A COMPATIBILITY CONDITION BETWEEN INVARIANT RIEMANNIAN METRICS AND LEVI-WHITEHEAD DECOMPOSITIONS ON A COSET SPACE

BY

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0. Introduction. Let $M = G/K$ be an effective coset space of a connected Lie group by a compact subgroup. Then there may be many $G$-invariant riemannian metrics on $M$. But one expects the algebraic structure of the pair $(G, K)$ to have a strong influence on the curvatures of $M$ relative to any $G$-invariant riemannian metric. For example

(1) if $G$ is semisimple with finite center and $K$ is a maximal compact subgroup, then it is classical from symmetric space theory that all $G$-invariant riemannian metrics on $M$ have every sectional curvature $\leq 0$;

(2) if $G$ is commutative then every $G$-invariant riemannian metric on $M$ is flat; and

(3) if $G$ is noncommutative and nilpotent then [7] every $G$-invariant riemannian metric on $M$ has sectional curvatures of both signs.

Those results are proved by choosing an $\text{ad}_G (K)$-stable complement $\mathfrak{N}$ to the Lie algebra $\mathfrak{k}$ of $K$ inside the Lie algebra $\mathfrak{g}$ of $G$, and by performing calculations in $\mathfrak{g}$ and in $\mathfrak{g}$ in a manner justified by embedding $G$ in the orthonormal frame bundle of $M$. But at certain crucial parts of those calculations one must have $\mathfrak{g}$ either semisimple or nilpotent. The idea in this paper is to create a setup in which the calculations can still be carried out, by requiring that the complement $\mathfrak{N}$ split as

$$\mathfrak{N} = (\mathfrak{N} \cap \mathfrak{g}) + (\mathfrak{N} \cap \mathfrak{l})$$ orthogonal direct sum,

where

$$\mathfrak{g} = \mathfrak{R} + \mathfrak{L}$$ is a Levi-Whitehead decomposition

and

$$\mathfrak{N} \cap \mathfrak{R}$$ contains the nilpotent radical of $\mathfrak{g}$.

§2 is a study of the circumstances under which $\mathfrak{N}$ can be chosen, and the Levi-Whitehead decomposition $\mathfrak{g} = \mathfrak{R} + \mathfrak{L}$ can be chosen, so that $M$ has such an orthogonal splitting. We describe those circumstances by the condition (2.2) that the invariant riemannian metric on $M$ be "consistent" with $\mathfrak{g} = \mathfrak{R} + \mathfrak{L}$.

Given the consistency condition (2.2), our main result (Theorem 3.9) says that every unit vector $X \in \mathfrak{N} \cap \mathfrak{R}$, orthogonal to the nilpotent radical $\mathfrak{R}$ of $\mathfrak{g}$, is a
direction of negative mean curvature on \( M \). Our applications (§4) essentially consist of observing that, if \( M \) has mean curvature \( \geq 0 \) everywhere, then the consistency condition implies \( \mathcal{N} = (\mathcal{M} \cap \mathcal{R}) \), i.e. \( \mathcal{N} = \mathcal{R} + (\mathcal{R} \cap \mathcal{K}) \) semidirect sum. The most striking of the applications is Theorem 4.4, which says:

\[
\text{Let } M \text{ be a connected Riemannian manifold that has a solvable transitive group of isometries. Then the following conditions are equivalent.}
\]

(i) \( M \) has mean curvature \( \geq 0 \) everywhere.
(ii) \( M \) has every sectional curvature \( \geq 0 \).
(iii) \( M \) has every sectional curvature zero.
(iv) \( M \) is isometric to the product of an euclidean space and a flat riemannian torus.

Theorem 4.7 adds negative curvature conditions in case \( M \) has a transitive nilpotent group of isometries, extending the results of [7] to mean curvature.

1. Definitions and notation. \( \mathfrak{g} \) is a real Lie algebra. We have the nilpotent radical \( \mathcal{R} \) and the solvable radical \( \mathcal{R} \), characteristic nilpotent and solvable ideals in \( \mathfrak{g} \) defined by

\[
\mathcal{R} \text{ is the union of the nilpotent ideals of } \mathfrak{g},
\]
\[
\mathcal{S} \text{ is the union of the solvable ideals of } \mathfrak{g}.
\]

The basic facts on \( \mathcal{R} \) and \( \mathcal{S} \) are the following.

(1.1) If \( C \) is a fully reducible group of automorphisms of \( \mathfrak{g} \), then there are \( C \)-invariant semisimple subalgebras \( \mathcal{L} \subset \mathfrak{g} \) that map isomorphically onto \( \mathfrak{g}/\mathcal{R} \) under the projection \( \varphi: \mathfrak{g} \to \mathfrak{g}/\mathcal{S} \), and any two such subalgebras are conjugate by an automorphism \( \text{ad}_\varphi(n) \) of \( \mathfrak{g} \) where \( n \in \mathcal{R} \) is left fixed by every \( c \in C \).

The existence is due to G. D. Mostow [3, Corollary 5.2], and the conjugacy statement is the result [5, Theorem 4] of E. J. Taft. In general a semisimple subalgebra \( \mathcal{L} \subset \mathfrak{g} \) such that \( \varphi: \mathcal{L} \subset \mathfrak{g}/\mathcal{R} \) is called a Levi factor of \( \mathfrak{g} \), and the Levi factors of \( \mathfrak{g} \) are just the maximal semisimple subalgebras.

(1.2) If \( \mathcal{L} \) is a Levi factor of \( \mathfrak{g} \), then \( \mathfrak{g} = \mathcal{S} + \mathcal{L} \) semidirect sum, \( \mathcal{S} + \mathcal{L} \) (semidirect) is an ideal in \( \mathfrak{g} \), and the derived algebra \( [\mathfrak{g}, \mathfrak{g}] \subset \mathcal{R} + \mathcal{L} \).

The first assertion is immediate and the second follows from \( \mathcal{R} \subset \mathcal{S} \). For the third, one notes that \( [\mathfrak{R}, \mathfrak{R}] \subset \mathcal{R} \) by Ado's Theorem and that \( \text{ad}_\varphi(\mathcal{L}) \) normalizes \( \text{ad}_\varphi(\mathfrak{R}) \) in the derivation algebra of \( \mathcal{R} \).

Let \( M = G/K \) be a coset space of a Lie group by a closed subgroup. \( \mathfrak{R} \subset \mathfrak{g} \) are the Lie algebras of \( K \subset G \). An \( \text{ad}_G(K) \)-invariant subspace \( \mathfrak{R} \subset \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{R} + \mathfrak{S} \) (vector space direct sum), is called an invariant complement for \( K \). If an invariant complement for \( K \) exists, then \( M = G/K \) is called a reductive coset space.

