PAIRS OF INNER FUNCTIONS ON FINITE RIEMANN SURFACES[1]

BY
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I. **Introduction.** A finite (open) Riemann surface R is a connected open proper subset of some compact Riemann surface X such that the boundary ∂R of R is also the boundary of its closure $\bar{R}$ and is the union of finitely many disjoint simple closed analytic curves $\Gamma_1, \dots, \Gamma_k$. It is clear that R inherits a conformal structure from X. Let $D_1, \dots, D_k$ be disjoint closed circular discs and identify the boundary of D_i with Γ_i, for $1 \leqq i \leqq k$. Then $R \cup D_1 \cup \cdots \cup D_k$ is a compact orientable surface R_1 which has a certain genus g. (Topologically, R_1 is a sphere with g handles.) This integer g is defined to be the *genus of R*.

The class of all continuous complex functions on $\bar{R}$ which are holomorphic in R will be denoted by $A(R)$.

For example, every bounded plane region R whose boundary consists of k disjoint simple closed analytic curves is a finite Riemann surface of genus 0, by virtue of its natural embedding in the Riemann sphere S. If R is the open unit disc, then $A(R)$ is what is usually called the *disc algebra*.

An *inner function* in $A(R)$ is, by definition, a member of $A(R)$ whose absolute value is 1 at every point of ∂R.

This paper deals with the following question, $Q(R)$, which may be asked about any finite Riemann surface R.

$Q(R)$. *Does $A(R)$ contain an unramified separating pair of inner functions?*

More explicitly, $Q(R)$ asks whether $A(R)$ contains two functions F and G such that

(i) $|F(p)| = |G(p)| = 1$ at every $p \in \partial R$,

(ii) the pair F, G separates points on the closure $\bar{R}$ of R, and

(iii) every point of $\bar{R}$ has a neighborhood in which at least one of the functions F, G is one-to-one.

Our question has the following background. If such a pair F, G exists, then $p \to (F(p), G(p))$ embeds R as an analytic submanifold V of the unit polydisc U^2 of C^2 (the space of 2 complex variables) and ∂R is carried homeomorphically

Received by the editors April 30, 1968 and, in revised form, August 6, 1968 and October 1, 1968.

(1) Research partially supported by NSF Grant GP-6764.

into the torus T^2, the distinguished boundary of U^2. Under these conditions, Stout [8] has shown that every $f \in A(R)$, when transferred to V, can be extended to a uniformly continuous holomorphic function in U^2. This in turn implies that *the Banach algebra $A(R)$ is doubly generated whenever the answer to $Q(R)$ is affirmative.*

We shall see that $Q(R)$ has an affirmative answer for some surfaces R, but that there are many cases where the answer is negative. (Stout [8] showed that $A(R)$ always contains a *triple* with the desired properties.) Whether $A(R)$ is always doubly generated is still an open question [3, p. 347].

Here is one of our principal results; it will be proved in §IV.

THEOREM A. *If R is a finite Riemann surface of genus g whose boundary consists of disjoint curves $\Gamma_1, \dots, \Gamma_k$, and if $A(R)$ contains an unramified separating pair of inner functions F and G, then there exist positive integers s and t such that*

(a) *F has multiplicity s on each Γ_i and G has multiplicity t on each Γ_i,*

(b) *s and t are relatively prime, and*

(c) $(sk-1)(tk-1)-(k-1)=2g$.

A more explicit statement of (a) is that to every complex number λ with $|\lambda|=1$ there correspond s points p on each Γ_i at which $F(p)=\lambda$, and similarly for G and t. Conclusions (b) and (c) show that an affirmative answer to $Q(R)$ imposes rather severe restrictions on the topology of R. Let us discuss these briefly.

(A1) If the answer to $Q(R)$ is affirmative, then (c) implies that k divides $2g-2$ and that $(k-1)(k-2) \leqq 2g$, since the left side of (c) attains its minimum (for given g, k) when $s=t=1$. In particular, to each g there correspond only finitely many values of k for which (c) can hold. But even if (c) can be satisfied by integers s, t, there may not exist a relatively prime solution. For example, if $g=23$ and $k=4$, then (c) holds with $s=t=2$, and this is the only solution; thus the answer to $Q(R)$ is negative.

