = 0.$$

Let $g_n(x) = \sum_1^n |f_r(x)|$ for each $x \in X$. Then, by repeated applications of Lemma 1.6,

$$\phi(g_n^p)^{1/p} \leqq \sum_1^n \phi(|f_r|^p)^{1/p}.$$

By hypothesis there exists $b \in C(S)$ such that

$$\bigvee_{n=1}^{\infty} \sum_1^n \phi(|f_r|^p)^{1/p} = b.$$

So $\phi(g_n^p) \leqq b^p$ for all n.

Let $g(x) = \sup g_n(x)$ for each $x \in X$. By Proposition 3.3 of [6]

$$\phi(g^p) = \bigvee_{n=1}^{\infty} \phi(g_p^n) \leqq b^p.$$

Let $E = \{x : g(x) = +\infty\}$ then $m^*E = 0$. For $x \notin E$, $\sum_1^\infty |f_r(x)|$ is convergent. Let $f(x) = \sum_1^\infty f_r(x)$ for $x \notin E$ and $f(x) = 0$ for $x \in E$. So $|f|^p \leqq g^p$ and thus $f \in \mathscr{L}^p(X)$.

We have $\{|f - \sum_1^n f_r|^p\}$, $n = 1, 2, \ldots$, converges pointwise to zero, except in E, and each term is dominated by $(|f| + |g|)^p \leqq 2^p |g|^p$. By Proposition 3.5 of [6] it follows that $\mathrm{LIM}_n\, \phi(|f - \sum_1^n f_r|^p)$ exists and is zero. It follows from Lemma 1.1 of [6] that $\mathrm{LIM}_n\, \phi(|f - \sum_1^n f_r|^p)^{1/p} = 0$.

PROPOSITION 1.2. *If $1 \leqq p < \infty$ then $L^p(X)$ is a Banach space with respect to $\|\ \ \|_p$.*

Let $\sum_{r=1}^\infty f_r$ be such that $\sum_1^\infty \|f_r\|_p < \infty$. Then it follows from Proposition 1.1 that there exists $f \in \mathscr{L}^p(X)$ such that

$$\mathrm{LIM}\, \phi\left(\left|f - \sum_1^n f_r\right|^p\right)^{1/p} = 0.$$

It follows from the proof of Proposition 1.1 that we can suppose

$$f(x) = \lim_n \sum_1^n f_r(x)$$

and that $\sum_1^\infty |f_r(x)|$ is convergent almost everywhere.

Choose a positive real number ε. There exists $N=N(\varepsilon)$ such that $\sum_{r=N}^{\infty} \|f_r\|_p<\varepsilon$. For any $l>m>N$ we have

$$\phi\left(\left|\sum_{r=m}^{l} f_r\right|^p\right)^{1/p} \leqq \sum_{r=m}^{l} \phi(|f_r|^p)^{1/p}.$$

So

$$\left\|\sum_{r=m}^{l} f_r\right\|_p \leqq \sum_{r=m}^{l} \|f_r\|_p \leqq \varepsilon.$$

By Proposition 3.4 of [6]

$$\phi\left(\lim_{l\to\infty}\left|\sum_{r=m}^{l} f_r\right|^p\right) \leqq \operatorname{LIM\,INF}_{l} \phi\left(\left|\sum_{r=m}^{l} f_r\right|^p\right) \leqq \varepsilon^p.$$

So

$$\phi\left(\left|f-\sum_{1}^{m-1} f_r\right|^p\right) \leqq \varepsilon^p.$$

Thus $\|f-\sum_1^{m-1} f_r\|_p \leqq \varepsilon$ whenever $m>N=N(\varepsilon)$. Thus $L^p(X)$ is a Banach space.

2. **Modular measures.** To avoid some rather dull pathology we shall assume for the rest of this paper that each set of infinite measure contains a set of finite nonzero measure, that is, if $A\in\mathscr{B}$ such that $mA=+\infty$ then there exists $F\subset A$ such that $0<mF<mA$.

LEMMA 2.1. *Let $g\in\mathscr{L}^\infty(X,\mathscr{B},m)$ such that $\int_X gf\,dm=0$ for each $f\in\mathscr{L}^1(X,\mathscr{B},m)$. Then $[g]=0$.*

Let $E=\{x\in X: g(x)>0\}$. Then $g(2\chi_E-1)=g^++g^-=|g|$. For each $f\in\mathscr{L}^1(X,m)$ the function $(2\chi_E-1)f\in\mathscr{L}^1(X,m)$ and so

$$\int_X |g|f\,dm = \int_X g(2\chi_E-1)f\,dm=0.$$

Let $A_n=\{x\in X: |g|(x)\geqq 1/n\}$. If $mA_n\neq 0$ then there exists a set $F\subset A_n$ such that $0<mF<+\infty$. Then $\int_X |g|\chi_F\,dm=0$ and so $mF/n=0$, that is, $mF=0$. Therefore the assumption that $mA_n\neq 0$ is false and so $[g]=0$.

LEMMA 2.2. *If there exists a function $\pi\colon C(S)\to L^\infty(X,m)$ such that*

$$\int_X \pi(a)f\,dm = a\int_X f\,dm \quad \textit{for all } a\in C(S)$$

and all $f\in L^1(X,m)$ then π is unique and π is an algebra homomorphism.

Suppose π_1 and π_2 are two such functions. For each $a\in C(S)$ and all $f\in L^1(X,m)$

$$\int_X (\pi_1(a)-\pi_2(a))f\,dm = a\int_X f\,dm-a\int_X f\,dm = 0.$$

So, by Lemma 2.1, $\pi_1(a)=\pi_2(a)$. Thus π is unique.