K is called a reductive subgroup of \( G \) in case the group \( \text{ad}_G(K) \) of linear transformations of \( \mathfrak{g} \) is fully reducible. If \( K \) is a reductive subgroup of \( G \), then \( \text{ad}_G(K)\mathfrak{R} \)
$=\mathfrak{k}$ implies that $M=G/K$ is a reductive coset space. The converse fails in the example

$$G = SL(2, \mathbb{R}) \quad \text{and} \quad K = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$ 

However, compact subgroups, and semisimple subgroups with only finitely many components, are reductive subgroups.

2. The compatibility condition. We can now define the compatibility conditions with which we will operate. $M=G/K$ is a coset space of a Lie group by a closed subgroup. $\mathfrak{l}$ is a Levi factor of $\mathfrak{g}$ and $\mathfrak{m}$ is an invariant complement for $K$. If

$$(2.1) \quad \mathfrak{m} = (\mathfrak{m} \cap \mathfrak{t}) + (\mathfrak{m} \cap \mathfrak{s}) \quad \text{and} \quad \mathfrak{r} = (\mathfrak{r} \cap \mathfrak{t}) + (\mathfrak{r} \cap \mathfrak{s})$$

then we say that $\mathfrak{l}$ splits $\mathfrak{m}$ and that $\mathfrak{m}$ is split by the Levi-Whitehead decomposition $\mathfrak{g}=\mathfrak{r}+\mathfrak{l}$. Suppose further that we have a $G$-invariant pseudo-riemannian metric $ds^2$ on $M$. Represent $ds^2$ by an $ad_G(K)$-invariant inner product $\langle , \rangle$ on $\mathfrak{m}$. If

$$(2.2) \quad \mathfrak{m} = (\mathfrak{m} \cap \mathfrak{t}) + (\mathfrak{m} \cap \mathfrak{s}) \quad \text{and} \quad \langle \mathfrak{m} \cap \mathfrak{r}, \mathfrak{m} \cap \mathfrak{l} \rangle = 0,$$

then we say that $\mathfrak{l}$ splits $\mathfrak{m}$ orthogonally and that $ds^2$ is consistent with the Levi-Whitehead decomposition $\mathfrak{g}=\mathfrak{r}+\mathfrak{l}$.

2.3. Proposition. Let $K$ be a closed reductive subgroup of a Lie group $G$. Then for every $ad_G(K)$-invariant Levi factor $\mathfrak{l}$ of $\mathfrak{g}$, there exists an invariant complement $\mathfrak{m}$ for $K$ such that $\mathfrak{l}$ splits $\mathfrak{m}$. If $ds^2$ is a $G$-invariant pseudo-riemannian metric on $G/K$, $\varphi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{k}$ is the projection, and the representations of $K$ on $\mathfrak{g}/(\mathfrak{m} \cap \mathfrak{s})$ and $\mathfrak{g}/(\mathfrak{m} \cap \mathfrak{r})$ are disjoint, then it is automatic that $\mathfrak{l}$ splits $\mathfrak{m}$ orthogonally and $ds^2$ is consistent with $\mathfrak{g}=\mathfrak{r}+\mathfrak{l}$.

Proof. Mostow’s result (1.1) provides an $ad_G(K)$-invariant Levi factor $\mathfrak{l}$ of $\mathfrak{g}$. As $K$ is reductive in $G$ we have $ad_G(K)$-invariant direct sum decompositions

$$\mathfrak{r} = \mathfrak{r}_1 + (\mathfrak{r} \cap \mathfrak{g}) \quad \text{and} \quad \mathfrak{g}/\mathfrak{r} = \mathfrak{r}_2 + \mathfrak{r}/\mathfrak{g}.$$

Now define

$$\mathfrak{m}_2 = \mathfrak{r} \cap \mathfrak{r}_1^{-1}(\mathfrak{m}_2) \quad \text{and} \quad \mathfrak{m} = \mathfrak{r}_1 + \mathfrak{m}_2.$$ 

Then $ad_G(K)\mathfrak{m}_1=\mathfrak{m}_1$, so $\mathfrak{m}$ is an $ad_G(K)$-invariant subspace of $\mathfrak{g}$ that satisfies (2.1). If $x \in \mathfrak{m} \cap \mathfrak{r}$ then $\varphi(x) \in \mathfrak{m}_2 \cap \mathfrak{g}$ so $x \in \mathfrak{r}_2$; then $x \in \mathfrak{r}_1 \cap (\mathfrak{r} \cap \mathfrak{g})=0$ so $x=0$; thus $\mathfrak{m} \cap \mathfrak{r}=0$. On the other hand

$$\dim \mathfrak{m} = \dim \mathfrak{m}_1 + \dim(\mathfrak{r} \cap \mathfrak{g})$$

and

$$\dim \mathfrak{g}/\mathfrak{r} = \dim \mathfrak{m}_2 + \dim \mathfrak{m}/\mathfrak{g} = \dim \mathfrak{m}_2 + \dim \varphi(\mathfrak{g})$$

$$= \dim \mathfrak{m}_2 - \dim (\mathfrak{r} \cap \mathfrak{g}) + \dim \mathfrak{g}$$

so

$$\dim \mathfrak{g} = \dim \mathfrak{m} + \dim \mathfrak{g}/\mathfrak{r} = \dim \mathfrak{m} + \dim \mathfrak{g}.$$
Thus \( \mathfrak{N} \) is a vector space complement to \( \mathfrak{S} \) in \( \mathfrak{W} \). Now \( \mathfrak{N} \) is an invariant complement for \( K \) such that \( \mathfrak{L} \) splits \( \mathfrak{W} \).

Let \( ds^2 \) and the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{W} \) be given. The representation of \( K \) on \( \mathfrak{W}/(\mathfrak{L} \cap \mathfrak{S}) \) is the representation \( \text{ad}_\mathfrak{g}|_K \) on \( \mathfrak{W}_1 \); the representation of \( K \) on \( \mathfrak{L}/(\mathfrak{L} \cap \varphi^{-1}\varphi\mathfrak{S}) \) is \( \text{ad}_\mathfrak{g}|_K \) on \( \mathfrak{W}_2 \). If those two are disjoint then necessarily \( \langle \mathfrak{W}_1, \mathfrak{W}_2 \rangle = 0 \). Q.E.D.

We reformulate the metric portion of Proposition 2.3.