(A2) If $g=0$ (in which case R is conformally equivalent to a plane region) and if (c) holds, then $k=1$ or $k=2$, by (A1). Thus,

If R is a plane region (bounded by finitely many analytic curves) which is not conformally equivalent to the unit disc or to an annulus, then $A(R)$ contains no unramified separating pair of inner functions.

(A3) If $g=0$ and $k=1$, then (c) becomes $(s-1)(t-1)=0$, so either $s=1$ or $t=1$. Thus,

If two finite Blaschke products separate points on the closed unit disc $\overline{U}$ and if their derivatives have no common zero in U, then one of them must be one-to-one.

(A4) If $g=0$ and $k=2$ (the case of an annulus) then (c) becomes $(2s-1)(2t-1)=1$, hence $s=t=1$. So if F, G is an unramified separating pair of inner functions on an annulus R, each must have multiplicity 1 on each boundary curve, hence each has multiplicity 2 on R. Such pairs do exist. They are all described in §V.

(A5) For $g=1, 2, 3$, the following table exhibits all solutions of (c) (with $s \leqq t$, to avoid duplication):

g	1	1	1	2	2	2	3	3	3	3
k	1	2	3	1	1	2	1	1	2	4
s	2	1	1	2	3	1	2	3	1	1
t	3	2	1	5	3	3	7	4	4	1

Note that one of these violates (b).

(A6) If $k=1$ or 2, then no matter what g is, one can find s and t so that (b) and (c) hold: If $k=1$, take $s=2$, $t=1+2g$. If $k=2$, take $s=1$, $t=1+g$.

Theorem A has a partial converse (also proved in §IV):

THEOREM B. *If g, k, s, t are positive integers which satisfy conditions* (b) *and* (c) *of Theorem* A *then there is a finite Riemann surface R of genus g, bounded by k curves, such that $A(R)$ contains an unramified separating pair of inner functions F and G which satisfies condition* (a) *of Theorem* A.

However, there do exist surfaces R whose conformal structure is such that the answer to $Q(R)$ is negative although the topological conditions imposed by Theorem A are satisfied. In §V there are two examples of this, one with $g=1$ and $k=2$, the other with $g=2$ and $k=1$.

That these two examples are, in a sense, minimal is shown by the next theorem which is the only one I have found (for $g>0$) in which topological information about R is enough to gurantee an affirmative answer to $Q(R)$.

THEOREM C. *If a finite Riemann surface R has $g=1$ and $k=1$ (i.e., if R is a torus with one hole) then the answer to $Q(R)$ is affirmative.*

Every finite Riemann surface R can be embedded in its "double" $\tilde{R}$; this is a compact Riemann surface which consists essentially of two copies of R. For the details of this construction, see [2, pp. 581–2] or [7, p. 217]. If R has genus g and k boundary components one verifies easily (by thinking of spheres with handles and holes) that the genus of $\tilde{R}$ is $2g+k-1$. In the situation of Theorem C, the genus of $\tilde{R}$ is therefore 2. It is known [2, p. 577], [7, p. 294] that every compact Riemann surface of genus 2 is hyperelliptic, i.e. that it carries a meromorphic function of multiplicity 2. Thus Theorem C is a special case of the following result (proved in §IV).

THEOREM D. *If the boundary of a finite Riemann surface is connected (i.e., $k=1$) and if the double of R is hyperelliptic, then the answer to $Q(R)$ is affirmative.*

Conditions (a) and (b) of Theorem A come from a purely topological fact which is proved in §II. The result of §III (dealing with unramified separating pairs of meromorphic functions on compact Riemann surfaces) will be applied to the double of R in the proof of Theorem A.

II. **Homeomorphisms into a torus.** In this section, h will be a homeomorphism of a simple closed curve into the torus T^2. It will be convenient to regard h as a continuous mapping of the real line into T^2 such that $h(x_1)=h(x_2)$ if and only if x_1-x_2 is an integer. With such an h are associated two real continuous functions α and β such that

$$h(x) = (e^{2\pi i\alpha(x)}, e^{2\pi i\beta(x)}) \qquad (-\infty < x < \infty)$$

and two integers

$$s(h) = \alpha(1)-\alpha(0), \qquad t(h) = \beta(1)-\beta(0);$$

these are the winding numbers of the components of h.