For each $a, b \in C(S)$ and all $f \in L^1(X, m)$

$$\int_X \pi(ab) f\, dm = ab \int_X f\, dm = a \int_X \pi(b) f\, dm = \int_X \pi(a)\pi(b) f\, dm.$$

Then, by Lemma 2.1, $\pi(ab) = \pi(a)\pi(b)$. A similar argument shows that π is linear.

We observe that if a is a positive element of $C(S)$ then $\pi(a) = \pi(a^{1/2}a^{1/2}) = \pi(a^{1/2})^2 \geqq 0$. So π is a positive linear map and thus bounded with $\|\pi\| = \|\pi(1)\| = 1$.

DEFINITION 2.1. Let m be a $C(S)$-valued measure on $(X, \mathscr{B})$. If there exists an algebra homomorphism $\pi: C(S) \to L^\infty(X, m)$ such that

$$\int_X \pi(a) f\, dm = a \int_X f\, dm$$

for all $a \in C(S)$ and all $f \in L^1(X, m)$ then m is a *modular measure* (with respect to π).

It follows from Lemma 2.2 that π is unique.

It is clear that when m is modular with respect to π then m^* is also modular with respect to π.

When m is a modular measure with respect to π we write $a \cdot [f]$ for $\pi(a)[f]$. It is then apparent that $L^p(X)$ is a module over $C(S)$. Further, when $\phi: L^1(X) \to C(S)$ is defined by $\phi(f) = \int_X f\, dm$, ϕ is a module homomorphism of $L^1(X)$ into $C(S)$.

PROPOSITION 2.1. *Let m be a $C(S)$-valued measure on $(X, \mathscr{B})$ which is modular with respect to π. Then there exists an idempotent $e \in C(S)$ such that*

(i) *$\pi(a) = 0$ if and only if $a = ea$.*

(ii) *$\int_X f\, dm \in (1-e)C(S)$ for each $f \in L^1(X, m)$.*

(iii) *π is an algebra isomorphism of $(1-e)C(S)$ into $L^\infty(X, m)$.*

Let $\mathscr{F} = \{a \in C(S) : 0 \leqq a \leqq 1$ and $a \int_X f\, dm = 0$ for all $f \in L^1(X, m)\}$. Let $e = \bigvee \mathscr{F}$ then $1 \geqq e$ and so $e \geqq e^2$. If $b \in \mathscr{F}$ then $b^{1/2} \in \mathscr{F}$ and so $e^2 \geqq b$. Thus $e^2 \geqq e \geqq e^2$, that is, e is an idempotent.

For each positive $f \in L^1(X, m)$, $e \int_X f\, dm = \bigvee \{b \int_X f\, dm : b \in \mathscr{F}\} = 0$, and so $e \in \mathscr{F}$. Since $\int_X \pi(e) f\, dm = e \int_X f\, dm = 0$ for all $f \in L^1(X, m)$, it follows from Lemma 2.1 that $\pi(e) = 0$.

Suppose $\pi(a) = 0$. Because π is a lattice homomorphism $\pi(|a|) = |\pi(a)| = 0$. So

$$|a| \int_X f\, dm = \int_X \pi(|a|) f\, dm = 0$$

for all $f \in L^1(X, m)$. Hence $|a|/(\|a\|+1) \in \mathscr{F}$ and thus $e|a| = |a|$. So $a = ea$.

Since e is an element of $\mathscr{F}$ it follows that

$$\int_X f\, dm \in (1-e)C(S) \quad \text{for each } f \in L^1(X, m).$$

We observe that $\pi((1-e)a) = 0$ if and only if $(1-e)a = e(1-e)a = 0$. This completes the proof.

Proposition 2.2. *Let X be a compact Hausdorff space and $\pi: C(S) \to C(X)$ an algebra homomorphism. Let $T: C(X) \to C(S)$ be a positive linear map such that $T(\pi(a)f) = aTf$ for all $a \in C(S)$ and $f \in C(X)$. Let m be the unique quasi-regular $C(S)$-valued Borel measure on X such that*

$$\int_X f\,dm = Tf \quad \text{for all } f \in C(X).$$

Let N be the natural map of $C(X)$ into $L^\infty(X, m)$. Then m is a modular measure with respect to $N \circ \pi$.

The existence, uniqueness and quasi-regularity of m follows from Theorem 4.1 of [6].

Let f^* be a positive bounded lower semicontinuous function on X and let a be a positive element of $C(S)$. Then

$$\begin{aligned} a\int_X f^*\,dm &= a\left[\bigvee\left\{\int_X g\,dm : 0 \leqq g \leqq f^* \text{ and } g \in C(X)\right\}\right] \\ &= \bigvee\left\{a\int_X g\,dm : 0 \leqq g \leqq f^* \text{ and } g \in C(X)\right\} \\ &= \bigvee\{aTg : 0 \leqq g \leqq f^* \text{ and } g \in C(X)\} \\ &= \bigvee\left\{\int_X \pi(a)g\,dm : 0 \leqq g \leqq f^* \text{ and } g \in C(X)\right\} \\ &= \int_X \pi(a)f^*\,dm \end{aligned}$$

since $\langle \pi(a)g : 0 \leqq g \leqq f^* \text{ and } g \in C(X), \leqq\rangle$ is an increasing net which converges pointwise to $\pi(a)f^*$.

Let $B^\infty(X)$ be the bounded Borel functions on X and let $\mathscr{A}_0$ be the subalgebra of $B^\infty(X)$ consisting of all functions of the form $f^* - g^*$ where f^* and g^* are positive bounded lower semicontinuous functions.

