A REPRESENTATION FOR A CLASS OF LATTICE ORDERED GROUPS

BY

F. D. PEDERSEN(1)

In a lattice ordered group (l-group), the set of regular subgroups forms a root system in the complete lattice of all convex l-subgroups. Conrad, Harvey, and Holland [4] have shown that an abelian l-group can be represented as an l-group of real-valued functions on any plenary subset of its root system. This paper is concerned with further investigation of the root system leading to a representation for a class of l-groups based on the decomposition of plenary subsets into connected parts (Definition 1). To accomplish this, the concept of Γ-indecomposable l-groups (Definition 8) is introduced. The major theorem (Theorem 15) then presents necessary and sufficient conditions that an l-group be representable as a full subdirect sum of a cardinal sum of Γ-indecomposable l-groups of the first kind.

1. Here we present some of the basic notation, definitions, and theorems relative to the study of l-groups. The uninitiated reader might also want to refer to either [1] or [5] whereas a person knowledgeable in this field might prefer to skip this section.

(i) The positive elements of an l-group G are denoted by G^+. From [5, p. 70], it can be deduced that for g, $h \in G^+$, there exist $\bar{g}$ and $\bar{h}$ with $\bar{g} \wedge \bar{h} = 0$ such that $g = g \wedge h + \bar{g}$ and $h = g \wedge h + \bar{h}$.

(ii) $C(A)$ denotes the convex l-subgroup generated by a nonvoid subset $A \subseteq G$. For convenience, $C(\{g\}) = C(g)$.

(iii) $\Gamma(G)$ denotes the lattice of all convex l-subgroups of G. By [3, Introduction], $\Gamma(G)$ forms a complete distributive sublattice of the lattice of all subgroups of G. Normal convex l-subgroups are called *l-ideals*. A *regular subgroup* is an element of $\Gamma(G)$ which is maximal with respect to not containing some $0 \neq g \in G$. For each nonzero $g \in G$, the completeness of $\Gamma(G)$ assures the existence of at least one regular subgroup maximal without containing g [3, Proposition 3.3]. Similarly, if $x \notin K \in \Gamma(G)$, then there exists $H \in \Gamma(G)$ which is maximal without containing x and such that $K \subseteq H$. Thus, as noted in [3], the regular subgroups of G generate the lattice $\Gamma(G)$.

(iv) Given $M \in \Gamma(G)$, $M \neq G$, M is a *prime subgroup* if M satisfies any one of the following equivalent conditions [3, Theorem 3.2]:

(a) If $A, B \in \Gamma(G)$ such that $A \cap B \subseteq M$, then $A \subseteq M$ or $B \subseteq M$.

(b) If $a, b \in G^+$ and $a, b \notin M$, then $a \wedge b \notin M$.

Received by the editors September 14, 1967 and, in revised form, May 28, 1968.

(1) This work is in part a portion of the author's doctoral dissertation written under the direction of Professor A. H. Clifford, Tulane University [6].

(c) M is an intersection of a chain of regular subgroups.

(d) The convex l-subgroups of G which contain M form a chain.

Also, if M is normal, then each of the above is equivalent to

(e) G/M is totally ordered.

(v) $\Gamma_0(G)$ denotes the set of all prime subgroups of G. $\Gamma_1(G)$ denotes the set of all regular subgroups of G. It follows from property (iv-c) that $\Gamma_1(G) \subseteq \Gamma_0(G)$.

(vi) A *root system* is a partially ordered set S with the property that for any $s' \in S$, the set of all $s \in S$ such that $s' \leqq s$ is a totally ordered subset. $\Gamma_0(G)$ and $\Gamma_1(G)$ are root systems by property (iv-d).

(vii) For a prime subgroup M, it follows from property (iv-b) that if $0 \leqq a \wedge b \in M$, then $a \in M$ or $b \in M$. Thus, as an immediate extension of [4, Lemma 4.5] to the nonabelian case, we have

LEMMA 0. *If* $0 < x$ *and* $0 \leqq y$ *are disjoint elements of* G (i.e., $x \wedge y = 0$) *and if* $C \in \Gamma_1(G)$ *is maximal without containing* x, *then* $y \in C$ *and no* $H \in \Gamma_1(G)$ *maximal without* y *is contained in* C.

(viii) A subset $\Delta \subseteq L_1(G)$ is *plenary* if Δ is an upper class (a lattice dual ideal) in $\Gamma_1(G)$ such that $\bigcap \{H : H \in \Delta\} = \{0\}$.

(ix) For $H \subseteq G$, $H \neq \varnothing$, the *polar* of H, H^*, is the set of all $x \in G$ such that $|x| \wedge |y| = 0$ for all $y \in H$. H^* is a convex l-subgroup [5] or [7].

