IRREDUCIBLE MODULE HOMOMORPHISMS OF A VON NEUMANN ALGEBRA INTO ITS CENTER(1)

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1. **Introduction.** A von Neumann algebra $\mathscr{A}$ can be considered as a module over its center $\mathscr{Z}$. The norm of $\mathscr{A}$ induces a norm on the module $\mathscr{A}$. Whenever we talk of the module $\mathscr{A}$ it will always be this specific module over $\mathscr{Z}$. In this article we study the set $\mathscr{A}^\sim$ of bounded module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. In an earlier article we studied those module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$ which are continuous in the σ-weak topology of $\mathscr{A}$ and $\mathscr{Z}$ respectively. In that paper we discovered a specific form for such homomorphisms and showed that a type I algebra could be characterized in terms of such functionals. These results were analogues of results known for factor algebras. For factor algebras multipliers are scalars and the mappings are scalar-valued functionals while in algebras with arbitrary centers the multipliers are central elements and the mappings are module homomorphisms into the center.

There are always module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. A kind which is particularly simple although fundamental may be constructed as follows. Let $\mathscr{Z}'$ be the commutator of $\mathscr{Z}$ and let E be an abelian projection in $\mathscr{Z}'$ with central support P. There is an isomorphism of $\mathscr{Z}P$ onto $E\mathscr{Z}'E$ given by $A \to AE$. For each A in $\mathscr{A}$ we denote the inverse image in $\mathscr{Z}P$ of EAE under this isomorphism by $\tau_E(A)$. Then the function τ_E on $\mathscr{A}$ is a homomorphism into $\mathscr{Z}$.

In general $\mathscr{Z}$ is a most suitable range for module homomorphisms. The following Hahn-Banach type theorem illustrates this. Let $\mathscr{B}$ be a normed space which is a module over a commutative AW^*-algebra $\mathscr{Z}$. Let $\mathscr{C}$ be any submodule of $\mathscr{B}$ and let ϕ be a bounded module homomorphism of $\mathscr{C}$ into $\mathscr{Z}$. There is a bounded module homomorphism ψ of $\mathscr{B}$ into $\mathscr{Z}$ such that $\psi(C)=\phi(C)$ for every C in $\mathscr{C}$ and such that $\|\psi\|=\|\phi\|$ [19], [24]. From this theorem many homomorphisms may be constructed.

A module homomorphism ϕ of $\mathscr{A}$ into $\mathscr{Z}$ will be called a functional of the module $\mathscr{A}$. A functional ϕ of the module $\mathscr{A}$ is said to be hermitian if $\phi(A^*)=\phi(A)^*$ for every A in $\mathscr{A}$. Every bounded functional of the module A can be written as a linear combination of two bounded hermitian functionals. A functional ϕ of the module $\mathscr{A}$ is said to be positive if ϕ maps $\mathscr{A}^+$ into $\mathscr{Z}^+$. Since

$$|\phi(A)|^2 = \phi(A)^*\phi(A) \leqq \phi(A^*A)\phi(1),$$

every positive functional ϕ is bounded with bound $\|\phi(1)\|$. Every bounded hermitian

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functional of the module $\mathscr{A}$ may be written as the difference of two positive functionals of the module $\mathscr{A}$ [19], [24].

In this paper we study the positive functionals of the module $\mathscr{A}$. The set $\mathscr{S}$ of positive functionals of $\mathscr{A}^\sim$ of norm not exceeding 1 is compact in a naturally defined topology in $\mathscr{A}^\sim$. The set $\mathscr{S}$ has extreme points and $\mathscr{S}$ is the closure (in this topology) of the convex hull of its extreme points. Here though the convexity is expressed in terms of multiplication by elements in $\mathscr{Z}$. We show that every linear functional f on $\mathscr{A}$ which is σ-weakly continuous when restricted to $\mathscr{Z}$ can be expressed as the composition of f with an element of $\mathscr{A}^\sim$.

A positive functional ϕ in $\mathscr{A}^\sim$ normalized so that $\phi(1)=1$ gives rise to a representation of $\mathscr{A}$ as a *-subalgebra of the algebra of all bounded linear operators on an AW^*-module M_ϕ over the center $\mathscr{Z}$ ([6], [17], [28]). We study the representations that arise from an extreme point ϕ of $\mathscr{S}$. By presenting a specific form for the representation we are able to obtain the analogue of Kadison's theorem on strict irreducibility. If $A \to A^\wedge$ denotes the Gelfand transform of $\mathscr{Z}$ onto the algebra of all continuous complex-valued functions on the spectrum Z of $\mathscr{Z}$, then the analogue of Kadison's theorem allows us to conclude that $A \to \phi(A)^\wedge(\zeta)$ is a pure state of $\mathscr{A}$ for every ζ in $\mathscr{Z}$. In a certain sense this result illustrates the advantage of a global theory over a decomposition theory. By an additional construction we are able to find an extreme point ϕ such that the kernel of the canonical representation of $\mathscr{A}$ on a Hilbert space induced by $A \to \phi(A)^\wedge(\zeta)$ (ζ fixed but arbitrary in Z) is the smallest closed two-sided ideal $[\zeta]$ in $\mathscr{A}$ containing ζ. So $[\zeta]$ is a minimal primitive ideal of $\mathscr{A}$.

We then define a vector state of the $\mathscr{A}$ as a module. This definition comes from ideas in a previous paper [12]. The set of elements in $\mathscr{A}^\sim$ obtained as pointwise limits of these vector states is called the vector state space. The set of pointwise limits in $\mathscr{A}^\sim$ of the set of extreme points ϕ of the positive elements of the unit sphere of $\mathscr{A}^\sim$ which satisfy $\phi(1)=1$ is called the pure state space of the module $\mathscr{A}$. We then compare the set of all ϕ in the unit sphere of $\mathscr{A}^\sim$ such that $\phi(1)=1$ with the pure state space and the vector state space of the module $\mathscr{A}$. These structures have exactly the same relations as the corresponding structures of scalar functionals as given by Glimm ([3], [4]). Here the ideal of completely continuous operators is replaced by the ideal generated by the abelian projections of $\mathscr{A}$.

2. **Existence of extreme points.** Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ and let $\mathscr{A}^\sim$ be the space of bounded functionals of the module $\mathscr{A}$. Let $\mathscr{Z}_*$ be the set of all σ-weakly continuous functionals on $\mathscr{Z}$. For each f in $\mathscr{Z}_*$ and A in $\mathscr{A}$ define the seminorm $p_{f,A}=p$ of $\mathscr{A}^\sim$ by $p(\phi)=|f(\phi(A))|$. The family $\{p_{f,A} \mid f \in \mathscr{Z}_*, A \in \mathscr{A}\}$ of seminorms of $\mathscr{A}^\sim$ defines a topology on $\mathscr{A}^\sim$ under which $\mathscr{A}^\sim$ is a locally convex Hausdorff topological linear space. We call this topology the weak-* topology of $\mathscr{A}^\sim$. If f is a weak-* continuous functional on $\mathscr{A}^\sim$, there are functionals $f_1, f_2, \ldots, f_n$ in $\mathscr{Z}_*$ and $A_1, A_2, \ldots, A_n$ in $\mathscr{A}$ such that

$$f(\psi) = \sum \{f_j(\psi(A_j)) \mid 1 \leqq j \leqq n\}$$

for every $\psi \in \mathscr{A}^\sim$. Since every positive functional g in $\mathscr{Z}_*$ is of the form $g(A) = (Ax, x)$ for some vector x of the Hilbert space H of $\mathscr{Z}$, we have that there are vectors $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ in H and $B_1, B_2, \ldots, B_m$ in $\mathscr{Z}$ such that

$$f(\psi) = \sum \{(\psi(B_j)x_j, y_j) \mid 1 \leqq j \leqq m\}.$$

PROPOSITION 2.1. *Let $\mathscr{A}$ be a von Neumann algebra. Let $\mathscr{A}_1^\sim$ be the unit sphere of the set $\mathscr{A}^\sim$ of bounded functionals of the module $\mathscr{A}$ and let $\mathscr{S}$ be the set of positive elements of $\mathscr{A}_1^\sim$. The sets $\mathscr{A}_1^\sim$ and $\mathscr{S}$ are compact in the weak-* topology of $\mathscr{A}^\sim$.*

Proof. Let $\mathscr{Z}_A = \mathscr{Z}$ for every $A \in \mathscr{A}$. Let $\prod \{\mathscr{Z}_A \mid A \in \mathscr{A}\}$ be the product space of $\{\mathscr{Z}_A \mid A \in \mathscr{A}\}$ supplied with the product topology induced by the σ-weak topology on each $\mathscr{Z}_A$. Let Φ be a function of $\mathscr{A}^\sim$ into $\prod \mathscr{Z}_A$ given by $\Phi(\phi)_A = \phi(A)$. The function Φ is an isomorphism of $\mathscr{A}^\sim$ onto $\Phi(\mathscr{A}^\sim)$ which is bicontinuous when $\mathscr{A}^\sim$ is supplied with the weak-* topology. Let $\mathscr{N} = \prod \{\mathscr{N}_A \mid A \in \mathscr{A}\}$ be the subset of $\prod \mathscr{Z}_A$ defined by the relation

$$\mathscr{N}_A = \{B \in \mathscr{Z}_A \mid \|B\| \leqq \|A\|\}.$$

The set $\mathscr{N}$ is compact in $\prod \mathscr{Z}_A$. Since $\|\Phi(\phi)_A\| \leqq \|A\|$ whenever $\phi \in \mathscr{A}_1^\sim$, it is sufficient to show that $\Phi(\mathscr{A}_1^\sim)$ is closed in $\mathscr{N}$ in order to show $\mathscr{A}_1^\sim$ is compact in the weak-* topology. Let $\{\psi_n\}$ be a net in $\mathscr{A}_1^\sim$ such that $\{\Phi(\psi_n)\}$ converges to an element ρ in $\mathscr{N}$. Let f be an element of $\mathscr{Z}_*$, A_1 and A_2 be elements of $\mathscr{A}$, and C_1 and C_2 be elements of $\mathscr{Z}$. Since the nets

$$\{f(\psi_n(C_1A_1))\}, \qquad \{f(\psi_n(C_2A_2))\} \quad \text{and} \quad \{f(\psi_n(C_1A_1 + C_2A_2))\}$$

converge to

$$f(C_1\rho_{A_1}), \qquad f(C_2\rho_{A_2}) \quad \text{and} \quad f(\rho_{(C_1A_1+C_2A_2)})$$

respectively, we have that

$$f(C_1\rho_{A_1} + C_2\rho_{A_2}) = f(\rho_{(C_1A_1+C_2A_2)}).$$

Because f is arbitrary, we have that

$$C_1\rho_{A_1} + C_2\rho_{A_2} = \rho_{(C_1A_1+C_2A_2)}.$$

Therefore, the function $A \to \rho_A$ is a module homomorphism ϕ of $\mathscr{A}$ into $\mathscr{Z}$. But $\|\phi(A)\| \leqq \|A\|$ and therefore ϕ is an element of $\mathscr{A}_1^\sim$. This proves $\Phi(\mathscr{A}_1^\sim)$ is closed in $\mathscr{N}$.

