AN APPLICATION OF HOLMGREN'S THEOREM AND CONVEXITY WITH RESPECT TO DIFFERENTIAL OPERATORS WITH FLAT CHARACTERISTIC CONES([1])

BY

E. C. ZACHMANOGLOU

1. **Introduction.** We consider linear partial differential operators of order m in n-dimensional space

$$P(x, D) = \sum_{|\alpha| \leq m} a^\alpha(x) D^\alpha,$$

where $x = (x_1, \ldots, x_n)$ is a point in R_n, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers with $|\alpha| = \sum_{j=1}^n \alpha_j$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $D_j = \partial/\partial x_j$. The principal part $P_m(x, D)$ is the homogeneous part of order m,

$$P_m(x, D) = \sum_{|\alpha| = m} a^\alpha(x) D^\alpha.$$

At a fixed point $x^0 \in R_n$ the (real) zeroes of $P_m(x^0, \xi)$ form a cone in R_n which is called the (real) characteristic cone of $P(x, D)$ at x^0.

In this paper we study only differential operators with flat characteristic cones. Specifically we assume that at each point of an open set Ω the characteristic cone of $P(x, D)$ is contained in the orthogonal complement of a fixed subspace W of R_n different from $\{0\}$. In §2, by a repeated application of Holmgren's theorem (as extended to distribution solutions by Hormander [1]), we show that if $P(x, D)$ has analytic coefficients in Ω and if u is a distribution solution of $P(x, D)u = 0$ in Ω which vanishes in an open subset Ω_0 of Ω, then u must also vanish in the W-extension of Ω_0 in Ω consisting of all points of Ω which can be connected to points of Ω_0 by polygonal paths in Ω with sides defining vectors in W.

We consider next the problem of characterizing geometrically the open sets in R_n which are P-convex with respect to a differential operator $P(D)$ with constant coefficients. An open set Ω in R_n is called P-convex if to every compact subset K_1 of Ω there exists a compact subset K_2 of Ω such that for every $u \in \mathscr{E}'(\Omega)$ (distribution with compact support in Ω),

$$\operatorname{supp} P(-D)u \subset K_1 \quad \text{implies} \quad \operatorname{supp} u \subset K_2.$$

The importance of this concept lies in the following property of P-convex sets

---

proved by Malgrange [2]: The equation $P(D)u = f$ in Ω has a distribution (or a C^∞) solution u for every f in $C^\infty(\Omega)$ if and only if Ω is P-convex.

Obviously an open set Ω is P-convex if and only if every connected component of Ω is P-convex. It is well known (see [1] or [3]) that an open connected set Ω is P-convex with respect to every differential operator P if and only if Ω is convex. Furthermore, every open set Ω is P-convex if and only if P is elliptic. For a general differential operator some sufficient and some necessary conditions for P-convexity are also known (see [1] and [3]). However, a complete characterization of P-convex sets is only known when $n = 2$: An open connected set Ω in R_2 is P-convex if and only if every characteristic line intersects Ω in an open interval.

In §3 we show that if the characteristic cone of $P(D)$ lies in the orthogonal complement of a subspace W of R_n different from $\{0\}$ and if K is any compact subset of an open set Ω in R_n then for every $u \in \mathscr{E}'(\Omega)$

$$\text{supp } P(D)u \subset K \quad \text{implies} \quad \text{supp } u \subset \hat{K}(W, \Omega)$$

where $\hat{K}(W, \Omega)$ is the W-hull of K in Ω consisting of K and of the components of intersections of translations of W with the complement $\mathscr{C}K$ of K which are relatively compact in Ω. This result immediately yields the following sufficient condition for P-convexity: Ω is P-convex if for every compact subset K of Ω, $\hat{K}(W, \Omega)$ is compact. In §4 we show that for first order differential operators this condition is also necessary for P-convexity.

The author wishes to express his sincere thanks to Professor Francois Treves of Purdue University for his constant advice, criticism and encouragement of this work.

