

MODULE HOMOMORPHISMS OF A VON NEUMANN ALGEBRA INTO ITS CENTER⁽¹⁾

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1. Introduction. A tool for studying von Neumann algebras is the space of σ -weakly continuous linear functionals. There are certain limitations in the applicability of this tool because the center of the algebra is somehow submerged and on the whole it seems σ -weakly continuous functionals are more suitable for studying factor algebras. In this article we propose another tool, viz. the σ -weakly continuous module homomorphisms of the von Neumann algebra \mathcal{A} , considered as a module over its center \mathcal{Z} , into \mathcal{Z} . First it is shown that every σ -weakly continuous linear functional can be written as the composition of a σ -weakly continuous module homomorphism into \mathcal{Z} and a σ -weakly continuous linear functional on \mathcal{Z} . For these module homomorphisms there is obtained a specific form which generalizes the well-known form of σ -weakly continuous linear functionals ([1], [16]). Certain facts concerning decomposition theory result. Further \mathcal{A} is the dual (in terms of module homomorphisms) of the space of all σ -weakly continuous module homomorphisms of \mathcal{A} into \mathcal{Z} . As an example of the applicability of this tool a type I algebra is described as the second dual (in terms of module homomorphisms) of a module. Further applications are contained in a later paper.

It seems reasonable to conjecture from the ensuing proofs that considering the module structure might further illuminate the relation between a von Neumann algebra and an AW^* -algebra.

2. Extension of homomorphisms. Let \mathcal{Z} be a commutative von Neumann algebra. An algebraic \mathcal{Z} -module \mathcal{M} with norm $\|\cdot\|$ will be called a Banach \mathcal{Z} -module if (1) \mathcal{M} is a Banach space, and (2) $\|CB\| \leq \|C\| \|B\|$ for every $C \in \mathcal{Z}$ and $B \in \mathcal{M}$. A module homomorphism of \mathcal{M} into \mathcal{Z} will be called a functional and the set of all bounded functionals of the module \mathcal{M} will be denoted by \mathcal{M}^\sim . The space \mathcal{M}^\sim with the operator bound is again a Banach \mathcal{Z} -module. If \mathcal{A} is a von Neumann algebra with center \mathcal{Z} , then \mathcal{A} is a Banach \mathcal{Z} -module. Whenever we talk about the module \mathcal{A} , we shall have this particular module in mind. A functional ϕ of the module \mathcal{A} is said to be hermitian (resp. positive) if $\phi(A)$ is hermitian (resp. positive) in \mathcal{Z} whenever A is hermitian (resp. positive) in \mathcal{A} . A positive functional ϕ of the algebra (resp. module) \mathcal{A} is said to be normal if $\text{lub } \phi(A_n) = \phi(A)$ for any monotonely increasing net $\{A_n\}$ of hermitian elements of \mathcal{A} with least upper bound A .

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Then the positive functional ϕ is a normal functional of the algebra (respectively, module) \mathcal{A} if and only if ϕ (resp. the composition $f \cdot \phi$) is an element of the associated space \mathcal{A}_* of \mathcal{A} (resp. for every f in the associated space \mathcal{L}_* of \mathcal{L}) [2, I, §4, Theorem 1]. If E is an abelian projection with central support P of the commutator \mathcal{L}' of \mathcal{L} , then $CP \rightarrow CE$ is an isomorphism of $\mathcal{L}P$ onto $E\mathcal{L}'E$. For each A in \mathcal{L}' let $\tau_E(A)$ denote the inverse image of EAE under this isomorphism. The map τ_E restricted to \mathcal{A} is then easily seen to be a normal functional of the module \mathcal{A} .

The following theorem shows the relation between functionals of \mathcal{A} and the module \mathcal{A} .

THEOREM 1. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} and let f be a normal functional on \mathcal{A} . Then there is a normal functional ϕ of the module \mathcal{A} with $\phi(1)$ equal to the support P of f restricted to \mathcal{Z} such that $f=f \cdot \phi$. The functional ϕ is unique in the following sense: if ψ is a functional of the module \mathcal{A} with $f=f \cdot \psi$, then $P\psi(A) = \phi(A)$ for every A in \mathcal{A} .*

Proof. We may assume $f \neq 0$. Let Φ be the canonical representation of \mathcal{A} on a Hilbert space H induced by f and let x be a cyclic vector under $\Phi(\mathcal{A})$ in H such that $f(A) = w_x \cdot \Phi(A) = (\Phi(A)x, x)$ for every A in \mathcal{A} [2, I, §4]. There is a central projection Q such that $\mathcal{A}(1 - Q)$ is the kernel of Φ and such that Φ carries $\mathcal{A}Q$ isomorphically onto $\Phi(\mathcal{A})$. We observe that $1 - P$ is in the kernel of Φ and so $Q \leq P$. However, $f(P - Q) = w_x(\Phi(P - Q)) = 0$; therefore $P = Q$. Now the inverse θ of Φ restricted to $\mathcal{A}P$ is a σ -weakly continuous homomorphism of $\Phi(\mathcal{A})$ onto $\mathcal{A}P$ [2, I, §4, Corollaries, Theorem 2]. Let E be the abelian projection of the commutator $\Phi(\mathcal{L})'$ of $\Phi(\mathcal{L})$ on H corresponding to the subspace closure $\{Ax \mid A \in \Phi(\mathcal{L})\}$ [2, I, §2, no. 1, and §6, Proposition 4, Corollary 2]. Let $\tau = \tau_E$. Then we can easily verify that the functional $\phi = \theta \cdot \tau \cdot \Phi$ of the module \mathcal{A} into $\mathcal{L}P$ is normal, and that $f(A) = f(\Phi(A))$.

Now let ψ be a functional of the module \mathcal{A} such that $f \cdot \psi = f$. There is a vector y in the Hilbert space of \mathcal{A} such that $f = w_y$ on \mathcal{L} [2, III, §1, Corollary to Theorem 4]. We have that $(\psi(A)By, Cy) = (\phi(A)By, Cy)$ for every B and C in \mathcal{L} and every A in \mathcal{A} . If F is the projection corresponding to closure $\{By \mid B \in \mathcal{L}\}$, then $\psi(A)F = \phi(A)F$ for every A in \mathcal{A} . Because the central support of F is P , we have that $\psi(A)P = \phi(A)$ for every A in \mathcal{A} . Q.E.D.

