ASYMPTOTIC BEHAVIOR OF MEROMORPHIC FUNCTIONS WITH EXTREMAL DEFICIENCIES

BY

ALLEN WEITSMAN[1]

Introduction. This paper continues and completes the preceding one of A. Edrei. I shall adopt the terminology, the bibliographical references and all the notations and conventions of Edrei's paper. Whenever necessary, I shall refer to it as [L]. In view of my frequent use of specific formulae of this paper, as well as of [2], I shall write, for instance, [L, (2.9)] or [2, (2.9)] to denote, respectively, formula (2.9) of [L] or of [2]. Other references will be denoted in the same way as is done in [L].

One of the aims of my investigation is the completion of the proof of Theorem A of [L]. Since the relation [L, (7)] is already proved I have only to examine [L, (8)].

Using Theorem 2 of [L], Edrei had previously proved [L, (8)] for values of μ belonging to the sequence

$$\{1/2+1/2q\} \qquad (q = 1, 2, \dots).$$

[Notices Amer. Math. Soc. **14** (1967), Abstracts 643–23 (p. 248) and 644–72 (p. 380).]

The methods which I develop here enable me to prove [L, (8)] for all μ in the interval $(\frac{1}{2}, 1)$. They may be summarized as follows:

I. Consider the sets $E_0(r_m)$ and $E_\infty(r_m)$ which appear in Theorem 1 of [L]. The limits of their measures have been determined but it is still possible that these sets be the union of many disjoint intervals. I first show that in some sense each of the sets $E_0(r_m)$ and $E_\infty(r_m)$ is "essentially" an interval.

II. This enables me to return to the distribution of the zeros and poles lying in the annuli

$$R'_m < r = |z| \leqq R''_m \qquad (R'_m < r_m < R''_m) \tag{1}$$

where R'_m, r_m, and R''_m are quantities satisfying [L, (2.7)]. I prove that almost all the poles in (1) have arguments close to some quantity ω_m and almost all the zeros have arguments close to $\omega_m+\pi$.

III. This knowledge about the zeros and poles of f in (1) is sufficient to determine the asymptotic behavior of $f(z)$ on some circles in the annuli (1).

IV. Theorem 2 of [L] shows that these arguments may be applied to $f'(z)$. The asymptotic evaluation mentioned above, applied to $f'(z)$, indicates that there exists a circle in the annulus (1) such that $f'(z)$ is very small on a single arc $\mathscr{C}_m$ of

Received by the editors February 5, 1968.

[1] The research of the author was supported in part by a grant from the National Science Foundation GP-7407. It was done, under the guidance of Professor Edrei, in partial fulfillment of the Ph.D. requirements at Syracuse University.

the circumference. By an obvious integration we then verify that $f(z)$ is practically constant on $\mathscr{C}_m$. On the complementary arc $f(z)$ is very large so that $f(z)$ can have only one finite deficient value.

1. **Statement of the main results.** In addition to the notations of [L] I require the following ones, which will enable me to conveniently refer to some sets which appear in my proofs.

Throughout this paper, I denote by C the set of all the arguments θ such that $-\pi<\theta\leqq\pi$.

Since we are only interested in the circular arrangement of the elements of C, the points $\theta=-\pi$ and $\theta=\pi$ will be "identified" and, more generally, all the values $\theta+2k\pi$ ($k=0, \pm 1, \pm 2, \pm 3, \dots$) will be considered as different numerical representations of a single element of C.

Beside C, I introduce

I. The sector

$$\mathscr{S}(\omega, \gamma; R', R'') = \{z: \omega-\gamma < \arg z \leqq \omega+\gamma;\ R' < |z| \leqq R''\}.$$

II. Put $\theta=\arg z$. The "interval" $\omega-\gamma<\theta\leqq\omega+\gamma$, considered as a subset of C, will be denoted by $\Gamma(\omega, \gamma)$.

III. I extend Nevanlinna's notation and denote by $n(\mathscr{D}, f)$ the number of poles of $f(z)$ which fall in the bounded set $\mathscr{D}$. (Multiple poles are counted as often as indicated by their multiplicity.)

With these conventions, we obtain a natural complement to Theorem 1 of [L].

THEOREM 1. *Let $f(z)$ be a meromorphic function of lower order μ $(0<\mu<1)$ and let*

$$\limsup_{r\to\infty;\, r\notin\mathscr{E}} \frac{N(r, 1/f)}{T(r,f)} = u, \qquad \limsup_{r\to\infty;\, r\notin\mathscr{E}} \frac{N(r,f)}{T(r,f)} = v, \tag{1.1}$$

where $\mathscr{E}$ is any fixed set of density zero.

Assume that u and v satisfy

$$u<1, \qquad v<1 \tag{1.2}$$

and

$$\sin^2 \pi\mu = u^2+v^2-2uv\cos \pi\mu. \tag{1.3}$$

Then, with every sequence $\{r_m\}$ of Pólya peaks of order μ of $T(r, f)$, it is possible to associate four sequences $\{\omega_m\}$, $\{\eta_m\}$, $\{\rho'_m\}$, and $\{\rho''_m\}$ having the following properties:

$$0<\eta_m<\pi \qquad (m=1, 2, \dots), \qquad \eta_m\to\pi \quad \textit{as } m\to\infty, \tag{1.4}$$

$$\rho'_m\to+\infty, \qquad r_m/\rho'_m\to+\infty, \qquad \rho''_m/r_m\to+\infty \quad \textit{as } m\to\infty, \tag{1.5}$$

and

$$\begin{aligned} n(\mathscr{S}(\omega_m, \eta_m; \rho'_m, \rho''_m), 1/f) &= o(T(r_m, f)),\\ n(\mathscr{S}(\omega_m+\pi, \eta_m; \rho'_m, \rho''_m), f) &= o(T(r_m, f)), \end{aligned} \tag{1.6}$$

as $m\to\infty$.

Theorem 1 enables us to obtain an asymptotic evaluation of $f(z)$ which leads to

THEOREM 2. *Let the assumptions and notations of Theorem* 1 *be unchanged, and let* $s(0)$ *and* $s(\infty)$ *be the quantities defined by* [L, (2.4)] *and* [L, (2.5)], *and let* ε $(0<\varepsilon<\frac{1}{2}\min\{\sigma(\infty),\sigma(0)\})$ *be given.*

Then there exists a sequence $\{\omega_m\}$, *a positive sequence* $\{\sigma_m\}$ $(\sigma_m\to+\infty)$ *and a constant* $K>0$, *such that*

$$\begin{aligned} \log|f(re^{i\theta})| &> KT(r,f) \qquad (\theta\in\Gamma(\omega_m, s(\infty)/2-\varepsilon)),\\ \log|f(re^{i\theta})| &< -KT(r,f) \qquad (\theta\in\Gamma(\omega_m+\pi, s(0)/2-\varepsilon)), \end{aligned} \tag{1.7}$$

provided

(i) $r\to+\infty$ *in the intervals* $\sigma_m^{-1}r_m<r\leqq\sigma_m r_m$;

(ii) r *avoids in each of these intervals an exceptional set* $\mathscr{E}_m$ *of measure not greater than* $\sigma_m^{-2}r_m$.

From this theorem we deduce at once that:

The values of r for which the inequalities (1.7) *are valid have upper density one.*

Theorem 2 and the well-known relations between a function and its derivative lead to

THEOREM 3. *Let* $f(z)$ *be a meromorphic function satisfying the conditions of Theorem* 1.

Then, if $F'(z)=f(z)$, *and if* $F(z)$ *is meromorphic, it has at most two deficient values.*

Theorem 3 is not vacuous because the meromorphic function

$$F(z) = \frac{\prod_{j=1}^{\infty}(1+zn^{-1/\mu})}{\prod_{j=1}^{\infty}(1-zn^{-(\mu+1)/\mu})} \qquad (\tfrac{1}{2}<\mu<1)$$

satisfies the conditions $\delta(0,F)=1-\sin\pi\mu$, $\delta(\infty,F)=1$ [see for example R. Nevanlinna, *Eindeutige analytische Funktionen*, p. 232], and hence, in view of Theorem 2 of [L], $u(F')=\sin\pi\mu$, $v(F')=0$.