Now we show that $\mathscr{S}$ is weak-* compact in $\mathscr{A}^\sim$. Let $\{\psi_n\}$ be a net in $\mathscr{S}$ converging in the weak-* topology to a point ψ in $\mathscr{A}_1^\sim$. But if A is a positive element of $\mathscr{A}$, then

$$f(\psi(A)) = \lim_n f(\psi_n(A)) \geqq \liminf f(\psi_n(A)) \geqq 0$$

for every positive σ-weakly continuous f functional of $\mathscr{Z}$. Thus $\psi(A) \geqq 0$ for every

$A \geqq 0$. This proves that $\mathscr{S}$ is closed in $\mathscr{A}_1^\sim$. So $\mathscr{S}$ is compact in the weak-* topology. Q.E.D.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. The space $\mathscr{A}^\sim$ of bounded functionals on the module $\mathscr{A}$ is a locally convex linear topological space with the weak-* topology. A linear functional f on $\mathscr{A}^\sim$ is said to be hermitian if $f(\phi)$ is real for every hermitian functional ϕ in $\mathscr{A}^\sim$. If $\mathscr{K}$ is a nonvoid convex weak-* closed subset of $\mathscr{A}^\sim$ and if ϕ is an element of the complement of $\mathscr{K}$, there is a weak-* continuous functional f of $\mathscr{A}^\sim$ such that

$$\text{lub}\,\{\text{Re}\,f(\psi) \mid \psi \in \mathscr{K}\} < \text{Re}\,f(\phi).$$

Here Re α denotes the real part of the number α. Suppose ϕ is hermitian and the elements of $\mathscr{K}$ are hermitian. Let $f(\psi)=\sum_j (\psi(A_j)x_j, y_j)$ where $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ are vectors of the Hilbert space of $\mathscr{A}$ and $A_1, A_2, \ldots, A_n$ are elements of $\mathscr{A}$. Let $g(\psi)=\sum (\psi(A_j^*)y_j, x_j)$. The functional $h(\psi)=(f(\psi)+g(\psi))/2$ is a weak-* continuous hermitian functional on $\mathscr{A}^\sim$ which coincides with Re f on $\mathscr{K} \cup \{\phi\}$. This means that there is a weak-* continuous hermitian functional h of $\mathscr{A}^\sim$ such that

$$\text{lub}\,\{h(\psi) \mid \psi \in \mathscr{K}\} < h(\phi).$$

Let $\mathscr{Z}$ be a commutative von Neumann algebra and let Z be the spectrum of $\mathscr{Z}$. If C is an element of $\mathscr{Z}$ whose Gelfand transform $C^\wedge$ on Z has range contained in the open interval (0, 1), then C is said to lie strictly between 0 and 1. If C lies strictly between 0 and 1 we write $0<C<1$. If M is a $\mathscr{Z}$-module, a subset $\mathscr{K}$ of M will be called $\mathscr{Z}$-convex if $CA+(1-C)B$ is in $\mathscr{K}$ whenever A and B are in $\mathscr{K}$ and C is in $\mathscr{Z}$ with $0 \leqq C \leqq 1$. A point A of a $\mathscr{Z}$-convex subset $\mathscr{K}$ of M is said to be an extreme point of $\mathscr{K}$ if $CB+(1-C)D=A$ implies $B=D=A$ whenever B and D are elements of $\mathscr{K}$ and C is an element of $\mathscr{Z}$ strictly between 0 and 1.

THEOREM 2.2. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ and let $\mathscr{S}$ be the set of positive functionals of norm not exceeding* 1 *of the module $\mathscr{A}$. If $\mathscr{K}$ is a nonvoid $\mathscr{Z}$-convex weak-* compact subset of $\mathscr{S}$, then $\mathscr{K}$ is the weak-* closure of the smallest $\mathscr{Z}$-convex subset of $\mathscr{K}$ containing the extreme points of $\mathscr{K}$.*

Proof. Let B be an element of $\mathscr{A}^+$. The set $\{\phi(B) \mid \phi \in \mathscr{K}\}$ is a monotonely increasing net in $\mathscr{Z}^+$ which is bounded above. Let $B_0=\text{lub}\,\{\phi(B) \mid \phi \in \mathscr{K}\}$. The $\mathscr{Z}$-convex set $S(B)=\{\phi \in \mathscr{K} \mid \phi(B)=B_0\}$ is nonvoid and contains an extreme point of $\mathscr{K}$. This was demonstrated in Theorem 7 [12] for an analogous situation and virtually the same demonstration applies here.

Let $\mathscr{K}'$ be the weak-* closure of the smallest $\mathscr{Z}$-convex subset of $\mathscr{K}$ containing the set of extreme points of $\mathscr{K}$. We show that $\mathscr{K}'=\mathscr{K}$ by arguing by contradiction. Suppose there is an element ϕ in the complement of $\mathscr{K}'$ with respect to $\mathscr{K}$. There is a weak-* continuous hermitian functional f of $\mathscr{A}^\sim$ such that

$$\text{lub}\,\{f(\psi) \mid \psi \in \mathscr{K}'\} < f(\phi).$$

Let

$$T = \{\theta \in \mathscr{K} \mid f(\theta) = \operatorname{lub}\{f(\psi) \mid \psi \in \mathscr{K}\}\}.$$

Since $\mathscr{K}$ is a weak-* compact set and since f is weak-* continuous, the set T is a nonvoid weak-* compact subset of $\mathscr{K}$. We show that T is $\mathscr{Z}$-convex. Let P be a projection in $\mathscr{Z}$. We have that

$$(\psi(A)x, y) = (P\psi(A)x, y) + ((1-P)\psi(A)x, y)$$

for every $\psi \in \mathscr{A}^{\sim}$, $A \in \mathscr{A}$, and x and y in the Hilbert space of $\mathscr{A}$. Thus $f(\psi) = f(P\psi) + f((1-P)\psi)$ for every ψ in $\mathscr{A}^{\sim}$. Now let θ be an element of T. We have that $f(P\theta) = \operatorname{lub}\{f(P\psi) \mid \psi \in \mathscr{K}\}$. Indeed, if there is a ψ in $\mathscr{K}$ with $f(P\theta) < f(P\psi)$ we have that

$$f(P\psi + (1-P)\theta) = f(P\psi) + f((1-P)\theta) > f(P\theta) + f((1-P)\theta) = f(\theta).$$

However, the function at $P\psi + (1-P)\theta$ is an element of $\mathscr{K}$. We have reached a contradiction. So,

$$f(P\theta) = \operatorname{lub}\{f(P\psi) \mid \psi \in \mathscr{K}\}.$$

This means that $f(P\theta) = f(P\psi)$ for any two elements θ and ψ in T and any central projection P. Now let C be any element in $\mathscr{Z}^+$. Let $\varepsilon > 0$ be given; let $\{P_j \mid 1 \leqq j \leqq n\}$ be mutually orthogonal projections of $\mathscr{Z}$ and let $\{\alpha_j \mid 1 \leqq j \leqq m\}$ be nonnegative scalars such that $\|C - \sum \alpha_j P_j\| < \varepsilon$. If θ and ψ are elements of T, then

$$|f(C\theta) - f(C\psi)| \leqq \left| f(C\theta) - f\left(\left(\sum \alpha_j P_j\right)\theta\right)\right| + \left| f\left(\left(\sum \alpha_j P_j\right)\psi\right) - f(C\psi)\right| \leqq 2\varepsilon\|f\|.$$

Since ε is arbitrary, we see that $f(C\theta) = f(C\psi)$. Thus the set T is $\mathscr{Z}$-convex. Now by the remarks made at the beginning of this proof we can conclude that T has an extreme point ϕ_0. We show that ϕ_0 is an extreme point of $\mathscr{K}$. Indeed, let ϕ_1 and ϕ_2 be elements of $\mathscr{K}$ such that $C\phi_1 + (1-C)\phi_2 = \phi_0$ for some central element C strictly between 0 and 1. Let D be a positive central element; let $\varepsilon > 0$ be given and let $\{P_j \mid 1 \leqq j \leqq n\}$ be mutually orthogonal central projections such that $\|D - \sum \alpha_j P_j\| \leqq \varepsilon$ for suitable nonnegative scalars $\{\alpha_j \mid 1 \leqq j \leqq n\}$. Because

$$f(P_j\phi_0) = \operatorname{lub}\{f(P_j\theta) \mid \theta \in \mathscr{K}\} \quad \text{for } j = 1, 2, \ldots, n$$

we have that

$$f\left(\left(\sum \alpha_j P_j\right)\phi_1\right) = \sum \alpha_j f(P_j\phi_1) \leqq \sum \alpha_j f(P_j\phi_0) = f\left(\left(\sum \alpha_j P_j\right)\phi_0\right).$$

So we have that

$$f(D\phi_1) \leqq f\left(\left(\sum \alpha_j P_j\right)\phi_1\right) + \varepsilon\|f\| \leqq f\left(\left(\sum \alpha_j P_j\right)\phi_0\right) + \varepsilon\|f\| \leqq f(D\phi_0) + 2\varepsilon\|f\|.$$

Since $\varepsilon > 0$ is arbitrary, we have that $f(D\phi_1) \leqq f(D\phi_0)$. So for every central projection Q we may conclude that

$$f(CQ\phi_0) = f(CQ\phi_1) \quad \text{and} \quad f((1-C)Q\phi_0) = f((1-C)Q\phi_2),$$

since the sum of the two positive numbers

$$f(CQ\phi_0)-f(CQ\phi_1) \quad \text{and} \quad f((1-C)Q\phi_0)-f((1-C)Q\phi_2)$$

is zero. The elements C and $1-C$ are invertible in $\mathscr{Z}^+$. Given $\varepsilon>0$, there are mutually orthogonal central projections $\{Q_j \mid 1\leqq j\leqq n\}$ and nonnegative numbers $\{\alpha_j \mid 1\leqq j\leqq n\}$ such that $\|C^{-1}-\sum \alpha_j Q_j\|\leqq\varepsilon$. Therefore,

$$\begin{aligned}|f(\phi_1)-f(\phi_0)| &\leqq \left|f\big((1-(\textstyle\sum \alpha_j Q_j)C)\phi_1\big)\right|+\left|f\big(((\textstyle\sum \alpha_j Q_j)C-1)\phi_0\big)\right| \\ &\leqq 2\|f\| \left\|1-(\textstyle\sum \alpha_j Q_j)C\right\| \\ &\leqq 2\|f\| \, \|C\| \left\|C^{-1}-\textstyle\sum \alpha_j Q_j\right\| \leqq 2\|f\| \, \|C\|\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have that $f(\phi_1)=f(\phi_0)$. Similarly we find that $f(\phi_2)=f(\phi_0)$. This proves that both ϕ_1 and ϕ_2 are elements of T. Because ϕ_0 is an extreme point of T, the element ϕ_0 is equal to ϕ_1 and ϕ_2. Hence ϕ_0 is an extreme point of $\mathscr{K}$. However, ϕ_0 cannot be in the set $\mathscr{K}'$. This is a contradiction. Therefore, we must have that $\mathscr{K}=\mathscr{K}'$. Q.E.D.

In the final section of this paper we shall present some facts about the closure of the smallest $\mathscr{Z}$-convex subset of $\mathscr{S}$ containing the extreme points of $\mathscr{S}$ in the topology of pointwise convergence on $\mathscr{S}$ where $\mathscr{Z}$ is taken with the uniform topology.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. A positive functional ϕ of the module $\mathscr{A}$ is said to majorize a positive functional ψ if $\phi-\psi$ is a positive functional of the module $\mathscr{A}$. If ϕ majorizes ψ, we write $\phi\geqq\psi$. A positive functional ϕ is said to be $\mathscr{Z}$-irreducible if given any positive functional ψ majorized by ϕ then there is a positive element C in $\mathscr{Z}$ such that $C\phi=\psi$. In [12] we proved the following theorem:

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let $\mathscr{S}$ be the set of all positive functionals of the module $\mathscr{A}$ with norm not exceeding 1. *Let $\phi \in \mathscr{S}$. The following are equivalent*:

(1) *ϕ is an extreme point of $\mathscr{S}$*; *and*

(2) $\phi(1)$ *is a projection and ϕ is $\mathscr{Z}$-irreducible.*

3. **Functionals σ-weakly continuous on the center.** In this section we examine the positive functionals of a von Neumann algebra which are σ-weakly continuous when restricted to the center.