THEOREM. (a) *If h is a homeomorphism of a simple closed curve into T^2 then either*

(i) $s(h)=t(h)=0$, *or*

(ii) $s(h)$ *and* $t(h)$ *are relatively prime.*

(b) *If h_1 and h_2 are homeomorphisms of a simple closed curve into T^2 whose ranges do not intersect, and if both satisfy* (ii), *then there exists $\eta=\pm 1$ such that*

$$s(h_1) = \eta s(h_2) \quad and \quad t(h_1) = \eta t(h_2).$$

Proof. (a) Put $s=s(h)$, $t=t(h)$, assume that (i) does not hold, and let d be the greatest common divisor of s and t. For $-\infty<x<\infty$, let $\gamma(x)$ be the point $(\alpha(x),\beta(x))$ in the plane. Let Δ be the distance between the points $A'=\gamma(0)$ and $A''=\gamma(1)$. Since (i) does not hold, $\Delta>0$. We now appeal to a theorem of Paul Lévy [6] and Heinz Hopf [5] which asserts that every plane continuum which contains A' and A'' also contains points B' and B'' such that the chord $B'B''$ is parallel to $A'A''$ and the distance between B' and B'' is Δ/d. (This depends on the fact that d is a positive *integer*!) Applying this to the continuum $\gamma([0,1])$, we see that there are points x' and x'' on $[0, 1]$ with $\gamma(x')=B'$, $\gamma(x'')=B''$. It follows that

$$\alpha(x'')-\alpha(x') = s/d, \qquad \beta(x'')-\beta(x') = t/d.$$

These differences are integers. Hence $h(x')=h(x'')$, which implies that $\{x', x''\}=\{0, 1\}$, since h identifies no other pair of points of $[0, 1]$. Thus $\{B', B''\}=\{A', A''\}$, so that $d=1$. This proves (a).

Now suppose h_1 and h_2 are as in (b). Associate γ_i with h_i $(i=1, 2)$ as above. Each γ_i is a periodic curve, in the sense that

$$\gamma_i(x+1)-\gamma_i(x) = \gamma_i(1)-\gamma_i(0) \qquad (-\infty < x < \infty).$$

Note that $\gamma_i(1)\neq\gamma_i(0)$. Let L_i be the line through the points $\gamma_i(0)$ and $\gamma_i(1)$. The range of γ_i intersects L_i at infinitely many equally spaced points and lies within a strip bounded by two lines parallel to L_i. If L_1 and L_2 were not parallel, the ranges of γ_1 and γ_2 would therefore intersect. But then the same would be true of h_1 and h_2, contrary to our hypothesis. So L_1 and L_2 are parallel. By elementary analytic geometry this means that $s(h_1)t(h_2)=s(h_2)t(h_1)$ and since both h_1 and h_2 satisfy conclusion (ii) of part (a), the proof of (b) is complete.(2)

III. **A theorem about compact surfaces.** We begin this section with a brief summary of some well-known facts (see [7], for instance) about nonconstant meromorphic functions f on a compact Riemann surface X. Every such f has at most finitely many *ramification points*; these are points of X which have no neighborhood in which f is one-to-one. To each ramification point p there corresponds an integer $r=r(p)>1$ with the following property: Every neighborhood of p contains a neighborhood V of p such that f is precisely r-to-1 in $V-\{p\}$. At such a point, $f-f(p)$ has a zero of order r if $f(p)\neq\infty$; if $f(p)=\infty$, f has a pole of order r at p. With f there is associated an integer n, the so-called *multiplicity* of f. It has the property that if z is any point of the Riemann sphere S which is not the image of a ramification point of f, then $f^{-1}(z)$ consists of n distinct points of X.

A simple counting argument [2, p. 511], [4, p. 143] based on triangulations of X and S and on the fact that the Euler characteristic of a compact orientable surface of genus g is $2-2g$ leads to the formula

$$2(n+g-1) = \sum (r(p)-1)$$

where n is the multiplicity of f, g is the genus of X, and the sum (which will be called the *total ramification* of f) is extended over all ramification points p of f.

This "ramification formula" will be used several times in the sequel.

The theorem which follows(3) may be regarded as dealing with the possible embeddings of a compact Riemann surface as an analytic submanifold of $S\times S$.

THEOREM. *If F and G are meromorphic functions on a compact Riemann surface X, such that*

(i) *the pair F, G separates points on X,*

(ii) *F and G have no ramification point in common,*

and if n and m are the multiplicities of F and G, respectively, then the genus of X is $(n-1)(m-1)$.