Consider the collection of all algebras $\mathscr{A}$ such that (1) $\mathscr{A}_0 \subset \mathscr{A} \subset B^\infty(X)$ and (2) $f \in \mathscr{A}$ and $a \in C(S)$ implies $a\int_X f\,dm = \int_X \pi(a)f\,dm$. Since $\mathscr{A}_0$ is in the collection, the collection is not empty. By Zorn's Lemma there is a maximal such algebra, $\mathscr{M}$, say. By appealing to the generalised Dominated Convergence Theorem proved in [6] we can show that the algebra of all functions which are pointwise limits of uniformly bounded sequences of functions in $\mathscr{M}$ is in the collection. It follows from the maximality of $\mathscr{M}$ that the pointwise limit of a uniformly bounded sequence of functions in $\mathscr{M}$ is in $\mathscr{M}$. Hence $\mathscr{M} = B^\infty(X)$. It now follows that m is modular with respect to $N \circ \pi$.

This provides us with examples of modular measures. We shall now give a simple example of a measure which is not modular. In the introduction we defined measures l and m in the special case where $X = S = \{a, b\}$ a two-point space. It is easy to verify that these measures are both modular but we shall show that $l + 2m$

is not modular. Suppose $l+2m$ is modular with respect to π. Let x and y be distinct real numbers and let $\pi(x, y)=(c, d)$. Let $f\colon \{a, b\} \to \boldsymbol{R}$ be defined by $f(a)=1$ and $f(b)=0$. Then, by the supposed modularity of $l+2m$,

$$\int_{\{a,b\}} (c, d) f \, d(l+2m) = (x, y) \int_{\{a,b\}} f \, d(l+2m).$$

So

$$cf(a)(1, 2)+df(b)(2, 1) = (x, y)[f(a)(1, 2)+f(b)(2, 1)].$$

Thus $(c, 2c)=(x, 2y)$ and so $x=y$ which is a contradiction. It follows that $l+m$ is not modular.

Let e be an element of $C(S)$. We call e a *projection* or *idempotent* when $e=e^2$, that is, when e is the characteristic function of a clopen subset of S. Let $\{E_j : j\in J\}$ be a family of clopen subsets of S and let E be the closure of $\bigcup E_j$. Then E is a clopen set and is the least upper bound of $\{E_j : j \in J\}$ in the Boolean algebra of clopen subsets of S. Further, χ_E is the least upper bound in $C(S)$ of the family $\{\chi_{E_j} : j \in J\}$.

We say $\{e_j : j \in J\}$ is a family of *orthogonal projections* in $C(S)$ when the functions e_j are idempotents such that $e_j e_k=0$ when $j\neq k$, that is, they are characteristic functions of pairwise disjoint clopen subsets of S.

PROPOSITION 2.3. *Let $\{e_j : j \in J\}$ be a family of orthogonal projections in $C(S)$ and let $e=\bigvee \{e_j : j \in J\}$. Let m be a modular measure with respect to π. Let $1 \leq p < \infty$ and $f \in \mathscr{L}^p(X)$. If $e_j\cdot [f]=\pi(e_j)[f]=0$ for each $j \in J$ then $e\cdot [f]=0$.*

We recall from the concluding remark of Lemma 2.2 that since π is an algebra homomorphism it is a positive linear map.

We observe

$$e_j \int_X |f|^p \, dm^* = \int_X |\pi(e_j)f|^p \, dm^* = 0.$$

Thus

$$e \int_X |f|^p \, dm^* = \bigvee_{j\in J} e_j \int_X |f|^p \, dm^* = 0.$$

But

$$\int_X |\pi(e)f|^p \, dm^* = \int_X \pi(e)|f|^p \, dm^* = e \int_X |f|^p \, dm^*.$$

The result follows.

COMPLEX L^p-SPACES. We define $\mathscr{L}^p(X, m, \boldsymbol{C})$ to be the set of complex valued measurable functions f on X such that $|f| \in \mathscr{L}^p(X, m)$ and define the pseudonorm of f to be the pseudonorm of $|f|$ as an element of $\mathscr{L}^p(X, m)$. It causes no clash with earlier notation if we denote the pseudonorm on $\mathscr{L}^p(X, m, \boldsymbol{C})$ by $\| \quad \|_p$. Let $\boldsymbol{L}^p(X)$

be the quotient of $\mathscr{L}^p(X, m, \boldsymbol{C})$ by the kernel of the pseudonorm $\| \quad \|_p$. Each element of $\boldsymbol{L}^p(X)$ is of the form $\xi + i\eta$ where ξ and η are elements of $L^p(X)$ and

$$\max (\|\xi\|_p, \|\eta\|_p) \leqq \|\xi + i\eta\|_p \leqq \|\xi\|_p + \|\eta\|_p,$$

so a sequence in $\boldsymbol{L}^p(X)$ converges iff its real and imaginary parts converge. Thus $\boldsymbol{L}^p(X)$ is a complex Banach space in which $L^p(X)$ is naturally embedded. Further, when m is a modular measure, $\boldsymbol{L}^p(X)$ is a module over $\boldsymbol{C}(S)$, the ring of continuous complex-valued functions on S.

Let $\phi(f) = \phi(\mathscr{R}(f)) + i\phi(\mathscr{I}(f))$ for $f \in \boldsymbol{L}^1(X)$. Then ϕ is a module homomorphism of $\boldsymbol{L}^1(X)$ into $\boldsymbol{C}(S)$ when m is a modular measure.