2. **An application of Holmgren's theorem.** Holmgren's theorem asserts the uniqueness of the classical solution of the Cauchy problem when the initial surface is not characteristic with respect to the differential operator. A C^1 surface is said to be characteristic at a point with respect to $P(x, D)$ if its normal belongs to the characteristic cone of $P(x, D)$ at that point. Hormander [1] has extended Holmgren's theorem to distribution solutions.

LEMMA 1 (HORMANDER). *In an open set Ω of R_n let $P(x, D)$ be a differential operator with analytic coefficients and assume that the hyperplanes $x_n = $ constant are nowhere characteristic with respect to $P(x, D)$ in Ω. If u is a distribution in Ω such that $P(x, D)u = 0$ in $\Omega_c = \{x : x \in \Omega, x_n < c\}$ and if $\Omega_c \cap \text{supp } u$ is relatively compact in Ω, then $u = 0$ in Ω_c.*

We need a slightly modified form of this lemma which is easily obtained by an analytic transformation of coordinates.

LEMMA 2. *In an open set Ω of R_n let $P(x, D)$ be a differential operator with analytic coefficients. Let $F(x)$ be a real-valued analytic function defined in R_n, having the form*

$$F(x) = f(x') - x_n, \qquad x' = (x_1, \ldots, x_{n-1}),$$

and suppose that the level surfaces of F are nowhere characteristic with respect to $P(x, D)$ in Ω. If u is a distribution in Ω such that $P(x, D)u = 0$ in $\Omega_c = \{x : x \in \Omega, F(x) > c\}$ for some c and if $\Omega_c \cap \operatorname{supp} u$ is relatively compact in Ω, then $u = 0$ in Ω_c.

DEFINITION 1. Let Ω be an open set in R_n, Ω_0 an open subset of Ω and W a subspace of R_n different from $\{0\}$. The *W-extension of Ω_0 in Ω*, denoted by $\tilde{\Omega}_0(W, \Omega)$ is the union of Ω_0 and the set of points of Ω which can be connected to points of Ω_0 by polygonal paths in Ω with sides defining vectors in W.

Clearly $\tilde{\Omega}_0(W, \Omega)$ is an open subset of Ω. Moreover, it is easy to see that the following is an equivalent definition:

$\tilde{\Omega}_0(W, \Omega)$ is the union of all connected components of intersections of translations of W with Ω which intersect Ω_0.

THEOREM 1. *Let $P(x, D)$ be a differential operator with analytic coefficients in an open set Ω and suppose that at each point $x \in \Omega$ the characteristic cone of $P(x, D)$ is contained in the orthogonal complement of a fixed subspace W of R_n different from $\{0\}$. Then every distribution u in Ω satisfying the equation $P(x, D)u = 0$ in Ω and vanishing in an open subset Ω_0 of Ω must also vanish in $\tilde{\Omega}_0(W, \Omega)$.*

Explicitly the hypothesis of the theorem means that at every point $x \in \Omega$,

$$P_m(x, \xi) = 0 \quad \text{implies} \quad \xi \in W^{\perp}.$$

Proof of Theorem 1. We consider first the special case in which W is a one-dimensional subspace of R_n. If W is generated by the vector l, the hypothesis of the theorem implies that at every point $x \in \Omega$,

$$\langle \xi, l \rangle \neq 0 \Rightarrow P_m(x, \xi) \neq 0.$$

By a rotation of coordinates we may assume that $l = (0, \dots, 0, 1)$.

Now let $x^1 \in \tilde{\Omega}_0(W, \Omega)$. We will show that any distribution u in Ω, satisfying $P(x, D)u = 0$ in Ω and vanishing in Ω_0, must also vanish in a neighborhood of x^1. By definition of $\tilde{\Omega}_0(W, \Omega)$, there is a point $x^0 \in \Omega_0$ such that the straight line segment $x^0 x^1$ is contained in Ω and is parallel to the x_n-axis. Thus

$$x^1 = (x'^0, x_n^1), \qquad x^0 = (x'^0, x_n^0),$$

and we may assume without loss of generality that $x_n^1 > x_n^0$. Clearly there are constants $\varepsilon > 0$ and $r > 0$ such that the cylinder