(x) G is a *lexicographic extension* of $H \in \Gamma(G)$ if

(a) H is normal in G;

(b) $x \in G^+ - H$ implies $y < x$ for all $y \in H^+$; and

(c) G/H is totally ordered.

$K \in \Gamma(G)$ is *bounded* if there exists $x \in G^+$ such that $y < x$ for all $y \in K^+$. If $K \in \Gamma(G)$ is a lexicographic extension of a proper l-ideal of itself, and if K is unbounded in G, then $G = K \oplus K^*$ [2].

(xi) A *value* of $0 \neq g \in G$ is a regular subgroup which is maximal without containing g. Γg denotes the set of all values for $g \in G$. If $0 \neq g \in G$ has only one value, then we call both g and its value *special*. If g is special with G_α as its value, then there exists $N \in \Gamma(G)$ with $C(g)$ a lexicographic extension of N such that $G_\alpha = C(g)^* \oplus N$. For each $G_\alpha \in \Gamma_1(G)$, the notation G^α refers to $\bigcap \{H \in \Gamma(G); G_\alpha \text{ is proper in } H\}$. Thus, G^α is the unique smallest element of $\Gamma(G)$ which properly contains G_α [3, Theorem 3.1]. If g is special with G_α as its value, then $G^\alpha = C(g)^* \oplus C(g)$ [3]. $G_\alpha \in \Gamma_1(G)$ is called essential if there exists $0 \neq g \in G$ such that all values of g are contained in G_α.

(xii) Using the terminology of (xi), an equivalent definition for a plenary subset $\Delta \subseteq \Gamma_1(G)$ is

(a) If $G_\beta \in \Delta$ and $G_\beta \subseteq G_\alpha$, then $G_\alpha \in \Delta$; and

(b) Each $0 \neq g \in G$ has a value in Δ [4].

(xiii) $\Pi\{H_i : i \in I\}$ denotes the *large cardinal sum* of the l-groups, H_i, with I as an arbitrary index set. $\Sigma\{H_i : i \in I\}$ denotes the *small cardinal sum*. Order is defined by $0 \leqq x$ if and only if $0 \leqq x_i$ for all $i \in I$. A *full subdirect sum* is a subdirect sum which contains the small cardinal sum. For each $i \in I$, $\bar{H}_i$ denotes the l-isomorphic copy of H_i in a full subdirect sum of the H_i [7]. G is said to be *cardinally indecomposable* if G can not be represented as a cardinal sum of two nonzero convex l-subgroups of G.

2. Definition 1. $G_\alpha, G_\beta \in \Gamma_1(G)$ are connected ($G_\alpha \sim G_\beta$) if and only if there exists $G_\delta \in \Gamma_1(G)$ such that $G_\alpha, G_\beta \subseteq G_\delta$.

Proposition 2. *$\sim$ is an equivalence relation on $\Gamma_1(G)$.*

Proof. For transitivity, let $G_\alpha \sim G_\beta$, $G_\beta \sim G_\gamma$. By definition there exists G_δ containing G_α, G_β and G_ε containing G_β, G_γ. Since $\Gamma_1(G)$ is a root system, G_δ and G_ε are comparable. Thus, max $\{G_\delta, G_\varepsilon\}$ contains G_α, G_γ.

Definition 3. A *connected part* of $\Gamma_1(G)$ is an equivalence class of $\Gamma_1(G)$ under $\sim$ and will be denoted by Γ_i.

Note that if we restrict $\sim$ to a plenary subset $\Delta \subseteq \Gamma_1(G)$, then the resulting classes will be of the form $\Delta_i = \Gamma_i \cap \Delta$ for all i such that $L_i \cap \Delta \neq \varnothing$. It should also be noted that the Δ_i-classes are maximal directed subsets.

Definition 4. (a) Δ, a plenary subset, is said to be *connected* if there is only one equivalence class. (In other words, a plenary subset Δ is connected if and only if Δ is directed.)

(b) G is *Γ-indecomposable* if there exists a plenary subset of $\Gamma_1(G)$ which is connected.

(c) An element D in Δ_i is *dominating* with respect to Δ_i if for each G_α in Δ_i either $D \subseteq G_\alpha$ or $G_\alpha \subseteq D$.

(d) G is *Γ-indecomposable of the first kind* if there exists a connected plenary subset of $\Gamma_1(G)$ which contains a dominating element.

(e) G is *Γ-indecomposable of the second kind* if G is Γ-indecomposable but there does not exist a connected plenary subset of $\Gamma_1(G)$ with a dominating element.