In [7] a pre-Kaplansky-Hilbert module over $\boldsymbol{C}(S)$ is defined to be a module K over $\boldsymbol{C}(S)$ equipped with a $\boldsymbol{C}(S)$-valued "inner product" such that for x, y, z in K and a, b in $\boldsymbol{C}(S)$

(1) $(x, y) = (y, x)^*$

(2) $(ax + bz, y) = a(x, y) + b(z, y)$

(3) $(x, x) \geqq 0$ with $(x, x) = 0$ only if $x = 0$.

LEMMA 2.3. *Let m be a modular measure. Then $\boldsymbol{L}^2(X)$ is a pre-Kaplansky-Hilbert module over $\boldsymbol{C}(S)$ when the inner product is defined by $(f, g) = \phi(fg^*)$.*

3. **Ample measures.** A Kaplansky-Hilbert module over $\boldsymbol{C}(S)$ is a pre-Kaplansky-Hilbert module over $\boldsymbol{C}(S)$ satisfying three completeness conditions, see [4].

On inspecting the definition of a Kaplansky-Hilbert module we see that when m is a modular measure the pre-Kaplansky-Hilbert module $\boldsymbol{L}^2(X)$ satisfies two of the completeness conditions.

(1) $\boldsymbol{L}^2(X)$ is a Banach space with respect to the norm induced by the inner product.

(2) Let $\{e_j : j \in J\}$ be a family of orthogonal projections in $C(S)$ with

$$e = \bigvee \{e_j : j \in J\}.$$

Then it follows from Lemma 2.1 that if $\xi \in \boldsymbol{L}^2(X)$ and $e_j \cdot \xi = 0$ for all $j \in J$ then $e \cdot \xi = 0$.

So $\boldsymbol{L}^2(X)$ will be a Kaplansky-Hilbert module iff it satisfies the third completeness condition i.e.

(3) Let $\{e_j : j \in J\}$ be a family of orthogonal projections in $C(S)$ with $\bigvee e_j = 1$. Let $\{\xi_j : j \in J\}$ be a norm bounded family in $\boldsymbol{L}^2(X)$. Then there exists $\xi \in \boldsymbol{L}^2(X)$ such that $e_j \cdot \xi = e_j \cdot \xi_j$ for all $j \in J$.

DEFINITION 3.1. A $C(S)$-valued measure m is *ample* (with respect to π) when m is modular (with respect to π) and $\boldsymbol{L}^2(X)$ is a Kaplansky-Hilbert module.

It is clear that a modular measure m is ample iff condition (3) above is satisfied.

DEFINITION 3.2. Let $\mathscr{L}$ be any lattice. $\mathscr{L}$ satisfies the *upper* (*lower*) *countable chain condition* if, when $\mathscr{S}$ is a nonempty subset of $\mathscr{L}$ which is bounded above (below), then there is a countable set $\mathscr{S}_0 \subset \mathscr{S}$ such that $\mathscr{S}_0$ and $\mathscr{S}$ have the same set of upper (lower) bounds. $\mathscr{L}$ satisfies the *countable chain condition* when $\mathscr{L}$ satisfies both the upper and lower countable chain conditions.

In the special case where $\mathscr{L}$ is a Boolean algebra this is equivalent to the condition that each set of pairwise disjoint elements of $\mathscr{L}$ be countable. This equivalence is proved by Halmos in §14 of [2].

The following proposition gives a convenient sufficient condition for a measure to be ample.

PROPOSITION 3.1. *Let m be a $C(S)$-valued modular measure. If the Boolean algebra of clopen subsets of S satisfies the countable chain condition then m is ample.*

Let $\{e_j : j \in J\}$ be a family of orthogonal projections in $C(S)$ with $\bigvee \{e_j : j \in J\} = 1$. By hypothesis J is countable and so may be assumed to be the set of natural numbers.

Let $\{f_j\}$ $j=1, 2, \ldots$ be a sequence in $\mathscr{L}^2(X)$ and suppose there exists a constant M such that $\|f_j\|_2 \leqq M$ for each j. Then

$$\sum_{j=1}^{n} \phi(e_j \cdot f_j^2)^{1/2} = \sum_{j=1}^{n} e_j \phi(f_j^2)^{1/2} \leqq M \sum_{j=1}^{n} e_j.$$

Thus $\sum_{j=1}^{n} \phi(e_j \cdot f_j^2)^{1/2} \leqq M1$. By Proposition 1.1 there exists $f \in \mathscr{L}^2(X)$ such that

$$\operatorname*{LIM}_{n} \phi\left(\left| f - \sum_{1}^{n} e_j \cdot f_j \right|^2\right)^{1/2} = 0.$$

Thus

$$\operatorname*{LIM}_{n} e_r \phi\left(\left| f - \sum_{1}^{n} e_j \cdot f_j \right|^2\right)^{1/2} = 0.$$

Thus

$$\operatorname*{LIM}_{n} \phi(|e_r \cdot f - e_r \cdot f_r|^2)^{1/2} = 0.$$

So $e_r \cdot [f] = e_r \cdot [f_r]$.

It now follows that $L^2(X)$ satisfies all the completeness conditions necessary for it to be a Kaplansky-Hilbert module so m is an ample measure.

PROPOSITION 3.2. *Let S be extremally disconnected. The Boolean algebra of clopen subsets of S satisfies the countable chain condition if and only if $C(S)$ satisfies the condition.*

It is clear that if $C(S)$ satisfies the countable chain condition then so does the Boolean algebra of idempotents in $C(S)$.