If f is a positive functional on a C^*-algebra $\mathscr{A}$, let L_f be the closed left-ideal of $\mathscr{A}$ given by

$$L_f = \{A \in \mathscr{A} \mid f(A^*A) = 0\}.$$

The space $\mathscr{A}-L_f$ is a prehilbert space with the inner product

$$(A-L_f, B-L_f) = f(B^*A).$$

Let $H(f)$ be the completion of $\mathscr{A}-L_f$. The representation π of $\mathscr{A}$ on $H(f)$ which extends the left multiplication of $\mathscr{A}$ on $\mathscr{A}-L_f$ is called the canonical representation of $\mathscr{A}$ induced by f. There is a vector x in $H(f)$ which is cyclic under $\pi(\mathscr{A})$ such that $(\pi(A)x, x)=f(A)$ for every A in $\mathscr{A}$.

THEOREM 3.1. *Let f be a positive functional of a von Neumann algebra $\mathscr{A}$. Suppose that the restriction g of f to the center $\mathscr{Z}$ of $\mathscr{A}$ is σ-weakly continuous. There is a unique positive functional ϕ of module $\mathscr{A}$ such that $f=g\cdot\phi$ and such that $\phi(1)$ is equal to the support of g.*

Proof. Let P be the support of g. Let π be the canonical representation of $\mathscr{A}$ on a Hilbert space H induced by f. Let x be an element of H cyclic under $\pi(\mathscr{A})$ such that $f(A)=(\pi(A)x, x)$ for every A in $\mathscr{A}$. The representation π restricted to $\mathscr{Z}$ is σ-weakly continuous. Indeed, let $\{A_n\}$ be a monotonely increasing net in $\mathscr{Z}^+$ with least upper bound A. Then $\{(A_n-A)^*(A_n-A)\}$ converges σ-weakly to 0. So

$$\{g((A_n-A)^*(A_n-A))\}$$

converges to 0. This means that $\lim \pi(A_n-A)x=0$. Therefore, $\lim \pi(A_n)Bx=\pi(A)Bx$ for every $B\in\pi(\mathscr{A})$. Since the net $\{\pi(A_n)\}$ is bounded, the net $\{\pi(A_n)\}$ converges strongly to $\pi(A)$. This proves π is σ-weakly continuous on $\mathscr{Z}$. This shows that $\pi(\mathscr{Z})$ is a von Neumann algebra on H [1, Chapter I, §3, Theorem 2, Corollary 2].

The algebra $\mathscr{Z}P$ is isomorphic to $\pi(\mathscr{Z})$ under the map π. Let π^{-1} denote the inverse of this map. Now let E be the abelian projection of the commutator $\pi(\mathscr{Z})'$ of $\pi(\mathscr{Z})$ on H corresponding to the subspace

$$\text{closure } \{Ax \mid A\in\pi(\mathscr{Z})\}.$$

We have that

$$f(A) = (\pi(A)x, x) = (\tau_E(\pi(A))x, x)$$

for every A in $\mathscr{A}$. Then define $\phi(A)=\pi^{-1}(\tau_E(\pi(A)))$. We have that ϕ is a positive functional of the module $\mathscr{A}$ such that $\phi(1)=P$. Also we see that

$$g(\phi(A)) = (\pi(\phi(A))x, x) = (\pi(A)x, x) = f(A)$$

for every A in $\mathscr{A}$.

Assume that ψ is a positive functional of the module $\mathscr{A}$ such that $g\cdot\psi=f$. If $P\psi\neq\phi$, then there is an element A in $\mathscr{A}^+$ such that $P\psi(A)\neq\phi(A)$. There is a nonzero projection Q in $\mathscr{Z}P$ and an $\varepsilon>0$ such that either

$$Q\psi(A)+\varepsilon Q \leqq \phi(A) \quad \text{or} \quad Q\phi(A)+\varepsilon Q \leqq Q\psi(A).$$

However, we have that $g(Q\phi(A))=g(Q\psi(A))$ and so $g(\varepsilon Q)=0$ in either case. This means $Q=0$. This is a contradiction. Therefore $P\psi=\phi$. Q.E.D.

A positive functional f of a C^*-algebra $\mathscr{A}$ with center $\mathscr{Z}$ is said to be centrally reducible if for every positive functional g of $\mathscr{A}$ majorized by f there is an element C in $\mathscr{Z}^+$ such that $f(CA)=g(A)$ for every A in $\mathscr{A}$. These centrally reducible functionals have been the object of much study ([5], [8], [25], [26]). The next theorem concerns these functionals.

THEOREM 3.2. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let f be a positive functional on $\mathscr{A}$ whose restriction g to the center $\mathscr{Z}$ is σ-weakly continuous. The functional f is centrally reducible if and only if the unique positive functional ϕ of the module $\mathscr{A}$ with $g \cdot \phi = f$ and with $\phi(1)$ equal to the support P of g is $\mathscr{Z}$-irreducible.*

Proof. Suppose f is centrally reducible. Let ψ be a positive functional of the module $\mathscr{A}$ which is majorized by ϕ. Then $g \cdot \psi$ is majorized by $g \cdot \phi$. There is a C in $\mathscr{Z}^+$ with $g(C\phi(A)) = g(\psi(A))$ for every A in $\mathscr{A}$. By the same argument as contained in Theorem 3.1, we find that $C\phi(A) = P\psi(A)$ for every A in $\mathscr{A}$. Because $0 \leqq \psi(1-P) \leqq \phi(1-P) = 0$ we have that $P\psi = \psi$. Therefore $C\phi = \psi$. This proves ϕ is $\mathscr{Z}$-irreducible.

Conversely, let ϕ be $\mathscr{Z}$-irreducible. Let h be a positive functional on $\mathscr{A}$ majorized by f. The restriction of h to the center of $\mathscr{A}$ is majorized by g. Therefore, h is weakly continuous on $\mathscr{Z}$. By the Radon-Nikodym theorem there is a positive element B in $\mathscr{Z}P$ such that $g(BA) = h(A)$ for every A in $\mathscr{Z}$. There is by Theorem 3.1 a positive functional of the module $\mathscr{A}$ such that $h \cdot \psi = h$. Thus $g(B\psi(A)) = h(A)$ for every A in $\mathscr{A}$. Hence, for every A in $\mathscr{A}^+$ we find that $\phi(A) - B\psi(A) \geqq 0$. This means that ϕ majorizes $B\psi$. There is a C in $\mathscr{Z}^+$ such that $C\phi = B\psi$. Thus we find that $f(CA) = h(A)$ for every A in $\mathscr{A}$. This proves f is centrally reducible. Q.E.D.

Now let f be a positive functional of the von Neumann algebra with center $\mathscr{Z}$. Suppose the restriction g of f to $\mathscr{Z}$ is weakly continuous. Let ν be the so-called spectral measure on the spectrum Z of $\mathscr{Z}$ such that $g(A) = \int A^\wedge(\zeta)\, d\nu(\zeta)$ for every $A \in \mathscr{Z}$. Here $A^\wedge$ denotes the Gelfand transform of A. Let ϕ denote the unique positive functional of the module $\mathscr{A}$ such that $\phi(1)$ is the support P of g and such that $f = g \cdot \phi$. Then $f(A) = \int \phi(A)^\wedge(\zeta)\, d\nu(\zeta)$. We note that

(1) $\{\zeta \in Z \mid P^\wedge(\zeta) = 1\}$ is the support of the spectral measure ν;

(2) $f_\zeta(A) = \phi(A)^\wedge(\zeta)$ is a positive functional of $\mathscr{A}$ whose kernel contains $[\zeta]$;

(3) for each fixed A in $\mathscr{A}$, the function $\zeta \to f_\zeta(A)$ is continuous on $\mathscr{Z}$.

In §4 we shall show that

(4) f_ζ is irreducible if ϕ is $\mathscr{Z}$-irreducible.

If ν is a spectra measure and $\{f_\zeta \mid \zeta \in Z\}$ is a family of functions satisfying properties (1)–(3) (respectively, (1)–(4)) then the relation $f(A) = \int f_\zeta(A)\, d\nu(\zeta)$ defines a positive functional (respectively, a centrally reducible functional) which is weakly continuous on $\mathscr{Z}$ [26].

4. **Representations on AW^*-modules.** In this section we study the representations induced by positive module homomorphisms. Our main result will be an analogue of Kadison's Theorem [13] on strictly irreducible representations.

Let $\mathscr{A}$ be a von Neumann algebra. A positive functional ϕ of the module $\mathscr{A}$ will be called a state (or expectation) of the module $\mathscr{A}$ if $\phi(1) = 1$. Then if ψ is a positive functional of the module $\mathscr{A}$, there is a state ϕ of the module $\mathscr{A}$ such that $\psi = \psi(1)\phi$ [19], [24]. A state ϕ of the module $\mathscr{A}$ is said to be a pure state if it is an extreme point of the set of positive functionals of norm not exceeding 1 of the module $\mathscr{A}$.

Proposition 4.1. *Let $\mathscr{A}$ be a von Neumann algebra. Let E be a projection in $\mathscr{A}$ and let P be the central support of E. There is a pure state of the module $\mathscr{A}$ such that $\phi(E)=P$.*

Proof. Let $\mathscr{Z}$ be the center of $\mathscr{A}$. The set $\mathscr{K}$ of states ϕ of the module $\mathscr{A}$ such that $\phi(E)=P$ is a $\mathscr{Z}$-convex weak*-compact subset of the set $\mathscr{S}$ of positive functionals of norm not exceeding 1 of the module $\mathscr{A}$. The set $\mathscr{K}$ is nonvoid. Indeed, let F_1 be an abelian projection in the commutator $\mathscr{Z}'$ of $\mathscr{Z}$ with central support P which is majorized by E. Let F_2 be an abelian projection in $\mathscr{Z}'$ of central support $1-P$. Then $F=F_1+F_2$ is an abelian projection of central support 1. This means that τ_F restricted to $\mathscr{A}$ is a state. Also $\tau_F(E)=P$, i.e. τ_F is an element of $\mathscr{K}$.

Let ϕ be an extreme point of $\mathscr{K}$ (Theorem 2.2). We show ϕ is an extreme point of $\mathscr{S}$. Let ϕ_1 and ϕ_2 be two functionals in $\mathscr{S}$ and let C be a central element strictly between 0 and 1 such that

$$C\phi_1+(1-C)\phi_2 = \phi.$$

We have that $\phi_j(1)\leqq 1$ and thus, $\phi_j(E)\leqq\phi_j(P)\leqq P$ $(j=1, 2)$. Therefore,

$$C\phi_1(1)+(1-C)\phi_2(1) = 1 \quad\text{and}\quad C\phi_1(E)+(1-C)\phi_2(E) = P$$

imply that $\phi_1(1)=\phi_2(1)=1$ and $\phi_1(E)=\phi_2(E)=P$. Thus, both ϕ_1 and ϕ_2 are elements of $\mathscr{K}$. Because ϕ is an extreme point of $\mathscr{K}$, we have that $\phi_1=\phi_2=\phi$. Q.E.D.