Theorem 5. *If G is Γ-indecomposable, then G is cardinally indecomposable.*

Proof. Assume $G = A \oplus B$. Since any G_α which is regular is also prime, and since $A \cap B = \{0\} \subseteq G_\alpha$, it follows that G_α contains either A or B but not both. We need only consider positive elements of G; and any $g \in G^+$ can be expressed uniquely as $x + y$ where $0 \leqq x \in A$, $0 \leqq y \in B$. Since any $x \in A^+$ is disjoint from any $y \in B^+$, Lemma 0 quoted in §1 may be applied; and, thus, the values for $x \in A$ are noncomparable to the values for $y \in B$. Thus, any plenary subset Δ can be written as the union of two disjoint sets, Δ_1 and Δ_2, such that Δ_1 contains values of elements in A and Δ_2 contains values of elements in B. But, since Δ is assumed to be connected, Δ is contained in one of Δ_1 or Δ_2. Thus, either the elements of A or the elements of B have no values in Δ, which implies that either A or B is zero since Δ is plenary.

Definition 6. $0 < g \in G$ is a *weak unit* if $0 < h \wedge g$ for all $h \in G^+$.

A *nonunit* is an element $0 < x$ for which there exists $0 < y$ such that $x \wedge y = 0$.

THEOREM 7. *If G is an l-group, then the following are equivalent:*

(a) *G is Γ-indecomposable of the first kind;*

(b) *G contains a weak unit which is special;*

(c) *$\Gamma_1(G)$ contains a dominating element;*

(d) *G is a lexicographic extension of a proper l-ideal.*

Proof. (a) $\rightarrow$ (b). Let Δ be a connected plenary subset of $\Gamma_1(G)$ with dominating element, G_α. Since the values of $0 < x \in G^\alpha - G_\alpha$ with respect to Δ form a trivially ordered subset of Δ, x has only G_α as value in Δ; and, thus, x is special [4, Theorem 3.7]. Assume $y \wedge x = 0$. It follows from the lemma quoted in §1 that y has no value comparable to G_α. Since G_α is dominating, y has no value in Δ; and, thus, $y = 0$. Therefore, x is a weak unit by definition.

(b) $\rightarrow$ (c). Let x be a weak unit which is also special with G_α as its only value. Then, $G_\alpha = C(x)^* \oplus N$ where N is the maximal convex l-subgroup of $C(x)$. Since x is a weak unit, $C(x)^* = \{0\}$; and, thus, $G_\alpha = N \subset C(x)$. Let $G_\beta \in \Gamma_1(G)$. If $x \notin G_\beta$, then x has a value containing G_β. Since G_α is the only value for x, $G_\beta \subseteq G_\alpha$. If $x \in G_\beta$, then $G_\alpha \subseteq C(x) \subseteq G_\beta$. Hence G_α is dominating in $\Gamma_1(G)$.

(c) $\rightarrow$ (d). Let M_0 be the convex l-subgroup generated by all nonunits. M_0 has the following properties [3, p. 111]:

(1) G is a lexicographic extension of M_0;

(2) M_0 is a prime l-ideal or $M_0 = G$;

(3) M_0 is the smallest convex l-subgroup of G that is comparable to all convex l-subgroups of G.

Let G_α be a dominating element of $\Gamma_1(G)$. Using the properties of M_0, it suffices to show that G_α is comparable to all convex l-subgroups of G. Let $K \in \Gamma(G)$; and assume $K \nsubseteq G_\alpha$. For $x \in G_\alpha^+ - K$, $x \notin K$ implies there exists a value for x, say G_β, such that $G_\beta \supseteq K$. Since G_α is dominating in $\Gamma_1(G)$, G_α is comparable to G_β. Thus, $x \in G_\alpha$ and $x \notin G_\beta$ imply that $G_\beta \subset G_\alpha$. Thus, $K \subseteq G_\beta \subset G_\alpha$ or $K \subset G_\alpha$.

(d) $\rightarrow$ (a). If G is a lexicographic extension of a proper l-ideal M, then $M \supseteq M_0$ (notation as in preceding proof); and moreover, there exists $G_\alpha \in \Gamma_1(G)$ with $G_\alpha \supseteq M_0$. Therefore, G_α is comparable to all convex l-subgroups of G. Thus G_α is dominating in $\Gamma_1(G)$; and $\Delta = \Gamma_1(G)$ satisfies condition (a).

In general, a maximal chain in a root system is not cofinal; but a maximal chain in a connected part, Δ_i, will be cofinal in Δ_i.