Proof. Let E be the set of ramification points of F. Since $F^{-1}(F(E))$ is a finite set, there is a complex number λ such that $G^{-1}(\lambda)$ contains no ramification point of G and no point of $F^{-1}(F(E))$. Replacing G by $1/(G-\lambda)$ we may therefore add the following to our hypotheses, without loss of generality.

(2) The referee has pointed out that this can also be proved by the methods of algebraic topology.

(3) Although (as was pointed out to me after completion of the present paper) this is contained in a theorem about algebraic curves stated on p. 385 of Hensel and Landsberg's book *Theorie der algebraischen Funktionen einer Variabeln* (1902), it seems worthwhile to include the following proof since it is self-contained and quite elementary.

G has m simple poles, at points $p_1, \ldots, p_m$. *If* $P=\{p_1, \ldots, p_m\}$, *then* $F(P)$ *and* $F(E)$ *are* (obviously finite) *disjoint subsets of* S.

Note also that G identifies the points of P, so that F separates them, i.e. $F(P)$ consists of m distinct points.

If $z \in S-F(E)$ then $F^{-1}(z)$ consists of n distinct points, say $\alpha_1(z), \ldots, \alpha_n(z)$. For such z, define

$$D(z) = \prod_{1 \leqq i<j \leqq n} [G(\alpha_i(z))-G(\alpha_j(z))]^2.$$

D is evidently holomorphic in $S-(F(E) \cup F(P))$. Each factor of the product is bounded in the complement of any neighborhood of $F(P)$, and is bounded away from O in the complement of any neighborhood of $F(E)$ since G separates points which are identified by F. Hence D extends to a function which is holomorphic in $S-F(P)$ and whose zeros lie in $F(E)$.

Now consider a point p_k, one of the poles of G. Since F is unramified at p_k there are neighborhoods V of p_k and W of $z_k=F(p_k)$ such that F is a homeomorphism of V onto W. For $z \in W$, define $\alpha_1(z) \in V$ so that $F(\alpha_1(z))=z$, and form the product

$$\Phi(z) = \prod_{j=2}^{n} [G(\alpha_1(z))-G(\alpha_j(z))]^2 \qquad (z \in W).$$

Both Φ/D and D/Φ are bounded in some neighborhood of z_k; so are

$$G(\alpha_2(z)), \ldots, G(\alpha_n(z));$$

since G has a pole of order 1 at $\alpha_1(z_k)$ and since α_1 is one-to-one in W, it follows that Φ (and hence D) has a pole of order $2(n-1)$ at z_k. The same is true at every point of $F(P)$.

Thus D is a rational function on S, with a total of $2m(n-1)$ poles, if they are counted according to their orders. Hence D has $2m(n-1)$ zeros on S. We saw earlier that these lie in $F(E)$.

Fix a point $\zeta \in F(E)$. Then $F^{-1}(\zeta)$ consists of fewer than n points x_k. There are simply connected neighborhoods W of ζ, V_k of x_k, and positive integers $r_k=r(x_k)$ such that (1) $F(V_k)=W$, (2) the V_k are pairwise disjoint, (3) F is an r_k-to-1 map of $V_k-\{x_k\}$ onto $W-\{\zeta\}$, and (4) G is one-to-one in V_k if $r_k>1$.

Note that (4) depends on the hypothesis that F and G have no common ramification point.

Choose one x_k for which $r=r(x_k)>1$. For $z \in W-\{\zeta\}$ let $\alpha_1(z), \ldots, \alpha_r(z)$ be the distinct points of $F^{-1}(z)$ in V_k, and define

$$\Psi_k(z) = \prod_{1 \leqq i<j \leqq r} [G(\alpha_i(z))-G(\alpha_j(z))]^2.$$

We claim that Ψ_k *has a zero of order* $r-1$ *at* ζ.

Once this is established, the proof is easily completed. Since G separates the points of $F^{-1}(\zeta)$ it follows that D has a zero at ζ of order $\sum (r(x_k)-1)$, the sum being extended over all $x_k \in F^{-1}(\zeta)$. *The total ramification of F is therefore equal to the total number of zeros of D*; the latter is $2m(n-1)$, as noted earlier. Hence

$$2m(n-1) = 2(g+n-1)$$

where g is the genus of X. Thus $g=(m-1)(n-1)$ as asserted by the theorem.