2.4. Proposition. Let \( M = G/K \) be a homogeneous pseudo-riemannian manifold with metric \( ds^2 \), where \( K \) is a reductive subgroup of \( G \). Let \( \emptyset \) be the base point, \( M_\emptyset \) the tangent space at \( \emptyset \), \( \chi \) the linear isotropy representation of \( K \) on \( M_\emptyset \), and \( R_\emptyset \) the subspace of \( M_\emptyset \) spanned by vector fields from elements of the solvable radical of \( \emptyset \). Suppose that the representations of \( K \) induced by \( \chi \), on \( R_\emptyset \) and on \( M_\emptyset/R_\emptyset \), are disjoint. Then \( M_\emptyset = R_\emptyset + R_\emptyset^1 \) and \( ds^2 \) is consistent with every Levi-Whitehead decomposition \( \emptyset = \mathfrak{R} + \mathfrak{L} \) for which \( \text{ad}_\emptyset(K) \cdot \mathfrak{L} = \mathfrak{L} \).

For, in the notation of the proof of Proposition 2.3, \( R_\emptyset \) is spanned by the vector fields from \( \mathfrak{W}_1 \) while \( R_\emptyset^1 \) is spanned by those from \( \mathfrak{W}_2 \).

In the riemannian case we will be able to arrange that \( \mathfrak{W}_1 = \mathfrak{R} \cap \mathfrak{R} \) contain the nilpotent radical \( \mathfrak{R} \). For that, we need a technical lemma.

2.5. Lemma. In a connected nilpotent Lie group every compact subgroup is central.

Proof. Let \( N \) be the connected nilpotent Lie group, \( \pi: \tilde{N} \to N \) the universal Lie group covering, and \( \Gamma \) the kernel of \( \pi \). Then \( \Gamma \) is a discrete central subgroup of \( \tilde{N} \). Let \( C \) be a maximal compact subgroup of \( N \) and \( \tilde{C} = \pi^{-1}(C) \). Then \( C \) is a torus group, \( \tilde{C} \) is a simply connected commutative subgroup of \( \tilde{N} \), and \( \Gamma \subset \tilde{C} \) such that \( \tilde{C} = \tilde{C}/\Gamma \) compact. As \( \Gamma \) is central in \( \tilde{N} \), and as \( \tilde{N} \) is nilpotent, now \( \tilde{C} \) is central in \( \tilde{N} \), so \( C \) is central in \( N \). If \( E \) is any compact subgroup of \( N \) we have \( n \in N \) such that \( nEn^{-1} \subset C \), so \( E \subset n^{-1}Cn = C \), proving \( E \) central in \( N \). Q.E.D.

2.6. Proposition. Let \( M = G/K \) be an effective coset space of a connected Lie group by a compact subgroup. Then, for any \( \text{ad}_\emptyset(K) \)-invariant Levi factor \( \emptyset \) of \( \emptyset \), there is an invariant complement \( \mathfrak{W} \) for \( K \) such that

\[
(2.7) \quad \mathfrak{W} = (\mathfrak{W} \cap \mathfrak{R}) + (\mathfrak{W} \cap \emptyset), \quad \mathfrak{R} = (\mathfrak{W} \cap \mathfrak{R}) + (\mathfrak{R} \cap \mathfrak{R}), \quad \text{and} \quad \mathfrak{R} \subset \mathfrak{W} \cap \mathfrak{R}.
\]

In particular, if \( G \) is a group of isometries for a riemannian metric on \( M \), and if the metric is consistent with the Levi-Whitehead decomposition \( \emptyset = \mathfrak{R} + \emptyset \), then in addition we can choose \( \mathfrak{W} \) so that \( \mathfrak{W} \cap \mathfrak{R} \) has a subspace \( \mathfrak{A} \) such that

\[
(2.8) \quad \mathfrak{W} = \mathfrak{R} + \mathfrak{A} + (\mathfrak{W} \cap \emptyset), \quad \mathfrak{R} \cap \mathfrak{R} = \mathfrak{R} + \mathfrak{A}, \quad \text{orthogonal direct sums}.
\]

Proof. Following Proposition 2.3 we take \( \mathfrak{W} = \mathfrak{W}_1 + \mathfrak{W}_2 \), invariant complement for \( K \) split by \( \emptyset \), where \( \mathfrak{W}_1 \) is any \( \text{ad}_\emptyset(K) \)-invariant complement to \( \mathfrak{R} \cap \mathfrak{R} \) in \( \mathfrak{R} \).
Let $N$ be the analytic subgroup of $G$ for $\mathfrak{g}$. Then $N$ is closed in $G$, so $N \cap K$ is compact. Let $T$ be a maximal compact subgroup of $N$. It contains $N \cap K$ and is central in $N$ by Lemma 2.5. Thus $T$ is unique, hence normal in $G$. As $T$ is a torus and $G$ is connected now $T$ is central in $G$. Thus $N \cap K$ is central in $G$. But $G$ acts effectively on $M$, so $K$ contains no nontrivial normal subgroup of $G$. That proves $N \cap K = \{1\}$. In particular $\mathfrak{r} \cap \mathfrak{r} = \{0\}$. Thus $\mathfrak{s} = \mathfrak{r} + \mathfrak{u} + (\mathfrak{r} \cap \mathfrak{r})$, $ad_{\mathfrak{g}} (K)$-invariant direct sum, where $\mathfrak{u}$ is any invariant complement to $\mathfrak{r} + (\mathfrak{r} \cap \mathfrak{r})$. For (2.7) we just choose $\mathfrak{r} = \mathfrak{r} + \mathfrak{u}$.

Suppose further that $ds^2$ is consistent with $\mathfrak{g} = \mathfrak{r} + \mathfrak{u}$. Then we have another choice, say $\mathfrak{m}^* = \mathfrak{m}^*_1 + \mathfrak{m}^*_2$, of invariant complement for $K$, with $\langle \mathfrak{m}^*_1, \mathfrak{m}^*_2 \rangle = 0$ and $\mathfrak{r} = \mathfrak{m}^*_1 + (\mathfrak{r} \cap \mathfrak{r})$. If $\psi : \mathfrak{r} \simeq \mathfrak{r}/\mathfrak{r}$ and $\psi^* : \mathfrak{m}^* \simeq \mathfrak{r}/\mathfrak{r}$ are induced by the projection $\mathfrak{g} \to \mathfrak{r}/\mathfrak{r}$, now $\psi^{-1} \psi^* : \mathfrak{m}^* \to \mathfrak{m}$ is a linear isometry carrying $\mathfrak{m}^*_1$ to $\mathfrak{r}$. Thus $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$, and we obtain (2.8) by choosing $\mathfrak{u}$ to be the ortho-complement of $\mathfrak{r}$ in $\mathfrak{m}_1$. Q.E.D.

We will view the space $\mathfrak{a}$ of (2.8) as the "gap" between nilpotent and solvable radicals of $\mathfrak{g}$, taken modulo $\mathfrak{r}$.

3. Mean curvature along the gap between the nilpotent and solvable radicals. We compute the mean curvature of a homogeneous riemannian manifold along a direction in the solvable radical complementary to the nilpotent radical. This is done by specializing the following general calculation to the case where the riemannian metric is consistent with a Levi-Whitehead decomposition.