THEOREM 8. *$\Gamma_1(G)$ is connected and its special elements form a cofinal subset if and only if $G = \bigcup \{G_i : i \in \Lambda\}$ where $\{G_i : i \in \Lambda\}$ is a chain of convex l-subgroups each of which is Γ-indecomposable of the first kind.*

Proof. Assume that $\Gamma_1(G)$ is connected and that its special elements are cofinal in $\Gamma_1(G)$. For each special G_i, choose $0 < g_i$, special, and with $G_i \in \Gamma g_i$. Note that $G_i \cap C(g_i) = M_i$ is the maximal convex l-subgroup of $C(g_i)$; and, moreover, $C(g_i)$ is a lexicographic extension of M_i. Thus, Theorem 6 implies that each $C(g_i)$ is Γ-indecomposable of the first kind. It will now be shown that G is the union of any

$\{C(g_i) : i \in I\}$ where the corresponding $\{G_i : i \in I\}$ is a chain of special G_i's which is cofinal in $\Gamma_1(G)$. Suppose G_i is proper in G_j and $g_i \notin C(g_j)$; then, since G_i is the only value for g_i, $C(g_j) \subseteq G_i \subset G_j$—a contradiction. Therefore, $g_i \in C(g_j)$, or $C(g_i) \subset C(g_j)$. Thus, $\{C(g_i) : i \in I\}$ is chain order isomorphic to $\{G_i : i \in I\}$. Let

$$K = \bigcup \{C(g_i) : i \in I\}.$$

Then K is a convex l-subgroup of G. If $K \neq G$, there exists $G_\alpha \in \Gamma_1(G)$ such that $G_\alpha \supseteq K$. Since $\{G_i : i \in I\}$ is cofinal in $\Gamma_1(G)$, $G_\alpha \subseteq G_i$ for some $i \in I$. This implies that $g_i \notin K$—contrary to the definition of K.

Conversely, assume that $G = \bigcup \{G_i : i \in \Lambda\}$ where each G_i is a convex l-subgroup of G which is Γ-indecomposable of the first kind and $\{G_i : i \in \Lambda\}$ is a chain. For $0 < x \in G$, $x \in G_i$ for some G_i; and there exists a y such that $x \leqq y$ and y is a special element which has a special value G'_y which is dominating in $\Gamma_1(G_i)$. Since y is special, the convex l-subgroup generated by y in G_i is a lexicographic extension of a proper l-ideal. Since G_i is convex in G, the convex l-subgroup generated by y is the same for G as it is for G_i. Thus, $C(y)$ is a lexicographic extension of a proper l-ideal. Therefore, y is special with value $G_y \in \Gamma_1(G)$. This implies that all values of x in $\Gamma_1(G)$ are contained in some special G_y—which, in turn, implies that the special elements are cofinal. To show $\Gamma_1(G)$ is connected it suffices to show two special elements G_α, G_δ are connected since they are cofinal in $\Gamma_1(G)$. Choose $z_1, z_2 > 0$, special, such that $G_\alpha \in \Gamma z_1$ and $G_\delta \in \Gamma z_2$. Since G is the union of the G_i, some G_j contains both z_1 and z_2. G_j is assumed to be Γ-indecomposable; thus, there exists $z_3 > 0$ in G_j such that z_3 is special and $z_3 \geqq z_1, z_2$. But, the condition that $z_3 \geqq z_1, z_2$ immediately implies that the values for z_1 and the values for z_2 are contained in the value for z_3. Thus, if G_β is the value for z_3 in $\Gamma_1(G)$, then $G_{\beta'} \supseteq G_\alpha, G_\delta$—or any two special elements of $\Gamma_1(G)$ are connected.

3. Definition 9. Let Δ be a plenary subset of $\Gamma_1(G)$ and Δ_i a connected part of Δ. Then the union of the set of all x having all values with respect to Δ in Δ_i, together with zero, is denoted by H_i; that is

$$H_i = \{x \in G : G_\alpha \in \Gamma x \cap \Delta \to G_\alpha \in \Delta_i\} \cup \{0\}.$$

Proposition 10. *H_i is a convex l-subgroup of G; and, moreover,*

$$H_i = \bigcap \{G_\delta : G_\delta \in \Delta - \Delta_i\}.$$

Proof. Since the set of all convex l-subgroups forms a complete lattice, it need only be shown that $H_i = \bigcap \{G_\delta : G_\delta \in \Delta - \Delta_i\}$. Let $x \in H_i$ and $G_\delta \in \Delta - \Delta_i$. If $x \notin G_\delta$, then Δ plenary implies that there exists a value G_α of x in Δ such that $G_\alpha \supseteq G_\delta$. But $\Delta - \Delta_i$ being an upper class implies $G_\alpha \notin \Delta_i$, or $x \notin H_i$. Hence $x \in G_\delta$. Conversely, if $x \notin H_i$, then x has a value $G_\delta \in \Delta - \Delta_i$, and $x \notin G_\delta$.