We now determine the form of a normal functional in terms of the trace class (cf. [9, §4]).

THEOREM 2. *Let \mathcal{A} be a von Neumann algebra on a Hilbert space H . If ϕ is a normal functional of the module \mathcal{A} , then there is a positive element B in the trace class of the commutator \mathcal{L}' of the center \mathcal{Z} of \mathcal{A} such that $\phi(A) = \text{Tr}(AB)$.*

Proof. There is a set $\{P_i \mid i \in S\}$ of mutually orthogonal projections of \mathcal{L} of sum 1 such that each algebra $\mathcal{L}P_i$ on P_iH has a separating unit vector x_i [2, I, §2]. Since the functional $w_{x_i} \cdot \phi = f_i$ restricted to $\mathcal{A}P_i$ is normal, there is a normal functional g_i of the commutator $\mathcal{L}'P_i$ of $\mathcal{L}P_i$ on P_iH that coincides with f_i on $\mathcal{A}P_i$. However, by

Theorem 1 there is a normal functional ψ_i of the module $\mathcal{L}'P_i$ such that $\psi_i(P_i) \leq P_i$ and $g_i \cdot \psi_i = g_i$. There is a y_i in P_iH such that $g_i(A) = (Ay_i, y_i)$ for every A in $\mathcal{L}P_i$. Then $(Ay_i, y_i) = f_i(A) = (A\phi(1)x_i, x_i)$ for every A in $\mathcal{L}P_i$. So $(\phi(1)\psi_i(A)Bx_i, Cx_i) = (\phi(A)Bx_i, Cx_i)$ for every A in $\mathcal{A}P_i$ and B and C in $\mathcal{L}P_i$. Because x_i is separating for $\mathcal{L}P_i$ we see that $\phi(A) = \phi(1)\psi_i(A)$ for every A in $\mathcal{A}P_i$. There is a unique positive element B_i in the trace class of $\mathcal{L}'P_i$ such that $\phi(1)\psi_i(A) = \text{Tr}(AB_i)$ for every A in $\mathcal{L}'P_i$ [9, Theorem 4.13]. But for every finite subset π of S we have that $\text{Tr}(\sum_{\pi} B_i) \leq \phi(1)$. So $B = \sum B_i$ is a positive element of the trace class of \mathcal{L}' such that $\text{Tr}(AB) = \phi(A)$ for every A in \mathcal{A} since $\text{Tr}(AB)P_i = \text{Tr}((AP_i)B_i) = \phi(AP_i) = \phi(A)P_i$ for every P_i . Q.E.D.

COROLLARY. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . If ϕ is a normal functional of the module \mathcal{A} , there is a monotone decreasing sequence $\{B_n\}$ of elements in \mathcal{L}^+ and an orthogonal sequence $\{E_n\}$ of abelian projections of the commutator \mathcal{L}' of \mathcal{L} such that (1) $\lim B_n = 0$ (uniformly); (2) $\sum B_n$ is bounded above; and (3) $\phi(A) = \sum B_n \tau_{E_n}(A)$ for every $A \in \mathcal{A}$.*

Proof. This results from [9, §2 and §4].

REMARK. Theorem 2 gives rise to a representation of a normal functional f on a von Neumann algebra \mathcal{A} as an integral over the spectrum Z of the center \mathcal{Z} of \mathcal{A} . Let ν be the so-called spectral measure on Z induced by the relation $f(A) = \int_Z A^\wedge(\zeta) d\nu(\zeta)$ ($A \in \mathcal{L}$). Here A^\wedge denotes the Gelfand transform of A in \mathcal{L} . There is a positive element B in the trace class of the commutator of \mathcal{L} such that $f(A) = f(\text{Tr}(AB)) = \int \text{Tr}(AB)^\wedge(\zeta) d\nu(\zeta)$. In the same notation as the preceding corollary let $f_{n\zeta}(A) = (B_n \tau_{E_n}(A))^\wedge(\zeta)$ for every $\zeta \in Z$. Then $f_{n\zeta}$ is a positive functional on \mathcal{A} whose kernel contains the ideal ζ in Z . If \mathcal{A} is discrete, the nonzero $f_{n\zeta}$ are irreducible (cf. [8, Proposition 2.1]). If \mathcal{A} is continuous, then $f_{n\zeta}(1)^{-1}f_{n\zeta}$ is in the pure state space of \mathcal{A} whenever $f_{n\zeta} \neq 0$ [4, Theorem 3]. Now for fixed $A \in \mathcal{A}^+$ $\{\sum \{B_n \tau_{E_n}(A) \mid 1 \leq n \leq m\}\}_m$ is a monotonely increasing sequence which converges strongly to $\text{Tr}(AB)$. There is a subset N of ν -measure 0 such that

$$\lim_m \sum \{f_{n\zeta}(A) \mid 1 \leq n \leq m\} = \text{Tr}(BA)^\wedge(\zeta)$$

for every $\zeta \notin N$. The restriction that $A \in \mathcal{A}^+$ can certainly be removed and so, we obtain the formula $\sum_n \int f_{n\zeta}(A) d\nu(\zeta) = \int \text{Tr}(BA)^\wedge(\zeta) d\nu(\zeta) = f(A)$ for every A in \mathcal{A} by the dominated convergence theorem.