3.1. Lemma. Let $(M, ds^2)$ be a connected $n$-dimensional riemannian homogeneous space. Let $\mathfrak{g} \in M$. Let $G$ be a connected transitive group of isometries of $M$ and let $M_e$ denote the subspace of $M_e$ consisting of tangent vectors $Y_e$ where $Y$ is in the derived algebra $[\mathfrak{g}, \mathfrak{g}]$. If $X_e \in M_e$ is a unit vector orthogonal to $M'_e$, then the mean curvature

\[(n-1)k(X_e) = \sum_i \langle [[X, E_i]_R + \frac{1}{2}[X, E_i]_R], X\rangle_{R_i}, E_i\rangle
\]

\[= \frac{1}{2} \sum_i ||[X, E_i]_R||^2
\]

\[-\frac{1}{2} \sum_{i,j} \langle [X, E_i]_R, E_i \rangle \cdot \langle [X, E_i]_R, E_i \rangle,
\]

where $\mathfrak{m} \subseteq \mathfrak{g}$ is an invariant complement to $K$, $X \in \mathfrak{m}$ represents $X_e$, $\langle , , \rangle$ is the inner product on $\mathfrak{m}$ from $ds^2$, and $\{E_i\}$ is any orthonormal basis of $\mathfrak{m}$ containing $X$.

Proof. We follow the method of Nomizu [4], using the notation

$\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ for the connection function,

$\mathcal{U} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ for the symmetric part of $\alpha$,

$\mathcal{R} : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ for the curvature tensor.

Now $(n-1)k(X) = \sum_{i \neq i_0} K_i = -\sum_{i \neq i_0} \langle \mathcal{R}(X, E_i)X, E_i \rangle = -\sum_{i} \langle \mathcal{R}(X, E_i)X, E_i \rangle$ where
\{E_i\} is an orthonormal basis of \(\mathcal{W}\), where \(X = E_{i_0}\), and where \(K_i\) is the sectional curvature of the tangent 2-plane spanned by \(X\) and \(E_i\). Thus

\[
(n-1)k(X) = -\sum_i \langle \mathcal{R}(X, E_i)X, E_i \rangle.
\]

Using [4, formulae 9.6 and 13.1] and correcting a misprint in the latter,

\[
\mathcal{R}(X, E_i)X = \alpha(X, \alpha(E_i, X)) - \alpha(E_i, \alpha(X, X)) - \alpha([X, E_i], X) - [[X, E_i], X].
\]

\[
\alpha(S, T) = \frac{1}{2} [S, T]_{\mathcal{R}} + \mathcal{U}(S, T).
\]

\[
\mathcal{U}(S, T) = -\frac{1}{2} \sum_j \langle [S, E_j]_{\mathcal{R}}, T \rangle + \langle [T, E_j]_{\mathcal{R}}, S \rangle E_j.
\]

Our hypothesis on \(X\) and \(X_\theta\) is that \(\langle X, [A, B]_{\mathcal{R}} \rangle = 0\) for all \(A, B \in \mathcal{W}\). In particular

\[
\mathcal{U}(X, S) = \mathcal{U}(S, X) = -\frac{1}{2} \sum_j \langle [X, E_j]_{\mathcal{R}}, S \rangle E_j.
\]

Thus \(\alpha(X, X) = 0\). Substituting that into (3.3) we have

\[
(n-1)k(X) = -\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i \rangle
\]

\[
+ \sum_i \langle \alpha([X, E_i]_{\mathcal{R}}, X), E_i \rangle
\]

\[
+ \sum_i \langle [[X, E_i], X], E_i \rangle.
\]

In order to evaluate the right-hand side of (3.5) we define coefficients \(b_{jk}\) by

\[
[X, E_j]_{\mathcal{R}} = \sum_k b_{jk} E_k.\]

Then, using (3.4),

\[
2 \sum_i \langle [X, \mathcal{U}(E_i, X)]_{\mathcal{R}}, E_i \rangle = -\sum_{i,j} \langle [X, [X, E_j]_{\mathcal{R}}], E_i \rangle E_i
\]

\[
= -\sum_{i,j} \langle [X, E_j]_{\mathcal{R}}, E_i \rangle^2 = -\sum_{i,j} b_{jk}^2
\]

\[
= -\sum_j \left( \sum_k b_{jk}^2 \right) = -\sum_j \|[X, E_j]_{\mathcal{R}}\|^2
\]

\[
= -\sum_i \|[X, E_i]_{\mathcal{R}}\|^2 = \sum_i \langle [X, E_i]_{\mathcal{R}}, [E_i, X]_{\mathcal{R}} \rangle
\]

\[
= \sum_i \langle [X, E_i]_{\mathcal{R}}, [E_i, X]_{\mathcal{R}} \rangle E_i, E_i
\]

\[
= -2 \sum_i \langle \mathcal{U}(X, [E_i, X]_{\mathcal{R}}), E_i \rangle.
\]

In other words

\[
-\frac{1}{2} \sum_i \langle [X, \mathcal{U}(E_i, X)]_{\mathcal{R}}, E_i \rangle - \frac{1}{2} \sum_i \langle \mathcal{U}(X, [E_i, X]_{\mathcal{R}}), E_i \rangle = 0.
\]
Using (3.4) and (3.6) we compute

\[-\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i \rangle\]

\[= -\frac{1}{2} \sum_i \langle [X, [E_i, X]_{\mathbb{R}}], E_i \rangle - \frac{1}{2} \sum_i \langle \mathcal{U}(X, \mathcal{U}(E_i, X)), E_i \rangle\]

\[= -\frac{1}{2} \sum_i \langle [X, \mathcal{U}(E_i, X)]_{\mathbb{R}}, E_i \rangle - \frac{1}{2} \sum_i \langle \mathcal{U}(X, [E_i, X]_{\mathbb{R}}), E_i \rangle\]

That gives us the first summand of the right-hand side of (3.5):

\[-\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i \rangle = -\frac{1}{2} \sum_i \langle [X, E_i]_{\mathbb{R}}, [X, E_i]_{\mathbb{R}}, E_i \rangle\] (3.7)

The second summand of the right-hand side of (3.5) is, again using (3.4),

\[\sum_i \langle \alpha([X, E_i]_{\mathbb{R}}, X), E_i \rangle\]

\[= \frac{1}{2} \sum_i \langle [[X, E_i]_{\mathbb{R}}, X]_{\mathbb{R}}, E_i \rangle - \frac{1}{2} \sum_{i,j} \langle [[X, E_i]_{\mathbb{R}}, [X, E_j]_{\mathbb{R}}], E_i \rangle, E_j \rangle, E_i \rangle\]

\[= \frac{1}{2} \sum_i \langle [[X, E_i]_{\mathbb{R}}, X]_{\mathbb{R}}, E_i \rangle - \frac{1}{2} \sum_i \| [X, E_i]_{\mathbb{R}} \|^2.\]