We shall use the following properties from the notes of P. F. Conrad (proofs may be found in [6, Appendix]);

(1) If $G_{H_i}=\{M : M \in \Gamma_0(G) \text{ and } M \not\supseteq H_i\}$, then $\sigma: G_{H_i} \to \Gamma_0(H_i)$, defined by $\sigma(M)=M \cap H_i$, is a one to one and onto mapping.

(2) $G_\alpha \in G_{H_i} \cap \Gamma_1(G)$ if and only if $\sigma(G_\alpha) \in \Gamma_1(H_i)$. Furthermore, $G_\alpha \in \Gamma x \cap G_{H_i}$, $x \in H_i$, if and only if $\sigma(G_\alpha)$ is a value for x in $\Gamma_1(H_i)$.

PROPOSITION 11. *σ is one to one from Δ_i to $\Gamma_1(H_i)$ if and only if $G_\alpha \in \Delta_i$ implies $G_\alpha \not\supseteq H_i$. If this is the case, $\sigma(\Delta_i)$ is a plenary subset of $\Gamma_1(H_i)$.*

Proof. We need only show that under these conditions the image of Δ_i is plenary. Let $0<x \in H_i$; then all the values of x in Δ are in Δ_i. Thus, there exists $G_\alpha \in \Gamma x \cap \Delta_i$ with $\sigma(G_\alpha)$ as a value for x in $\Gamma_1(H_i)$—property 2, above. Suppose $\sigma(G_\alpha) \in \Gamma_1(H_i)$ and $0<x \notin \sigma(G_\alpha)$ for $x \in H_i$. Then $x \notin G_\alpha$, and there exists $G_\beta \in \Delta_i$ such that $G_\beta \in \Gamma x$, $G_\alpha \subseteq G_\beta$. By property 2 above, $\sigma(G_\beta)$ is a value for x in $\Gamma_1(H_i)$. Since $G_\alpha \subseteq G_\beta$, $\sigma(G_\alpha) \subseteq \sigma(G_\beta)$. Therefore $\sigma(\Delta_i)$ is plenary in $\Gamma_1(H_i)$.

COROLLARY 12. *If no G_α in Δ_i contains H_i, then H_i is Γ-indecomposable.*

PROPOSITION 13. *If no G_α in Δ_i contains H_i, then H_i is unbounded in G.*

Proof. Assume H_i is bounded. Thus, there exists x in G^+ such that $y<x$ for all $y \in H_i^+$. Since Δ is plenary, Δ_i contains a value G_α for x. By hypothesis, G_α does not contain H_i; and hence, there exists $y \in H_i^+ - G_\alpha$. Consider first the case where G_α is maximal in Δ_i. Then y is special; and $G_\alpha = C(y)^* \oplus N$ where N is the maximal l-ideal of $C(y)$. Moreover, $G^\alpha = C(y)^* \oplus C(y)$. Then, $0<x \in G^\alpha$ implies the existence of $0 \leqq x_1 \in C(y)^*$, $0 \leqq x_2 \in C(y)$, such that $x_1+x_2=x$; and $x \notin C(y)^*$ implies that $0<x_2$. Moreover, since $x_2 \in C(y)$, there exists $n>0$ such that $0<x_2<ny$. Thus, using the fact that $x_1 \wedge ny=0$, $ny=x \wedge ny=(x_1+x_2) \wedge ny=(x_1 \wedge ny)+(x_2 \wedge ny)=x_2$, which is a contradiction. Assume, on the other hand, that G_α is not maximal in Δ_i. Then, there exists $G_\beta \in \Delta_i$ with $G^\alpha \subseteq G_\beta$. By hypothesis, there exists $z \in H_i^+ - G_\beta$. Since $z<x$ implies the contradiction $z \in G^\alpha \subseteq G_\beta$, H_i must be unbounded.

PROPOSITION 14. *If H_i is unbounded and Δ_i contains a dominating element, then $G=H_i \oplus H_i^*$. Moreover, H_i is a proper lexicographic extension of an l-ideal; and the mapping σ from Δ_i to $\Gamma_1(H_i)$ is one to one.*

Proof. Let G_α be a dominating element of Δ_i. It will be shown that H_i is not contained in G_α. This fact immediately implies that no element of Δ_i contains H_i—or that the mapping from Δ_i to $\Gamma_1(H_i)$ is one to one. Thus, the image of Δ_i will provide H_i with a connected plenary subset with dominating element; and H_i is, therefore, a proper lexicographic extension of an l-ideal. Then, by applying the theorem mentioned in §1, part (x), $G=H_i \oplus H_i^*$ can be concluded.