Suppose the Boolean algebra of clopen subsets of S satisfies the countable chain condition. Let $\{a_j : j \in J\}$ be a family of functions in $C(S)$ which is bounded above by a constant M. For each real number r we define the following sets:

$$O(j, r) = \{s \in S : a_j(s) > r\}, \qquad K(j, r) = \operatorname{cl}(O(j, r)), \qquad K(r) = \operatorname{cl}\Big(\bigcup_{j \in J} O(j, r)\Big).$$

Then $K(r)$ and $K(j, r)$ are clopen and $K(r) = \bigvee_{j \in J} K(j, r)$.

Observe that $r_1 < r_2$ implies $K(r_1) \supset K(r_2)$. If $r > M$ and $K(r) \neq \varnothing$ then $a_j(s) > M$ for some j and s, which is false. So an upper bounded finite-valued function can be defined by

$$a(s) = \sup \{\rho : \rho \text{ is rational and } s \in K(\rho)\}.$$

We show that a is continuous and the least upper bound of $\{a_j : j \in J\}$.

If $r < a(s)$ then $s \in K(r)$. If $s \in K(t)$ then $a(s) \geqq t$ so $a(s) < t$ implies $s \notin K(t)$. For each open interval (r, t),

$$\{s : r < a(s) < t\} = \bigcup_{n=1}^{\infty} \left(K\left(r+\frac{1}{n}\right) - K\left(t-\frac{1}{n}\right)\right)$$

and so a is continuous.

If $a_j(s) > \rho$ where ρ is rational then $s \in K(\rho)$ and so $a(s) \geqq \rho$. Thus a is an upper bound for $\{a_j : j \in J\}$. Let b be another upper bound and suppose $a(s_0) > b(s_0)$ for some $s_0 \in S$. Let ρ be a rational strictly between $a(s_0)$ and $b(s_0)$. Then $s_0 \in K(\rho)$. Let U be the open set $\{s \in S : a(s) > \rho > b(s)\}$. We have $a_j(s) \leqq b(s) < \rho$ for each $s \in U$ and so $U \cap O(j, \rho) = \varnothing$ for each $j \in J$. Thus s_0 is not in the closure of $\bigcup_{j \in J} O(j, \rho)$ that is, $s_0 \notin K(\rho)$. This contradiction shows that a is the least upper bound.

Let ρ be rational. Because the Boolean algebra of clopen subsets of S satisfies the countable chain condition we can choose a countable set $J(\rho) \subset J$ such that

$$\bigvee \{K(j, \rho) : j \in J(\rho)\} = \bigvee \{K(j, \rho) : j \in J\}.$$

Let $I = \bigcup \{J(\rho) : \rho \text{ is rational}\}$ then I is countable and $K(\rho) = \bigvee \{K(j, \rho) : j \in I\}$ for each rational ρ. It follows, by repeating the construction in the first part of the proof, that a is the least upper bound of $\{a_j : j \in I\}$ which is a countable subset of $\{a_j : j \in J\}$. This establishes the result.

When $C(S)$ is isomorphic to a commutative von Neumann algebra over a separable Hilbert space or, equivalently, when $C(S) \simeq L^\infty(X, \mu)$ where μ is a finite real valued measure, then $C(S)$ satisfies the countable chain condition.

Let $B^\infty[0, 1]$ be the bounded Borel measurable functions on $[0, 1]$ and let $\mathcal{M}$ be the functions in $B^\infty[0, 1]$ which vanish outside a meagre set. Then $B^\infty[0, 1]/\mathcal{M}$ is isomorphic to $C(S)$ where S is the Boolean space of the Boolean algebra of regular open subsets of $[0, 1]$. This Stone algebra satisfies the countable chain condition and is not isomorphic to any von Neumann algebra.

PROPOSITION 3.3. *Let m be a $C(S)$-valued measure which is not required to be modular. If $C(S)$ satisfies the countable chain condition then so also does $L^1(X)$. Further, if $L^1(X)$ satisfies the countable chain condition and m is a finite measure then $L^\infty(X)$ is a Stone algebra satisfying the countable chain condition.*

Let $(\xi_j : j \in J, \leqq)$ be an increasing net in $L^1(X)$ which is bounded above by an element of $L^1(X)$. Then $(\phi(\xi_j) : j \in J, \leqq)$ is an increasing net in $C(S)$ which has a least upper bound b. Because $C(S)$ satisfies the countable chain condition we can find a countable set $I \subset J$ such that b is the least upper bound of $\{\phi(\xi_j) : j \in I\}$.

Enumerate I, $\{i(1), i(2), \ldots\}$ and define, inductively, a countable subset $\{j(1), j(2), \ldots\}$ of J such that $\xi_{j(r)} \geqq \xi_{i(r)}$ for each r and $\{\xi_{j(r)}\}$ $r=1, 2, \ldots$ is a monotonic sequence. Then $\bigvee_{r=1}^{\infty} \phi(\xi_{j(r)})=b$. By the Monotone Convergence Theorem, Proposition 3.3 of [6], there exists $[f] \in L^1(X)$ such that if $f_j \in \xi_j$ for each j then $\{f_{j(r)}\}$ $r=1, 2, \ldots$ converges to f pointwise almost everywhere as $r \to \infty$ and $\phi(f)=b$.

Now we must show that $[f]$ is the least upper bound of the ξ_j. Clearly no smaller element of $L^1(X)$ will do. Let $n \in J$. Then define inductively, as above, an increasing sequence $\{\xi_{k(r)}\}$ $r=1, 2, \ldots$ such that

$$\xi_{k(r)} \geqq \xi_n \vee \xi_{j(r)} \quad \text{for } r = 1, 2, \ldots.$$

Then there is a $[g] \geqq \xi_n$ in $L^1(X)$ such that

$$\phi(g) = \bigvee_{r=1}^{\infty} \phi(\xi_{k(r)}) = b$$

and $[g] \geqq [f]$. Since $\phi(g-f)=b-b=0$ this implies $[f]=[g] \geqq \xi_n$.