We state some additional facts for type I algebras. Here we may assume $\mathcal{A} = \mathcal{L}'$. The support X of ν is open and closed and corresponds to a projection P in \mathcal{L} under the relation $\{\zeta \in Z \mid P^\wedge(\zeta) = 1\} = X$. Let E be an abelian projection of central support P . If $\zeta \in X$ and if $[\zeta]$ denotes the smallest closed two-sided ideal in \mathcal{A} containing ζ , then the factor space $\mathcal{A}E/[\zeta]$ becomes a Hilbert space $H(\zeta)$ with inner product $(A(\zeta), B(\zeta)) = \tau_E(B^*A)^\wedge(\zeta)$. Here $A(\zeta)$ denotes the image of A in $\mathcal{A}/[\zeta]$. Let Φ_ζ be the representation by left multiplication of \mathcal{A} with kernel $[\zeta]$ on $H(\zeta)$. There is a

sequence $\{P_k\}$ of mutually orthogonal projections in \mathcal{L} of sum P such that $\{\sum_n B_n P_k\}$ converges uniformly for each P_k (cf. [9, Theorem 4.1]). Let $X_k = \{\zeta \in \mathcal{Z} \mid P_k \hat{\zeta} = 1\}$. For each $\zeta \in \bigcup X_k$ there is a normal functional f_ζ on the algebra L of bounded linear operators of $H(\zeta)$ with $f_\zeta \cdot \Phi_\zeta(A) = \text{Tr}(BA) \hat{\zeta}$. Indeed, there is a normal functional $g_{n\zeta}$ on L with $g_{n\zeta} \cdot \Phi_\zeta = f_{n\zeta}$. Since the set $N = X - \bigcup X_k$ has ν -measure 0, we have that $f(A) = \int f_\zeta(A(\zeta)) \, d\nu(\zeta)$, [9, §4].

Let \mathcal{A} be a von Neumann algebra with center \mathcal{L} . The set \mathcal{A}_\sim of all functionals of the module \mathcal{A} which are continuous in the $\sigma(\mathcal{A}, \mathcal{A}_*)$ and $\sigma(\mathcal{L}, \mathcal{L}_*)$ topologies of \mathcal{A} and \mathcal{L} respectively is an algebraic \mathcal{L} -module. The module \mathcal{A}_\sim is a closed subspace of the space \mathcal{A}^\sim of all bounded functionals of the module \mathcal{A} because the uniform limit of a sequence of σ -weakly continuous functionals of \mathcal{A} is σ -weakly continuous. Thus \mathcal{A}_\sim is a Banach \mathcal{L} -module.

In passing we remark that analogues of theorems for functionals of \mathcal{A}_* may be formulated and proved for functionals of \mathcal{A}_\sim using Theorem 1. In particular, we may obtain a polar decomposition for functionals of \mathcal{A}_\sim [14] as well as a decomposition of functionals of \mathcal{A}_\sim into a linear combination of normal functionals. The proof of the latter statement is virtually identical with that of Theorem 4.12 of [9]. In particular, if ϕ is a hermitian functional of \mathcal{A}_\sim , then there is a projection $E \in \mathcal{A}$ such that $\phi_1(A) = \phi(AE)$ and $\phi_2(A) = -\phi(A(1-E))$ are normal functionals of the module \mathcal{A} . Also a Radon-Nikodym theorem [15] holds.

The next theorem is an analogue of a well-known result concerning linear functionals on a von Neumann algebra. The theorem though is potentially applicable to the theory of AW^* -algebras in a way precluded for linear functionals (cf. [13]) and so is interesting in its own right. We plan to explore this application in a future paper.

THEOREM 3. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{L} . For each A in \mathcal{A} define the function F_A on \mathcal{A}_\sim by $F_A(\phi) = \phi(A)$. Then the function $A \rightarrow F_A$ defines an isometric isomorphism of the module \mathcal{A} onto the module $(\mathcal{A}_\sim)^\sim$.*

Proof. First it is easy to see that $F_A \in (\mathcal{A}_\sim)^\sim$ for each $A \in \mathcal{A}$ and that the function $A \rightarrow F_A$ is a module homomorphism of \mathcal{A} into $(\mathcal{A}_\sim)^\sim$. Now let \mathcal{L}' be the commutator of \mathcal{L} ; let $\varepsilon > 0$ be given. There is an abelian projection E in \mathcal{L}' such that $\|E|A|E\| \geq \|A\| - \varepsilon$ where $|A| = (A^*A)^{1/2}$. Let U be a partial isometry of \mathcal{A} such that $A = U|A|$ is the polar decomposition of A . Defining $\phi(B) = \tau_E(U^*B)$ for $B \in \mathcal{A}$, we see that $\|\phi(B)\|^2 \leq \|\tau_E(U^*U)\| \|\tau_E(B^*B)\| \leq \|B\|^2$ for every $B \in \mathcal{A}$. Thus ϕ is in the unit sphere of \mathcal{A}_\sim . However $\|F_A(\phi)\| \geq \|A\| - \varepsilon$. Because $\varepsilon > 0$ is arbitrary $\|F_A\| \geq \|A\|$. Since it is obviously true that $\|F_A\| \leq \|A\|$, we have that $\|F_A\| = \|A\|$. Hence the map $A \rightarrow F_A$ is an isometry.

We complete the proof by showing $A \rightarrow F_A$ is onto $(\mathcal{A}_\sim)^\sim$. Let f be an element of $(\mathcal{A}_\sim)^\sim$. For each ϕ in \mathcal{L}'_\sim let $F(\phi)$ denote the restriction of ϕ to \mathcal{A} . Then F is a continuous homomorphism of the \mathcal{L}'_\sim -module \mathcal{L}'_\sim onto the \mathcal{L} -module \mathcal{A}_\sim . This means that $f \cdot F$ is an element of $(\mathcal{L}'_\sim)^\sim$. There is an element A in \mathcal{L}' such that

$f \cdot F(\phi) = \phi(A)$ for every ϕ in \mathcal{L}'_{\sim} [9, Theorems 4.10 and 4.13]. We show that $A \in \mathcal{A}$. Indeed, if $A \notin \mathcal{A}$ then there is a unitary operator V in the commutator \mathcal{A}' of \mathcal{A} such that $VAV^* \neq A$. There is an abelian projection G in \mathcal{L}' such that $GVAV^*G \neq GAG$. Let G' be the abelian projection in \mathcal{L}' given by $G' = V^*GV$. If $\phi = \tau_{G'} - \tau_G$, then $\phi(B) = 0$ for every B in \mathcal{A} ; but $\phi(A) = \tau_G(VAV^* - A) \neq 0$. Thus $0 = f \cdot F(\phi) = \phi(A) \neq 0$. This is impossible. So we conclude that $A \in \mathcal{A}$. Now by the preliminary remarks concerning the decomposition of elements of \mathcal{A}_{\sim} into a linear combination of normal functionals and by Theorem 2 we find that the range of F is \mathcal{A}_{\sim} . If $\phi \in \mathcal{A}_{\sim}$, then there is a $\psi \in \mathcal{L}'_{\sim}$ such that $F(\psi) = \phi$ and thus $f(\phi) = \psi(A) = \phi(A)$ since $A \in \mathcal{A}$. This completes the proof. Q.E.D.