This shows that $f=F'$ satisfies the conditions of Theorem 1.

It might be of interest to investigate whether there exist functions $f(z)$, satisfying the conditions of Theorem 1 with $0<\delta(\infty,f)<1$, and having a meromorphic integral. I am at present unable to answer this question.

As an immediate consequence of Theorem 3 and [L, Theorem 2] we now obtain [L, (8)] which I restate for completeness.

If $f(z)$ *is a meromorphic function of lower order* μ $(\frac{1}{2}<\mu<1)$, *if* $\delta(\infty,f)=1$, *and if* $\Delta(f)=2-\sin\pi\mu$, *then* $\nu(f)=2$.

Hence any function $f(z)$ satisfying the above conditions has precisely one finite deficient value τ, such that $\delta(\tau,f)=1-\sin\pi\mu$, and $f(z)-\tau$ has the asymptotic behavior described in Theorem 2.

2. **Structure of the sets $E_0(r_m)$ and $E_\infty(r_m)$.** Let E denote a measurable subset of C and let

$$\gamma = \tfrac{1}{2}\text{ meas } E. \tag{2.1}$$

Consider the function $\mathcal{M}(\omega) = \text{meas}\,\{E - \Gamma(\omega, \gamma)\}$ which is clearly a nonnegative function of ω, defined and continuous on C. Let $\tilde{\omega}$ be any one of the values of ω such that

$$\mathcal{M}(\tilde{\omega}) = \inf_{\omega \in C} \mathcal{M}(\omega) = \chi.$$

We shall say that $\tilde{\omega}$ *is a center of* E.

The inequalities $0 \leqq \mathcal{M}(\omega) \leqq 2\gamma$, $\mathcal{M}(\omega) \leqq 2(\pi - \gamma)$, are obvious.

If $\gamma = 0$ or $\gamma = \pi$, we have $\mathcal{M}(\omega) \equiv 0$ and $\chi = 0$ (trivially); in both cases, every $\omega \in C$ is a center of E. If $0 < \gamma < \pi$, the inequality $\chi > 0$ is possible; the quantity χ then represents, in some sense, the total measure of the "gaps" in E.

If $\gamma > 0$ and $\chi = 0$, we may think of E as being, apart from a set of zero measure, an interval on C. The following lemma shows that, for functions satisfying (1.3), the sets $E_\infty(r_m)$ and $E_0(r_m)$, tend, as $m \to +\infty$, toward this "single interval" structure.

Lemma 1. *Let $f(z)$ satisfy the hypotheses of Theorem* 1, *and let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, f)$. Let*

$$\gamma_m = \tfrac{1}{2}\text{ meas } E_\infty(r_m). \tag{2.2}$$

Then, there exists a sequence $\{\omega_m\}$ such that

$$\begin{aligned} \lim_{m \to \infty} \text{meas}\,\{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} &= 0, \\ \lim_{m \to \infty} \text{meas}\,\{E_0(r_m) - \Gamma(\omega_m + \pi, \pi - \gamma_m)\} &= 0. \end{aligned} \tag{2.3}$$

Before proving Lemma 1 we prove two elementary lemmas.

Lemma 2. *Let E be a measurable subset of C. Then, if w is any value, real or complex,*

$$\frac{1}{2\pi}\int_E |\log|1 - we^{i\theta}||\,d\theta \leqq \left\{\log(1 + |w|) + \left(1 + \log^+ \frac{1}{\text{meas } E}\right)\right\} \text{meas } E.$$

Proof. Put $\arg w + \theta = \phi$.

Then

$$|1 - we^{i\theta}| = |1 - |w|e^{i\phi}| = |e^{-i\phi} - |w|\,| \geqq |\sin \phi|, \tag{2.4}$$

which may be sharpened to

$$|1 - we^{i\theta}| \geqq 1, \tag{2.5}$$

if $\pi/2 \leqq |\phi| \leqq \pi$.

Now

$$|\sin \phi| \geqq (2/\pi)|\phi| \qquad (|\phi| \leqq \pi/2) \tag{2.6}$$

and hence, if meas $E=\mathscr{K}$ and

$$\alpha = \min\{\pi, \mathscr{K}\}, \tag{2.7}$$

we observe, with Edrei and Fuchs [7, p. 338], that (2.4), (2.5), (2.6), and (2.7) imply

$$\begin{aligned} I(E) = \frac{1}{2\pi}\int_E \log^+ \left|\frac{1}{1-we^{i\theta}}\right| d\theta &\leqq \frac{1}{\pi}\int_0^{\alpha/2} \log\left(\frac{1}{\sin\phi}\right) d\phi \\ &\leqq \frac{1}{\pi}\int_0^{\alpha/2} \log\left(\frac{\pi}{2\phi}\right) d\phi \leqq -\int_0^{\alpha/2\pi} \log t\, dt. \end{aligned} \tag{2.8}$$

By definition $\alpha/2\pi \leqq \mathscr{K}/2\pi \leqq 1$, so that (2.8) yields

$$I(E) \leqq -\int_0^{\mathscr{K}/2\pi} \log t\, dt = \frac{\mathscr{K}}{2\pi} + \frac{\mathscr{K}}{2\pi}\log\left(\frac{2\pi}{\mathscr{K}}\right) \leqq \mathscr{K}\left(1+\log^+\frac{1}{\mathscr{K}}\right). \tag{2.9}$$

We now obtain Lemma 2 by integrating over E the obvious relation

$$|\log|1-we^{i\theta}|| \leqq \log(1+|w|) + \log^+ |1/(1-we^{i\theta})|,$$

and using the estimate (2.9).

Lemma 3. *Let E be a measurable subset of C, and let*

$$\text{meas } E = 2\gamma. \tag{2.10}$$

Assume

$$\text{meas}\{E-\Gamma(0,\gamma)\} \geqq 2\xi. \tag{2.11}$$

Then, if t is restricted to the range

$$\sigma^{-1} \leqq t \leqq \sigma \qquad (1 < \sigma < +\infty), \tag{2.12}$$

we have

$$\frac{1}{2\pi}\int_{-\gamma}^{\gamma} \log|1+te^{i\theta}|\, d\theta - \frac{1}{2\pi}\int_E \log|1+te^{i\theta}|\, d\theta \geqq K = K(\sigma,\xi). \tag{2.13}$$

The constant K which appears in (2.13) *may be chosen equal to*

$$K(\sigma,\xi) = 2\xi\sin^2(\xi/2)/\pi\{4+\sigma(1+\sigma)^2\}, \tag{2.14}$$

which is clearly positive for $0<\xi\leqq\pi/2$.

Proof. Put

$$\{E-\Gamma(0,\gamma)\} = G_1, \qquad \{E\cap\Gamma(0,\gamma)\} = G_2, \tag{2.15}$$

so that $E=\{G_1\cup G_2\}$, $\{G_1\cap G_2\}=0$. Hence, in view of (2.10) and (2.11), we have

$$0 \leqq 2\xi \leqq 2\eta = \text{meas } G_1 \leqq 2(\pi-\gamma), \quad \text{meas } G_2 = 2(\gamma-\eta). \tag{2.16}$$

We now use the familiar remark that, for any fixed $t>0$, $\log|1+te^{i\theta}|$ is an even function of θ, strictly decreasing as θ varies from 0 to π. By (2.15) and (2.16), this leads to the obvious inequalities:

$$\int_{G_2} \log|1+te^{i\theta}|\,d\theta \leqq 2\int_0^{\gamma-\eta} \log|1+te^{i\theta}|\,d\theta,$$

and

$$\int_{G_1} \log|1+te^{i\theta}|\,d\theta \leqq 2\int_{\gamma}^{\gamma+\eta} \log|1+te^{i\theta}|\,d\theta = 2\int_{\gamma-\eta}^{\gamma} \log|1+te^{i(\phi+\eta)}|\,d\phi,$$

which, when added, yield

$$\text{(2.17)}\quad \int_E \log|1+te^{i\theta}|\,d\theta \leqq 2\int_0^{\gamma} \log|1+te^{i\theta}|\,d\theta - 2\int_{\gamma-\eta}^{\gamma} \log\left|\frac{1+te^{i\theta}}{1+te^{i(\theta+\eta)}}\right| d\theta.$$