$$U = \{x : |x' - x'^0| < r, x_n^0 < x_n < x_n^1 + \varepsilon\}$$

is contained in Ω and its base

$$\{x : |x' - x'^0| < r, x_n = x_n^0\}$$

is contained in Ω_0. Consider now the function

$$F(x) = (x_n^1 + \varepsilon) - (x_n^1 + \varepsilon - x_n^0)|x' - x'^0|^2/r^2 - x_n$$

which is defined and analytic in R_n. The level surfaces of F are paraboloids with common axis parallel to the x_n-axis. The normals to these surfaces always have nonzero components in the direction of the x_n-axis. Hence the hypothesis implies that the level surfaces of F are nowhere characteristic with respect to $P(x, D)$ in U. Moreover, if we set

$$U_{\varepsilon/2} = \{x : x \in U, F(x) > \varepsilon/2\},$$

then $U_{\varepsilon/2} \cap \operatorname{supp} u$ is relatively compact in U. It follows from Lemma 2 that $u = 0$ in $U_{\varepsilon/2}$ which is an open neighborhood of x^1.

In the general case in which W is a nonzero subspace of R_n, the hypothesis of the theorem implies that at each point $x \in \Omega$,

$$\xi \notin W^\perp \Rightarrow P_m(x, \xi) \neq 0.$$

The method of proof in this case consists of a repeated application of the method of proof of the above special case: Let x^1 be a point of Ω which can be joined to a point $x^0 \in \Omega_0$ by a straight line segment contained in Ω and such that the vector $x^0 x^1 \in W$. The paraboloidal surfaces with common axis the line passing through x^0 and x^1 (of the type constructed in the proof of the special case) are nowhere characteristic with respect to $P(x, D)$ in Ω. This follows from the hypothesis, and from the fact that the normals to these surfaces have nonzero components in the direction of $x^0 x^1 \in W$. It follows, as in the proof of the special case, that $u = 0$ in an open neighborhood of the segment $x^0 x^1$. Now let x^2 be a point in Ω such that the straight line segment $x^1 x^2$ is contained in Ω and the vector $x^1 x^2 \in W$. Since $u = 0$ in an open neighborhood of x^1 we can again conclude that $u = 0$ in an open neighborhood of the segment $x^1 x^2$. It should be clear now that repeating this argument a finite number of times completes the proof of the theorem.

It should be noted that the use of paraboloidal surfaces in proving uniqueness theorems for the Cauchy problem is standard. See for example the paper by F. John [4, §2].

EXAMPLE 1. If $P(x, D)$ is elliptic in Ω, then $W = R_n$ and Theorem 1 states the well-known fact that any solution of a homogeneous elliptic equation in Ω which vanishes in an open subset Ω_0 must also vanish in all connected components of Ω which contain pieces of Ω_0.

EXAMPLE 2. Consider the heat operator

$$P(D) = D_1^2 + \cdots + D_{n-1}^2 - D_n$$

in an open set Ω of R_n. Here W is the $(n-1)$-dimensional subspace of R_n containing the $x_1, \ldots, x_{n-1}$ axes. Theorem 1 states that if u is a solution of $P(D)u = 0$ in Ω and $u = 0$ in the open subset Ω_0 of Ω, then u must also vanish in $\tilde{\Omega}_0(W, \Omega)$ which in this case consists of all points of Ω which can be connected to points of Ω_0 by polygonal paths in Ω which lie in hyperplanes normal to the x_n-axis (characteristic hyperplanes).

EXAMPLE 3. Consider the first order differential operator in an open set Ω of R_n,

$$P(x, D) = \sum_{j=1}^{n} a^j D_j + c(x),$$

where a^j, $j = 1, \ldots, n$, are (complex valued) constants and $c(x)$ is analytic in Ω. In this case, W is the subspace of R_n generated by Re a and Im a. If u is a solution of $P(x, D)u = 0$ in $\Omega \subset R_n$ and if $u = 0$ in the open subset Ω_0 of Ω then $u = 0$ in $\tilde{\Omega}_0(W, \Omega)$ which in this case consists of all points of Ω which can be connected to points of Ω_0 by polygonal paths in Ω with sides defining vectors which are linear combinations of Re a and Im a.