Let $\mathscr{A}$ be a von Neumann algebra and let ϕ be a state of $\mathscr{A}$. Let

$$L_\phi = \{A\in\mathscr{A} \mid \phi(A^*A) = 0\}.$$

The factor set $\mathscr{A}-L_\phi$ is a module over $\mathscr{Z}$ which is supplied with an inner product $(A-L_\phi, B-L_\phi)=\phi(B^*A)$ with values in $\mathscr{Z}$. The space $\mathscr{A}-L_\phi$ can then be embedded in a faithful AW^*-module M_ϕ over $\mathscr{Z}$ obtained by completing $\mathscr{A}-L_\phi$ in the following way. The set M_ϕ is the norm completion of the set of all $\{A_n-L_\phi, P_n\}_n$, where $\{P_n\}$ is a set of orthogonal central projections of sum 1 and $\{A_n-L_\phi\}$ is a set of elements of $\mathscr{A}-L_\phi$ with $\{\phi(A_n^*A_n)\}$ bounded in $\mathscr{Z}$, supplied with the norm induced by the inner product

$$\langle\{A_n-L_\phi, P_n\}, \{B_m-L_\phi, Q_m\}\rangle = \sum_{m,n} \phi(B_m^*A_n)P_nQ_m.$$

There is a bounded homomorphism π_ϕ of $\mathscr{A}$, which is also a module homomorphism over $\mathscr{Z}$, into the algebra $L(M_\phi)$ of all bounded module homomorphisms of M_ϕ onto itself that extends the left multiplication representation of $\mathscr{A}$ on $\mathscr{A}-L_\phi$. This map π_ϕ is called the canonical representation of $\mathscr{A}$ on M_ϕ induced by ϕ. For the operators T in $L(M_\phi)$ an involution $T\to T^*$ of $L(M_\phi)$ is defined. We have the relation $\langle TA, B\rangle=\langle A, T^*B\rangle$ for A and B in M_ϕ. The involution also satisfies the relation $\|T^*T\|=\|T\|^2$. Finally, the representation π_ϕ preserves adjoints in the sense that $\pi_\phi(A^*)=\pi_\phi(A)^*$ for every A in $\mathscr{A}$ ([17], [6], [28]).

If $\mathscr{Z}$ is a commutative von Neumann algebra on a Hilbert space H and if $\mathscr{Z}'$ is the commutator of $\mathscr{Z}$ on H, then for any abelian projection E of $\mathscr{Z}'$ of central support 1 the module $\mathscr{Z}'E$ is an AW^*-module over $\mathscr{Z}$. The inner product is defined to be $\langle A, B\rangle = \tau_E(B^*A)$ for A and B in $\mathscr{Z}'E$ [17].

A specific form for M_ϕ is now obtained.

PROPOSITION 4.2. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ and let ϕ be a state of the module $\mathscr{A}$. There is a Hilbert space H and a representation π of $\mathscr{A}$ on H with the following properties*:

(1) *π is faithful on $\mathscr{Z}$*;

(2) *$\pi(\mathscr{Z})$ is a von Neumann algebra on H*;

(3) *there is an abelian projection E in the commutator $\pi(\mathscr{Z})'$ of $\pi(\mathscr{Z})$ on H such that $\pi(\phi(A)) = \tau_E(\pi(A))$; and*

(4) *there is a function Φ of M_ϕ onto the completion of the module $\pi(\mathscr{A})E$ in $\pi(\mathscr{Z})'E$ such that*

$$\Phi(A_1B_1 + A_2B_2) = \pi(A_1)\Phi(B_1) + \pi(A_2)\Phi(B_2)$$

for every A_1 and A_2 in $\mathscr{Z}$ and every B_1 and B_2 in M_ϕ;

$$\pi(\langle A, B\rangle) = \langle \Phi(A), \Phi(B)\rangle$$

for every A and B in M_ϕ; and $\Phi(\pi_\phi(A)B) = \pi(A)\Phi(B)$ for every A in $\mathscr{A}$ and B in M_ϕ. If ϕ is a pure state of the module $\mathscr{A}$, then

(5) *the commutator of $\pi(\mathscr{A})$ on H is equal to $\pi(\mathscr{Z})$.*

Proof. Let $\{P_n\}$ be a set of nonzero mutually orthogonal projections of $\mathscr{A}$ with sum equal to 1 such that each algebra $\mathscr{Z}P_n$ is σ-finite. Let x_n be a unit vector of the Hilbert space of $\mathscr{Z}P_n$ which separates $\mathscr{Z}P_n$ [1, I, §2, No. 1]. Let π_n be the canonical representation of $\mathscr{A}$ on the Hilbert space H_n induced by the positive functional $w_{x_n}\cdot\phi$ of $\mathscr{A}$. Here $w_x(A) = (Ax, x)$ for any vector x. Let y_n be a vector in H_n cyclic under $\pi_n(A)$ such that

$$(\pi_n(A)y_n, y_n) = w_{x_n}(\phi(A)).$$

Let π be the representation $\pi = \sum \oplus \pi_n$ on the Hilbert space $H = \sum \oplus H_n$.

We show that π is faithful on $\mathscr{Z}$. Indeed, if $A \in \mathscr{Z}$ and $\pi(A) = 0$, then $\pi(AP_n) = 0$ for every n. This means $\pi_n(AP_n) = 0$. However, the representation π_n is faithful on $\mathscr{Z}P_n$; hence $AP_n = 0$ for every n. This means $A = 0$. Thus π is faithful on $\mathscr{Z}$.

We prove now that π is σ-weakly continuous when restricted to $\mathscr{Z}$. Let $\{A_m\}$ be a monotonely increasing net $\mathscr{Z}^+$ with least upper bound A. We have (Proposition 3.1) that $\{\pi_n(A_m)\}_m$ converges strongly to $\pi_n(A)$ for each n. Now let x be an element in H and let $\varepsilon > 0$ be given. There is a finite subset $P_1, P_2, \ldots, P_k$ of $\{P_n\}$ of sum P such that $\|x - \pi(P)x\| \leqq \varepsilon$ because each $\pi(P_n)$ is the projection of H on H_n. Suppose that for $m \geqq m_0$ we have that

$$\|(\pi_j(A_m) - \pi_j(A))\pi(P_j)x\| \leqq \varepsilon k^{-1} \quad \text{for } j = 1, \ldots, k.$$

Then

$$\begin{aligned}\|\pi(A)x-\pi(A_m)x\| &\leqq \|(\pi(A)-\pi(A_m))(1-\pi(P))x\|+\|(\pi(A)-\pi(A_m))\pi(P)x\| \\ &\leqq 2\|A\|\varepsilon+\sum\{\|(\pi_j(A)-\pi_j(A_m))\pi_j(P_j)x\| \mid 1 \leqq j \leqq k\} \\ &\leqq (2\|A\|+1)\varepsilon.\end{aligned}$$

This proves that π is a σ-weakly continuous isomorphism of $\mathscr{Z}$.

By the proof of Theorem 3.1 there is for each n an abelian projection E_n' in the commutator $\pi_n(\mathscr{Z}P_n)'$ on H_n associated with the subspace

$$\text{closure}\,\{\pi(A)y_n \mid A \in \mathscr{Z}\}$$

such that

$$\tau_{E_n'}(\pi_n(A)) = \pi_n(\phi(AP_n)).$$

Since $\pi(\mathscr{Z})'\pi(P_n)$ is the commutator of $\pi(\mathscr{Z})\pi(P_n)$ on H_n, we have that there is an abelian projection E_n in the von Neumann algebra $\pi(\mathscr{Z})'$ on H majorized by $\pi(P_n)$ such that

$$\tau_{E_n}(\pi(AP_n)) = \pi(\phi(AP_n)).$$

Let E be the abelian projection in $\pi(\mathscr{Z})'$ given by $E=\sum E_n$. Then

$$\begin{aligned}\tau_E(\pi(A))\pi(P_n) &= \tau_{E_n}(\pi(AP_n)) = \pi(\phi(AP_n)) \\ &= \pi(\phi(A))\pi(P_n) \quad \text{for every } n.\end{aligned}$$

This proves that $\tau_E(\pi(A))=\pi(\phi(A))$ for every A in $\mathscr{A}$.

Let $\{A_n-L_\phi \mid n\in N\}$ and $\{B_m-L_\phi \mid m \in N'\}$ be two bounded sets in $\mathscr{A}-L_\phi$ and let $\{Q_n \mid n\in N\}$ and $\{R_m \mid m\in N'\}$ be two sets of mutually orthogonal central projections of sum 1 respectively. Then $\sum \pi(Q_n)\pi(A_n)E$ and $\sum \pi(R_m)\pi(B_m)E$ are elements of the AW^*-module $\pi(\mathscr{Z})'E$. We have that

$$\begin{aligned}\pi\left(\left\langle\sum Q_n(A_n-L_\phi), \sum R_m(B_m-L_\phi)\right\rangle\right) &= \pi\Big(\sum_{m,n} Q_nR_m\phi(B_m^*A_n)\Big) \\ &= \left\langle\sum \pi(Q_n)\pi(A_n)E, \sum \pi(R_m)\pi(B_m)E\right\rangle\end{aligned}$$

in the respective inner products of M_Φ and $\pi(\mathscr{Z})'E$. Therefore,

$$\Phi\left(\sum Q_n(A_n-L_\phi)\right) = \sum \pi(Q_n)\pi(A_n)E$$

defines a function of a uniformly dense submodule

$$\begin{aligned}M_1 = \Big\{\sum Q_n(A_n-L_\phi) \mid \{Q_n\} &\text{ is a set of mutually orthogonal} \\ &\text{central projections of sum } 1; \\ &\{A_n-L_\phi\} \text{ is a bounded set in } \mathscr{A}-L_\phi\Big\}\end{aligned}$$

of the module M_Φ into the submodule

$$M_2 = \Big\{\sum \pi(Q_n)\pi(A_n)E \mid \{\pi(Q_n)\} \text{ is a set of mutually orthogonal projections of } \pi(\mathscr{Z});\ \{\pi(A_n)E\} \text{ is a bounded subset of } \pi(\mathscr{A})E\Big\}$$

of the module $\pi(\mathscr{Z})'E$.

We have that Φ is a linear function of M_1 into M_2 such that $\Phi(AB)=\pi(A)\Phi(B)$ for every A in $\mathscr{Z}$ and B in M_1. The range of Φ is M_2. There is a unique extension of Φ to a map which we again call Φ of the norm completion M_ϕ of M_1 onto the closure of M_2 in $\pi(\mathscr{Z})'E$ such that

$$\Phi(A_1B_1+A_2B_2) = \pi(A_1)\Phi(B_1)+\pi(A_2)\Phi(B_2)$$

for every A_1 and A_2 in $\mathscr{Z}$ and B_1 and B_2 in M_ϕ and such that

$$\langle \Phi(A), \Phi(B)\rangle = \pi(\langle A, B\rangle)$$

for every A and B in M_ϕ. Since the closure of M_2 is precisely the AW^*-module generated by $\pi(\mathscr{A})E$ in $\pi(\mathscr{Z})'E$ [6, Lemma 4.1], we have that the range of Φ is the AW^*-module generated by $\pi(\mathscr{A})E$.

Finally, let $\{A_n-L_\phi\}$ be a bounded set in $\mathscr{A}-L_\phi$ and let $\{Q_n\}$ be a set of mutually orthogonal central projections of sum 1. Then

$$\begin{aligned}\Phi\big(\pi_\phi(A)\big(\sum Q_n(A_n-L_\phi)\big)\big) &= \Phi\big(\sum Q_n(AA_n-L_\phi)\big)\\ &= \sum \pi(Q_n)\pi(AA_n)E = \pi(A)\sum \pi(Q_n)\pi(A_n)E\\ &= \pi(A)\Phi\big(\sum Q_n(A_n-L_\phi)\big)\end{aligned}$$

for every A in $\mathscr{A}$. Thus we have that $\Phi(\pi_\phi(A)B)=\pi(A)\Phi(B)$ for every A in $\mathscr{A}$ and B in M_ϕ. This completes the proof of (4).