Consider now the positive function

$$\text{(2.18)}\qquad H(t,\theta,\eta) = \left|\frac{1+te^{i\theta}}{1+te^{i(\theta+\eta)}}\right|^2 = 1+\frac{2t\{\cos\theta-\cos(\theta+\eta)\}}{1+t^2+2t\cos(\theta+\eta)},$$

which appears in the last integral of (2.17). From (2.16) we deduce $\gamma+\eta/2\leqq\pi-\eta/2$, $\eta/2\leqq\gamma-\eta/2$, and hence

$$\text{(2.19)}\quad \cos\theta-\cos(\theta+\eta) = 2\sin(\theta+\eta/2)\sin\eta/2 \geqq 2\sin^2(\eta/2) \qquad (\gamma-\eta\leqq\theta\leqq\gamma).$$

Combining (2.18), (2.12) and (2.19), we find

$$H(t,\theta,\eta)-1 \geqq \frac{4\sigma^{-1}\sin^2(\eta/2)}{(1+\sigma)^2} = \zeta,$$

$$\log H(t,\theta,\eta) \geqq \log(1+\zeta) > \frac{\zeta}{1+\zeta} \geqq \frac{4\sin^2(\eta/2)}{4+\sigma(1+\sigma)^2},$$

$$\int_{\gamma-\eta}^{\gamma} \log H(t,\theta,\eta)\,d\theta \geqq \frac{4\eta\sin^2(\eta/2)}{4+\sigma(1+\sigma)^2},$$

and, since $0<\xi\leqq\eta$, it is obvious that (2.17), (2.18) and (2.19) imply (2.13) and (2.14). This completes the proof of Lemma 3.

Proof of Lemma 1. The assumptions of Lemma 1 coincide with those of Theorem 1 so that (1.1), (1.2) and (1.3) hold. Then, in view of assertion II of [L, Theorem 1], we have

$$\text{(2.20)}\quad 0<\beta = \lim_{m\to\infty}\gamma_m = \frac{1}{\mu}\cos^{-1}v = \frac{s(\infty)}{2} < \pi \qquad \left(0<\cos^{-1}v\leqq\frac{\pi}{2}\right),$$

where $2\gamma_m = \operatorname{meas} E_\infty(r_m)$.

Let ω_m be a center of $E_\infty(r_m)$; we first examine the implications of

$$\text{(2.21)}\qquad \limsup_{m\to\infty} \operatorname{meas}\{E_\infty(r_m)-\Gamma(\omega_m,\gamma_m)\} \neq 0.$$

From (2.21) we deduce the existence of a constant $\xi>0$ and of an unbounded sequence $\mathscr{M}$, of positive integers, such that

$$\text{meas}\,\{E_\infty(r_m)-\Gamma(\omega_m,\gamma_m)\} \geqq 2\xi \qquad (m\in\mathscr{M}). \tag{2.22}$$

Let $\sigma>1$ be a given, fixed quantity and let $a=|a|e^{i\psi}$ be any one of the zeros of $f(z)$ such that

$$\sigma^{-1}r_m < |a| \leqq \sigma r_m. \tag{2.23}$$

In view of the extremal character of the centers ω_m, the inequalities (2.22) remain true if ω_m is replaced by any other point of C; in particular

$$\text{meas}\,\{E_\infty(r_m)-\Gamma(\psi+\pi,\gamma_m)\} \geqq 2\xi \qquad (m\in\mathscr{M}). \tag{2.24}$$

The transformation of the set C defined by

$$\phi = \theta-\psi-\pi \qquad (\theta\in C), \tag{2.25}$$

is a "translation" which leaves C invariant and transforms the subsets of C without affecting their measures. In particular, the sets $E_\infty(r_m)$, $\Gamma(\psi+\pi,\gamma_m)$ are transformed, respectively, into sets $\tilde{E}_m$ and $\Gamma(0,\gamma_m)$ and the inequalities (2.24) become

$$\text{meas}\,\{\tilde{E}_m-\Gamma(0,\gamma_m)\} \geqq 2\xi \qquad (m\in\mathscr{M}).$$

Hence, in view of (2.23) and (2.25), Lemma 3 yields

$$\begin{aligned}\frac{1}{2\pi}\int_{E_\infty(r_m)}\log\left|1-\frac{r_me^{i\theta}}{a}\right|d\theta &= \frac{1}{2\pi}\int_{\tilde{E}_m}\log\left|1+\frac{r_m}{|a|}e^{i\phi}\right|d\phi\\ &\leqq -K+\frac{1}{\pi}\int_0^{\gamma_m}\log\left|1+\frac{r_me^{i\theta}}{|a|}\right|d\theta\\ &\qquad (m\in\mathscr{M};\ K=K(\sigma,\xi)),\end{aligned} \tag{2.26}$$

where the positive constant K depends on no parameters other than σ and ξ.

There are $\tilde{n}_m=n(\sigma r_m,1/f)-n(\sigma^{-1}r_m,1/f)$ zeros of $f(z)$ characterized by the inequalities (2.23). Since our assumptions imply the validity of assertion II of [L, Theorem 1], we deduce from [L, (2.9)] and [L, (2.11)]

$$\lim_{m\to\infty;\,m\in\mathscr{M}}\frac{\tilde{n}_m}{T(r_m,f)} = \mu u(\sigma^\mu-\sigma^{-\mu}). \tag{2.27}$$

We denote by a_j the zeros of $f(z)$ and by b_j its poles and, in the following inequality, confine our attention, and our summations, to the zeros satisfying (2.23). Then (2.26) and (2.27) yield

$$\begin{aligned}&\sum\frac{1}{2\pi}\int_{E_\infty(r_m)}\log\left|1-\frac{r_me^{i\theta}}{a_j}\right|d\theta\\ &\quad\leqq\frac{1}{\pi}\int_0^{\gamma_m}\left\{\sum\log\left|1+\frac{r_me^{i\theta}}{|a_j|}\right|\right\}d\theta-K_1T(r_m,f)u(1+o(1))\\ &\qquad\qquad (m\to+\infty,\ m\in\mathscr{M}),\end{aligned} \tag{2.28}$$

where $K_1=K(\sigma,\xi)\mu(\sigma^\mu-\sigma^{-\mu})$.

We now examine a proof of Edrei [2, pp. 87–94] and consider, in particular, the fundamental inequality [2, (2.18)]. With our notations this inequality implies

$$
\begin{aligned}
m(r_m, f) \leqq & \sum_{0<|a_j|\leqq R_m} \frac{1}{2\pi}\int_{E_\infty(r_m)} \log\left|1-\frac{r_m e^{i\theta}}{a_j}\right| d\theta \\
& - \sum_{0<|b_j|\leqq R_m} \frac{1}{2\pi}\int_{E_\infty(r_m)} \log\left|1-\frac{r_m e^{i\theta}}{b_j}\right| d\theta \qquad (2.29)\\
& + 15\frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \\
& \qquad (m\to\infty,\ r_m \leqq \tfrac{1}{2}R_m,\ T(r) = T(r,f)),
\end{aligned}
$$

where it is understood that, subject to the restriction $R_m \geqq 2r_m$, the size of the error term is not affected by the choice of R_m.

The arguments in [2, p. 90] may be repeated with the following minor modification: instead of using [2, (2.20)] to estimate all the terms of the first sum in the right-hand side of (2.29), we use (2.28) to evaluate the contribution of all the a_j such that $\sigma^{-1}r_m < |a_j| \leqq \sigma r_m$.