Using (3.7) now, the sum of the first two summands of the right-hand side of (3.5) is

\[-\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i \rangle + \sum_i \langle \alpha([X, E_i]_{\mathbb{R}}, X), E_i \rangle\]

\[= +\frac{1}{2} \sum_i \langle [[X, E_i]_{\mathbb{R}}, X]_{\mathbb{R}}, E_i \rangle - \frac{1}{2} \sum_i \| [X, E_i]_{\mathbb{R}} \|^2\]

\[-\frac{1}{2} \sum_{i,j} \langle [X, E_i]_{\mathbb{R}}, E_j \rangle \cdot \langle [X, E_j]_{\mathbb{R}}, E_i \rangle.\] (3.8)

Adding \( \sum_i \langle [X, E_i]_{\mathbb{R}}, X]_{\mathbb{R}}, E_i \rangle \) to both sides of (3.8), our assertion (3.2) follows from (3.5). Q.E.D.

We apply Lemma 3.1 to the gap between the nilpotent and solvable radicals of \( G \).

3.9. Theorem. Let \((M, ds^2)\) be a riemannian homogeneous space, \( G \) a transitive Lie group of isometries, \( K \) the isotropy subgroup at a point \( \Theta \in M \), and \( \mathfrak{L} \) an \( \text{ad}_G(K) \)-invariant Levi factor of \( \Theta \), such that \( ds^2 \) is consistent with the Levi-Whitehead decomposition \( \Theta = \mathfrak{R} + \mathfrak{L} \). Choose an invariant complement \( \mathfrak{R} = \mathfrak{R} + \mathfrak{A} + (\mathfrak{R} \cap \mathfrak{L}) \)
for $K$ that satisfies (2.8). Let $X \in \mathfrak{M}$ be a unit vector, $X_0 \in M_0$ the corresponding unit tangent vector.

1. If $X \perp [\mathfrak{M}, \mathfrak{M} \cap \Lambda]$ then the mean curvature $k(X_0) \leq 0$, and $k(X_0) = 0$ if and only if (a) $X \in \mathfrak{M}$ and (b) $[X, \mathfrak{M}] = 0$.

2. If $X \in \mathfrak{M}$ then $k(X_0) < 0$.

Proof. By choice of $\mathfrak{M}$ and by $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M} + \mathfrak{M}$ we have an orthogonal direct sum decomposition

$$\mathfrak{M} = \mathfrak{R} + \mathfrak{B} + (\mathfrak{R} \cap \Lambda), \quad \mathfrak{R} + \mathfrak{B} = \mathfrak{R} \cap \mathfrak{R}, \quad \mathfrak{R} \subset \mathfrak{R},$$

$$[\mathfrak{M}, \mathfrak{M}]_{\mathfrak{M}} = \mathfrak{R} + (\mathfrak{R} \cap \Lambda), \quad \mathfrak{M} \subset \mathfrak{M}.$$ (3.10)

Let $X \in \mathfrak{B}$. Then we have an orthonormal basis $\{E_i\}$ of $\mathfrak{M}$ containing $X$, such that each $E_i$ is in $\mathfrak{R}$, $\mathfrak{R} \cap \Lambda$ or $\mathfrak{B}$. We apply Lemma 3.1 with that basis.

Define coefficients by $[X, E_i]_{\mathfrak{M}} = \sum_k a_{jk} E_k$ and let $A = (a_{jk})$. Then

$$\langle [X, E_i]_{\mathfrak{M}}, E_i \rangle = a_{ii} \quad \text{and} \quad \langle [X, E_i]_{\mathfrak{M}}, E_i \rangle = a_{ii}$$

so

$$\sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_i \rangle \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle = \text{trace } (A^* A).$$

Take polar decomposition $A = ST$ with $S$ symmetric and $T$ orthogonal. Then $A^* A = S^{-1} S$. Let $S = (s_{ii})$, so

$$\text{trace } (A^* A) = \text{trace } (S^{-1} S) = \sum_{i,j} s_{ii}^2 \geq 0,$$

and note that $\sum s_{ii}^2 = 0$ if and only if $S = 0$, which is equivalent to $A = 0$. Thus

$$-\frac{1}{2} \sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_i \rangle \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle \leq 0$$

with equality if and only if $[X, \mathfrak{M}]_{\mathfrak{M}} = 0$.

That takes care of the last summand of (3.2). For the first two summands we define

$$k_t = \langle [[X, E_t]_{\mathfrak{B}} + \frac{1}{2} [X, E_t]_{\mathfrak{B}}], X \rangle_{\mathfrak{M}}, E_t \rangle - \frac{1}{2} \| [X, E_t]_{\mathfrak{B}} \|^2.$$ (3.12)

If $E_i \in \mathfrak{B}$ then $\langle [\mathfrak{M}, \mathfrak{M}]_{\mathfrak{B}}, \mathfrak{B} \rangle = 0$ implies $k_t = -\frac{1}{2} \| [X, E_t]_{\mathfrak{B}} \|^2$. If $E_i \in \mathfrak{R} \cap \Lambda$ then $[\mathfrak{M}, X]_{\mathfrak{B}} \subseteq [\mathfrak{M}, \mathfrak{M}]_{\mathfrak{B}} \subseteq (\mathfrak{R} \cap \Lambda) \perp (\mathfrak{R} \cap \Lambda)$ implies $\langle [\mathfrak{M}, X]_{\mathfrak{B}}, E_t \rangle = 0$ so

$$k_t = -\frac{1}{2} \| [X, E_t]_{\mathfrak{B}} \|^2.$$ (3.13)

Thus

$$\text{if } E_t \in \mathfrak{B} + (\mathfrak{R} \cap \Lambda) \text{ then } k_t = -\frac{1}{2} \| [X, E_t]_{\mathfrak{B}} \|^2 \leq 0.$$ (3.14)

If $E_t \in \mathfrak{R}$ then $E_t \in \mathfrak{R}$ so $[X, E_t]_{\mathfrak{M}} \in \mathfrak{M} \subset \mathfrak{M}$. Thus
As \((\text{ad } X)\mathfrak{g}^* \subset \mathfrak{g}^*\) we can stipulate that, for numbers \(\{\lambda_i\}\) such that \(\{\lambda_o, \lambda_i\}\) are the eigenvalues of \((\text{ad } X)_{|\mathfrak{g}^*}\), each \(E_i \in \mathfrak{g}^*\) is contained in the sum of the subspaces of \(\mathfrak{g}^*\) on which (for some \(b_i = b_i\)) \(\text{ad } X - \lambda_i\) and \(\text{ad } X - \lambda_o\) are nilpotent. That stipulation made, \([\|E_i, F_j\|^2 \geq |\lambda_i|^2\) and \(\langle \text{ad } (X)^2E_i, E_j\rangle \leq |\lambda_i|^2\). So (3.14) implies

\[(3.15) \quad \text{if } E_i \in \mathfrak{g}^* \text{ then } k_i \leq -\frac{1}{3} |\lambda_i|^2 \leq 0.\]