3. Characterization of type I algebras. It is known that a C^* -algebra that can be expressed as the second dual of a Banach space is a von Neumann algebra with minimal projections ([13], [7], [19]). So linear functionals are suitable for characterizing type I factor algebras but unsuitable for characterizing algebras with arbitrary centers. The object of this section is to bring the center into consideration by considering module homomorphisms and thereby to obtain a description of type I algebras.

THEOREM 4. *Let \mathcal{L} be a commutative von Neumann algebra and let \mathcal{A} be a C^* -algebra with center \mathcal{L} . Let \mathcal{M} be a Banach \mathcal{L} -module. If \mathcal{A} is the dual \mathcal{M}' of \mathcal{M} , then \mathcal{A} is a von Neumann algebra and every functional $A \rightarrow \phi(A)$ of \mathcal{A} into \mathcal{L} for fixed ϕ in \mathcal{M} is a σ -weakly continuous functional of the module \mathcal{A} .*

NOTE. We write $\phi(A)$ to indicate the functional $A \in \mathcal{A}$ applied to $\phi \in \mathcal{M}$.

Proof. Let \mathcal{B} be the linear manifold in the dual \mathcal{A}^* of \mathcal{A} generated by the set $\{f \cdot \phi \mid f \in \mathcal{L}', \phi \in \mathcal{M}\}$. We show that the dual \mathcal{B}' of \mathcal{B} is identified with \mathcal{A} . Then by Sakai's Theorem [13, Theorem 1 and Remark 1] \mathcal{A} is a von Neumann algebra and the functionals of \mathcal{B} are σ -weakly continuous on \mathcal{A} . In fact the closure of \mathcal{B} in \mathcal{A}^* is equal to the associated space of \mathcal{A} [12, 22.6]. This means that the elements of \mathcal{M} are σ -weakly continuous. We proceed to identify \mathcal{B}' with \mathcal{A} . Let Φ be an element of \mathcal{B}' . Let H be a Hilbert space on which \mathcal{L} may be represented as a von Neumann algebra. The functional $w_{x,y} \cdot \phi$ is in \mathcal{B} for every x and y in H and $\phi \in \mathcal{M}$. Here $w_{x,y}(A) = (Ax, y)$. Setting $\langle x, y \rangle_{\phi} = \Phi(w_{x,y} \cdot \phi)$ we obtain for each $\phi \in \mathcal{M}$ a bilinear form on H . Since $|\langle x, y \rangle_{\phi}| \leq \|x\| \|y\| \|\phi\| \|\Phi\|$, the form is continuous and hence there is a unique bounded linear operator A_{ϕ} on H such that $\langle x, y \rangle_{\phi} = (A_{\phi}x, y)$ for every x and y in H . If B is an arbitrary element in the commutator \mathcal{L}' of \mathcal{L} on H , then $(A_{\phi}Bx, y) = (BA_{\phi}x, y)$ for every x and y in H because $w_{Bx,y} \cdot \phi = w_{x,B^*y} \cdot \phi$. This means A_{ϕ} is in the double commutator of \mathcal{L} which is \mathcal{L} . The map $\phi \rightarrow A_{\phi}$ is easily seen to be a module homomorphism of \mathcal{M} into \mathcal{L} . It is bounded because

$$\|A_{\phi}\| = \text{lub} \{|\Phi(w_{x,y} \cdot \phi)| \mid \|x\| \leq 1, \|y\| \leq 1\} \leq \|\Phi\| \|\phi\|.$$

By hypothesis there is a unique A_0 in \mathcal{A} such that $\phi(A_0) = A_{\phi}$ for every $\phi \in \mathcal{M}$. Then for every $g \in \mathcal{B}$ we have that $\Phi(g) = g(A_0)$.

For each $A \in \mathcal{A}$ let G_A be the element of \mathcal{B}^* defined by $G_A(g) = g(A)$. The map $A \rightarrow G_A$ of \mathcal{A} into \mathcal{B}^* is a linear function of \mathcal{A} whose range is \mathcal{B}^* by the preceding paragraph. We complete the proof by showing that this map is an isometry. Because \mathcal{B} is a subset of \mathcal{A}^* we have that $\|G_A\| \leq \|A\|$. On the other hand the two relations

$$\text{lub} \{ \|\phi(A)\| \mid \phi \in \mathcal{M}, \|\phi\| \leq 1 \} = \|A\|$$

and

$$\text{lub} \{ |f \cdot \phi(A)| \mid f \in \mathcal{L}_*, \|f\| \leq 1 \} = \|\phi(A)\|$$

imply the relation $\|A\| \leq \|G_A\|$. So $\|A\| = \|G_A\|$. Q.E.D.

Before proceeding we make some observations. Assume that \mathcal{M} is a Banach modulo over a commutative von Neumann algebra \mathcal{L} whose second dual $\mathcal{M}^{\sim\sim}$ is a C^* -algebra \mathcal{A} with center \mathcal{L} . Let \mathcal{M}' be the Banach \mathcal{L} -module \mathcal{M} reduced module $\mathcal{K} = \bigcap \{ \text{kernel } \phi \mid \phi \in \mathcal{M}^{\sim} \}$ with the norm $\|A'\| = \text{glb} \{ \|A+B\| \mid B \in \mathcal{K} \}$. Here A' is the image in \mathcal{M}' of A in \mathcal{M} under the canonical homomorphism. It is standard to show that \mathcal{M}^{\sim} is identified with (i.e. is isometric isomorphic to) the dual \mathcal{M}'^{\sim} under the correspondence $\phi \rightarrow \phi'$ where $\phi'(A') = \phi(A)$. By replacing \mathcal{M} by \mathcal{M}' if necessary we may assume that $\mathcal{K} = (0)$. For each $A \in \mathcal{M}$ the function $\phi \rightarrow \phi(A)$ defines an element F_A of $\mathcal{M}^{\sim\sim} = \mathcal{A}$. The map $A \rightarrow F_A$ is an isomorphism of \mathcal{M} into a submodule \mathcal{N} of the module \mathcal{A} . Setting $\phi'(A) = \phi(F_A)$ for each $\phi \in \mathcal{M}^{\sim}$ and $A \in \mathcal{M}$, we obtain a functional ϕ' of \mathcal{M}^{\sim} because $\|F_A\| \leq \|A\|$. It satisfies the relation

$$\begin{aligned} \|\phi\| &= \text{lub} \{ \|\phi(A)\| \mid A \in \mathcal{N}, \|A\| \leq 1 \} \\ &\geq \text{lub} \{ \|\phi'(A)\| \mid A \in \mathcal{M}, \|A\| \leq 1 \} = \|\phi'\| \\ &\geq \text{lub} \{ \|\phi'(A)\| \mid A \in \mathcal{A}, \|A\| \leq 1 \} \geq \|\phi\|. \end{aligned}$$