It follows from this result and the fact that $L^1(X)$ is a lattice that each nonempty subset of $L^1(X)$ which is bounded above has a countable subset with the same least upper bound.

Suppose $L^1(X)$ satisfies the countable chain condition and m is a finite measure. Then $L^\infty(X) \subset L^1(X)$ and so $L^\infty(X)$ satisfies the countable chain condition. Since $L^\infty(X) \cong C(E)$, where E is a compact Hausdorff space, and, since $L^\infty(X)$ is boundedly σ-complete, $L^\infty(X)$ is a Stone algebra. This completes the proof.

We have shown that a sufficient condition for a modular measure to be ample is that it takes its values in a Stone algebra satisfying the countable chain condition. This condition is simple and convenient for applications but is not necessary. We give a further sufficient condition for a measure to be ample. Since no use is made of this result we suppress the rather tedious proof.

DEFINITION 3.3. $C(S)$ satisfies the *local countable chain condition* if there is a family of pairwise disjoint clopen sets $\{E_j : j \in J\}$ such that

(1) $C(E_j)$ satisfies the countable chain condition.

(2) $\bigvee \{E_j : j \in J\}=S$.

Kaplansky shows in §10 of [3] that any von Neumann algebra satisfies the local countable chain condition.

PROPOSITION 3.4. *Let X, S, π and T satisfy the hypotheses of Proposition 2.2 and suppose that $C(S)$ satisfies the local countable chain condition. Let m be the measure constructed in Proposition 2.2. Then there is a σ-algebra $\mathscr{B}$ which contains the Borel sets and a measure l on $\mathscr{B}$ such that $lE=mE$ for each Borel set E and l is ample.*

If $C(S)$ is isomorphic to a von Neumann algebra in Proposition 3.4 then m is regular. The regularity of m can be used to show that each element of $\mathscr{B}$ is of the form $B \cup N$ where B is a Borel set and $N \subset E$ where E is a Borel set and $mE=0$. It follows that $L^2(X, m)=L^2(X, l)$ and so m is ample.

4. **Absolute continuity and the generalized Radon-Nikodym Theorem.** In this section l and m are modular $C(S)$-valued measures on a measurable space $(X, \mathscr{B})$. They are finite measures, that is, $lX \in C(S)$ and $mX \in C(S)$. When f is any real valued $\mathscr{B}$-measurable function let $[f]_l$ ($[f]_m$) be the set of $\mathscr{B}$-measurable functions differing from f only on sets of zero l-measure (m-measure).

Suppose $E \in \mathscr{B}$ and $mE=0$ implies $lE=0$. Then $[f]_m \subset [f]_l$ and so a natural map $N: L^\infty(X, m) \to L^\infty(X, l)$ can be defined by $N[f]_m=[f]_l$. N is an algebra homomorphism onto $L^\infty(X, l)$.

Definition 4.1. l is *absolutely continuous* with respect to m (modulo π) when

(1) If $E \in \mathscr{B}$ and $mE=0$ then $lE=0$.

(2) The map $\pi: C(S) \to L^\infty(X, m)$ is an algebra homomorphism such that m is modular with respect to π and l is modular with respect to $N \circ \pi$.

Lemma 4.1. *Let l be absolutely continuous with respect to m (modulo π) then $l+m$ is modular with respect to π.*

We have $(l+m)E=0$ iff $lE=0$ and $mE=0$ and thus $(l+m)E=0$ iff $mE=0$. So $[g]_m=[g]_{l+m}$ whenever g is measurable. Thus $L^\infty(X, m)=L^\infty(X, l+m)$. Also $\int_X g\, d(l+m)$ exists iff $\int_X g\, dl$ and $\int_X g\, dm$ exist. So $\mathscr{L}^p(X, l+m)=\mathscr{L}^p(X, l) \cap \mathscr{L}^p(X, m)$ for $1 \leqq p < \infty$.

Let $a \in C(S)$ and suppose $\pi(a)=[g]_m=[g]_{l+m}$ where g is bounded. Let $h \in \mathscr{L}^1(X, l+m)=\mathscr{L}^1(X, l) \cap \mathscr{L}^1(X, m)$. Then

$$\begin{aligned} \int_X \pi(a)h\, d(l+m) &= \int_X gh\, dl + \int_X gh\, dm \\ &= \int_X N \circ \pi(a)h\, dl + \int_X \pi(a)h\, dm \\ &= a \int_X h\, dl + a \int_X h\, dm \\ &= a \int_X h\, d(l+m). \end{aligned}$$

Definition 4.2. l is *amply absolutely continuous* with respect to m (modulo π) when l is absolutely continuous with respect to m (modulo π) and the π-modular measure $l+m$ is ample.

We note that if $C(S)$ satisfies the countable chain condition and $l+m$ is a modular measure then $l+m$ is ample. So when $C(S)$ satisfies the countable chain condition Definition 4.2 is superfluous.