We thus obtain

$$
\begin{aligned}
T(r_m) \leqq & \frac{1}{\pi}\int_0^{\gamma_m}\left\{\sum_{0<|a_j|\leqq R_m} \log\left|1+\frac{r_m e^{i\theta}}{|a_j|}\right|\right\} d\theta + \int_0^{\pi-\gamma_m}\left\{\sum_{0<|b_j|\leqq R_m} \log\left|1+\frac{r_m e^{i\theta}}{|b_j|}\right|\right\} d\theta \\
& + 15\frac{r_m}{R_m} T(2R_m) + o(T(r_m)) - K_1 u T(r_m) \qquad (m\to+\infty,\ m\in\mathscr{M},\ r_m \leqq \tfrac{1}{2}R_m),
\end{aligned}
$$

instead of [2, (2.22)].

In view of (2.20), $0<\gamma_m<\pi$ $(m>m_0)$ and we obtain, as in [2],

$$
\begin{aligned}
T(r_m) \leqq & \int_0^{R_m} N_0(t)P(t, r_m, \gamma_m)\, dt + \int_0^{R_m} N_\infty(t)P(t, r_m, \pi-\gamma_m)\, dt - K_1 u T(r_m) \qquad (2.30)\\
& + A\frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \qquad (m\to+\infty,\ m\in\mathscr{M}),
\end{aligned}
$$

where A (>0) is an absolute constant and the symbols N_0, N_∞, P have the same meaning as in [2] or in [L, (4.8)].

The main difference between (2.30) and [L, (4.8)] is the presence, in (2.30), of the negative quantity $-K_1 u T(r_m)$. The arguments which, in [L], lead to [L, (4.12)] now yield

$$
\sin \pi\mu \leqq u \sin \beta\mu + v \sin(\pi-\beta)\mu - K_1 u \sin \pi\mu. \qquad (2.31)
$$

Using the Cauchy-Schwarz inequality, as in [L, (5.1)], we deduce from (2.31)

$$
(K_1 u + 1)\sin^2 \pi\mu \leqq u^2 + v^2 - 2uv\cos \pi\mu,
$$

and hence, by (1.3), $u=0$.

We have thus shown that, if $u \neq 0$, the relation (2.21) is impossible and therefore

$$\lim_{m\to\infty} \text{meas}\,\{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} = 0. \tag{2.32}$$

This is the first of the relations (2.3).

From (2.32) and (2.20) we deduce

$$\lim_{m\to\infty} \text{meas}\,\{E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = \lim_{m\to\infty} (2\gamma_m) = s(\infty). \tag{2.33}$$

The sets $E_0(r_m)$ and $E_\infty(r_m)$ are disjoint by definition so that C may be represented as the union of three pairwise disjoint sets:

$$C = \{E_0(r_m) \cup E_\infty(r_m) \cup E_1(r_m)\}. \tag{2.34}$$

Then, by [L, (2.6)]

$$\lim_{m\to\infty} \text{meas}\, E_1(r_m) = 0. \tag{2.35}$$

From (2.34) we deduce

$$\text{meas}\,\{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} + \text{meas}\,\{E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m)\} + \text{meas}\,\{E_1(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = 2\gamma_m,$$

and hence, by (2.33) and (2.35)

$$\lim_{m\to\infty} \text{meas}\,\{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = 0. \tag{2.36}$$

Finally, $\Gamma(\omega_m\, \gamma_m)$ and $\{(\pi+\omega_m, \pi-\gamma_m)$ are disjoint and their union is C. Therefore

$$\text{meas}\,\{E_0(r_m) \cap \Gamma(\pi+\omega_m, \pi-\gamma_m)\} + \text{meas}\,\{E_0(r_m) \cap \Gamma(\omega_m, \gamma_m)\} = \text{meas}\, E_0(r_m)$$

which, in view of (2.36), yields

$$\lim_{m\to\infty} \text{meas}\,\{E_0(r_m) \cap \Gamma(\pi+\omega_m, \pi-\gamma_m)\} = \lim_{m\to\infty} \text{meas}\, E_0(r_m),$$

and proves the second relation in (2.3).

We have thus completed the proof of Lemma 1 in the case $u \neq 0$. If $u=0$, we certainly have $v \neq 0$ (by (1.3)) and hence Lemma 1 follows from the consideration of the function $1/f$, instead of f.

3. **Arguments of the zeros and poles of $f(z)$.** Lemma 1 gives a precise meaning to step I of the general argument outlined in the Introduction. The following Lemma 4 clarifies, in a similar manner, step II.

LEMMA 4. *Let the assumptions and notations of Lemma* 1 *be unchanged. Let* σ *and* η *be given, fixed quantities such that*

$$1 < \sigma, \qquad 0 < \pi - \eta < \min\,(s(0), s(\infty)). \tag{3.1}$$

[$s(0)$ *and* $s(\infty)$ *are defined as in* [L, *Theorem* 1]].

I. *Then, if* $K(\sigma, \xi)$ *is the constant in* (2.14), *and if*

$$2K_2 = K(\sigma, (\pi-\eta)/2), \tag{3.2}$$

we have

$$\frac{1}{2\pi}\int_{E_\infty(r_m)} \log\left|1-\frac{r_m e^{i\theta}}{a}\right| d\theta \leqq \frac{1}{\pi}\int_0^{\gamma_m} \log\left|1+\frac{r_m e^{i\theta}}{|a|}\right| d\theta - K_2, \tag{3.3}$$

provided

$$a \in \mathscr{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m) \tag{3.4}$$

and $m > m_2$.

The bound m_2, *which depends on* f, $\{r_m\}$, σ *and* η, *holds uniformly for all* a *satisfying* (3.4).

II. *The counting functions of all zeros and poles of* $f(z)$ *satisfy the relations*

$$n(\mathscr{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f) = o(T(r_m, f)), \tag{3.5}$$

$$n(\mathscr{S}(\omega_m+\pi, \eta; \sigma^{-1}r_m, \sigma r_m), f) = o(T(r_m, f)), \tag{3.6}$$

as $m \to \infty$.

Proof. The assumptions of Lemma 4 coincide with those of Lemma 1 as well as with those of [L, Theorem 1, assertion II]. Hence if $\{r_m\}$ is a sequence of Pólya peaks, of order μ, of $T(r) = T(r, f)$, we see that (2.2) and (2.3) hold and that

$$0 < \lim_{m\to\infty} 2\gamma_m = s(\infty) < 2\pi, \qquad 0 < \lim_{m\to\infty} 2(\pi-\gamma_m) = s(0) < 2\pi. \tag{3.7}$$

Consider the sets

$$\begin{aligned} C_{m1} &= \{E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m)\}, \\ C_{m2} &= \{E_\infty(r_m) - \Gamma(\omega_m, \gamma_m)\} = \{E_\infty(r_m) - C_{m1}\}, \\ C_{m3} &= \{\Gamma(\omega_m, \gamma_m) - E_\infty(r_m)\} = \{\Gamma(\omega_m, \gamma_m) - C_{m1}\}, \end{aligned} \tag{3.8}$$

and notice that, in view of the first of the relations (2.3), we have

$$\lim_{m\to\infty} \text{meas } C_{m1} = \lim_{m\to\infty} \text{meas } E_\infty(r_m) = s(\infty), \tag{3.9}$$

and hence, as $m \to \infty$,

$$(\text{meas } C_{m2} + \text{meas } C_{m3}) = c_m \to 0. \tag{3.10}$$

Let $\int$ indicate integration of some measurable function defined on C; then, by (3.8),

$$\int_{E_\infty(r_m)} - \int_{\Gamma(\omega_m, \gamma_m)} = \int_{C_{m2}} - \int_{C_{m3}} = \zeta_m. \tag{3.11}$$

In particular, if we apply (3.11) to the function

$$\log|1-r_m e^{i\theta}/a| \qquad (|a| \leqq \sigma r_m), \tag{3.12}$$

we obtain, in view of (3.10) and Lemma 2,

$$|\zeta_m| \leqq 4\pi c_m\{\log(1+\sigma)+1+\log(1/c_m)\} \qquad (m > m_o(\sigma)). \tag{3.13}$$