Combining (3.13) and (3.15) we have \(\sum k_i \leq 0\). Adding that inequality to (3.11), and applying Lemma 3.1, we conclude

\[(3.16) \quad k(X_e) \leq 0 \text{ with equality if and only if } [X, \mathfrak{g}]_{\mathfrak{g}} = 0.\]

If \([X, \mathfrak{g}]_{\mathfrak{g}} = 0\) then \([X, \mathfrak{g}] = \mathfrak{g}\). As \(\mathfrak{g} = \mathfrak{r}\) and \([X, \mathfrak{g}] \subset \mathfrak{r}\) it follows that \([X, \mathfrak{g}] = 0\). Then \(\mathfrak{g} = X\mathfrak{r} + \mathfrak{r}\) is a nilpotent subalgebra of \(\mathfrak{g}\). But \(X \in \mathfrak{g}\) and \([\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{r}\), so \(\mathfrak{g}\) is a nilpotent ideal in \(\mathfrak{g}\). As \(\mathfrak{r}\) is the maximal nilpotent ideal of \(\mathfrak{g}\) it follows that \(X \in \mathfrak{g}\). This fact and (3.16) imply the first statement of Theorem 3.9. If \(X \in \mathfrak{g}\) then \(X \not\in \mathfrak{r}\), so \(k(X_e) < 0\). That completes the proof of Theorem 3.9. Q.E.D.

4. Application to manifolds of nonnegative mean curvature. We first apply Theorem 3.9 to homogeneous riemannian manifolds.

4.1. Theorem. Let \((M, ds^2)\) be a connected homogeneous riemannian manifold, \(G\) a transitive Lie group of isometries, \(K\) an isotropy subgroup, and \(\mathfrak{s}\) an \(\text{ad}_G(K)\)-invariant Levi factor of \(\mathfrak{g}\) such that \(ds^2\) is consistent with the Levi-Whitehead decomposition \(\mathfrak{g} = \mathfrak{r} + \mathfrak{s}\).

1. If \((M, ds^2)\) has mean curvature \(\geq 0\) everywhere, then the solvable radical \(\mathfrak{r}\) and the nilpotent radical \(\mathfrak{g}\) of \(\mathfrak{g}\) satisfy \(\mathfrak{g} = \mathfrak{r} + (\mathfrak{s} \cap \mathfrak{r})\) semidirect sum.

2. If \((M, ds^2)\) has mean curvature \(> 0\) everywhere, then the derived group \([G, G]\) of \(G\) is transitive on \(M\).

Proof. If \((M, ds^2)\) has mean curvature \(\geq 0\) everywhere, then, in the notation (2.8), Theorem 3.9 says \(\mathfrak{r} = 0\), so \(\mathfrak{r} \cap \mathfrak{r} = \mathfrak{r}\); thus \(\mathfrak{g} = (\mathfrak{r} \cap \mathfrak{r}) + (\mathfrak{s} \cap \mathfrak{r}) = \mathfrak{r} + (\mathfrak{s} \cap \mathfrak{r})\). If further \((M, ds^2)\) has mean curvature \(> 0\) everywhere, then Theorem 3.9 says \(\mathfrak{r} = [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}\), so the derived group \(G' = [G, G]\) has an open orbit \(G'(\mathfrak{g}) \subset M\). As \(G'(\mathfrak{g})\) is complete and \(M\) is connected, \(G'(\mathfrak{g}) = M\), so \(G'\) is transitive on \(M\). Q.E.D.

4.2. Corollary. Let \((M, ds^2)\) be a connected riemannian homogeneous manifold, \(G\) a transitive Lie group of isometries of \(M\), \(\mathfrak{g} \subset M\), and \(K\) the isotropy subgroup of \(G\) at \(\mathfrak{g}\). Let \(R_\mathfrak{g}\) denote the subspace of the tangent space \(M_\mathfrak{g}\) consisting of vectors \(Y_\mathfrak{g}\) where \(Y\) is contained in the solvable radical \(\mathfrak{r}\) of \(\mathfrak{g}\). Suppose that the linear isotropy representation of \(K\) splits into disjoint representations on \(R_\mathfrak{g}\) and \(R_\mathfrak{g}^*\).

1. If \((M, ds^2)\) has mean curvature \(\geq 0\) everywhere, then \(\mathfrak{g}\) is related to the nilpotent radical \(\mathfrak{r}\) of \(\mathfrak{g}\) by \(\mathfrak{r} = \mathfrak{r} + (\mathfrak{s} \cap \mathfrak{r})\).
2. If \((M, ds^2)\) has mean curvature \(> 0\) everywhere, then the derived group of \(G\) is transitive on \(M\).

**Proof.** Let \(\mathfrak{L}\) be any \(\text{ad}_G (K)\)-invariant Levi factor of \(\mathfrak{G}\). Proposition 2.4 says that \(ds^2\) is consistent with the Levi-Whitehead decomposition \(\mathfrak{G} = \mathfrak{R} + \mathfrak{L}\). Our assertions now follow from Theorem 4.1. Q.E.D.

In order to apply Theorem 4.1 to the case of a transitive solvable group of isometries, we must first prove the following lemma about a simply transitive nilpotent group of isometries. Note that the lemma extends the positive curvature portion of [7].

**4.3. Lemma.** Let \((N, ds^2)\) be a connected nilpotent Lie group with a left invariant riemannian metric. Then the following conditions are equivalent.

(i) \((N, ds^2)\) has mean curvature \(\geq 0\) everywhere.

(ii) \((N, ds^2)\) has every sectional curvature zero.

(iii) \(N\) is commutative.