This means that $\phi \rightarrow \phi'$ is an isometric isomorphism of \mathcal{M}^{\sim} onto \mathcal{M}'^{\sim} . Because it is well known that the dual of the closure of \mathcal{N} is equal to \mathcal{M}'^{\sim} , we may assume that \mathcal{M} is embedded in \mathcal{A} . For every A in \mathcal{M} we have that

$$(1) \quad \|A\| = \text{lub} \{ \|\phi(A)\| \mid \phi \in \mathcal{M}^{\sim}, \|\phi\| \leq 1 \}$$

because $\mathcal{M}^{\sim\sim} = \mathcal{A}$ and for every $\phi \in \mathcal{M}^{\sim}$ we have that

$$(2) \quad \begin{aligned} \|\phi\| &= \text{lub} \{ \|\phi(A)\| \mid A \in \mathcal{M}, \|A\| \leq 1 \} \\ &\leq \text{lub} \{ \|\phi(A)\| \mid A \in \mathcal{A}, \|A\| \leq 1 \} \leq \|\phi\|. \end{aligned}$$

So we are always justified in assuming that \mathcal{M} is embedded in \mathcal{A} and that (1) and (2) hold.

THEOREM 5. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{L} . If \mathcal{M} is a closed submodule of the module \mathcal{A} whose second dual $\mathcal{M}^{\sim\sim}$ is equal to \mathcal{A} , then \mathcal{M} is σ -weakly dense in \mathcal{A} .*

Proof. We use the same notation as Theorem 4. Every $f \in \mathcal{B}$ has the form $f = \sum \{f_i \cdot \phi_i \mid 1 \leq i \leq n\}$ where $f_i \in \mathcal{L}_*$ and $\phi_i \in \mathcal{M}^\sim$. Because every f_i may be written as a linear combination of positive functionals in \mathcal{L}_* , we may assume every f_i is positive. Then $\sum f_i = w_x$ for some $x \in H$. By the Radon-Nikodym theorem there is a $C_i \in \mathcal{L}^+$ such that $f_i = w_{C_i x}$ ($1 \leq i \leq n$). Hence we may assume that f has the form $f = w_x \cdot \phi$ for some $\phi \in \mathcal{M}^\sim$.

We now obtain a contradiction by assuming that the complement of the weak closure \mathcal{N} of the unit sphere of \mathcal{M} with respect to the unit sphere \mathcal{A}_1 of \mathcal{A} contains an element B . Standard separation theorems [12, 14.2] give an $f \in \mathcal{A}_*$ such that $\text{lub} \{\text{Re } f(A) \mid A \in \mathcal{N}\} < \text{Re } f(B)$. Here $\text{Re } \alpha$ denotes the real part of α . Because \mathcal{B} is uniformly dense in \mathcal{A}_* and because $\mathcal{N} \subset \mathcal{A}_1$, there is by the preceding paragraph an $x \in H$ and a $\phi \in \mathcal{M}^\sim$ such that $\text{lub} \{\text{Re } w_x \cdot \phi(A) \mid A \in \mathcal{N}\} < \text{Re } w_x \cdot \phi(B)$. We show that $\{\text{Re } \phi(A) \mid A \in \mathcal{N}\} = X$ is monotonely increasing in \mathcal{L} . Here $\text{Re } \phi(A) = (\phi(A) + \phi(A)^*)/2$. For A_i ($i=1, 2$) in \mathcal{N} let $U_i^* |\phi(A_i)| = \phi(A_i)$ be the polar decomposition of $\phi(A_i)$ in \mathcal{L} . The element $\phi(U_i A_i)$ is in X because $U_i A_i \in \mathcal{N}$. Note that $\text{Re } \phi(A_i) \leq |\phi(A_i)| = \phi(U_i A_i)$. There is a central projection P such that $P\phi(U_1 A_1) \leq P\phi(U_2 A_2)$ and $(1-P)\phi(U_2 A_2) \leq (1-P)\phi(U_1 A_1)$ and so the element $A = PU_2 A_2 + (1-P)U_1 A_1$ in \mathcal{N} satisfies the relation $\text{Re } \phi(A_i) \leq \phi(A)$ for $i=1, 2$. This shows X is monotonely increasing. Because X is bounded above in \mathcal{L} , it is possible to find a net $\{A_n\}$ in \mathcal{N} such that $\{\text{Re } \phi(A_n)\}$ converges strongly to the least upper bound C_0 of X . Because \mathcal{N} is compact in the weak topology, there is a subnet $\{A_{n_j}\}$ which converges weakly to an element $A_0 \in \mathcal{N}$. However $\text{Re } \phi(A_0) = C_0$ due to the weak continuity of ϕ (Theorem 4). By multiplying $\phi(A_0)$ by an appropriate partial isometry in \mathcal{L} we may assume that $\phi(A_0) = C_0$. For every projection P in \mathcal{L} we have $\|P\phi\| = \|PC_0\|$. On the contrary, if $\|PC_0\| < \|P\phi\|$, then $Q(C_0 + \varepsilon) \leq (\|PC_0\| + \varepsilon)Q \leq Q\phi(A)$ for suitable $A \in \mathcal{M} \cap \mathcal{A}_1$, $\varepsilon > 0$ and central projection Q with $0 < Q \leq P$ because $\|P\phi\| = \text{lub} \{\|P\phi(C)\| \mid C \in \mathcal{M} \cap \mathcal{A}_1\}$. This contradicts the definition of C_0 . Therefore $\|P\phi\| = \|PC_0\|$ for every central projection P . Now if $U^* |\phi(B)| = \phi(B)$ is the polar decomposition of $\phi(B)$, then $\text{Re } w_x(\phi(B)) = w_x(\text{Re } \phi(B)) \leq w_x(|\phi(B)|) = w_x(\phi(UB))$. There is no loss of generality in assuming $\phi(B) \in \mathcal{L}^+$. We see that $\phi(B) \leq C_0$; otherwise there is a projection $P \neq 0$ in \mathcal{L} and an $\varepsilon > 0$ such that $P(C_0 + \varepsilon) \leq P\phi(B)$. This means that $\|PC_0\| < \|P\phi\|$ and that is impossible. However $w_x \cdot \phi(B) \leq w_x(C_0) = w_x(\phi(A_0))$ is obtained. Because $A_0 \in \mathcal{N}$ we have reached a contradiction. Therefore $\mathcal{N} = \mathcal{A}_1$. Q.E.D.