Now let

$$a = |a|e^{i\psi} \tag{3.14}$$

satisfy the condition (3.4) so that

$$\sigma^{-1} \leqq r_m/|a| < \sigma, \tag{3.15}$$

$$\psi = \omega_m+\kappa\eta \qquad (-1 < \kappa \leqq 1). \tag{3.16}$$

With a suitable choice of $m_1(\sigma, \eta)$ we may, in view of (3.1), assume

$$0 < \pi-\eta < \min(2\gamma_m, 2\pi-2\gamma_m) \qquad (m > m_1(\sigma, \eta)). \tag{3.17}$$

The change of variable $\phi=\theta-\pi-\psi$ leads to

$$\int_{\Gamma(\omega_m,\gamma_m)} \log\left|1-\frac{r_m e^{i\theta}}{a}\right| d\theta = \int_{\Gamma(\tilde{\omega}_m,\gamma_m)} \log\left|1+\frac{r_m e^{i\phi}}{|a|}\right| d\phi, \tag{3.18}$$

where

$$\tilde{\omega}_m = \omega_m-\psi-\pi. \tag{3.19}$$

Using (3.16) in (3.19), we find $\tilde{\omega}_m=-\pi-\kappa\eta$, and therefore

$$-\pi < \tilde{\omega}_m < -\pi+\eta \quad \text{if } -1 < \kappa < 0,$$
$$\pi-\eta \leqq 2\pi+\tilde{\omega}_m \leqq \pi \quad \text{if } 0 \leqq \kappa \leqq 1.$$

Hence if ω_m' is defined by the relations

$$\omega_m' = \tilde{\omega}_m \qquad \text{if } -1 < \kappa < 0,$$
$$\omega_m' = 2\pi+\tilde{\omega}_m \quad \text{if } 0 \leqq \kappa \leqq 1,$$

we always have

$$\Gamma(\tilde{\omega}_m, \gamma_m) = \Gamma(\omega_m', \gamma_m), \tag{3.20}$$

with

$$0 < \pi-\eta \leqq |\omega_m'| \leqq \pi. \tag{3.21}$$

Before applying Lemma 3 to the last integral in (3.18), we require the following elementary remark:

$$\operatorname{meas}\{\Gamma(\omega_m', \gamma_m)-\Gamma(0, \gamma_m)\} = \min\{|\omega_m'|, 2\gamma_m, 2(\pi-\gamma_m)\} \qquad (|\omega_m'| \leqq \pi, 0 < \gamma_m < \pi). \tag{3.22}$$

In order to verify this relation consider the set

$$\mathscr{G}(\omega) = \{\Gamma(\omega, \gamma) - \Gamma(0, \gamma)\}, \tag{3.23}$$

for γ fixed and ω variable.

A. First assume $0 < 2\gamma \leqq \pi$ and, as ω increases from 0 to π, follows the positions of the points $\omega-\gamma$, $\omega+\gamma$ in the interval $(-\gamma, 2\pi-\gamma]$. Obviously

$$\begin{aligned} \mathscr{G}(\omega) &= (\gamma, \omega+\gamma] && \text{if } 0 < \omega < 2\gamma, \\ \mathscr{G}(\omega) &= (\omega-\gamma, \omega+\gamma] && \text{if } 2\gamma < \omega \leqq \pi. \end{aligned} \tag{3.24}$$

B. Similarly, if $\pi < 2\gamma < 2\pi$, then

$$\begin{aligned} \mathscr{G}(\omega) &= (\gamma, \omega+\gamma] && \text{if } 0 < \omega \leqq 2(\pi-\gamma), \\ \mathscr{G}(\omega) &= (\gamma, 2\pi-\gamma) && \text{if } 2(\pi-\gamma) < \omega \leqq \pi. \end{aligned} \tag{3.25}$$

The relations (3.23), (3.24) and (3.25) yield

$$\text{meas } \mathscr{G}(\omega) = \text{meas } \mathscr{G}(-\omega) = \min(|\omega|, 2\gamma, 2(\pi-\gamma)) \qquad (|\omega| \leqq \pi, 0 < \gamma \leqq \pi),$$

and (3.22) follows.

In view of (3.17) and (3.21) we have

$$0 < \pi-\eta \leqq \min(|\omega_m'|, 2\gamma_m, 2\pi-2\gamma_m) \qquad (m > m_1(\sigma, \eta)),$$

which used in (3.22) yields

$$\text{meas } \{\Gamma(\omega_m', \gamma_m) - \Gamma(0, \gamma_m)\} \geqq \pi-\eta > 0. \tag{3.26}$$

Consider the constant $K(\sigma, \xi)$ defined by (2.14) and set

$$2K_2 = 2K_2(\sigma, \eta) = K(\sigma, \tfrac{1}{2}(\pi-\eta)) > 0.$$

By (3.15), (3.26) and Lemma 3, we find

$$\frac{1}{2\pi} \int_{\Gamma(\omega_m', \gamma_m)} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \leqq -2K_2 + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \tag{3.27}$$

$$(m > m_1(\sigma, \eta)).$$

In view of (3.18) and (3.20) the left-hand side of (3.27) may be replaced by

$$\frac{1}{2\pi} \int_{\Gamma(\omega_m, \gamma_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta.$$

If we combine the resulting inequality with (3.11), we find

$$\frac{1}{2\pi} \int_{E_\infty(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta \leqq -2K_2 + \frac{\zeta_m}{2\pi} + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \tag{3.28}$$

$$(m > m_1(\sigma, \eta)).$$

Now by (3.10) and (3.13)

$$\lim_{m\to\infty} \zeta_m = 0,$$

uniformly for all a satisfying (3.4). Hence, if m_2 is chosen large enough, the inequality $m > m_2$ implies $m > m_1(\sigma, \eta)$, $\zeta_m/2\pi \leqq K_2$, and (3.3) follows from (3.28). We have thus proved assertion I of Lemma 4.

Proof of assertion II of Lemma 4. The parameters σ, η, as well as the sequence $\{\omega_m\}$, are fixed. Explicit reference to all these quantities is unnecessary and we simplify our notation by setting

$$\tilde{n}_m = n(\mathscr{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f).$$

Assume that (3.5) is false. Then there exists some constant $\xi > 0$ and some unbounded sequence $\mathscr{M}$, of positive integers such that

$$\tilde{n}_m > \xi T(r_m) \qquad (m \in \mathscr{M}).$$

This yields a contradiction as may be seen by a repetition, with minor modifications, of the proof of Lemma 1:

(i) start from (2.29). Consider its right-hand side and use (3.3) (instead of (2.28)) to estimate the contribution of the $\tilde{n}_m$ terms involving the zeros of $f(z)$ in

$$\mathscr{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m);$$

(ii) we are thus led to an inequality such as (2.30) with $-K_1uT(r_m)$ replaced by $-K_2\xi T(r_m)$, and finally to

$$\sin \pi\mu \leqq u \sin \beta\mu + v \sin (\pi-\beta)\mu - K_2\xi \sin \pi\mu$$

(instead of (2.31)).

Hence $\xi = 0$, a contradiction which proves (3.5). The relation (3.6) is obtained by applying our arguments to $1/f$ instead of f. This completes the proof of Lemma 4.

4. **Proof of Theorem 1.** Let $l > 2$ be a fixed integer. By Lemma 4 it is possible to determine m_l so that $m > m_l$ implies

$$n(\mathscr{S}(\omega_m, \pi-1/l; r_m/l, lr_m), 1/f) + n(\mathscr{S}(\omega_m+\pi, \pi-1/l; r_m/l, lr_m), f) < T(r_m)/l.$$

We then set

$$\eta_m = \pi - 1/l, \qquad \rho'_m = r_m/l, \qquad \rho''_m = lr_m \tag{4.1}$$

for

$$m_l < m \leqq m_{l+1} \qquad (l = 3, 4, 5, \ldots). \tag{4.2}$$

Theorem 1 is now obvious since the quantities defined by (4.1) and (4.2) clearly satisfy the relations (1.4), (1.5) and (1.6).