**Proof.** As (iii) \(\Rightarrow\) (ii) \(\Rightarrow\) (i) trivially we need only check that (i) \(\Rightarrow\) (iii). So assume that \((N, ds^2)\) has mean curvature \(\geq 0\) everywhere. In the context of Theorem 3.9,

\[
G = N, \quad K = \{1\}, \quad \mathfrak{L} = 0, \quad \mathfrak{M} = \mathfrak{R},
\]

and consistency of \(ds^2\) with \(\mathfrak{G} = \mathfrak{R} + \mathfrak{L}\) is automatic. Now Theorem 3.9 says that there is no noncentral element \(X \in \mathfrak{R}\) such that \(X \not\subset [\mathfrak{R}, \mathfrak{M}]\). But nilpotence of \(\mathfrak{R}\) implies that in the lower central series

\[
\mathfrak{R} = \mathfrak{R}_0 \supset \mathfrak{R}_1 \supset \cdots \supset \mathfrak{R}_n \supset \cdots \supset \mathfrak{R}_{s+1} = 0, \quad \mathfrak{R}_{n+1} = [\mathfrak{R}, \mathfrak{R}_k],
\]

any vector space complement to \(\mathfrak{R}_1 = [\mathfrak{R}, \mathfrak{R}]\) generates \(\mathfrak{R}\). Let \([\mathfrak{R}, \mathfrak{R}]^1\) be the complement. As it consists of central elements of \(\mathfrak{R}\) (our application of Theorem 3.9), it must be all of \(\mathfrak{R}\). Thus \(N\) is commutative. Q.E.D.

Now we have a general result on the curvature of riemannian solvmanifolds.

**4.4. Theorem.** Let \((M, ds^2)\) be a connected riemannian manifold that has a solvable transitive group of isometries. Then the following conditions are equivalent.

(i) \((M, ds^2)\) has mean curvature \(\geq 0\) everywhere.

(ii) \((M, ds^2)\) has every sectional curvature \(\geq 0\).

(iii) \((M, ds^2)\) has every sectional curvature zero.

(iv) \((M, ds^2)\) is isometric to the product of an euclidean space and a flat riemannian torus.

**Proof.** As (iv) \(\Rightarrow\) (iii) \(\Rightarrow\) (ii) \(\Rightarrow\) (i) trivially we need only check that (i) \(\Rightarrow\) (iv). So assume that \((M, ds^2)\) has mean curvature \(\geq 0\).
$G$ denotes the closure of a solvable transitive group of isometries of $(M, ds^2)$ in the full group of isometries. So $G$ is a solvable transitive Lie group of isometries. Let $K$ be an isotropy subgroup. $\mathfrak{g}$ is its own solvable radical $\mathfrak{r}$, so the $ad_G(K)$-invariant Levi factor $\mathfrak{u} = 0$, and $ds^2$ is consistent with $\mathfrak{g} = \mathfrak{r} + \mathfrak{u} = \mathfrak{r}$. Our invariant complement $\mathfrak{r}$ for $K$ satisfying (2.8) now has the form $\mathfrak{r} = \mathfrak{r} + \mathfrak{a}$, and Theorem 3.9 says $\mathfrak{a} = 0$.

Let $N$ be the analytic subgroup of $G$ for $\mathfrak{g}$. Now we have an open orbit $N(\mathfrak{g}) \subset M$. As $N(\mathfrak{g})$ is complete and $M$ is connected, the two are equal. Thus $N$ is transitive on $M$. As $G$ acts effectively on $M$, also $N$ acts effectively, so Lemma 2.5 says $K \cap N = \{1\}$. That proves $N$ simply transitive on $M$. Lemma 4.3 says that $(M, ds^2)$ is flat and $N$ is commutative. It follows [6, Théorème 4] that $(M, ds^2)$ is the product of an euclidean space and a flat riemannian torus. Q.E.D.

4.5. Corollary. Let $(M, ds^2)$ be a connected riemannian Einstein manifold that has a solvable transitive group of isometries. Then either $(M, ds^2)$ has vanishing Ricci tensor and is isometric to the product of an euclidean space with a flat riemannian torus, or $(M, ds^2)$ has negative definite Ricci tensor.

**Proof.** The Einstein homogeneous hypothesis says that $(M, ds^2)$ has constant mean curvature, say $k$. If $k \geq 0$ then Theorem 4.4 says that $(R_{ij}) = 0$ and that $(M, ds^2)$ is the product of an euclidean space with a flat riemannian torus. If $k < 0$ then $(R_{ij})$ is negative definite. Q.E.D.

For examples of the latter case of Corollary 4.5, let $(M, ds^2)$ be a noncompact irreducible riemannian symmetric space, $G$ the largest connected group of isometries, $K$ an isotropy subgroup, and $G = NAK$ an Iwasawa decomposition. Then $S = NA$ is a simply transitive solvable Lie group of isometries of $(M, ds^2)$, and $(M, ds^2)$ is a connected riemannian Einstein manifold with negative definite Ricci tensor. G. Jensen [2] has shown that this example is essentially exhaustive in dimensions $\leq 4$.

The following lemma is similar to results of G. Jensen [2].

4.6. Lemma. Let $G$ be a Lie group, let $ds^2$ be a left invariant riemannian metric on $G$, and let $X$ be a nonzero central element of the Lie algebra $\mathfrak{g}$. Then the mean curvature $k(X) \geq 0$, and $k(X) = 0$ if and only if $X$ is orthogonal to the derived algebra of $\mathfrak{g}$.

**Proof.** We use the notation of the proof of Lemma 3.1. Note $\mathfrak{r} = \mathfrak{g}$. We take $X$ to be a unit vector and $\{E_i\}$ to be an orthonormal basis of $\mathfrak{g}$ that contains $X$. Then (3.3) holds. As $X$ is central in $\mathfrak{g}$, the analog of (3.4) is

$$U(S, X) = U(X, S) = -\frac{1}{2} \sum_i \langle [S, E_i], X \rangle E_i.$$
We still have \( \mathcal{H}(X, X) = 0 \), so \( \alpha(X, X) = 0 \) and (3.5) holds. But \([X, E_i] = 0\) simplifies (3.5) to
\[
(n - 1)k(X) = -\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i \rangle
\]
\[
= \frac{1}{2} \sum_{i, j} \langle \alpha(X, \langle [E_i, E_j], X \rangle), E_i \rangle
\]
\[
= \frac{1}{2} \sum_{i, j} \langle [E_i, E_j], X \rangle \cdot \langle \alpha(X, E_i), E_i \rangle
\]
\[
= -\frac{1}{2} \sum_{i, j, k} \langle [E_i, E_j], X \rangle \langle [E_j, E_k], X \rangle E_i
\]
\[
= -\frac{1}{2} \sum_{i, j} \langle [E_i, E_j], X \rangle \langle [E_j, E_i], X \rangle
\]
\[
= \frac{1}{2} \sum_{i, j} \langle [E_i, E_j], X \rangle^2.
\]
Thus \( k(X) \geq 0 \), and \( k(X) = 0 \) if and only if each \( \langle [E_i, E_j], X \rangle = 0 \), which is equivalent to \( \langle \mathfrak{g}, \mathfrak{g}, X \rangle = 0 \). Q.E.D.

We now combine Lemmas 4.3 and 4.6, extending our calculations [7] from sectional curvature to mean curvature, and sharpening Theorem 4.4 in the case of a nilpotent group. After hearing the result, G. Jensen gave another proof of Theorem 4.7 [2, Theorem 4].