Assume that $H_i \subseteq G_\alpha$. Let $0<g \in G^\alpha - G_\alpha$. Since H_i was assumed unbounded, there exists $0<h \in H_i$ such that $h \not\leqq g$. Also $h \not> g$ since this would imply that $g \in H_i \subseteq G_\alpha$, contradicting the choice of g. Thus, g and h are not comparable. Let $g=g \wedge h+\bar{g}$, $h=g \wedge h+\bar{h}$ with $\bar{h} \wedge \bar{g}=0$. The incomparability of g and h implies, moreover, that $\bar{g}$ and $\bar{h}$ are greater than 0. Since $g \wedge h \in H_i \subseteq G_\alpha$, $\bar{g} \in G^\alpha - G_\alpha$; and, thus, $\bar{h} \wedge \bar{g} = 0$ implies that $\bar{h}$ has no value comparable to G_α. But, $\bar{h} \in H_i$ implies that all values of $\bar{h}$ in Δ are comparable to the dominating element G_α of Δ_i; a contradiction.

THEOREM 15. *If G is an l-group, then the following are equivalent*:

(a) *there exists* Δ *plenary such that each* Δ_i *contains a cofinal chain of dominating special elements*;

(b) *there exists* Δ *plenary such that each* Δ_i *contains a dominating element and each* H_i *is unbounded*;

(c) *there exists* Δ *plenary such that for each i*, $G = H_i^* \oplus H_i$ *and* H_i *is* Γ*-indecomposable of the first kind*;

(d) *G is l-isomorphic to a full subdirect sum of* Γ*-indecomposable l-groups of the first kind.*

It is to be noted that the Γ-indecomposable l-groups mentioned in (d) are the H_i obtained from the choice of Δ as in (a) or (b). The proof of Theorem 15 depends upon the following lemma—the proof of which will appear later in this section.

LEMMA 16. *If* $\Pi = \Pi\{G_i : i \in I\}$, *where each* G_i *is an l-group, and if G is a full subdirect sum of* Π, *then each* $G_\alpha \in \Gamma_1(G)$ *is classified by means of the projections* π_i *from G onto the* G_i *into one of the following two types*:

[*first type*]: *there exists one* G_j *such that* $\pi_i(G_\alpha) = G_i$ *for all* $i \neq j$, $\pi_j(G_\alpha)$ *is regular in* G_j, *and* $G_\alpha = (\bar{G}_j \cap G_\alpha) \oplus \bar{G}_j^*$.

[*second type*]: $\pi_i(G_\alpha) = G_i$ *for all i.*

Moreover, the set of all G_α *of the first type is plenary, and no* G_β *of the second type is connected to a* G_α *of the first type.*

Proof of Theorem 15. (a) $\to$ (b). It need only be shown that H_i is unbounded or, by Proposition 13, that no G_α in Δ_i contains H_i. By hypothesis on Δ_i, it suffices to show that no dominating special element G_β in Δ_i contains H_i. Let $0 < x \in G$ whose only value is G_β, then $x \in H_i$. Since $x \notin G_\beta$, we can conclude that $G_\beta \not\supseteq H_i$.

(b) $\to$ (c). This follows from Proposition 14.

(c) $\to$ (d). Let Δ be chosen as in (c). Then each $g \in G$ has a unique expression as $g^i + g_i$, $g^i \in H_i^*$, $g_i \in H_i$. Define ϕ_i by $\phi_i(g) = g_i$. Then ϕ_i is an l-homomorphism for each i. Let ϕ denote the induced l-homomorphism from G to $\Pi\{H_i : i \in I(\Delta)\}$. We need only show that ϕ is one to one. If $\phi(g) = 0$, $g \geqq 0$, then $g_i = 0$ for all i. Let us now assume that the values for $x \in H_i$, i in the index set of Δ, are cofinal in Δ_i for each i. The assumption will be proved in the following paragraph. Suppose $g \neq 0$, then g has a value $G_\beta \in \Delta_i$ for some i. From the foregoing assumption, $G_\beta \subseteq G_\alpha \in \Gamma x$, $0 < x \in H_i$. Thus, $0 < x \wedge g = x \wedge (g^i + g_i) = (x \wedge g^i) + (x \wedge g_i)$ which implies $g_i > 0$ since $g^i \wedge x = 0$—a contradiction. $g = 0$ implies ϕ is one to one and an l-isomorphism into. Since for each $g_i \in H_i \subseteq G$, $\phi(g_i)$ is equal to the element 0 in all except the ith coordinate and is g_i in the ith coordinate, $\phi(G)$ contains $\Sigma \oplus H_i$.