5. **Preliminary steps leading to Theorem 2.** Let the assumptions of Theorem 2 be satisfied. Since they include those of Theorem 1, the existence and the properties of the four sequences $\{\omega_m\}$, $\{\eta_m\}$, $\{\rho'_m\}$, $\{\rho''_m\}$ may be taken for granted. In particular,

consider the left-hand sides of the two relations (1.6) and let n_m denote their sum; by Theorem 1

$$n_m/T(r_m) = \delta_m \to 0 \qquad (m \to +\infty, T(t) = T(t, f)). \tag{5.1}$$

We define

$$\sigma_m^2 = \tfrac{1}{2} \min \{r_m/\rho'_m, r_m/R'_m, \rho''_m/r_m, R''_m/r_m, 1/\sqrt{\delta_m}\}, \tag{5.2}$$

where R'_m and R''_m are the quantities in [L, Theorem 1]; by (1.5), [L, (2.7)], and (5.1) this implies

$$\lim_{m \to \infty} \sigma_m = +\infty, \tag{5.3}$$

as well as

$$\lim_{m \to +\infty} \frac{n_m \sigma_m^2}{T(r_m)} = 0. \tag{5.4}$$

Given ε $(0 < \varepsilon < \frac{1}{2} \min \{s(0), s(\infty)\})$, we define η by the relation

$$\pi - \eta = \varepsilon/2, \tag{5.5}$$

and from now on write

$$\begin{aligned} \mathscr{S}_{0m} &= \mathscr{S}(\omega_m, \eta; \sigma_m^{-2} r_m, \sigma_m^2 r_m), \\ \mathscr{S}_{\infty m} &= \mathscr{S}(\pi + \omega_m, \eta; \sigma_m^{-2} r_m, \sigma_m^2 r_m), \\ \mathscr{A}_m &= \{z : \sigma_m^{-2} r_m < |z| \leqq \sigma_m^2 r_m\}, \\ \mathscr{K}_{0m} &= \{\mathscr{A}_m - \mathscr{S}_{0m}\}, \qquad \mathscr{K}_{\infty m} = \{\mathscr{A}_m - \mathscr{S}_{\infty m}\}. \end{aligned} \tag{5.6}$$

By (5.2), (5.4), (5.5) and (5.6)

$$\lim_{m \to +\infty} \frac{(n(\mathscr{S}_{0m}, 1/f) + n(\mathscr{S}_{\infty m}, f))\sigma_m^2}{T(r_m)} = 0. \tag{5.7}$$

We propose to study the asymptotic behavior of $f(z)$ as $z \to \infty$ by values such that

$$\sigma_m^{-1} r_m < r \leqq \sigma_m r_m \qquad (z = re^{i\theta}), \tag{5.8}$$

and

$$\theta \in \Gamma\left(\omega_m, \frac{s(\infty)}{2} - \varepsilon\right) = \Gamma_m. \tag{5.9}$$

Consider the fundamental representation [2, (2.6)]; for our purposes this relation may be rewritten in the form

$$\begin{aligned} \log |f(z)| = \log &\left| \prod_{a_j \in \mathscr{A}_m} \left(1 - \frac{z}{a_j}\right) \right| - \log \left| \prod_{b_j \in \mathscr{A}_m} \left(1 - \frac{z}{b_j}\right) \right| \\ &+ \log \left| \prod_{0 < |a_j| \leqq \sigma_m^{-2} r_m} \left(1 - \frac{z}{a_j}\right) \right| - \log \left| \prod_{0 < |b_j| \leqq \sigma_m^{-2} r_m} \left(1 - \frac{z}{b_j}\right) \right| \\ &+ \log (|c| r^q) + S(z, \sigma_m^2 r_m) \qquad \left(0 < |z| = r \leqq \frac{\sigma_m^2 r_m}{2}\right), \end{aligned} \tag{5.10}$$

where c ($\neq 0$) and q (an integer) are constants and the "error term" $S(z, \sigma_m^2 r_m)$ satisfies the inequality

$$|S(z, \sigma_m^2 r_m)| \leqq 15 \frac{r}{\sigma_m^2 r_m} T(2\sigma_m^2 r_m).$$

By (5.2), (5.8) and [L, (2.9)], it follows that

$$|S(z, \sigma_m^2 r_m)| \leqq 30\sigma_m^{-(1-\mu)} T(r) \qquad (m > m_0). \tag{5.11}$$

Since for any nonrational meromorphic function f,

$$\log r = o(T(r, f)) \qquad (r \to +\infty),$$

it is obvious that (5.3) and (5.11) yield

$$|\log(|c| r^q)| + |S(z, \sigma_m^2 r_m)| = o(T(r)) \qquad (r \to +\infty). \tag{5.12}$$

Let $L_m(z)$ denote the sum of the third and fourth terms in the right-hand side of (5.10). In order to estimate $L_m(z)$ we observe that if $|z|$ satisfies (5.8) and $|a| \leqq \sigma_m^{-2} r_m$, then $|z|/|a| > \sigma_m$ and therefore

$$0 < -\log 2 + \log(r/|a|) < \log|1 - z/a| < \log(r/|a|) + \log 2 \qquad (m > m_0),$$

which yields

$$\left| \sum_{0 < |a_j| \leqq \sigma_m^{-2} r_m} \log \left| 1 - \frac{z}{a_j} \right| \right| \leqq (\log 2 + 3 \log \sigma_m) n\left(\sigma_m^{-2} r_m, \frac{1}{f}\right) + N\left(\sigma_m^{-2} r_m, \frac{1}{f}\right) + O(\log r) \qquad (m \to \infty). \tag{5.13}$$

There is a similar formula involving the poles of $f(z)$.

By [L, (2.9)] and (5.8)

$$T(\sigma_m^{-2} r_m) < 2\sigma_m^{-\mu} T(r) \qquad (m > m_0). \tag{5.14}$$

We now use (5.14) in (5.13), and in the analogous inequality for poles, and take into account [L, (2.10)], [L, (2.11)] and (5.3). This yields

$$L_m(z) = o(T(r)) \qquad (m \to +\infty). \tag{5.15}$$

Denote by $\Lambda_m(z)$ the sum of the two first terms in the right-hand side of (5.10); in view of (5.12) and (5.15) we have

$$\log|f(z)| = \Lambda_m(z) + o(T(r)), \tag{5.16}$$

uniformly as $r \to \infty$ *in the intervals* (5.8).

The next two sections are devoted to the study of $\Lambda_m(z)$.

6. **Bounds for the primary factors.** Consider a zero a of $f(z)$ such that

$$a = |a| e^{i\psi}, \qquad a \in \mathscr{K}_{0m} \tag{6.1}$$

and let z satisfy the conditions (5.8) and (5.9).

By (6.1) and (5.5) there exists a determination of ψ such that $\omega_m+\pi-\varepsilon/2<\psi \leqq \omega_m+\pi+\varepsilon/2$, and by (5.9) $\omega_m-s(\infty)/2+\varepsilon<\theta\leqq\omega_m+s(\infty)/2-\varepsilon$.

Hence

$$(-s(\infty)+\varepsilon)/2 < \theta+\pi-\psi < (s(\infty)-\varepsilon)/2,$$

which, in view of the fact that $\log|1+te^{i\phi}|$ decreases as $|\phi|$ increases from 0 to π, yields

$$\begin{aligned}\log\left|1-\frac{z}{a}\right| &= \log\left|1+\frac{r\exp[i(\theta+\pi-\psi)]}{|a|}\right| \\ &> \log\left|1+\frac{re^{i\lambda}}{|a|}\right| \qquad (a\in\mathscr{K}_{0m},\ \theta\in\Gamma_m),\end{aligned} \tag{6.2}$$

where

$$\lambda = (s(\infty)-\varepsilon)/2, \qquad 0<\lambda<\pi. \tag{6.3}$$

Similarly, if b is a pole of $f(z)$ lying in $\mathscr{K}_{\infty m}$, and if $\theta\in\Gamma_m$, then

$$\log|1-re^{i\theta}/b| < \log|1-re^{i\lambda}/|b||. \tag{6.4}$$

The inequalities (6.2) and (6.4), and our definition of $\Lambda_m(z)$, yield

$$\begin{aligned}\Lambda_m(z) > \log&\left|\prod_{a_j\in\mathscr{A}_m}\left(1+\frac{re^{i\lambda}}{|a_j|}\right)\right| - \log\left|\prod_{b_j\in\mathscr{A}_m}\left(1-\frac{re^{i\lambda}}{|b_j|}\right)\right| \\ &-\log\left|\prod_{a_j\in\mathscr{S}_{0m}}\left(1+\frac{re^{i\lambda}}{|a_j|}\right)\right| + \log\left|\prod_{b_j\in\mathscr{S}_{\infty m}}\left(1-\frac{re^{i\lambda}}{|b_j|}\right)\right| \\ &+\log\left|\prod_{a_j\in\mathscr{S}_{0m}}\left(1-\frac{z}{a_j}\right)\right| - \log\left|\prod_{b_j\in\mathscr{S}_{\infty m}}\left(1-\frac{z}{b_j}\right)\right| \qquad (\theta\in\Gamma_m).\end{aligned} \tag{6.5}$$