4.7. THEOREM. Let \((M, ds^2)\) be a connected riemannian manifold that has a nilpotent transitive group of isometries. Then the following conditions are equivalent.

(i) \((M, ds^2)\) has mean curvature \( \geq 0 \) everywhere.
(ii) \((M, ds^2)\) has mean curvature \( = 0 \) everywhere.
(iii) \((M, ds^2)\) has mean curvature \( \leq 0 \) everywhere.
(iv) \((M, ds^2)\) has every sectional curvature \( \geq 0 \).
(v) \((M, ds^2)\) has every sectional curvature \( = 0 \).
(vi) \((M, ds^2)\) has every sectional curvature \( \leq 0 \).
(vii) \((M, ds^2)\) is isometric to the product of an euclidean space and a flat riemannian torus.

Proof. Let \( N \) denote the identity component of the closure of a nilpotent transitive group of isometries. Then \( N \) is a connected nilpotent transitive Lie group of isometries of \((M, ds^2)\). Its isotropy subgroups are central by Lemma 2.5, hence trivial; thus \( N \) is simply transitive on \((M, ds^2)\). Now we may view \( ds^2 \) as a left invariant riemannian metric on \( N \).

Lemma 4.3 says that (i) implies (v); so (ii) implies (v).

We use Lemma 4.6 to prove that (iii) implies (v). Let \( \mathfrak{Z} \) be the last nonzero term of the lower central series of \( \mathfrak{g} \). Then \( \mathfrak{Z} \) is central in \( \mathfrak{g} \). Let \( 0 \neq X \in \mathfrak{Z} \). Assume (iii), so \( k(X) \leq 0 \). Lemma 4.6 says \( k(X) \geq 0 \). Thus \( k(X) = 0 \) and Lemma 4.6 says \( \langle \mathfrak{g}, \mathfrak{g}, X \rangle = 0 \). If \( \mathfrak{g} \) is noncommutative then \( X \in \mathfrak{Z} < [\mathfrak{g}, \mathfrak{g}] \) and so \( \langle \mathfrak{g}, \mathfrak{g}, X \rangle \...
That proves \( N \) commutative, so every sectional curvature of \((N, ds^2)\) is zero. Thus (iii) implies (v).

Now (i), (ii) and (iii) each implies (v). It follows that (iv) and (vi) each implies (v). But (v) implies (vii) by [6, Théorème 4], and (vii) clearly implies each of (i), (ii), (iii), (iv), (v) and (vi). Q.E.D.

4.8. Corollary. Let \((M, ds^2)\) be a connected riemannian Einstein manifold that has a nilpotent transitive group of isometries. Then \((M, ds^2)\) is isometric to the product of an euclidean space and a flat riemannian torus.

Proof. We have the hypothesis of Theorem 4.7 as well as condition (i), (ii) or (iii); thus we have condition (vii) of Theorem 4.7. Q.E.D.

Our last application is a refinement of [8, Corollary 5.8].

4.9. Theorem. Let \((M, ds^2)\) be a compact connected locally homogeneous riemannian manifold with mean curvature \( \geq 0 \) everywhere. Suppose that the fundamental group \( \pi_1(M) \) has a solvable subgroup of finite index.

Let \( \pi: \tilde{M} \to M \) be the universal riemannian covering, \( G \) the largest connected group of isometries of \((\tilde{M}, \pi^* ds^2)\), \( K \) an isotropy subgroup of \( G \), \( \mathfrak{g} \) an \( \text{ad}_G(K) \)-invariant Levi factor of \( \mathfrak{g} \) and \( L \) its analytic subgroup of \( G \), and \( \mathfrak{r} \) and \( \mathfrak{r} \) the solvable and nilpotent radicals of \( \mathfrak{g} \) and \( R \) and \( N \) their analytic subgroups of \( G \). Let \( \Gamma \) denote the group \( \cong \pi_1(M) \) of deck transformations of \( \tilde{M} \to M \). Suppose that \( \pi^* ds^2 \) is consistent with \( \mathfrak{g} = \mathfrak{r} + \mathfrak{g} \).

1. \( L \) is compact.
2. \( R = N \cdot (K \cap R)_0 \) semidirect product, where \((K \cap R)_0 \) is a torus group whose Lie algebra \( \mathfrak{k} \cap \mathfrak{r} \) acts effectively on \( \mathfrak{r} \) in the adjoint representation.
3. \( \Gamma \) has a torsion free normal nilpotent subgroup \( \Delta \) of finite index, \( \Delta \subset N \cdot Z_L(N)_0 \) where \( Z_L(N) \) is the centralizer of \( N \) in \( L \), and \( \Delta \) projects isomorphically to a discrete subgroup with compact quotient in \( N \).

Proof. Compactness of \( L \) is part of [8, Corollary 5.8], and the decomposition \( R = N \cdot (K \cap R)_0 \) follows from Theorem 4.1. In the proof of [8, Corollary 5.8] it is shown that \( \Gamma \) has a nilpotent subgroup \( \Delta \) of finite index, and that the identity component of the closure of \( R \Delta \) in \( G \) has form

\[
F = R \cdot U
\]

Enlarge \( U \) to a maximal torus \( T \) of \( F \). Then \( T = T_N \cdot T_{R/N} \cdot U \) local direct product, where \( T_N \) is a maximal torus of \( N \) and \( T_{R/N} \) is an \( \text{ad}_G(R) \)-conjugate of \((K \cap R)_0\). These constructions are not changed if \( \Delta \) is cut down to a subgroup of finite index. So we first cut \( \Delta \) down to \( \Delta \cap F \), then \([8, (4.5)]\) to a torsion free group, and finally to a normal subgroup of \( \Gamma \).

Let \( V \) be the kernel of the action of \( T \) on \( N \), i.e. the centralizer \( Z_T(N) \). Then \( T_N \subset V \subset T_N \cdot U \). Let \( F^* = F/V \); then \( F^* = N^*. T^* \) where \( N^* = N/T_N \) simply connected nilpotent group and

\[
T^* = (T_{R/N} \cdot U)/\left((T_{R/N} \cdot U) \cap V\right).
\]
Δ* is the projection of Δ to F*. As Δ is a discrete subgroup with compact quotient in F, the same is true of Δ* in F*. By construction of F* and the fact that T̄ is central in F, conjugation represents T* faithfully as a group of automorphisms of N*, so L. Auslander’s result [1] says that Δ* ∩ N* has finite index in Δ*. Again cutting Δ down, we may assume Δ* ⊂ N*, i.e. that Δ ⊂ N·Z_u(N)₀ ⊂ N·Z_L(N)₀.
Q.E.D.

REFERENCES

1. L. Auslander, Bieberbach’s theorems on space groups and discrete uniform subgroups of Lie groups, Ann. of Math. 71 (1960), 579–590.