Now assume ϕ is a pure state. Let η be the inverse of π restricted to $\mathscr{Z}$. Let A be a positive element in the unit sphere of the commutator, $\pi(\mathscr{A})'$ of $\pi(\mathscr{A})$ on H. Let $\tau=\tau_E$. The relation

$$\eta(\tau(A\pi(B))) = \psi(B)$$

defines a functional of the module $\mathscr{A}$. For every B in $\mathscr{A}$ we have that

$$\psi(B^*B) = \eta(\tau(A^{1/2}\pi(B^*B)A^{1/2})) \geqq 0$$

and

$$\begin{aligned}\psi(B^*B) &= \eta(\tau(\pi(B^*B)^{1/2}A\pi(B^*B)^{1/2}))\\ &\leqq \eta(\tau(\pi(B^*B)))\|A\| \leqq \phi(B^*B).\end{aligned}$$

So ψ is a positive functional majorized by ϕ. There is a C in $\mathscr{Z}^+$ such that $C\phi=\psi$ (cf. §2). So for every B_1 and B_2 in $\mathscr{A}$ we have that

$$\tau(\pi(B_2)^*(A-\pi(C))\pi(B_1)) = 0.$$

This means that

$$((A-\pi(C))\pi(B_1)y_n, \pi(B_2)y_m) = 0$$

for every y_n and y_m. However, the closure of the linear span of

$$\{\pi(B)y_n \mid B \text{ in } \mathscr{A}, \text{ all } y_n\}$$

is H. Thus $A=\pi(C)$. Therefore $\pi(\mathscr{A})'$ is equal to $\pi(\mathscr{Z})$. Q.E.D.

Before continuing we present a brief discussion of a certain trace that is particularly useful. Let $\mathscr{A}$ be a type I algebra with center $\mathscr{Z}$. There is a locally compact space X and a positive measure ν on X of support X such that $\mathscr{Z}$ is isometric *-isomorphic to the algebra $L^\infty(X, \nu)$ of all essentially bounded complex-valued measurable functions on X. Identify $\mathscr{Z}$ with $L^\infty(X, \nu)$. There is a function Tr of $\mathscr{A}^+$ into the set of all positive finite or infinite valued measurable functions on X with the following properties:

(1) $\mathrm{Tr}\,(C_1A_1+C_2A_2)=C_1\,\mathrm{Tr}\,(A_1)+C_2\,\mathrm{Tr}\,(A_2)$ for C_1, C_2 in $\mathscr{Z}^+$ and A_1, A_2 in $\mathscr{A}^+$;

(2) $\mathrm{Tr}\,(U^*AU)=\mathrm{Tr}\,(A)$ for every A in $\mathscr{A}^+$ and every unitary U in $\mathscr{A}$;

(3) if $\{A_n\}$ is a monotonely increasing net in $\mathscr{A}^+$ with least upper bound A, then $\{\mathrm{Tr}\,(A_n)\}$ has least upper bound $\mathrm{Tr}\,(A)$;

(4) $\mathrm{Tr}\,(E)=\tau_E(E)$ for every abelian projection E in $\mathscr{A}$.

If $\mathscr{P}=\{A\in\mathscr{A}^+ \mid \mathrm{Tr}\,(A)\in\mathscr{Z}^+\}$, then $\mathscr{P}$ is the set of all positive elements of a two-sided ideal $\mathscr{T}$ in $\mathscr{A}$ called the trace class of $\mathscr{A}$. In particular every abelian projection is a member of $\mathscr{T}$. The function Tr on $\mathscr{P}=\mathscr{T}\cap\mathscr{A}^+$ may be extended to a linear function Tr of $\mathscr{T}$ into $\mathscr{Z}$ which is also a module homomorphism. For every $A\in\mathscr{T}$ the function $B\to\mathrm{Tr}\,(AB)$ is a function of $\mathscr{A}^\sim$ which is also continuous in the respective σ-weak topologies. We have that $\mathrm{Tr}\,(BA)=\mathrm{Tr}\,(AB)$ for every A in $\mathscr{T}$ and B in $\mathscr{A}$. Also we have that $\|B\|^2\leqq\|\mathrm{Tr}\,(B^*B)\|$ for every B in $\mathscr{T}$ [9, §4].

Let M be an AW^*-module over the commutative AW^*-algebra $\mathscr{Z}$ and let $\mathscr{B}$ be a subalgebra of the algebra $L(M)$ of all bounded linear operators on M. The algebra $\mathscr{B}$ is said to be irreducible on M if given A in $L(M)$ and $C_1, C_2, \dots, C_n$ in M then there is a B in $\mathscr{B}$ such that $BC_j=AC_j$ for every $j=1, 2, \dots, n$.

THEOREM 4.3. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let ϕ be a pure state of the module $\mathscr{A}$. Then the module M_ϕ induced by ϕ is equal to $\mathscr{A}-L_\phi$ and $\pi_\phi(\mathscr{A})$ is irreducible on M_ϕ.*

Proof. Let π be the representation relative to ϕ of $\mathscr{A}$ on the Hilbert space H constructed in Proposition 4.2. Then π enjoys properties (1)–(5) of this proposition. Let E be an abelian projection of the commutator $\mathscr{B}$ of $\pi(\mathscr{Z})$ on H such that $\tau_E(\pi(A))=\pi(\phi(A))$. We show that $\pi(\mathscr{A})E=\mathscr{B}E$. This means that the module M_ϕ is $\mathscr{A}-L_\phi$. The algebra of all bounded linear operators on $\mathscr{B}E$ is identified with $\mathscr{B}$ acting on $\mathscr{B}E$ by left multiplication [17, Theorem 8]. Given $B_1, B_2, \dots, B_m$ and B in $\mathscr{B}$ we show that there is an A in $\mathscr{A}$ with $\pi(A)B_jE=BB_jE$ for $j=1, 2, \dots, m$. We may also show that A can be chosen to be self-adjoint if B is self-adjoint. The proof essentially consists of showing that E is a regular projection with respect to $\pi(\mathscr{A})$ [27] using a construction known for pure states (cf. [2, §2.8]).

As a preliminary step assume that $B_1E, B_2E, \ldots, B_mE$ are partial isometric operators $V_1, V_2, \ldots, V_m$ respectively. Assume also that the range projections $F_1, F_2, \ldots, F_m$ of the $V_1, V_2, \ldots, V_m$ are mutually orthogonal. We show that there is an element B' in $\mathscr{B}$ such that $B'V_i = BV_i$ $(1 \leqq i \leqq m)$ and such that $\|B'\|^2 \leqq 2 \sum \|V_i^*B^*BV_i\|$. We show that B' may be chosen to be self-adjoint if B is self-adjoint. Let G_i be the range projection of BV_i $(1 \leqq i \leqq m)$. Since G_i is equivalent to the domain projection of BV_i, which is majorized by E, the projection G_i is abelian (cf. [1, III, §1]). Let G be the least upper bound of the set

$$\{F_i \mid 1 \leqq i \leqq m\} \cup \{G_i \mid 1 \leqq i \leqq m\}.$$

The projection $G - \sum F_i$ may be written as the sum of mutually orthogonal abelian projections $F_{m+1}, F_{m+2}, \ldots, F_p$ (cf. [9, Theorem 2.1]). Let

$$B' = \sum \{F_jBF_i \mid 1 \leqq i \leqq m;\ 1 \leqq j \leqq p\}$$

if B is not self-adjoint and let

$$B' = \sum \{F_jBF_i \mid 1 \leqq i \leqq p;\ 1 \leqq j \leqq m\} + \sum \{F_jBF_i \mid 1 \leqq i \leqq m;\ m+1 \leqq j \leqq p\}$$

if B is self-adjoint. In this case B' is self-adjoint. In either case

$$B'V_i = \sum \{F_jBV_i \mid 1 \leqq j \leqq p\} = BV_i$$

for $i = 1, 2, \ldots, m$. In the first case

$$\begin{aligned} \mathrm{Tr}\,(B'^*B') &= \sum \{\mathrm{Tr}\,(F_iB'^*B'F_i) \mid 1 \leqq i \leqq m\} \\ &= \sum \{\tau_{F_i}(B'^*B') \mid 1 \leqq i \leqq m\}. \end{aligned}$$

In the second case we have that

$$\begin{aligned} \mathrm{Tr}\,(B'^*B') = \mathrm{Tr}\,(B'^2) &= \sum \{\mathrm{Tr}\,(F_iB'^2F_i) \mid 1 \leqq i \leqq m\} \\ &\quad + \sum \{\mathrm{Tr}\,(F_jB'F_iB'F_j) \mid m+1 \leqq i \leqq p;\ 1 \leqq j \leqq p\} \\ &= \sum \{\mathrm{Tr}\,(F_iB'^2F_i) \mid 1 \leqq i \leqq m\} \\ &\quad + \sum \{\mathrm{Tr}\,(F_jB'F_iB'F_j) \mid m+1 \leqq i \leqq p;\ 1 \leqq j \leqq m\} \\ &\leqq 2 \sum \{\tau_{F_i}(B'^2) \mid 1 \leqq i \leqq m\} \end{aligned}$$

since $F_jB'(\sum \{F_i \mid m+1 \leqq i \leqq p\})B'F_j \leqq F_jB'^2F_j$.

We have that

$$\|\tau_{F_i}(B'^*B')\| = \|F_iB'^*B'F_i\| = \|V_i^*B'^*B'V_i\|.$$

Thus in either case we conclude that

$$\|B'^*B'\| \leqq \|\mathrm{Tr}\,(B'^*B')\| \leqq 2 \sum \|V_i^*B^*BV_i\|.$$

This verifies the existence of B' in $\mathscr{B}$. So we may assume that

$$\|B\| \leqq (2m)^{1/2}\alpha \quad \text{where } \alpha = \max\{\|BV_i\| \mid 1 \leqq i \leqq m\}.$$

By an application of Tomita's results [27, Theorem 6] we may find a nonzero projection F in $\mathscr{B}$ majorized by E and an element A in $\pi(\mathscr{A})$ such that $\|A\| \leqq 2(2m)^{1/2}\alpha$ and $AV_jF = BV_jF$ for $j=1, 2, \ldots, m$. Indeed given a unit vector x in the Hilbert space of $\mathscr{B}$ such that $Ex=x$, then we may construct by induction a decreasing sequence $\{F'_n\}$ of abelian projections and a sequence of elements $\{A_n\}$ in $\pi(\mathscr{A})$ such that

(1) $\|F'_n x - F'_{n+1}x\| \leqq 4^{-n+1}$ and $\|x - F'_1 x\| \leqq 4^{-1}$;

(2) $\|A_n\| \leqq 2^{-n+1}(2m)^{1/2}\alpha$; and

(3) $\operatorname{lub}\{\|(\sum\{A_j : 1 \leqq j \leqq n\} - B)V_iF'_n\| : 1 \leqq i \leqq m\} \leqq 2^{-n}\alpha$ for every $n=1, 2, \ldots$.

Then $A = \sum A_n$ and $F = \operatorname{glb} F'_n \neq 0$. If B is self-adjoint then A may be chosen self-adjoint. Let $\{P_n \mid n \in D\}$ be a maximal set of mutually orthogonal nonzero projections in $\pi(\mathscr{Z})$ with the property: for each P_n there is an element A_n in $\pi(\mathscr{A})P_n$ such that $\|A_n\| \leqq 2(2m)^{1/2}\alpha$ and such that $A_nV_jE = BV_jEP_n$. We see that $\sum P_n = 1$; otherwise, the projection $P = 1 - \sum P_n$ is nonzero. There is a nonzero projection F majorized by EP and an element A in $\pi(\mathscr{A})$ such that $\|A\| \leqq 2(2m)^{1/2}\alpha$ and $AV_jF = BV_jF$. But there is a nonzero projection Q in $\pi(\mathscr{Z})$ majorized by P such that $QE = F$. This contradicts the maximality of the set $\{P_n\}$. Therefore, the least upper bound of the set $\{P_n\}$ is 1. There is a set $\{Q_n \mid n \in D\}$ of mutually orthogonal projections in $\mathscr{Z}$ such that $\pi(Q_n) = P_n$ for each $n \in D$. Since π is norm decreasing, there is for each A_n an element B_n in $\mathscr{A}P_n$ of norm not exceeding $3(2m^{1/2})\alpha$ such that $\pi(B_n) = A_n$. There is an A in $\mathscr{A}$ such that $AQ_n = B_n$ for each n in D. For each $j=1, 2, \ldots, m$ we have $\pi(A)V_jE = BV_jE$ because $\pi(A)V_jEP_n = BV_jEP_n$ for every n in D.