If $a\in\mathscr{A}_m$ and r satisfies (5.8), we have $\sin\lambda\leqq|1+re^{i\lambda}/|a||<2\sigma_m^3$, and hence, by (5.4) and [L, (2.9)],

$$\begin{aligned}\sum_{a_j\in\mathscr{S}_{0m}}\left|\log\left|1+\frac{re^{i\lambda}}{|a_j|}\right|\right| + \sum_{b_j\in\mathscr{S}_{\infty m}}\left|\log\left|1-\frac{re^{i\lambda}}{|b_j|}\right|\right| \\ \leqq n_m(\log 2+3\log\sigma_m) = o(T(r)) \qquad (m>m_0,\ r\to+\infty).\end{aligned} \tag{6.6}$$

The two last terms of (6.5) are estimated by the following straight-forward application of the lemma of Boutroux-Cartan: if $z\in\mathscr{A}_m$ and if z avoids finitely many disks with sum of diameters equal to $\sigma_m^{-2}r_m/2$, we have

$$\prod_{a_j\in\mathscr{S}_{0m}}|z-a_j| \geqq \left(\frac{\sigma_m^{-2}r_m}{8e}\right)^n \qquad (n=n(\mathscr{S}_{0m},1/f)),$$

$$(1+\sigma_m^4)^n \geqq \prod_{a_j\in\mathscr{S}_{0m}}\left|1-\frac{z}{a_j}\right| \geqq (8e\sigma_m^4)^{-n}.$$

The same bounds hold for the polynomial formed with the poles b_j ($\in \mathscr{S}_{\infty m}$). Hence, the arguments used in the proof of (6.6), yield

$$
\begin{aligned}
(6.7)\quad \left|\log\left|\prod\left(1-\frac{z}{a_j}\right)\right|\right| + \left|\log\left|\prod\left(1-\frac{z}{b_j}\right)\right|\right| \\
\leqq n_m(\log(8e)+4\log\sigma_m) = o(T(r)) \qquad (r\to\infty),
\end{aligned}
$$

provided r (confined to $\mathscr{A}_m$) avoids a set $\mathscr{E}_m$, of measure not greater than $\sigma_m^{-2}r_m$.

7. **Proof of Theorem 2.** Let $I_1(r)$ denote the first term in the right-hand side of (6.5). The elementary identity

$$
(7.1)\qquad \log\left|1+\frac{z}{|a|}\right| = \mathscr{Re}\left\{z\int_{|a|}^{+\infty}\frac{dt}{t(z+t)}\right\} \qquad (a\neq 0),
$$

valid if z is not real and negative, shows that

$$
\begin{aligned}
(7.2)\quad I_1(r) = \mathscr{Re}\Bigg\{z\int_{\sigma_m^{-2}r_m}^{\sigma_m^2 r_m}\frac{n(t,1/f)-n(\sigma_m^{-2}r_m,1/f)}{t(z+t)}\,dt \\
+z\int_{\sigma_m^2 r_m}^{+\infty}\frac{n(\sigma_m^2r_m,1/f)-n(\sigma_m^{-2}r_m,1/f)}{t(z+t)}\,dt\Bigg\} \qquad (z=re^{i\lambda}),
\end{aligned}
$$

and using again (7.1)

$$
\begin{aligned}
(7.3)\quad I_1(r) = \mathscr{Re}\left\{z\int_{\sigma_m^{-2}r_m}^{\sigma_m^2 r_m}\frac{n(t,1/f)}{t(z+t)}\,dt\right\}+n(\sigma_m^2r_m,1/f)\log\left|1+\frac{z}{\sigma_m^2r_m}\right| \\
-n(\sigma_m^{-2}r_m,1/f)\log\left|1+\frac{z}{\sigma_m^{-2}r_m}\right| \qquad (z=re^{i\lambda}).
\end{aligned}
$$

Now $\log|1+z/\sigma_m^2r_m| = O(r/\sigma_m^2r_m)$, and

$$
\log\left|1+\frac{z}{\sigma_m^{-2}r_m}\right| = \log\left(\frac{r\sigma_m^2}{r_m}\right)+o(1),
$$

uniformly as $r\to+\infty$ in the intervals $(\sigma_m^{-1}r_m, \sigma_m r_m]$. Hence, obvious estimates using [L, (2.9)] and [L, (2.11)] show that (7.3) reduces to

$$
\begin{aligned}
(7.4)\quad I_1(r) = \mathscr{Re}\left\{z\int_{\sigma_m^{-2}r_m}^{\sigma_m^2 r_m}\frac{n(t,1/f)}{t(z+t)}\,dt\right\}+o(T(r)) \\
(z=re^{i\lambda},\ r\to+\infty,\ \sigma_m^{-1}r_m<r\leqq\sigma_m r_m).
\end{aligned}
$$

Rewriting (7.4) in the form

$$
\frac{I_1(r)}{T(r)} = \mathscr{Re}\left\{z\int_{\sigma_m^{-2}r_m}^{\sigma_m^2 r_m}\frac{n(t,1/f)}{T(t)}\frac{T(t)}{T(r)}\frac{dt}{t(z+t)}\right\}+o(1),
$$

we obtain, in view of [L, (2.9)] and [L, (2.11)],

$$
\begin{aligned}
(7.5)\quad \frac{I_1(r)}{T(r)} &= \mathscr{Re}\left\{z\mu u\int_{\sigma_m^{-2}r_m}^{\sigma_m^2 r_m}\left(\frac{t}{r}\right)^{\mu}\frac{dt}{t(z+t)}\right\}+o(1) \\
&= \mathscr{Re}\left\{\mu u e^{i\lambda}\int_0^{\infty}\frac{x^{\mu-1}\,dx}{x+e^{i\lambda}}\right\}+o(1),
\end{aligned}
$$

as $r \to +\infty$ in the intervals $(\sigma_m^{-1} r_m, \sigma_m r_m]$. The value of the last integral in (7.5) is well known to be $\pi e^{i\lambda(\mu-1)}/\sin \pi\mu$, and hence we are finally led to

$$\frac{I_1(r)}{T(r)} = \frac{1}{T(r)} \log \left| \prod_{a_j \in \mathscr{A}_m} \left(1 + \frac{re^{i\lambda}}{|a_j|}\right) \right| = \frac{\pi\mu u}{\sin \pi\mu} \cos \lambda\mu + o(1). \tag{7.6}$$

The same method yields

$$\frac{1}{T(r)} \log \left| \prod_{b_j \in \mathscr{A}_m} \left(1 - \frac{re^{i\lambda}}{|b_j|}\right) \right| = \frac{\pi\mu v}{\sin \pi\mu} \cos (\pi - \lambda)\mu + o(1). \tag{7.7}$$

Combining (5.16), (6.5), (6.6), (6.7), (7.6) and (7.7) we obtain, uniformly in z,

$$\log |f(z)| \geqq \frac{\pi\mu T(r)}{\sin \pi\mu} \{u \cos (\lambda\mu) - v \cos (\pi - \lambda)\mu\} + o(T(r)), \tag{7.8}$$

provided

(i) $|z| = r \to +\infty$ in the intervals $(r_m \sigma_m^{-1}, r_m \sigma_m]$;