Lemma 4.2. *Let q be an ample modular measure and let $H: L^2(X) \to C(S)$ be a module homomorphism which is continuous when $L^2(X)$ and $C(S)$ are given their norm topologies. Then there is a unique $\xi \in L^2(X)$ such that*

$$Hf = \int_X f\xi\, dq \quad \text{for all } f \in L^2(X).$$

Each element of $\boldsymbol{L}^2(X)$ can be expressed uniquely in the form $f+ig$ where f and g are in $L^2(X)$. So $\boldsymbol{h}: \boldsymbol{L}^2(X) \to \boldsymbol{C}(S)$ can be defined by $\boldsymbol{h}(f+ig)=Hf+iHg$. Then $\boldsymbol{h}$ is a module homomorphism of $\boldsymbol{L}^2(X)$, regarded as a module over $\boldsymbol{C}(S)$, into $\boldsymbol{C}(S)$. We have

$$\begin{aligned}\|\boldsymbol{h}(f+ig)\| &\leqq \|Hf\|+\|Hg\| \\ &\leqq \|H\|\left\{\left\|\left(\int_X f^2\,dq\right)^{1/2}\right\|+\left\|\left(\int_X g^2\,dq\right)^{1/2}\right\|\right\} \\ &\leqq 2\|H\|\left\|\left(\int_X (f^2+g^2)\,dq\right)^{1/2}\right\|.\end{aligned}$$

So $\boldsymbol{h}$ is a bounded operator.

Since $\boldsymbol{L}^2(X)$ is a Kaplansky-Hilbert module it now follows from Theorem 5 of [4] that there is a unique $\zeta=\xi+i\eta$ such that

$$\boldsymbol{h}(z) = \int_X z\zeta^*\,dq \quad \text{for all } z \in \boldsymbol{L}^2(X).$$

Thus

$$Hf = \int_X f\xi\,dq - i\int_X f\eta\,dq \quad \text{for all } f \in L^2(X).$$

Since H is $C(S)$-valued the result follows.

We now come to the main result.

THEOREM 4.1. *Let l and m be finite $C(S)$-valued measures on the measurable space $(X, \mathscr{B})$. Let l be amply absolutely continuous with respect to m. Then there is a positive $\mathscr{B}$-measurable function f such that*

$$lE = \int_E f\,dm \quad \textit{for each } E \in \mathscr{B}.$$

Let $\pi: C(S) \to L^\infty(X, m)$ be such that m is π-modular and l is $N \circ \pi$-modular, where N is the natural map of $L^\infty(X, m)$ onto $L^\infty(X, l)$. By Lemma 4.1, $l+m$ is π-modular. Also $[g_m]=[g]_{l+m}$ for each $\mathscr{B}$-measurable g and $\mathscr{L}^2(X, l+m)= \mathscr{L}^2(X, l) \cap \mathscr{L}^2(X, m)$.

Let $[g]_{l+m} \in L^2(X, l+m)$ then $g \in \mathscr{L}^2(X, m)$. Appealing to Lemma 1.5 we have that $\int_X g\,dm$ exists and

$$\left|\int_X g\,dm\right| \leqq \left(\int_X g^2\,dm\right)^{1/2}(mX)^{1/2}.$$

So we can define $H: L^2(X, l+m) \to C(S)$ by $H[g]_{l+m}=\int_X g\,dm$. Since $l+m$ and m are modular with respect to π, H is a module homomorphism. Further

$$\|H[g]_{l+m}\| \leqq \|(mX)^{1/2}\|\left\|\left(\int_X g^2\,d(l+m)\right)^{1/2}\right\|.$$

Thus H is a bounded linear operator.

By Lemma 4.2 there exists $h \in \mathscr{L}^2(X, l+m)$ such that

$$H[g]_{l+m} = \int_X gh\, d(l+m).$$

Let $E_0 = \{x \in X : h(x) \leqq 0\}$ and

$$E_n = \{x \in X : h(x) \geqq 1+1/n\} \quad \text{for } n = 1, 2, \ldots.$$

Then

$$mE_0 = \int_{E_0} h\, d(l+m) \leqq 0$$

and

$$mE_n = \int_{E_n} h\, d(l+m) \geqq \left(1+\frac{1}{n}\right)(lE_n+mE_n)$$

so $0 \geqq (1+n^{-1})lE_n + n^{-1}mE_n$.

It follows that we may assume without loss of generality that $0 < h(x) \leqq 1$ for all $x \in X$. Then $1/h$ is a well defined measurable function.

Let $E \in \mathscr{B}$. Fix an arbitrary natural number n and let

$$F_r = \{x \in E : n/(r+1) < h(x) \leqq n/r\}$$

for each natural number r. For all $x \in F_r$ we have $r/n \leqq 1/h(x) < (r+1)/n$. So

$$\frac{r}{n} mF_r \leqq \int_{F_r} \frac{1}{h}\, dm \leqq \frac{r+1}{n} mF_r. \tag{1}$$

Also

$$\frac{n}{r+1}(l+m)F_r \leqq \int_{F_r} h\, d(l+m) = mF_r \leqq \frac{n}{r}(l+m)F_r.$$

Thus

$$rmF_r/n \leqq (l+m)F_r \leqq (r+1)mF_r/n. \tag{2}$$

From (1) and (2) we have

$$-\frac{1}{n} mF_r \leqq \int_X \chi_{F_r} \frac{1}{h}\, dm - (l+m)F_r \leqq \frac{1}{n} mF_r.$$

By this inequality and Proposition 3.3 of [6] we find

$$\int_E \frac{1}{h}\, dm \text{ exists and } \left\| (l+m)E - \int_E \frac{1}{h}\, dm \right\| \leqq \frac{1}{n} \|mX\|.$$

Since n is arbitrary this gives $lE = \int_E (1/h - 1)\, dm$.

Using Theorem 4.1 we obtain an analogue of the Hahn Decomposition Theorem. In the following lemmas and theorem X is a compact Hausdorff space and $C(X)$ the ring of continuous real valued functions on X, and $B_0^\infty(X)$ the ring of bounded Baire functions on X.