COROLLARY. *The module \mathcal{M}^\sim is equal to the module \mathcal{A}_\sim of all σ -weakly continuous functionals of the module \mathcal{A} .*

Proof. If $\phi \in \mathcal{A}_\sim$, then the restriction $\phi|_{\mathcal{M}}$ of ϕ to \mathcal{M} defines a bounded functional of the module \mathcal{M} . There is a $\psi \in \mathcal{M}^\sim$ such that $\psi = \phi|_{\mathcal{M}}$. Because $\psi \in \mathcal{A}_\sim$ (Theorem 4) and because \mathcal{M} is σ -weakly dense in \mathcal{A} , we have that $\psi = \phi$. So $\mathcal{A}_\sim \subset \mathcal{M}^\sim$. Since $\mathcal{M}^\sim \subset \mathcal{A}_\sim$ (Theorem 4), we have $\mathcal{A}_\sim = \mathcal{M}^\sim$. Q.E.D.

Let \mathcal{M} be a Banach module over the commutative von Neumann algebra \mathcal{L} ;

the topology induced on the dual \mathcal{M}^\sim of the module \mathcal{M} by the family of seminorms $p_{f,A}(\phi) = |f(\phi(A))|$ ($f \in \mathcal{L}_*$, $A \in \mathcal{M}$) will be called the weak topology of \mathcal{M}^\sim .

THEOREM 6. *Let \mathcal{M} be a Banach module over the commutative σ -finite von Neumann algebra \mathcal{L} . Then the unit sphere \mathcal{M}_1^\sim of \mathcal{M}^\sim is compact in the weak topology.*

Proof. Let x be a separating unit vector for \mathcal{L} . Let $\{\phi_n\}$ be a net in \mathcal{M}_1^\sim . A subnet $\{w_x \cdot \phi_n \mid n \in S\}$ of $\{\phi_n\}$ converges in the $\sigma(\mathcal{M}^*, \mathcal{M})$ -topology of the dual \mathcal{M}^* of \mathcal{M} to the functional f in the unit sphere of \mathcal{M}^* . Because $|f(CD^*B)| \leq \|B\| \|Cx\| \|Dx\|$ for each $B \in \mathcal{M}$ and $C, D \in \mathcal{L}$, the relation $\langle Cx, Dx \rangle_B = f(CD^*B)$ defines a continuous bilinear functional on a dense linear manifold of the subspace

$$K = \text{closure} \{Cx \mid C \in \mathcal{L}\}.$$

This bilinear functional may be uniquely extended to a bilinear functional $\langle y, z \rangle_B$ of K such that $|\langle y, z \rangle_B| \leq \|B\| \|y\| \|z\|$ for every y and z . There is a unique bounded linear operator $A' = A'_B$ on K such that $(A'y, z) = \langle y, z \rangle_B$ for all y and z . The projection E of the Hilbert space onto K is an abelian projection of the commutator \mathcal{L}' of \mathcal{L} . If $A \in \mathcal{L}$, then $(A'AE Cx, Dx) = (AEA' Cx, Dx)$ for every C and D in \mathcal{L} . Thus $(AE)A' = A'(AE)$, and because $A \in \mathcal{L}$ is arbitrary, we find that A' is an element of the commutator $E\mathcal{L}'E$ of $\mathcal{L}'E$ on K . There is unique $\phi(B)$ in \mathcal{L} , because the central support of E is 1, such that $\phi(B)E = A'_B$. By the uniqueness of $\phi(B)$ we may conclude that $B \rightarrow \phi(B)$ is a module homomorphism of \mathcal{M} into \mathcal{L} and by the relation $|\langle \phi(B)y, z \rangle| \leq \|B\| \|y\| \|z\|$ for all y and z in K we may conclude that $\|\phi(B)\| = \|\phi(B)E\| \leq \|B\|$. So $\phi \in \mathcal{M}_1^\sim$. From the fact that $\{\|\phi_n(B)\| \mid n \in S\}$ is bounded for fixed $B \in \mathcal{M}$ and from the fact that $\{Cx \mid C \in \mathcal{L}\}$ is dense in K , we see that $\{\phi_n(B)E \mid n \in S\}$ converges weakly to $\phi(B)E$. This implies that $\{\phi_n(B) \mid n \in S\}$ converges weakly to $\phi(B)$ for every $B \in \mathcal{M}$ [2, I, §4, Theorem 2]. Q.E.D.

A subset \mathcal{K} of a module \mathcal{M} over the commutative von Neumann algebra \mathcal{L} is said to be \mathcal{L} -convex if $CA + (1 - C)B$ is in \mathcal{K} for every A and B in \mathcal{K} and C in \mathcal{L} with $0 \leq C \leq 1$. A nonvoid \mathcal{L} -convex subset \mathcal{K}' of a \mathcal{L} -convex subset \mathcal{K} of \mathcal{M} is said to be a support of \mathcal{K} if $CA + (1 - C)B \in \mathcal{K}'$ for A and B in \mathcal{K} and C in \mathcal{L} with $0 < C < 1$ implies $A \in \mathcal{K}'$ and $B \in \mathcal{K}'$. By $0 < C < 1$ it is meant that C is strictly between 0 and 1, i.e. the spectrum of C is contained in the open interval $(0, 1)$. A point A of a \mathcal{L} -convex subset \mathcal{K} of \mathcal{M} is said to be an extreme point of \mathcal{K} if $\{A\}$ is a support of \mathcal{K} .