(ii) r avoids the exceptional sets $\mathscr{E}_m$ (meas $\mathscr{E}_m \leqq \sigma_m^{-2} r_m$);

(iii) $$\arg z = \theta \in \Gamma\left(\omega_m, \frac{s(\infty)}{2} - \varepsilon\right).$$

If we choose any K such that

$$0 < K < \frac{\pi\mu}{\sin \pi\mu} \left(u \cos \left\{\frac{s(\infty)\mu}{2} - \frac{\varepsilon\mu}{2}\right\} - v \cos \left\{\left(\pi - \frac{s(\infty)}{2}\right)\mu + \frac{\varepsilon\mu}{2}\right\}\right) = \tilde{K}, \tag{7.9}$$

we see that (7.8) and (6.3) imply the first of the inequalities (1.7). We must still verify that $\tilde{K} > 0$ since otherwise it will be impossible to find a K satisfying (7.9). The relations [L, (2.4)], [L, (2.5)] and [L, (2.6)] yield an explicit value of $\tilde{K}$:

$$\tilde{K} = \frac{\pi\mu}{\sin \pi\mu} \left\{u \sin \left(\frac{s(\infty)\mu}{2}\right) + v \sin \left(\frac{s(o)\mu}{2}\right)\right\} \sin \frac{\varepsilon\mu}{2} > 0.$$

The second inequality (1.7) is obtained by considering $1/f$ instead of f. Our proof of Theorem 2 is now complete.

8. **Proof of Theorem 3.** Assume that the Theorem is false. Then there exists a meromorphic function $F(z)$ of lower order μ $(0 < \mu < 1)$ having at least two finite, distinct, deficient values τ_1, τ_2 and such that $f(z) = F'(z)$ satisfies the conditions of Theorem 1.

By the elements of Nevanlinna's theory

$$m(r, f/F) + m(r, f/(F - \tau_1)) + m(r, f/(F - \tau_2)) = o(T(r, F)) \qquad (r \notin \mathscr{E}, r \to +\infty), \tag{8.1}$$

where $\mathscr{E}$ is an exceptional set of finite measure. It is well known that this relation implies

$$T(r, f) \leqq 2T(r, F)(1 + o(1)) \qquad (r \notin \mathscr{E}, r \to +\infty), \tag{8.2}$$

and also (since the relation [L, (9.3)] is valid with g replaced by F),

$$N(r, 1/f)+m(r, 1/(F-\tau_1))+m(r, 1/(F-\tau_2)) \leqq T(r,f)+o(T(r, F)) \quad (r \notin \mathscr{E}, r \to +\infty). \tag{8.3}$$

From the definition of deficient value, we deduce

$$m(r, 1/(F-\tau_k)) > \tfrac{1}{2}\delta(\tau_k, F)T(r, F) \qquad (r > r_0; k = 1, 2), \tag{8.4}$$

and hence, in view of (8.2) and (8.3), there exist two constants κ_1, κ_2 such that

$$0 < \kappa_1 < T(r,f)/T(r, F) < \kappa_2 < +\infty \qquad (r \notin \mathscr{E}, r > r_0). \tag{8.5}$$

Let J be any measurable subset of C such that meas $\{J\}=4\varepsilon>0$; then, by a lemma of Edrei and Fuchs [7, p. 322, Lemma III],

$$\frac{1}{2\pi}\int_J \log^+\left|\frac{1}{F-\tau_k}\right| d\theta = m\left(r, \frac{1}{F-\tau_k}; J\right) \leqq A_0T(2r, F)\varepsilon\left(1+\log^+\frac{1}{\varepsilon}\right) \quad (r > r_0; k = 1, 2), \tag{8.6}$$

where A_0 is an absolute constant.

From now on r will be restricted to the intervals $(r_m, 2r_m]$, and $\{r_m\}$ is the sequence of Pólya peaks (of $T(r,f)$) which appears in Theorems 1 and 2. By [L, (2.9)], (8.5) and the fact that the characteristic functions are increasing,

$$T(2r, F) < K_0T(r, F) \qquad (r_m < r \leqq 2r_m, r \notin \mathscr{E}, m > m_0); \tag{8.7}$$

the constant K_0 depends only on κ_1, κ_2 and μ.

Using (8.7) in (8.6) we obtain

$$m\left(r, \frac{1}{F-\tau_k}; J\right) \leqq A_0K_0T(r, F)\varepsilon\left(1+\log^+\frac{1}{\varepsilon}\right) \quad (r_m < r \leqq 2r_m, r \notin \mathscr{E}, m > m_0; k = 1, 2), \tag{8.8}$$

and choose ε $(0<\varepsilon<\frac{1}{2}\min(s(0), s(\infty)))$ so small that the right-hand side of (8.8) is less than

$$\tfrac{1}{4}\min\{\delta(\tau_1, F), \delta(\tau_2, F)\}T(r, F).$$

We use this value of ε in Theorem 2 and select a sequence $\{\tilde{r}_m\}$ such that

$$r_m < \tilde{r}_m \leqq 2r_m, \quad \tilde{r}_m \notin \mathscr{E}, \quad \tilde{r}_m \notin \mathscr{E}_m \qquad (m > m_0).$$

This is certainly possible because $\mathscr{E}$ is of finite measure and

$$\text{meas } \mathscr{E}_m \leqq \sigma_m^{-2}r_m = o(r_m) \qquad (m \to \infty).$$

The set

$$J_m = C-\{\Gamma(\omega_m, s(\infty)/2-\varepsilon) \cup \Gamma(\pi+\omega_m, s(0)/2-\varepsilon)\}$$

is of measure 4ε and hence (8.8) and our choice of ε and $\tilde{r}_m$ imply

$$m(\tilde{r}_m, 1/(F-\tau_k); J_m) < \tfrac{1}{4}\delta(\tau_k, F)T(\tilde{r}_m, F) \qquad (k = 1, 2). \tag{8.9}$$

Now (8.1), the first relation (1.7) and the elementary inequality

$$\log^+ |1/(F-\tau_k)| \leqq \log^+ |f/(F-\tau_k)| + \log^+ |1/f|,$$

yield

$$\text{(8.10)} \quad m(\tilde{r}_m, 1/(F-\tau_k); \Gamma(\omega_m, s(\infty)/2-\varepsilon)) = o(T(\tilde{r}_m, F)) \qquad (m \to \infty, k = 1, 2).$$

If we consider the inequalities (8.4) with $r = \tilde{r}_m$, and compare them with (8.9) and (8.10), we see that for m large enough, there will exist points

$$z_{1m} = \tilde{r}_m \exp(i\theta_{1m}), \qquad z_{2m} = \tilde{r}_m \exp(i\theta_{2m}),$$

such that $\theta_{1m}, \theta_{2m} \in \Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)$,

$$\text{(8.11)} \qquad |F(z_{1m}) - \tau_1| < \tfrac{1}{3}|\tau_2 - \tau_1|, \qquad |F(z_{2m}) - \tau_2| < \tfrac{1}{3}|\tau_2 - \tau_1|.$$

Let $\mathscr{C}_m$ denote the subinterval of $\Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)$ having end points θ_{1m}, θ_{2m}. Then, the obvious relation

$$|F(z_{1m}) - F(z_{2m})| = \left| \int_{\mathscr{C}_m} f(\tilde{r}_m e^{i\theta}) \tilde{r}_m e^{i\theta} \, d\theta \right|,$$

the second relation (1.7), and the fact that $\log \tilde{r}_m = o(T(\tilde{r}_m, F))$, imply

$$\text{(8.12)} \qquad |F(z_{1m}) - F(z_{2m})| < \tfrac{1}{3}(\tau_2 - \tau_1) \qquad (m > m_0).$$

The inequalities (8.11) and (8.12) are clearly incompatible. This contradiction shows that $F(z)$ cannot have the finite, distinct, deficient values τ_1, τ_2, and hence proves Theorem 3.

SYRACUSE UNIVERSITY,
SYRACUSE, NEW YORK