3. A sufficient condition for P-convexity.

DEFINITION 3. Let Ω be an open set in R_n, K a compact subset of Ω and W a subspace of R_n different from $\{0\}$. The *W-hull of K in Ω*, denoted by $\hat{K}(W, \Omega)$, is the union of K and the set of points of $\mathscr{C}K$ which cannot be connected to points of $\mathscr{C}\Omega$ or to infinity by polygonal paths in $\mathscr{C}K$ with sides defining vectors in W.

An equivalent definition is the following: $\hat{K}(W, \Omega)$ is the union of K and all components of intersections of translations of W with $\mathscr{C}K$ which are relatively compact subsets of Ω.

THEOREM 2. *Let $P(x, D)$ be a differential operator with analytic coefficients in an open neighborhood of the closure of an open set Ω in R_n and suppose that at each point of this neighborhood the characteristic cone of $P(x, D)$ is contained in the orthogonal complement of a fixed subspace W of R_n different from $\{0\}$. Let K be a compact subset of Ω. Then for every $u \in \mathscr{E}'(\Omega)$,*

$$\operatorname{supp} P(x, D)u \subset K \quad implies \quad \operatorname{supp} u \subset \hat{K}(W, \Omega).$$

Proof. The complement of $\hat{K}(W, \Omega)$ consists of all components of intersections of translations of W with $\mathscr{C}K$ which are either unbounded or intersect the complement of Ω. Such components certainly intersect the complement of every compact subset L of Ω containing K. It follows that the complement of $\hat{K}(W, \Omega)$ is contained in the W-extension of $\mathscr{C}L$ in $\mathscr{C}K$ for every compact subset L of Ω containing K. Let us denote this extension by U_L. Now for every $u \in \mathscr{E}'(\Omega)$ there is a compact subset L of Ω such that $u = 0$ in $\mathscr{C}L$. If, in addition, $P(x, D)u = 0$ in $\mathscr{C}K$, it follows from Theorem 1 that $u = 0$ in U_L. We have shown that every $u \in \mathscr{E}'(\Omega)$ such that $\operatorname{supp} P(x, D)u \subset K$ vanishes in an open set U_L containing the complement of $\hat{K}(W, \Omega)$. The proof is complete.

For elliptic differential operators, Theorem 2 yields the following well-known result:

COROLLARY 1. *Under the conditions of Theorem 2 and if $W = R_n$, i.e. if $P(x, D)$ is elliptic, the set $\hat{K}(W, \Omega)$ is the compact subset of Ω consisting of the union of K and all connected components of $\mathscr{C}K$ which are contained in Ω.*

Since $\hat{K}(W, \Omega)$ is always a bounded set contained in Ω, Theorem 2 yields immediately a sufficient condition for P-convexity.

THEOREM 3. *Let $P(D)$ be a differential operator in R_n with constant coefficients and suppose that the characteristic cone of $P(D)$ is contained in the orthogonal complement of a subspace W of R_n different from $\{0\}$. An open set Ω in R_n is P-convex if for every compact subset K of Ω, $\hat{K}(W, \Omega)$ is compact.*

COROLLARY 2. *Every open set Ω in R_n is P-convex with respect to every elliptic differential operator with constant coefficients.*

For differential operators of the first order we can show that the sufficient condition for P-convexity of Theorem 3 is also necessary.