Let us now assume that $B_1E, B_2E, \ldots, B_mE$ are arbitrary. Let $F_1, F_2, \ldots, F_m$ be the range projections of $B_1E, B_2E, \ldots, B_mE$ respectively. Let F be the least upper bound of $F_1, F_2, \ldots, F_m$. There are mutually orthogonal abelian projections $G_1, G_2, \ldots, G_p$ of sum F. Let $V_1, V_2, \ldots, V_p$ be partial isometries with range support $G_1, G_2, \ldots, G_p$ respectively and domain support majorized by E (cf. [1, Chapter III, §3, Lemma 1]). By the first part of the proof there is an element A in $\mathscr{A}$, which may be chosen to be self-adjoint if B is self-adjoint, such that $\pi(A)V_j = BV_j$ $(1 \leqq j \leqq p)$. We have that $GB_jE = B_jE$ $(1 \leqq j \leqq m)$. So

$$\begin{aligned} B_jE &= \sum\{G_kB_jE \mid 1 \leqq k \leqq p\} = \sum\{V_kV_k^*B_jE \mid 1 \leqq k \leqq p\} \\ &= \sum\{\tau_E(V_k^*B_j)V_k \mid 1 \leqq k \leqq p\}. \end{aligned}$$

Thus, we obtain

$$BB_jE = \sum \tau_E(V_k^*B_j)BV_k = \sum \tau_E(V_k^*B_j)\pi(A)V_k = \pi(A)B_jE$$

for $j=1, 2, \ldots, m$. Q.E.D.

In the corollary we use the following ideas. Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$; let ζ be a maximal ideal of $\mathscr{Z}$. The smallest closed two-sided ideal of

$\mathscr{A}$ containing ζ is denoted by $[\zeta]$. Then $[\zeta]$ is the closure of the set

$$\left\{\sum \{A_i B_i \mid 1 \leqq i \leqq n\} \mid A_i \in \zeta, B_i \in \mathscr{A}\ (1 \leqq i \leqq n); n = 1, 2, \dots\right\}.$$

Let $\mathscr{A}(\zeta)$ be the factor C^*-algebra $\mathscr{A}/[\zeta]$ and let $A(\zeta)$ denote the image of A in $\mathscr{A}(\zeta)$. Then J. Glimm proved that for each fixed A in $\mathscr{A}$ the function $\zeta \to \|A(\zeta)\|$ is continuous on the spectrum of $\mathscr{Z}$ [3, Lemma 10]. If P is a projection of $\mathscr{Z}$, then

$$\|AP\| = \text{lub}\,\{\|A(\zeta)\| \mid \zeta \text{ in the spectrum of } \mathscr{Z} \text{ and } P^\frown(\zeta) = 1\}.$$

The least upper bound is attained. If $A(\zeta)$ is a positive element in $\mathscr{A}(\zeta)$ for each ζ, then A is positive in $\mathscr{A}$.

Now assume $\mathscr{A}$ is a type I algebra. Let the notation be the same as the preceding paragraph. Let ζ be a fixed maximal ideal of $\mathscr{Z}$. Suppose E is an abelian projection in $\mathscr{A}$ such that $E(\zeta) \neq 0$. The space $H(\zeta) = \mathscr{A}E(\zeta)$ is a Hilbert space with the inner product $\langle AE(\zeta), BE(\zeta)\rangle = \tau_E(B^*A)^\frown(\zeta)$. The algebra $\mathscr{A}$ has a representation Ψ with kernel $[\zeta]$ on the algebra of all bounded operators on $H(\zeta)$ given by $\Psi(A)BE(\zeta) = AEB(\zeta)$, for every A and B in $\mathscr{A}$. The closed two-sided ideal I_a of $\mathscr{A}$ generated by the abelian projections of $\mathscr{A}$ maps onto the ideal of completely continuous operators of $H(\zeta)$. In particular if x is an arbitrary vector in $H(\zeta)$ there is an abelian projection F in $\mathscr{A}$ such that $\Psi(F)x = x$. The images of abelian projections under Ψ have dimension not exceeding 1 [3, §4].

Corollary. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ and let ϕ be a $\mathscr{Z}$-irreducible functional of the module $\mathscr{A}$. For every ζ in the spectrum of $\mathscr{Z}$ the functional $\phi(A)^\frown(\zeta)$ of $\mathscr{A}$ is irreducible. In particular if ϕ is an extreme point of the set of positive functionals of norm not exceeding* 1 *of the module $\mathscr{A}$, then $\phi(A)^\frown(\zeta)$ is irreducible on $\mathscr{A}$.*

Proof. We may assume that $\phi(1)^\frown(\zeta) \neq 0$. There is a projection P in $\mathscr{Z}$ which does not lie in the maximal ideal ζ of $\mathscr{Z}$ and a number $\alpha > 0$ such that $\phi(1)P \geqq \alpha P$. Let C be a positive element in $\mathscr{Z}P$ such that $C\phi(1) = P$. The functional $\psi = C\phi$ is a $\mathscr{Z}$-irreducible functional of the module $\mathscr{A}$. Indeed, if ψ majorizes the positive functional θ of the module $\mathscr{A}$, then $P\phi$ majorizes $P\phi(1)\theta$ and so ϕ majorizes $P\phi(1)\theta$. There is a D in $\mathscr{Z}^+$ such that $D\phi = P\phi(1)\theta$. Thus $D\psi = CD\phi = CP\phi(1)\theta = \theta$. This proves that ψ is $\mathscr{Z}$-irreducible. Since the functional $\psi(A)^\frown(\zeta)$ is equal to a nonzero scalar multiple $C^\frown(\zeta)$ of $\phi(A)^\frown(\zeta)$, it is sufficient to prove that $\psi(A)^\frown(\zeta)$ is irreducible.

Now let ψ_1 be any pure state of the module $\mathscr{A}$. The functional $P\psi + (1-P)\psi_1 = \psi'$ is a $\mathscr{Z}$-irreducible state of the module $\mathscr{A}$. Indeed, if θ is a positive functional of the module $\mathscr{A}$ majorized by ψ', then $P\psi = \psi$ majorizes $P\theta$ and $(1-P)\psi_1$ majorizes $(1-P)\theta$. There are elements C_1 and C_2 in $\mathscr{Z}^+$ with $C_1\psi = P\theta$ and $C_2\psi_1 = (1-P)\theta$. We may assume that $PC_1 = C_1$ and $(1-P)C_2 = C_2$. Setting $C = C_1 + C_2$ we have that $C\psi' = \theta$. So ψ' is a $\mathscr{Z}$-irreducible state of the module $\mathscr{A}$, i.e. ψ' is a pure state of $\mathscr{A}$. Since $\psi'(A)^\frown(\zeta) = \psi(A)^\frown(\zeta)$ for every A in $\mathscr{A}$ there is no loss of generality in assuming that ψ is a pure state of the module $\mathscr{A}$.

Let π be a representation of $\mathscr{A}$ on a Hilbert space H constructed in Proposition 4.2 relative to ϕ. Let E be a maximal abelian projection of the von Neumann algebra $\mathscr{B}$ generated by $\pi(\mathscr{A})$ on H such that $\tau_E(\pi(A))=\pi(\phi(A))$ for every A in $\mathscr{A}$. There is a homeomorphism η of the spectrum Z of the center $\mathscr{Z}$ of $\mathscr{A}$ onto the spectrum of Z_1 of $\pi(\mathscr{Z})$ such that $\pi(A)\hat{}(\eta(\zeta))=A\hat{}(\zeta)$ for every $\zeta\in Z$. Let ζ be a fixed element in Z and let $\eta(\zeta)=\zeta'$. Then

$$\phi(A)\hat{}(\zeta) = \tau_E(\pi(A))\hat{}(\zeta').$$

There is a homomorphism Ψ of $\mathscr{B}$ with kernel $[\zeta']$ into the algebra of all bounded linear operators on the Hilbert space $H(\zeta')=\mathscr{B}E(\zeta')$. The ideal generated by the set of all abelian projections of $\mathscr{B}$ maps onto the set of all completely continuous operators of $H(\zeta')$ under Ψ. Let $x_1, x_2,\dots, x_m$ be elements of $\mathscr{B}E(\zeta')$. There are elements $B_1, B_2,\dots, B_m$ in $\mathscr{B}$ with $x_j=B_jE(\zeta')$ for $j=1, 2,\dots, m$. Let B be an element in $\mathscr{B}$. There is an element A in $\pi(\mathscr{A})$ such that $AB_jE=BB_jE$ for $j=1, 2,\dots, m$ (Theorem 4.3). This means $\Psi(A)x_j=\Psi(B)x_j$ for $j=1, 2,\dots, m$. This proves that $\Psi(\pi(\mathscr{A}))$ is irreducible on $H(\zeta')$. Let x be the vector $E(\zeta')$ in $H(\zeta')$. We have that

$$\phi(A)\hat{}(\zeta) = \tau_E(\pi(A))\hat{}(\zeta') = (\Psi(\pi(A))x, x)$$

for every A in $\mathscr{A}$. This proves that $\phi(A)\hat{}(\zeta)$ is irreducible on $\mathscr{A}$. Q.E.D.

We now record some facts about the kernel of π_ϕ.

PROPOSITION 4.4. *Let $\mathscr{A}$ be a von Neumann algebra and let ϕ be a state of the module $\mathscr{A}$. The kernel of π_ϕ is contained in the strong radical (viz, the intersection of all two-sided maximal ideals) of $\mathscr{A}$. In particular, if $\mathscr{A}$ is finite or if $\mathscr{A}$ is σ-finite and of type* III, *then π_ϕ is faithful.*

Proof. Let A be an element of $\mathscr{A}$. Let $\mathscr{K}_A'$ be the uniform closure of the set

$$\Big\{\sum \{\alpha_i U_i^* A U_i \mid i = 1, 2,\dots, n\} \mid \alpha_1, \alpha_2,\dots, \alpha_n \text{ are positive of sum } 1;$$
$$U_1, U_2,\dots, U_n \text{ are unitary in } \mathscr{A};\ n = 1, 2,\dots\Big\}.$$

Then $\mathscr{K}_A'\cap\mathscr{Z}=\mathscr{K}_A$ is nonvoid for every A in $\mathscr{A}$. If $\mathscr{A}$ is finite, then $\mathscr{K}_A$ contains a single element $A^{\#}$. In this case if $A\in\mathscr{A}^+$ and $A^{\#}=0$, then $A=0$ [1, III, §5].

Assume first that $\mathscr{A}$ is finite. Set $\pi_\phi=\pi$ and let A be an element of $\mathscr{A}$ such that $\pi(A)=0$; then $\pi(A^*A)=0$. Since

$$\pi\Big(\sum \alpha_i U_i^* A^*A U_i\Big) = \sum \alpha_i\pi(U_i^*)\pi(A^*A)\pi(U_i) = 0$$

and since π is uniformly continuous, we have that $\pi((A^*A)^{\#})=0$. This means

$$0 = \phi((A^*A)^{\#}) = (A^*A)^{\#}.$$

Therefore, $A^*A=0$ and thus π is faithful.

Now assume that $\mathscr{A}$ is properly infinite. The radical of $\mathscr{A}$ is the ideal of $\mathscr{A}$ all of whose positive elements A satisfy the relation $\mathscr{K}_A=\{0\}$, [10, Proposition 2.4]. Therefore we readily conclude that $\pi(A)=0$ implies that A is in the radical of $\mathscr{A}$.