We return to Ψ_k. To simplify the writing, assume $\zeta=0$, $G(x_k)=0$. Then F has a zero of order $r>1$ at x_k. Our choice of V_k was made so that G is one-to-one in V_k. Hence G has a zero of order 1 at x_k, so G^r/F is holomorphic and $\neq 0$ in V_k, and there is a function H, holomorphic and $\neq 0$ in some neighborhood of 0 in the complex plane such that

$$F(p) = [G(p)H(G(p))]^r \qquad (p \in V_k).$$

Put $\tilde{G}(p)=G(p)H(G(p))$. For $z \in W-\{\zeta\}$, the numbers $\tilde{G}(\alpha_i(z))$ are the distinct rth roots of $z=F(p)$, for $1 \leqq i \leqq r$. Let η be a primitive rth root of 1, choose w so that $w^r=z$. Then $\tilde{G}(\alpha_i(z))=\eta^i \cdot w$ (except, perhaps, for a permutation of the subscripts), so that

$$\prod_{1 \leqq i<j \leqq r} [\tilde{G}(\alpha_i(z))-\tilde{G}(\alpha_j(z))]^2 = \prod_{1 \leqq i<j \leqq n} (\eta^i w-\eta^j w)^2$$
$$= w^{r(r-1)} \prod_{1 \leqq i<j \leqq r} (\eta^i-\eta^j)^2 = c_r z^{r-1},$$

where $c_r \neq 0$. Since $\tilde{G}(\alpha_i(z))/G(\alpha_i(z)) \to H(0) \neq 0$ as $z \to 0$, it follows that $\Psi_k(z)/z^{r-1}$ has a finite nonzero limit as $z \to 0$. This completes the proof.

Examples 2 and 4 in §V are relevant to this theorem.

IV. **Proofs of main results.**

Proof of Theorem A. We are given a finite Riemann surface R, bounded by disjoint curves $\Gamma_1, \ldots, \Gamma_k$, and an unramified separating pair of inner functions F and G in $A(R)$. Let h_i be the homeomorphism of Γ_i into T^2 given by

$$h_i(p) = (F(p), G(p)) \qquad (p \in \Gamma_i, 1 \leqq i \leqq k)$$

and put $s_i=s(h_i)$, $t_i=t(h_i)$, in the notation of §II.

Since inner functions in $A(R)$ are unramified at every point of ∂R (see [8, p. 367], for instance), we have $s_i \neq 0$, $t_i \neq 0$. The homeomorphisms h_i have disjoint ranges in T^2, since the pair F, G separates points on ∂R. We now conclude from the Theorem of §II that there are relatively prime integers s and t such that $s=|s_i|$ and $t=|t_i|$ for $1 \leqq i \leqq k$. This implies parts (a) and (b) of Theorem A.

Now extend R to its double $\tilde{R}$. Since $|F|=|G|=1$ on ∂R, the reflection principle furnishes meromorphic extensions $\tilde{F}$ and $\tilde{G}$, defined on $\tilde{R}$. These satisfy the hypotheses of the theorem of §III. Let n and m be the multiplicities of $\tilde{F}$ and $\tilde{G}$. Note that $|\tilde{F}|=|F|<1$ in R, $|\tilde{F}|=|F|=1$ on ∂R, and $|\tilde{F}|>1$ on the rest of $\tilde{R}$. Considering $\tilde{F}^{-1}(\lambda)$, for λ with $|\lambda|=1$, it therefore follows that $n=ks$. Similarly, $m=kt$. The genus of $\tilde{R}$ is $2g+k-1$. Hence §III gives

$$(sk-1)(tk-1) = 2g+k-1,$$

which concludes Theorem A.

Proof of Theorem B. We are given positive integers g, k, s, t such that s and t are relatively prime, and

$$(sk-1)(tk-1)-(k-1) = 2g.$$

Put $n=sk$, $m=tk$, choose distinct complex numbers $\alpha_1, \ldots, \alpha_m$ so that $0<|\alpha_j|<1$, and consider the irreducible polynomial

$$P(z, w) = w^n \prod_{j=1}^{m} (1-\bar{\alpha}_j z) - \prod_{j=1}^{m} (\alpha_j - z).$$

Let Y be the set of all $(z_0, w_0) \in C^2$ at which $P(z_0, w_0)=0$. If $w_0 \neq 0$, then $\partial P/\partial w=0$ on Y; if $w_0=0$, then $\partial P/\partial z \neq 0$ since the α_j are distinct. It follows (by the implicit function theorem) that every point of Y has a neighborhood (relative to Y) in which at least one of the projections $(z, w) \rightarrow z$ or $(z, w) \rightarrow w$ is one-to-one.