4. Characterization of P-convex sets for first-order differential operators.

LEMMA 3. *Let Ω be an open set in R_n and W a subspace of R_n different from $\{0\}$. Suppose that for some compact subset K of Ω the set $\hat{K}(W, \Omega)$ is not compact. Then there is a sequence of components of intersections of translations of W with $\mathscr{C}K$ which are relatively compact in Ω such that their distance from $\mathscr{C}\Omega$ tends to zero.*

Proof. Since $\hat{K}(W, \Omega)$ is not compact, there is a sequence $\{x^{(k)}\}$ of points in $\hat{K}(W, \Omega)$ converging to a point x^0 belonging to the complement of $\hat{K}(W, \Omega)$. Since K is compact, we may assume that $\{x^{(k)}\}$ is not contained in K. Let $\tau W'_k$ denote the component of the intersection of the translation of W with $\mathscr{C}K$ which is relatively compact in Ω and which contains the point $x^{(k)}$. Let $\tau W'_0$ denote the component of the intersection of the translation of W with $\mathscr{C}K$ which contains x^0. Since x^0 belongs to the complement of $\hat{K}(W, \Omega)$, $\tau W'_0$ must be either unbounded or it must intersect $\mathscr{C}\Omega$. It is sufficient to show that the first possibility cannot occur. If $\tau W'_0$ is unbounded, x^0 is the end point of an unbounded polygonal path which lies in $\tau W'_0$. This path is closed and does not intersect K, hence its distance from K is positive. This clearly implies that for sufficiently large k, $\tau W'_k$ are also unbounded, a contradiction.

THEOREM 4. *In R_n consider the first-order differential operator with constant coefficients*

$$P(D) = \sum_{j=1}^{n} a^j D_j + c,$$

and let W be the subspace of R_n generated by $\operatorname{Re} a$ and $\operatorname{Im} a$. An open set Ω in R_n is P-convex if and only if for every compact subset K of Ω, $\hat{K}(W, \Omega)$ is compact.

Proof. In view of Theorem 3, since the characteristic cone of $P(D)$ is the orthogonal complement of W, we need only to prove the necessity. Let K be a compact subset of Ω such that $\hat{K}(W, \Omega)$ is not compact. We must show that Ω is not P-convex. According to Lemma 3, there is a sequence $\{\tau W'_k\}$ of components of

intersections and translations of W with $\mathscr{C}K$ which are relatively compact in Ω such that their distance from $\mathscr{C}\Omega$ tends to zero. We note that for each k, the boundary of $\tau W_k'$ lies in the compact subset K of Ω. The proof will be complete if we can show that for each k there is a distribution $u_k \in \mathscr{E}'(\Omega)$ such that supp u_k is the closure of $\tau W_k'$ and supp $P(-D)u_k$ is the boundary of $\tau W_k'$. For simplicity, let us assume that Re a and Im a are linearly independent, so that W is two-dimensional. An obvious modification is necessary if Re a and Im a are linearly dependent. Let τW_k denote the translation of W containing $\tau W_k'$. Introduce new coordinates $(x_1', x_2', \ldots, x_n')$ with origin in τW_k such that x_1' and x_2' are coordinates for τW_k and set

$$v_k(x') = 1(x_1', x_2') \otimes \delta(x_3') \otimes \cdots \otimes \delta(x_n').$$

Clearly supp $v_k = \tau W_k$ and

$$\sum_{j=1}^{n} a^j D_j v_k = 0.$$

Now let w be any analytic solution of

$$-\sum_{j=1}^{n} a^j D_j w + c = 0 \quad \text{in } R_n,$$

and set

$$\tilde{u}_k = e^w v_k.$$

It is easy to verify that $P(-D)\tilde{u}_k = 0$ and supp $\tilde{u}_k = \tau W_k$. Finally, set

$$u_k = \chi(\tau W_k')\tilde{u}_k$$

where $\chi(\tau W_k')$ is the characteristic function of $\tau W_k'$. Clearly u_k has the required properties.

BIBLIOGRAPHY

1. L. Hormander, *Linear partial differential operators*, Academic Press, New York, 1963.

2. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier Grenoble **6** (1955–56), 271–355.

3. F. Treves, *Linear partial differential equations with constant coefficients*, Gordon and Breach, New York, 1966.

4. F. John, *On linear partial differential equations with analytic coefficients*, Comm. Pure Appl. Math. **2** (1949), 209–253.

PURDUE UNIVERSITY,
    LAFAYETTE, INDIANA