Now in the general case there is a projection P in the center of $\mathscr{A}$ such that $\mathscr{A}P$ is finite and $\mathscr{A}(1-P)$ is properly infinite. If A is an element in the kernel of π, then $AP=0$ and $A(1-P)$ is in the radical of $\mathscr{A}(1-P)$. But the radical of $\mathscr{A}(1-P)$ is the radical of $\mathscr{A}$. So the kernel of π is contained in the radical of $\mathscr{A}$. Q.E.D.

We now show that there are states which have faithful representations.

PROPOSITION 4.5. *Let $\mathscr{A}$ be a von Neumann algebra. There is a projection E in $\mathscr{A}$ of central support* 1 *such that every state ϕ of the module $\mathscr{A}$ with the property $\phi(E)=1$ has a faithful representation π_ϕ.*

Proof. First let $\mathscr{A}$ be semifinite. Let E be any finite projection of $\mathscr{A}$ of central support 1. Then let ϕ be a state of $\mathscr{A}$ such that $\phi(E)=1$. Let F be a projection of $\mathscr{A}$ with $\pi(F)=0$ where $\pi=\pi_\phi$. Suppose F has central support P. First assume that $F\leqq EP$. Since EP is finite, there is a set $\{P_i\}$ of mutually orthogonal central projections of sum P such that for each P_i there is a set

$$\{F_{ij} \mid 1 \leqq j \leqq n_i < +\infty\}$$

of mutually orthogonal projections with the properties:

$$F_{ij} \sim FP_i \quad \text{and} \quad F_i' = EP_i - \sum_j F_{ij} \prec FP_i \qquad [1, \text{III}, \S 1].$$

Since $\pi(FP_i)=0$ we have that $\pi(F_{ij})=0$ $(1\leqq j\leqq n_i)$ and $\pi(F_i')=0$. Indeed, if V is a partial isometric operator and $\pi(V^*V)=0$, then $0=\pi(V^*V)=\pi(V)^*\pi(V)$ implies $\pi(V)=0$. So $\pi(VV^*)=0$. Then we conclude that $\pi(EP_i)=0$ for every P_i. This means $P_i=0$ and thus $P=0$. So $F=0$.

In the general case there is a central projection P such that $FP\prec EP$ and $E(1-P)$ $\prec F(1-P)$. We have that $FP=0$ from the first part of the proof since we may assume $FP\leqq EP$. Also $\pi(E(1-P))=0$. So $1-P=0$. Thus, $F=0$.

Now let A be any element of $\mathscr{A}$ such that $\pi(A)=0$. Suppose $\varepsilon>0$ is given; let $F_1, F_2, \ldots, F_m$ be orthogonal projections and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive numbers with $0\leqq \sum \alpha_i F_i \leqq A^*A$ and $\|A^*A-\sum \alpha_i F_i\| \leqq \varepsilon$. Then $\pi(F_i)=0$ $(i=1, 2, \ldots, m)$ and so $F_i=0$ $(i=1, 2, \ldots, m)$. We obtain this from the first part. This shows $\|A^*A\|\leqq\varepsilon$. Since $\varepsilon>0$ is arbitrary, we have that $A=0$. This shows that π is faithful if $\mathscr{A}$ is semifinite.

Now let $\mathscr{A}$ be a purely infinite von Neumann algebra with no nonzero σ-finite central projections. There is a net $\{P_i\}$ of orthogonal central projections of sum 1 such that each P_i is least upper bound of a set S_i of equivalent mutually orthogonal σ-finite projections [1, III, §1, Lemma 7]. For each i let $E_i \in S_i$ and let $E=\sum E_i$. Then E is a projection of central support 1. If F is a projection of central support Q then $EQP_i\prec FP_i$ for each P_i [1, III, §8, Corollary 5]. So $EQ\prec F$.

Let ϕ be a state of the module $\mathscr{A}$ such that $\phi(E)=1$. We show that the kernel of $\pi_\phi=\pi$ is 0. It is sufficient to show that $\pi(F)=0$ implies $F=0$ whenever F is a projection. However, if F has central support Q then $EQ \prec F$. So $\pi(EQ)=0$ and thus $\phi(EQ)=Q=0$. This proves $F=0$.

Now let $\mathscr{A}$ be a purely infinite algebra. There is a projection P of $\mathscr{A}$ such that P is the least upper bound of σ-finite central projections and such that $1-P$ majorizes no nonzero σ-finite central projections. Now let F be any projection in $\mathscr{A}P$ of central support 1 and let E be a projection previously constructed for a purely infinite von Neumann algebra with no nonzero σ-finite central projections. Let ϕ be a state of $\mathscr{A}$ such that $\phi(E+F)=1$. The canonical representation π induced by ϕ has kernel equal to (0) [Proposition 4.4].

The general result for an arbitrary von Neumann $\mathscr{A}$ algebra now follows from the fact that there is a central projection P such that $\mathscr{A}P$ is semifinite and $\mathscr{A}(1-P)$ is purely infinite. Q.E.D.

THEOREM 4.6. *Let $\mathscr{A}$ be a von Neumann algebra. There is a pure state of the module $\mathscr{A}$ whose canonical representation is faithful.*

Proof. There is a projection E of $\mathscr{A}$ of central support 1 such that the canonical representation π_ϕ induced by a state ϕ of the module $\mathscr{A}$ is faithful whenever $\phi(E)=1$ (Proposition 4.5). By Proposition 4.1 there is a pure state ϕ of the module $\mathscr{A}$ such that $\phi(E)=1$. Q.E.D.

THEOREM 4.7. *Let $\mathscr{A}$ be a von Neumann algebra and let ζ be a maximal ideal of the center of $\mathscr{A}$. The smallest closed two-sided ideal $[\zeta]$ in $\mathscr{A}$ containing ζ is a primitive ideal.*

Proof. Let ϕ be a pure state of $\mathscr{A}$ whose canonical representation π_ϕ is faithful. The representation π of $\mathscr{A}$ on the space H satisfying properties (1)–(5) of Proposition 4.2 constructed relative to ϕ is faithful. Let $\zeta'=\pi(\zeta)$ and let $[\zeta']$ be the smallest closed two-sided ideal in the von Neumann algebra $\mathscr{B}$ generated by $\pi(\mathscr{A})$ on H which contains ζ'. There is an irreducible representation Ψ of $\pi(\mathscr{A})$ with kernel $\pi(\mathscr{A}) \cap [\zeta']$ (corollary, Theorem 4.3). However $\pi(\mathscr{A}) \cap [\zeta']$ is the smallest closed two-sided ideal J of $\pi(\mathscr{A})$ which contains ζ'. Indeed, if E is a projection in $\pi(\mathscr{A}) \cap [\zeta']$ then the Gelfand transform $P^\wedge$ of the central support P of E vanishes at the point ζ'. Thus the projection P is in the maximal ideal ζ' and so E is in the ideal J. Because J contains all projections of $\pi(\mathscr{A}) \cap [\zeta']$, the ideal $\pi(\mathscr{A}) \cap [\zeta']$ is contained in J. Therefore we have that $J=\pi(\mathscr{A}) \cap [\zeta']$. However, $\pi^{-1}(J)=[\zeta]$ since π is faithful. So the kernel of $\Psi\cdot\pi$ is $[\zeta]$. Q.E.D.

The set Prim $(\mathscr{A})$ of all primitive ideals of $\mathscr{A}$ supplied with the hull-kernel topology is called structure space of $\mathscr{A}$.

PROPOSITION 4.7. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let Z be the spectrum of $\mathscr{Z}$. The set $\{[\zeta] \mid \zeta \in Z\}$ is dense in the structure space of $\mathscr{A}$.*

Proof. Let X be a nonvoid open set in Prim $(\mathscr{A})$. There is an ideal I in $\mathscr{A}$ such that

$$X = \{J \in \text{Prim}\,(\mathscr{A}) \mid J \not\supset I\}.$$

Let J be an ideal in X and let $J \cap \mathscr{Z} = \zeta$. The ideal ζ is maximal in $\mathscr{Z}$. We have that $[\zeta] \not\supset I$ since $[\zeta] \subset J$. This proves that $[\zeta] \in X$. Thus $\{[\zeta] \mid \zeta \in Z\}$ is dense in Prim ($\mathscr{A}$). Q.E.D.

The set $\mathscr{A}^\wedge$ of unitary equivalence classes of irreducible representations of $\mathscr{A}$ with the topology induced by the map $\pi \to$ kernel π of $\mathscr{A}^\wedge$ into Prim ($\mathscr{A}$) is known to be a Baire space [2, 3.4.13]. A proof of this fact is obtainable from the preceding proposition.

The next theorem characterizes a pure state in terms of its kernel. It is the analogue of a theorem of Kadison [13].

THEOREM 4.8. *Let $\mathscr{A}$ be a von Neumann algebra. A state ϕ of the module $\mathscr{A}$ is a pure state if and only if the kernel of ϕ is the sum of the sets*

$$L_\phi = \{A \in \mathscr{A} \mid \phi(A^*A) = 0\} \quad \textit{and} \quad L_\phi^* = \{A \in \mathscr{A} \mid A^* \in L_\phi\}.$$

Proof. Suppose ϕ is a pure state of the module $\mathscr{A}$. Let π be a representation of $\mathscr{A}$ on a Hilbert space H which satisfies properties (1)–(5) of Proposition 4.2 with respect to ϕ. Let E be the abelian projection of the von Neumann algebra $\mathscr{B}$ generated by $\pi(\mathscr{A})$ such that $\pi(\phi(A)) = \tau_E(\pi(A))$ for every A in $\mathscr{A}$. Suppose A is a point of the kernel of ϕ. The range projection F of $\pi(A)E$ in $\mathscr{B}$ is an abelian projection orthogonal to E. There is a hermitian element C in $\mathscr{A}$ such that $\pi(C)\pi(A)E = \pi(A)E$ and $\pi(C)E = 0$ (Theorem 4.3). Thus, $A - CA \in L_\phi$ and $A^*C \in L_\phi$. So $A = (A - CA) + CA$ is an element of $L_\phi + L_\phi^*$. This proves that $L_\phi + L_\phi^*$ contains the kernel of ϕ. Because $|\phi(A)|^2 \leqq \phi(A^*A)$ for every A in $\mathscr{A}$, the kernel of ϕ contains $L_\phi + L_\phi^*$. So the kernel of ϕ is equal to $L_\phi + L_\phi^*$.

Conversely, let $L_\phi + L_\phi^*$ be the kernel of ϕ. Let C be a central element of $\mathscr{A}$ strictly between 0 and 1 and let ϕ_1 and ϕ_2 be two positive functionals of the module $\mathscr{A}$ of norm not exceeding 1 such that $C\phi_1 + (1 - C)\phi_2 = \phi$. First notice that ϕ_1 and ϕ_2 are states of $\mathscr{A}$. Then if $\phi(A) = 0$, there are elements B_1 and B_2 in L_ϕ and L_ϕ^* respectively such that $A = B_1 + B_2$. Because $\phi(B_1^*B_1) = \phi(B_2B_2^*) = 0$, we have that $\phi_1(B_j) = \phi_2(B_j) = 0$ for $j = 1, 2$. Thus $\phi_1(A) = \phi_2(A) = 0$. Now for arbitrary A in $\mathscr{A}$ there is a central element B in $\mathscr{A}$ such that $\phi(A - B) = 0$. Thus $\phi_1(A - B) = \phi_2(A - B) = 0$ and so $\phi_1(A) = \phi_2(A) = B = \phi(A)$. This proves ϕ is a pure state. Q.E.D.