LEMMA 4.3. *Let $T: C(X) \to C(S)$ be a bounded linear operator. Then there exist positive linear operators T^+ and T^- such that $T = T^+ - T^-$ and if $T = P - Q$ where P and Q are positive operators then $P \geqq T^+$ and $Q \geqq T^-$. Further, if $\pi: C(S) \to C(X)$ is an algebra homomorphism such that $T(\pi(a)f) = aTf$ for each $a \in C(S)$ and $f \in C(X)$ then $T^+(\pi(a)f) = aT^+f$ and $T^-(\pi(a)f) = aT^-f$ for each $a \in C(S)$ and $f \in C(X)$.*

For each positive $f \in C(X)$ the set $\{T(h) : 0 \leqq h \leqq f \text{ and } h \in C(X)\}$ is a subset of $C(S)$ which is bounded above and so has a least upper bound and so we define

$$T^+f = \bigvee \{Th : 0 \leqq h \leqq f \text{ and } h \in C(X)\}.$$

Then T^+ is additive on the positive cone of $C(X)$ and, when π exists, $T^+(\pi(a)f) = aT^+f$ for each positive $a \in C(S)$.

For an arbitrary element g of $C(X)$ we define

$$T^+g = T^+(g + \|g\|) - \|g\| T^+ 1.$$

Let $T^- = T^+ - T$. Then T^+ and T^- have the required properties.

LEMMA 4.4. *Let m be a $C(S)$-valued Baire measure on the compact Hausdorff space X. Suppose h is a bounded Baire measurable function on X such that $\int_X hg\, dm \geqq 0$ whenever g is a positive element of $C(X)$. Then $h \geqq 0$ except on a set of zero m-measure.*

Let $\phi(g) = \int_X hg\, dm$ for each $g \in C(X)$. Then ϕ is a positive linear $C(S)$-valued map. So by Theorem 4.1 of [6] there is a unique $C(S)$-valued Baire measure q such that $\phi(g) = \int_X g\, dq$ for each $g \in C(X)$.

Let $B_0^\infty(X)$ be the set of bounded Baire measurable functions on X and let

$$\mathscr{V} = \left\{ f \in B_0^\infty(X) : \int_X f\, dq = \int_X hf\, dm \right\}.$$

Then $C(X) \subset \mathscr{V}$. Also the pointwise limit of a uniformly bounded sequence of functions in $\mathscr{V}$ is a function in $\mathscr{V}$.

Consider the collection of all algebras $\mathscr{A}$ such that $C(X) \subset \mathscr{A} \subset \mathscr{V}$. Order this family by set inclusion. It follows by a Zorn's Lemma argument that there is a maximal algebra $\mathscr{M}$. The algebra $\bar{\mathscr{M}}$ whose elements are the pointwise limits of uniformly bounded sequences of functions in $\mathscr{M}$ is in the collection and contains $\mathscr{M}$. Since $\mathscr{M}$ is maximal we have $\mathscr{M} = \bar{\mathscr{M}}$. Hence $\mathscr{M} = B_0^\infty(X)$ and so $\mathscr{V} = B_0^\infty(X)$.

Let $D_n = \{x : h(x) \leqq -1/n\}$ then

$$\int_{D_n} h\, dm = \int_{D_n} dq = qD_n \geqq 0.$$

So $-mD_n/n \geqq 0$ and thus $m \bigcup_{n=1}^\infty D_n = 0$. The result follows.

THEOREM 4.2. *Suppose $C(S)$ satisfies the countable chain condition or is isomorphic to a von Neumann algebra. Let $\pi: C(S) \to C(X)$ be an algebra homomorphism. Let $T: C(X) \to C(S)$ be a bounded linear map such that, for all $a \in C(S)$ and*

$f \in C(X)$, $T\pi(a)f = aTf$. Let m^+ and m^- be the unique $C(S)$-valued Baire measures on X corresponding to T^+ and T^-. Then there is a Baire set A such that

$$m^+E = m^+E \cap A \quad \text{and} \quad m^-E = m^-E \cap (X-A)$$

for each Baire set E.

Let $|m| = m^+ + m^-$. Then it follows from Theorem 4.1 that there is a positive Baire measurable function g such that

$$\int_X f\,dm^+ = \int_X fg\,d|m| \quad \text{for all } f \in B_0^\infty(X).$$

So

$$\int_X f\,dm^- = \int_X f(1-g)\,d|m| \quad \text{for all } f \in B_0^\infty(X).$$

We can suppose without loss of generality that $1-g$ is positive. So $0 \leqq g \leqq 1$.

We have $Tf = \int_X f(2g-1)\,d|m|$ for $f \in C(X)$. Let $A = \{x \in X : g(x) \geqq \frac{1}{2}\}$. Then

$$Tf = \int_A f|2g-1|\,d|m| - \int_{X-A} f|2g-1|\,d|m|.$$

It follows from Lemma 4.3 that when f is a positive element of $C(X)$,

$$T^+f \leqq \int_A f|2g-1|\,d|m|.$$

So

$$\int_X \{(2g-1)\chi_A - g\} f\,d|m| \geqq 0 \quad \text{for all positive } f \text{ in } C(X).$$

Appealing to Lemma 4.4 we have

$$\chi_A g - \chi_{X-A} g - \chi_A \geqq 0 \tag{1}$$

except on a set of zero $|m|$-measure. Multiplying both sides of (1) by χ_A and recalling that $g \leqq 1$ we have $\chi_A g = \chi_A$ almost everywhere. Multiplying both sides of (1) by χ_{X-A} and recalling that $0 \leqq g$ we have $\chi_{X-A} g = 0$ almost everywhere. Thus $g = g(\chi_A + \chi_{X-A}) = \chi_A$ except on a set of zero $|m|$-measure.

The theorem follows.

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