If $z\neq 0$ and $w \neq 0$, then $(z, w) \in Y$ if and only if $(1/\bar{z}, 1/\bar{w}) \in Y$. This symmetry property shows that the closure of Y in $S \times S$ (the cartesian product of two Riemann spheres) is a compact Riemann surface X on which *z and w are meromorphic functions which separate points and have no common ramification point.* Since z has multiplicity n and w has multiplicity m, we see from §III (or by checking directly that the total ramification of z on X is $2m(n-1)$) that the genus of X is $g(X) =(m-1)(n-1)$.

Let R be that subset of X on which $|z|<1$. That R is connected is easily seen by examining a neighborhood of the point where $z=\alpha_1$, $w=0$. Thus R is a Riemann surface, and the above-mentioned symmetry property shows that X is the double of R. Furthermore, $|w|<1$ in R, and $|w|=|z|=1$ on ∂R. *Thus z and w form an unramified separating pair of inner functions in $A(R)$.*

It remains to be proved that R has the required topological properties. For this we appeal to §II. Since $p \rightarrow (z(p), w(p))$ is a homeomorphism of ∂R into T^2, and since z and w have multiplicities n and m, respectively, on ∂R, it follows from §II that the number of components of ∂R must be the greatest common divisor of m and n, and this is our prescribed k. Finally, we compute the genus $g(R)$ from $g(X)$, using the fact that X is the double of R:

$$(m-1)(n-1) = g(X) = 2g(R)+k-1,$$

which gives $g(R)=g$, as claimed.

Proof of Theorem D. We are given a finite Riemann surface R, bounded by a single simple closed curve Γ, whose double $\tilde{R}$ is hyperelliptic. We have to prove that $A(R)$ contains an unramified separating pair of inner functions.

To do this it is enough to exhibit one meromorphic function F on $\tilde{R}$, of multiplicity 2, such that $|F|<1$ in R and $|F|=1$ on Γ. For if $p_1, \ldots, p_m$ are the ramification points of F in R, the numbers $\alpha_i=F(p_i)$ are distinct, and there is a second meromorphic function G on $\tilde{R}$ such that

$$G^2 = \prod_{i=1}^{m} \frac{F-\alpha_i}{1-\bar{\alpha}_i F}.$$

The restrictions of F and G to $\bar{R}$ furnish the desired pair of inner functions.

If R is (conformally equivalent to) the unit disc, the theorem is trivial. So let us assume that R has positive genus.

By definition of "hyperelliptic" there is a meromorphic function ϕ on $\tilde{R}$, of multiplicity 2. Let λ_1, λ_2, λ_3 be distinct numbers in $\phi(\Gamma)$, let L be the linear fractional transformation which carries $\{\lambda_1, \lambda_2, \lambda_3\}$ to $\{-1, 0, 1\}$, and put $f = L \circ \phi$. Then f has multiplicity 2, and I claim that *the imaginary part of f has one sign in all of R and the opposite sign in $\tilde{R}-(R \cup \Gamma)$.*

Once this is established, the proof is completed by setting $F=(f-\alpha)/(f+\alpha)$, $\alpha = i$ or $\alpha = -i$.

Every $p \in \tilde{R}$ has a conjugate point $\tilde{p} \in \tilde{R}$, since $\tilde{R}$ is the double of R (see [7, p. 217]). The relation $\tilde{p}=p$ holds if and only if $p \in \Gamma$. Define f_1 on $\tilde{R}$ by $f_1(p)=\bar{f}(\tilde{p})$. Then f_1 is meromorphic on $\tilde{R}$, and since its multiplicity is also 2, a simple argument (which may be found in [1, p. 51]) shows that $f_1 = \Lambda \circ f$, where Λ is a linear fractional transformation. Since $f_1 = \bar{f}$ on Γ, we see that Λ has 3 fixed points, namely $-1, 0, 1$, so that Λ is the identity. Thus $f_1 = f$, and therefore f satisfies the symmetry relation

$$f(\tilde{p}) = \bar{f}(p) \qquad (p \in \tilde{R}).$$

In particular, f is real on Γ, except that it may have poles there.

Since R is connected, *it is now enough to show that $f(p)$ cannot be real if $p \in \tilde{R}-\Gamma$.* It is here that the proof will use the assumption that $\tilde{R}$ has positive genus. To see that this is needed consider the function $f(z)=z^2$ on the Riemann sphere in place of $\tilde{R}$, with the real axis in the role of Γ.