The following is a Krein-Milman Theorem for \mathcal{M}^\sim .

THEOREM 7. *Let \mathcal{M} be a Banach \mathcal{L} -module whose second dual $\mathcal{M}^{\sim\sim}$ is a von Neumann algebra \mathcal{A} with center \mathcal{L} . Let P be a nonzero projection in \mathcal{L} . Then there is an extreme point ϕ in the unit sphere \mathcal{S} of positive functionals of \mathcal{A} such that $P\phi \neq 0$.*

Proof. We may assume that P is σ -finite since every nonzero projection in \mathcal{L} majorizes a nonzero σ -finite projection. Let $\mathcal{H} = \{\text{Re } \phi \mid \phi \in \mathcal{A}_+ \text{ and } P\phi = \phi\}$. Here

Re $\phi(A) = (\phi(A) + \phi(A)^*)/2$. The set \mathcal{H} has a natural module structure over the ring of hermitian elements in \mathcal{L} . The seminorms $p_{f,A}$ ($f \in \mathcal{L}_*$, $A \in \mathcal{M}$) induce a weak topology on H in which the \mathcal{L} -convex subset $\mathcal{H}_1 = \{\text{Re } \phi \mid \phi \in \mathcal{A}_\sim, P\phi = \phi, \|\phi\| \leq 1\}$ is compact since the unit sphere of \mathcal{M}^\sim is compact in the weak topology and since the adjoint operation of \mathcal{L} is weakly continuous. Suppose that \mathcal{H}_1 has a nonzero extreme point ϕ' . We show that \mathcal{S} has a nonzero extreme point ψ such that $P\psi = \psi$. Let ϕ be an element of the unit sphere of \mathcal{A}_\sim such that $P\phi = \phi$ and $\text{Re } \phi = \phi'$. If $C\theta + (1-C)\theta' = \phi$ for θ, θ' in the unit sphere of \mathcal{A}_\sim and C strictly between 0 and 1 in \mathcal{L} , then $C \text{Re } \theta + (1-C) \text{Re } \theta' = \phi'$ and consequently $\phi' = \text{Re } \theta = \text{Re } \theta'$. This means that $\phi = \theta = \theta'$ and thus that ϕ is an extreme point of the unit sphere of \mathcal{A}_\sim . Using the polar decomposition for ϕ , we obtain a partial isometric operator U in $\mathcal{A}P$ such that $\psi(A) = \phi(UA)$ defines a functional in \mathcal{S} and such that $\psi(U^*A) = \phi(A)$ for every A in \mathcal{A} . If $\psi = C\theta + (1-C)\theta'$ for $\theta, \theta' \in \mathcal{S}$ and $C \in \mathcal{L}$ with $0 < C < 1$, then $\phi(A) = \theta(U^*A) = \theta'(U^*A)$ for every $A \in \mathcal{A}$. But $\psi(U^*U) = \psi(1)$ and so both θ and θ' vanish on the projection $1 - U^*U$. By the Cauchy-Schwarz inequality we have that $\theta(A) = \theta(U^*UA)$ and $\theta'(A) = \theta'(U^*UA)$ for every $A \in \mathcal{A}$. We then see that $\psi = \theta = \theta'$ and thus that ψ is a nonzero extreme point of \mathcal{S} such that $P\psi = \psi$. We complete the proof by showing that \mathcal{H}_1 has a nonzero extreme point.

Now let \mathcal{F} be a maximal family with the finite intersection property of compact supports of \mathcal{H}_1 . The set $\mathcal{H}_0 = \bigcap \mathcal{F}$ is nonvoid and evidently a compact support of \mathcal{H}_1 . We show that \mathcal{H}_0 contains a single point. On the contrary if ϕ and ψ are two distinct points in \mathcal{H}_0 , there is an element A in \mathcal{M} such that $\phi(A) \neq \psi(A)$. Indeed, an element in \mathcal{A}_\sim which vanishes under multiplication by $1 - P$ is uniquely determined by its values on the elements of \mathcal{M} . The set $\{\theta(A) \mid \theta \in \mathcal{H}_0\}$ is a monotonely increasing net in set of hermitian elements of \mathcal{L} and therefore the set

$$\mathcal{H}_0(A_0) = \{\theta \in \mathcal{H}_0 \mid \theta(A) = A_0 = \text{lub } \{\theta'(A) \mid \theta' \in \mathcal{H}_0\}\}$$

is nonvoid. This is an obvious variation of the statement concerning the set X in Theorem 5. The set $\mathcal{H}_0(A_0)$ is certainly \mathcal{L} -convex and weakly compact. It is a support of \mathcal{H}_0 for $C\theta(A) + (1-C)\theta'(A) = A_0$ for θ, θ' in \mathcal{H}_0 and C in \mathcal{L} with $0 < C < 1$ implies $\theta(A) = \theta'(A) = A_0$. However the compact support $\mathcal{H}_0(A_0)$ of \mathcal{H}_0 is a compact support of \mathcal{H}_1 which cannot contain both ϕ and ψ . This contradicts the maximality of \mathcal{F} . Hence we must conclude that \mathcal{H}_0 contains a single point ϕ_0 . We have that $\phi_0 \neq 0$. On the contrary, if E is an abelian projection in the commutator of \mathcal{L} which has central support P , then $\theta = \text{Re } \tau_E \in \mathcal{H}_1$ (Corollary, Theorem 5) and $(\theta - \theta)/2 = 0$. But $\theta = -\theta = 0$ is evidently not true. So $\phi_0 \neq 0$. Q.E.D.

Let \mathcal{A} be a von Neumann algebra with center \mathcal{L} ; a positive functional $\phi \in \mathcal{A}^\sim$ is said to be \mathcal{L} -irreducible on \mathcal{A} if given a positive functional ψ in \mathcal{A}^\sim with $\psi \leq \phi$ then there is a $C \in \mathcal{L}^+$ such that $C\phi = \psi$.