Since H_i is Γ-indecomposable of the first kind, by Theorem 7 there exists $G_\gamma \in \Gamma_1(H_i)$ such that G_γ is a value for a special element $0 < x \in H_i$. As in Proposition 11, $\sigma(G_\alpha) = G_\alpha \cap H_i = G_\gamma$ and G_α is the only value for x in G. Suppose $G_\beta \supseteq G_\alpha$ and $G_\beta \in \Gamma y$, $y > 0$. Let $y = y^i + y_i$, $0 \leqq y^i \in H_i^*$ and $0 \leqq y_i \in H_i$. Since $y^i \wedge x = 0$, y^i can have no value comparable to G_α. Thus, $G_\beta \in \Gamma y_i$, $y_i \in H_i$; or, the values for $x \in H_i$ are cofinal in Δ_i.

(d) $\rightarrow$ (a). Assume that G is a full subdirect sum of Γ-indecomposable l-groups, E_i, $i \in I$, each of which is Γ-indecomposable of the first kind. By Lemma 16, the regular subgroups of the first type form a plenary subset Δ of $\Gamma_1(G)$. Moreover, $\Delta_i = \{G_\alpha : \pi_i(G_\alpha) \in \Gamma_1(E_i)\}$ is just the set of all $G_\alpha \in \Gamma_1(G)$ such that $G_\alpha \not\supseteq \bar{E}_i$. Moreover, $\bigcup \{\Delta_i : i \in I\}$ is Δ. Thus the mapping $G_\alpha \rightarrow G_\alpha \cap \bar{E}_i$ is a one to one, onto, order preserving mapping from Δ_i to $\Gamma_1(\bar{E}_i)$. Since E_i is l-isomorphic to $\bar{E}_i$, Δ_i is connected with a cofinal sequence of dominating special elements by Theorem 7. Since any two elements from different Δ_i will necessarily be not comparable, Δ satisfies condition (a).

Proof of Lemma 16. Assume G_α is a regular subgroup of G which is not of the second type. The negation of the second type implies that there exists $i \in I$ such that $\pi_i(G_\alpha) \neq G_i$. Moreover, since G contains the small sum it can be extracted from [7] that $G = \bar{G}_i \oplus \bar{G}_i^*$. Since G_α is not of the second type $G_\alpha \not\supseteq \bar{G}_i$; and, thus, $\bar{G}_i^* \subseteq G_\alpha$ since G_α is prime. Hence, $G_\alpha = (G_\alpha \cap \bar{G}_i) \oplus \bar{G}_i^*$. Thus $\pi_i(G_\alpha) = G_\alpha \cap \bar{G}_i = \sigma(G_\alpha)$, in the notation of Proposition 11, and so is regular. The conclusion that no first type element is connected to a second type element follows from the fact that $G = G_\alpha \vee G_\beta$ where G_α is of the first type and G_β is of the second type.

The following corollary illustrates an alternative way of looking at Theorem 7.2 of [2]. Since a proof of the corresponding theorem is given in [2], the proof will not be given here. Before stating the corollary we need two definitions.

Definition 17. $0 < b \in G$ is a *basic element* if $C(b)$ is totally ordered.

Definition 18. G has a *basis* if for every $0 < x \in G$ there exists a basic element b such that $0 < b \leqq x$. The *basis group* for G is the convex l-subgroup generated by the basic elements in G.

Corollary 19. *If G is an l-group, then the following are equivalent*:

(a) *G has a basis and no maximal convex o-subgroup of the basis group is bounded*;

(b) *there exists Δ plenary such that each Δ_i is a chain of special elements of* $\Gamma_1(G)$;

(c) *G is l-isomorphic to a full subdirect sum of a cardinal sum of o-groups.*

It is noted that the o-groups referred to in (c) are the H_i derived from a choice of Δ as in (b).

4. Definition 20. For a plenary set Δ, the Δ_i-subgroup M_i is the convex l-subgroup formed by intersecting the elements of Δ_i; i.e., $M_i = \bigcap \{G_\alpha : G_\alpha \in \Delta_i\}$.

Proposition 21. *If M_i is a normal Δ_i-subgroup, then G/M_i is Γ-indecomposable.*

Proof. The natural l-homomorphism of G onto G/M_i preserves the lattice structure of the set of all $H \in \Gamma(G)$ such that $H \supseteq M_i$. Thus, Δ_i is mapped onto a plenary set for G/M_i and is connected.

PROPOSITION 22. *Every maximal element of* $\Gamma(G)$ *is normal if and only if each connected part of* $\Gamma_1(G)$, Γ_i, *is invariant under inner automorphisms of* G.