Being the double of a surface of positive genus, $\tilde{R}$ has genus at least 2. The ramification formula therefore shows that f has at least 6 ramification points. These cannot all be on Γ since the restriction of f to Γ has a local maximum or minimum at every ramification point on Γ and there cannot be 6 of these if f is at most 2-to-1 on Γ. So f has a ramification point $p_0 \notin \Gamma$. Then $\tilde{p}_0 \neq p_0$; since f is ramified at p_0 and the multiplicity of f is only 2, we must have $f(p_0) \neq f(\tilde{p}_0)$. Thus $f(p_0)$ is not real, $f(p_0) \neq \infty$.

Assume now, to arrive at a contradiction, that there is a real number x and a point $p \in \tilde{R}-\Gamma$ such that $f(p)=x$. Then $\tilde{p}$ and p lie in different components of $\tilde{R}-\Gamma$, and $f(\tilde{p})=x$. Let γ be the straight line interval with end points at x and $f(p_0)$. Then γ does not intersect $f(\Gamma)$, and $f^{-1}(\gamma)$ is the union of two arcs γ_1 and γ_2 in $\tilde{R}$, one from $\tilde{p}$ to p_0, the other from p to p_0. But then $\gamma_1 \cup \gamma_2$ connects p and $\tilde{p}$ within $\tilde{R}-\Gamma$, and this is impossible.

This completes Theorem D. As noted in the Introduction, Theorem C is a consequence of D.

V. Examples.

EXAMPLE 1. *The case of an annulus.* Fix r so that $0<r<1$, let R be the annulus consisting of all complex z such that $r<|z|<1/r$. As noted in (A4) we need only consider those inner functions $f\in A(R)$ which are one-to-one on each boundary curve and which therefore have multiplicity 2 in R.

Given such an f, associate to each $z\in R$ the other point $\psi(z)$ of R at which $f(\psi(z))=z$. Then ψ is a conformal map of R onto R which interchanges the two boundary components. Hence there is a complex number α, $|\alpha|=1$, such that $\psi(z)=\alpha/z$. The ramification of f occurs at the two square roots of α, and the zeros λ_1 and λ_2 of f (which determine f up to a multiplicative constant of absolute value 1) satisfy the relation $\lambda_1\lambda_2=\alpha$.

Conversely, if $\lambda_1\in R$, $\lambda_2\in R$, and $|\lambda_1\lambda_2|=1$, let $G(z,\lambda_i)$ be the Green's function of R with pole at λ_i, and consider the function

$$u(z) = G(z,\lambda_1)+G(z,\lambda_2) \qquad (z\in R).$$

The normal derivative of u satisfies

$$\frac{1}{2\pi}\int_{\Gamma_1}\frac{\partial u}{\partial n} = h(\lambda_1)+h(\lambda_2)$$

where Γ_1 is either of the two boundary curves and h is the harmonic function in R which is 1 on Γ_1 and 0 on the other boundary curve. Since $|\lambda_1\lambda_2|=1$ one computes easily that $h(\lambda_1)+h(\lambda_2)=1$, an integer, which implies that u has a single-valued harmonic conjugate v in R. Setting $f=\exp(-u-iv)$ we obtain an inner function of the required kind, with zeros at λ_1 and λ_2.

This information tells us that every unramified separating pair of inner functions in $A(R)$ is obtained in the following way (except for multiplication by constants of absolute value 1):

Choose λ_1, λ_2, λ_3, λ_4 in R such that

$$|\lambda_1\lambda_2| = |\lambda_3\lambda_4| = 1 \quad \textit{but} \quad \lambda_1\lambda_2 \neq \lambda_3\lambda_4,$$

and use the above construction to get an f_1 with zeros at λ_1, λ_2 and an f_2 with zeros at λ_3, λ_4.

EXAMPLE 2. *The case $g=1$, $k=2$.* We shall construct a finite Riemann surface R with $g=1$, $k=2$, for which the answer to $Q(R)$ is negative.

Consider 3 copies of the z-plane (more precisely, the Riemann sphere), slit along the straight line intervals from z_i to $\bar{z}_i$, $i=1,2,3$, where z_1, z_2, z_3 are points in the (open) upper half-plane whose real parts are all different. At the first slit connect the 3 sheets so that (traveling from left to right along the real axis) you go from sheet 1 to 2, from 2 to 3, from 3 to 1. Do the same at the second slit. At the third, connect 1 to 2, 2 to 1, 3 to 3.