THEOREM 8. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{L} . For a functional ϕ*

in the unit sphere \mathcal{S} of the set of positive functionals of \mathcal{A} the following are equivalent: (1) ϕ is an extreme point of \mathcal{S} ; and (2) $\phi(1)$ is a projection of \mathcal{Z} and ϕ is \mathcal{Z} -irreducible.

Proof. Let ϕ be an extreme point of \mathcal{S} . Let P be the support of the positive element $\phi(1)$ in the unit sphere of \mathcal{Z} . If there is a ζ in the spectrum Z of \mathcal{Z} such that $0 < \phi(1) \wedge (\zeta) < 1$, there is an $\varepsilon > 0$ and a projection Q majorized by P with $Q \wedge (\zeta) = 1$ and $\varepsilon Q \leq Q\phi(1) \leq (1 - \varepsilon)Q$. So $\|Q\phi\| \leq 1 - \varepsilon$ and $\theta = (1 - \varepsilon)^{-1}$. $Q\phi + (1 - Q)\phi$ is a point of \mathcal{S} . However $\phi = (1 - \varepsilon)\theta + \varepsilon(1 - Q)\phi$. This gives a contradiction. So we conclude that $\phi(1) = P$. Now let $\psi \in \mathcal{S}$ and $\psi \leq \phi$. Setting $\theta = \phi - \psi$ we assume that $\theta(1) \wedge (\zeta) \neq 0$ and $\theta(1) \wedge (\zeta) \neq 1$ for some $\zeta \in Z$. There is an $\varepsilon > 0$ and a central projection Q such that $Q \wedge (\zeta) = 1$ and $\varepsilon Q \leq Q\theta(1) \leq (1 - \varepsilon)Q$. This means that $B\theta(1) = Q$ and $D\psi(1) = Q$ for suitable B and D in $(\mathcal{Z}Q)^+$. Setting $C = ((1 - Q)/2) + Q\theta(1)$, we obtain a central element strictly between 0 and 1 satisfying the relation

$$\phi = C(B\theta + (1 - Q)\phi) + (1 - C)(D\psi + (1 - Q)\phi).$$

Because $B\theta + (1 - Q)\phi$ and $D\psi + (1 - Q)\phi$ are in \mathcal{S} , $D\psi = Q\phi$ and so $Q\psi = Q\psi(1)\phi$. If $\{Q_n\}$ is a maximal set of mutually orthogonal nonzero central projections such that $Q_n\psi(1)\phi = Q_n\psi$ for each Q_n , then $\sum Q_n = P$. This proves that $\psi = P\psi = \psi(1)\phi$. Thus, we see that ϕ is \mathcal{Z} -irreducible.

Conversely, let ϕ be \mathcal{Z} -irreducible and let $\phi(1)$ be a projection P . If $\phi = C\psi + (1 - C)\theta$ for $\psi, \theta \in \mathcal{S}$ and $0 < C < 1$ in \mathcal{Z} , then there is a $D \in \mathcal{Z}^+$ such that $C\psi = D\phi$. Then the two relations $C^{-1}DP = \psi(1)$ and $P = C\psi(1) + (1 - C)\theta(1)$ imply $P = \psi(1)$. So $\psi = C^{-1}D\phi = \phi$. Similarly $\theta = \phi$. This proves ϕ is an extreme point of \mathcal{S} . Q.E.D.

THEOREM 9. Let \mathcal{Z} be a commutative von Neumann algebra and let \mathcal{A} be a C^* -algebra with center \mathcal{Z} . Then \mathcal{A} is a type I von Neumann algebra if and only if the module \mathcal{A} is the second dual \mathcal{M}^{**} of a Banach \mathcal{Z} -module \mathcal{M} .

Proof. If \mathcal{A} is a type I von Neumann algebra, then \mathcal{A} is the second dual of the closed two-sided I_a of \mathcal{A} generated by the abelian projections of \mathcal{A} [9, §4].

Conversely, it is already known that \mathcal{A} is a von Neumann algebra (Theorem 4) and so it is sufficient to prove that every nonzero projection P of \mathcal{Z} majorizes a nonzero abelian projection. Let ϕ be an extreme point of the set of positive functionals \mathcal{S} of the unit sphere of \mathcal{A} such that $P\phi \neq 0$. Then it is easy to see that $P\phi$ is an extreme point of \mathcal{S} and so we assume that $P\phi = \phi$. Let x be a unit vector in the Hilbert space H of \mathcal{A} such that $\phi(1)x = x$ (Theorem 8). Let g be a positive functional on \mathcal{A} majorized by $w_x \cdot \phi$. If $A \in \mathcal{Z}^+$, then $g(A) \leq w_x(\phi(1)A) = w_x(A)$. There is an element C in \mathcal{Z}^+ such that $g(A) = (ACx, Cx)$ for all A in \mathcal{Z} . By Theorem 1 there is a normal functional ψ of the module \mathcal{A} such that $g = w_{Cx} \cdot \psi$ because g is normal. If E is the projection of H onto closure $\{Ax \mid A \in \mathcal{Z}\}$ then $C^2\psi(A)E \leq \phi(A)E$ for every A in \mathcal{A}^+ since $(C^2\psi(A)Ey, Ey) \leq (\phi(A)Ey, Ey)$ for every y in $E(H)$. However

the projection E is in the commutator of \mathcal{L} on H and so if Q is the central support of E we have that $C^2Q\psi(A) \leq Q\phi(A) \leq \phi(A)$ for every A in \mathcal{A}^+ . There is a $D \in \mathcal{L}^+$ such that $C^2Q\psi = D\phi$ (Theorem 8). Thus $g(A) = w_x \cdot \phi(DA)$ for all A in \mathcal{A} . This means that the projection in \mathcal{A} corresponding to closure $\{A'x \mid A' \in \mathcal{A}'\}$ where \mathcal{A}' is the commutator of \mathcal{A} is abelian in \mathcal{A} [8, §2]. This completes the proof. Q.E.D.

REMARK. Actually we have proved that a normal functional f on \mathcal{A} is centrally reducible (cf. [8, §2]) if and only if there is a \mathcal{L} -irreducible normal functional ϕ of the module \mathcal{A} such that $f \cdot \phi = f$.

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