Proof. First assume all maximal elements of $\Gamma(G)$ are normal. Let $G_\alpha \in \Gamma_i$ and $0 < x$ with $x + G_\alpha - x \neq G_\alpha$. Thus, $x \notin G_\alpha$ which implies that there exists $G_\beta \in \Gamma_x$ such that $G_\alpha \subseteq G_\beta$. If G_β is maximal in $\Gamma(G)$, then $x + G_\beta - x = G_\beta \supseteq G_\alpha$ and $x + G_\alpha - x$. Thus, $x + G_\alpha - x \in \Gamma_i$. If G_β is not maximal in $\Gamma(G)$, then there exists $G_\gamma \supset G_\beta$ and $x \in G_\gamma$. Thus, $x + G_\gamma - x = G_\gamma \supseteq G_\alpha$ and $x + G_\alpha - x$ which implies that $x + G_\alpha - x \in \Gamma_i$. For the converse assume G_α is maximal in $\Gamma(G)$. Thus, G_α is the maximal element of some Γ_i. Since Γ_i is invariant, for each $x \in G$, $-x + G_\alpha + x \in \Gamma_i$ and, thus, $G_\alpha \supseteq -x + G_\alpha + x$. Since x was arbitrary G_α is normal.

PROPOSITION 23. *If the maximal elements of* $\Gamma(G)$ *are normal, then the* Γ_i-*subgroups* M_i *are all normal.*

Proof. Suppose $y \in M_i$ and $x + y - x \notin M_i$. Then $x + y - x$ has a value in Γ_i since $M_i = \bigcap \{G_\alpha \in \Gamma_i\}$. $y \in M_i$ implies y does not have a value in Γ_i. Let $G_\alpha \in \Gamma_i$ such that $G_\alpha \in \Gamma x + y - x$. Thus, $-x + G_\alpha + x$ does not contain y which implies that $-x + G_\alpha + x \in \Gamma_j$ for $j \neq i$. Therefore Γ_i is not invariant under inner automorphisms and this contradicts Proposition 22.

THEOREM 24. *If* $\{M_j : j \in J\}$ *is a collection of normal* Δ_i-*subgroups such that* $\bigcap \{M_j : j \in J\} = \{0\}$, *then* G *is* l-*isomorphic to a subdirect sum of* Γ-*indecomposable* l-*groups.*

Proof. The natural homomorphism from G to G/M_j is an onto l-homomorphism, for each $j \in J$. Thus, the induced mapping $x \to x + M_j$ will be an l-homomorphism of G onto a subdirect sum of the G/M_j. Since $\bigcap \{M_j : j \in J\} = \{0\}$, this mapping will be one to one.

COROLLARY 25. *If* G *has a collection of normal primes* N_t, $t \in T$ *such that* $\bigcap \{N_t : t \in T\} = \{0\}$, *then* G *is* l-*isomorphic to a subdirect sum of* Γ-*indecomposable* l-*groups.*

Proof. Let $\Delta = \{G_\alpha \in \Gamma_1(G) : G_\alpha \supseteq N_t$ for some $t \in T\}$. Δ is plenary; and any Δ_i-subgroup, M_i, will be an intersection of a subcollection of the N_t. Thus, the Δ_i-subgroups will be normal and $\bigcap \{M_i : M_i$ the Δ_i-subgroup$\} = \bigcap \{N_t : t \in T\} = \{0\}$. Now, apply Theorem 24.

It is relatively easy to exhibit a counterexample to show that the converse of Corollary 25 is not true. Thus, since the hypothesis of the corollary is trivially equivalent to representability as a subdirect sum of o-groups it follows that representability as a subdirect sum of o-groups implies representability as a subdirect sum of Γ-indecomposable l-groups with the converse false.

COROLLARY 26. *If the maximal elements of* $\Gamma(G)$ *are normal, then* G *is l-isomorphic to a subdirect sum of* Γ*-indecomposable l-groups.*

Proof. Apply Propositon 23 and Theorem 24.

BIBLIOGRAPHY

1. G. Birhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, R. I., 1948.

2. P. F. Conrad, *Some structure theorems for lattice ordered groups*, Trans. Amer. Math. Soc. **99** (1961), 212–240.

3. ———, *The lattice of all convex l-subgroups of a lattice ordered group*, Czechoslovak Math. J. **15** (1965), 101–123.

4. P. F. Conrad, J. Harvey and C. Holland, *The Hahn embedding theorem for abelian lattice ordered groups*, Trans. Amer. Math. Soc. **108** (1963), 143–169.

5. L. Fuchs, *Partially ordered algebraic systems*, Addison-Wesley, Reading, Mass., 1963.

6. F. Pedersen, *Contributions to the theory of regular subgroups and prime subgroups of a lattice ordered group*, Tulane Dissertation, New Orleans, La., 1966.

7. F. Sik, *Zur Theorie der halbegeordneten Gruppen*, Czechoslovak Math. J. **6** (1956), 1–25. (Russian. German summary)

SOUTHERN ILLINOIS UNIVERSITY,
CARBONDALE, ILLINOIS