The result is a compact Riemann surface X on which z is a meromorphic function of multiplicity 3. There are 4 ramification points p with $r(p)=3$ and 2 with $r(p)=2$.

Thus the total ramification of z is 10, and the ramification formula tells us that X has genus 3.

We now need the following fact [2, p. 578]: *If a hyperelliptic compact Riemann surface carries a meromorphic function whose multiplicity is not larger than the genus of the surface, then this multiplicity must be even.*

Since z has multiplicity 3, it follows that our X is not hyperelliptic.

Now let R be the part of X which z projects onto the (open) lower half-plane. The symmetry of X relative to the real axis makes it evident that X is the double of R. Examination of the slits shows that the boundary of R has two components (i.e., $k=2$) and since $g(X)=2g(R)+k-1$ we see that R has genus 1.

If F and G were an unramified separating pair of inner functions in $A(R)$ they would extend (by the reflection principle) to meromorphic functions on X which would satisfy the hypotheses of §III. Their multiplicities m and n would therefore have to satisfy the relation $(m-1)(n-1)=3$, so that $m=2$, $n=4$ (or vice versa). But $m=2$ is impossible since X is not hyperelliptic.

EXAMPLE 3. *The case* $g=2$, $k=1$. We construct X as in Example 2, except that we add a fourth vertical slit, bisected by the real axis, along which we connect sheet 1 to 2, 2 to 1, 3 to 3. Again, z is a meromorphic function on X, of multiplicity 3; the total ramification of z is now 12, and the genus of X is therefore 4. The reasoning used in Example 2 shows again that X is not hyperelliptic. Also, X is the double of the part R of X on which the imaginary part of z is negative. Inspection shows that the boundary of R is now connected (i.e., $k=1$); therefore R has genus 2.

If F and G were an unramified separating pair of inner functions in $A(R)$, Theorem A would imply that there are relatively prime positive integers s and t such that $(s-1)(t-1)=4$ and such that F has multiplicity s, G has multiplicity t. (In general, these multiplicities are sk and tk, but now $k=1$.) The meromorphic extensions of F and G to X would have the same multiplicities. Since X is not hyperelliptic, this rules out $s=2$ (also $t=2$). The only other solution of the above equation in positive integers is $s=3$, $t=3$, and this pair is not relatively prime.

Thus the answer to $Q(R)$ is again negative.

EXAMPLE 4. We conclude with a compact Riemann surface X on which *a certain pair of meromorphic functions separates points, another pair has no common ramification point, but no pair has both of these properties.*

We take for X the Riemann-surface of the equation

$$w^4 = z^4-1.$$

Both z and w are meromorphic functions on X, of multiplicity 4. At the points of X where $z=1, i, -1, -i$, z is locally 4-to-1, but w has a *simple* zero at each of these points since the multiplicity of w is only 4. Every other finite value of z is attained at 4 distinct points of X, corresponding to the fourth roots of z^4-1. The function w/z attains the values $1, i, -1, -i$ only at the common poles of z and w. These 4 poles are therefore distinct, hence they have order 1. *It follows now that z and w have no common ramification point on X.*

Put $u=(z-1)/w$. This meromorphic function on X separates the poles of z (which were not separated by w). If $p_1 \neq p_2$ and $z(p_1)=z(p_2)=\alpha \neq \infty$, then $\alpha \neq 1$ since z attains the value 1 at a unique $p_0 \in X$; hence $w(p_1)$ and $w(p_2)$ are distinct fourth roots of α^4-1, so that $u(p_1) \neq u(p_2)$. *Thus z and u separate points on X.*

At the unique point p_0 where $z(p_0)=1$, the zeros of $z-1$ and w have orders 4 and 1, so u has a zero of order 3 at p_0. This shows that p_0 is a common ramification point of z and u, and since u has no other zero on X, it also shows that u has multiplicity 3. Since z has multiplicity 4 and total ramification 12, the ramification formula implies that X has genus 3. As in Example 2 we now conclude from the existence of u that X is not hyperelliptic (see also [7, pp. 294–5]). If F and G were a pair with both of the properties under discussion, their multiplicities would have to satisfy $(m-1)(n-1)=3$, by §III, and this is impossible since multiplicity 2 is ruled out.

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