5. **Pointwise convergence of states.** Let $\mathscr{A}$ be a von Neumann algebra. A net of states $\{\phi_n\}$ of the module $\mathscr{A}$ is said to converge pointwise to a state ϕ if $\{\phi_n(A)\}$ converges uniformly to $\phi(A)$ for every A in $\mathscr{A}$. The set $E(\mathscr{A})$ of states of the module $\mathscr{A}$ taken with the topology of pointwise convergence is called the state space of $\mathscr{A}$. The closure in the state space of the module $\mathscr{A}$ of the set of pure states in $\mathscr{A}$ is called the pure state space of the module $\mathscr{A}$. It is denoted by $P(\mathscr{A})$. An element ϕ in $E(\mathscr{A})$ is said to be a vector state if there is an abelian projection E in the commutator of the center of $\mathscr{A}$ such that $\phi(A) = \tau_E(A)$ for every A in $\mathscr{A}$. The closure in the space $E(\mathscr{A})$ of the set of vector states is called the vector state space of $\mathscr{A}$. It is denoted by $V(\mathscr{A})$.

We now study the structure of $P(\mathscr{A})$ and $V(\mathscr{A})$ using the theorems of Glimm [3, §4] as our guide.

THEOREM 5.1. *If $\mathscr{A}$ is a continuous von Neumann algebra, the state space, the pure state space, and the vector state space of the module $\mathscr{A}$ coincide.*

Proof. First we show that the vector state space $V(\mathscr{A})$ of the module $\mathscr{A}$ coincides with the state space $E(\mathscr{A})$ of the module $\mathscr{A}$. Let ϕ be an element of $E(\mathscr{A})$ and let $A_1, A_2, \dots, A_n$ be elements of $\mathscr{A}$. Assume $A_1 = 1$. Let $\mathscr{Z}'$ be the commutator of the center $\mathscr{Z}$ of $\mathscr{A}$ and let $[\zeta]$ denote the smallest closed two-sided ideal in $\mathscr{Z}'$ which contains the maximal ideal ζ of $\mathscr{Z}$. There is for each ideal ζ an irreducible representation Ψ_ζ of $\mathscr{Z}'$ with kernel $[\zeta]$ on the algebra of bounded linear operators of a Hilbert space $H(\zeta)$ such that $\Psi_\zeta(\mathscr{Z}')$ contains the ideal $C(H(\zeta))$ of completely continuous operators on $H(\zeta)$. Since $\mathscr{A}$ is a continuous algebra, the image $\Psi_\zeta(\mathscr{A})$ of $\mathscr{A}$ contains no minimal projections. So $\Psi_\zeta(\mathscr{A}) \cap C(H(\zeta)) = (0)$. There is unit vector x_ζ in $H(\zeta)$ such that

$$|\phi(A_j)\hat{}(\zeta) - (\Psi_\zeta(A_j)x_\zeta, x_\zeta)| < \tfrac{1}{2} \quad \text{for } j = 1, 2, \dots, n.$$

Indeed, the kernel of the functional $A \to \phi(A)\hat{}(\zeta)$ of $\mathscr{A}$ contains the ideal $\mathscr{A} \cap [\zeta]$. So there is a functional ϕ_ζ of $\Psi_\zeta(\mathscr{A})$ such that

$$\phi_\zeta \cdot \Psi_\zeta(A) = \phi(A)\hat{}(\zeta).$$

Then the statement in question simply states that the functional ϕ_ζ is the pointwise limit of vector states of $\Psi_\zeta(\mathscr{A})$ [2, 11.2.1]. There is an abelian projection E_ζ of $\mathscr{Z}'$ such that

$$(\Psi_\zeta(B)x_\zeta, x_\zeta) = \tau_{E_\zeta}(B)\hat{}(\zeta)$$

for every B in $\mathscr{Z}'$ (cf. [7, Theorem 1]). This means that there is a central projection P with $P\hat{}(\zeta) = 1$ such that $E_\zeta P$ has central support P and such that

$$\|\phi(A_j)P - \tau_{E_\zeta P}(A_j)\| < 1$$

for $j = 1, 2, \dots, n$. Thus there is an abelian projection E of central support 1 such that $\|\phi(A_j) - \tau_E(A_j)\| < 1$ for $j = 1, 2, \dots, n$. This shows that ϕ is the pointwise limit of vector states. Thus $E(\mathscr{A}) = V(\mathscr{A})$.

We now show that $E(\mathscr{A})$ is equal to the pure state space of the module $\mathscr{A}$. First let ψ be any pure state of the module $\mathscr{A}$ whose canonical representation π_ψ is faithful (Theorem 4.6). Let π be a faithful representation of $\mathscr{A}$ on a Hilbert space H such that the commutator $\pi(\mathscr{A})'$ of $\pi(\mathscr{A})$ on H is equal to $\pi(\mathscr{Z})$ and such that there is an abelian projection E in $\pi(\mathscr{Z})'$ of central support 1 with the property $\tau_E(\pi(A)) = \pi(\psi(A))$ for every A in $\mathscr{A}$ (Proposition 4.2). Let ϕ be an element of $E(\mathscr{A})$ and let $A_1, A_2, \dots, A_m$ be elements of $\mathscr{A}$. There is an abelian projection F of central support 1 in $\pi(\mathscr{Z})'$ such that

$$\|\pi \cdot \phi \cdot \pi^{-1}(\pi(A_j)) - \tau_F(\pi(A_j))\| < 1$$

for $j=1, 2, \ldots, m$. Indeed, if ζ is a maximal ideal in $\pi(\mathscr{Z})$, there is an irreducible representation Ψ_ζ of $\pi(\mathscr{Z})'$ with kernel $[\zeta]$ on a Hilbert space such that the image of $\pi(\mathscr{A})$ contains no completely continuous operators. The same reasoning as the previous paragraph therefore is applicable. So it is sufficient to show that $\pi^{-1}\cdot\tau_F\cdot\pi$ is a pure state of $\mathscr{A}$. We do this by showing that it is $\mathscr{Z}$-irreducible. Let θ be a positive functional of the module $\mathscr{A}$ majorized by $\pi^{-1}\cdot\tau_F\cdot\pi$. Then $\theta'=\pi\cdot\theta\cdot\pi^{-1}$ on $\pi(\mathscr{A})$ is majorized by τ_F on $\pi(\mathscr{A})$. Let ζ be a maximal ideal in $\pi(\mathscr{Z})$. There are positive functionals f and g on $\Psi_\zeta(\pi(\mathscr{A}))$ such that

$$f(\Psi_\zeta(A)) = \theta'(A)^\frown(\zeta) \quad \text{and} \quad g(\Psi_\zeta(A)) = \tau_F(A)^\frown(\zeta)$$

for A in $\pi(\mathscr{A})$. Then g majorizes f on $\Psi_\zeta(\pi(\mathscr{A}))$. However g is irreducible on $\Psi_\zeta(\pi(\mathscr{A}))$ and so there is an α_ζ in the complex field such that $f(A)=\alpha_\zeta g(A)$ for all A in $\Psi_\zeta(\pi(\mathscr{A}))$. But $\alpha_\zeta=\theta'(1)^\frown(\zeta)$. Since ζ is arbitrary we have that $\theta'=\theta'(1)\tau_F$ on $\pi(\mathscr{A})$. This proves that $\pi^{-1}\cdot\tau_F\cdot\pi$ is $\mathscr{Z}$-irreducible. Q.E.D.

We see that if π is a faithful representation of the continuous algebra $\mathscr{A}$ on a Hilbert space H with the property that the commutator of $\pi(\mathscr{A})$ is $\pi(\mathscr{Z})$ and that there is an abelian projection E with central support 1 in the commutator $\pi(\mathscr{Z})'$ of $\pi(\mathscr{Z})$ such that $\pi^{-1}\cdot\tau_E\cdot\pi$ is a pure state of $\mathscr{A}$, then the set

$$\{\pi^{-1}\cdot\tau_F\cdot\pi \mid F \text{ is an abelian projection of central support 1 in } \pi(\mathscr{Z})'\}$$

is pointwise dense in $E(\mathscr{A})$.

We now identify the pure state and vector state spaces of a type I algebra. We begin with the following theorem.

THEOREM 5.2. *If $\mathscr{A}$ is a type* I *von Neumann algebra, the vector state space $V(\mathscr{A})$ of the module $\mathscr{A}$ is equal to the pure state space $P(\mathscr{A})$ of the module $\mathscr{A}$.*

Proof. Since every vector state of the module $\mathscr{A}$ is a pure state of the module $\mathscr{A}$, we have that $V(\mathscr{A})\subset P(\mathscr{A})$ [12, Remark, Theorem 9].

Now let ϕ be a pure state of the module $\mathscr{A}$. Let $A_1, A_2, \ldots, A_m$ be elements of $\mathscr{A}$. For each maximal ideal ζ of the center of $\mathscr{A}$ there is an irreducible representation Ψ_ζ of $\mathscr{A}$ with kernel $[\zeta]$ on a Hilbert space $H(\zeta)$ such that $\Psi_\zeta(\mathscr{A})$ contains the completely continuous operators on $H(\zeta)$. The kernel of the function $A \to \phi(A)^\frown(\zeta)$ on $\mathscr{A}$ contains the ideal $[\zeta]$. There is thus a functional ϕ_ζ of $\Psi_\zeta(\mathscr{A})$ such that $\phi_\zeta(\Psi_\zeta(A)) =\phi(A)^\frown(\zeta)$ for every A. Since $\phi(A)^\frown(\zeta)$ is a pure state of $\mathscr{A}$ (corollary, Theorem 4.3), the functional ϕ_ζ is a pure state of $\Psi_\zeta(\mathscr{A})$. The pure state space of $\Psi_\zeta(\mathscr{A})$ is equal to the vector space of $\Psi_\zeta(\mathscr{A})$ (cf. [2, 3.4.1] due to R. V. Kadison). There is a unit vector x_ζ in $H(\zeta)$ such that

$$|\phi_\zeta(\Psi_\zeta(A_j))-(\Psi_\zeta(A_j)x_\zeta, x_\zeta)| < 1$$

for $j=1, 2, \ldots, m$. There is an abelian projection E_ζ in $\mathscr{A}$ such that

$$(\Psi_\zeta(A)x_\zeta, x_\zeta) = \tau_{E_\zeta}(A)^\frown(\zeta)$$

for every A in $\mathscr{A}$. By the same reasoning as Theorem 5.1 we obtain an abelian projection E in $\mathscr{A}$ of central support 1 such that

$$\|\phi(A_j)-\tau_E(A_j)\| < 1$$

for $j=1, 2, \ldots, m$. This means that $\phi \in V(\mathscr{A})$. Therefore, $P(\mathscr{A}) \subset V(\mathscr{A})$. This completes the proof.

Let $\mathscr{A}$ be a type I von Neumann algebra with center $\mathscr{Z}$. The uniformly closed *-subalgebra of $\mathscr{A}$ generated by the abelian projections of $\mathscr{A}$ is a two-sided ideal I_a in $\mathscr{A}$ [16]. If $A \in I_a^+$, there is a sequence $\{A_n\}$ of positive central elements and a sequence $\{E_n\}$ of orthogonal abelian projections such that

(1) $A_1 \geqq A_2 \geqq \cdots$;

(2) $\lim A_n = 0$ (uniformly);

(3) the central support of E_n has Gelfand transform equal to the characteristic function of the support for the Gelfand transform of A_n for each $n=1, 2, \ldots$;

(4) $A = \sum A_n E_n$; and

(5) the sequence $\{A_n\}$ is uniquely determined.

The sum $\sum A_n E_n$ is called a spectral decomposition of A.

Let $\mathscr{T}$ be the trace class of $\mathscr{A}$ and let Tr be the canonical trace of $\mathscr{A}$ (§4). For each A in $\mathscr{T}$ define the bounded module homomorphism Φ_A of I_a into $\mathscr{Z}$ by $\Phi_A(B) = \mathrm{Tr}\,(AB)$. Then if $\mathscr{T}$ is given the norm

$$\|A\|_1 = \|\mathrm{Tr}\,((A^*A)^{1/2})\|,$$

the function $A \to \Phi_A$ defines an order preserving isometric isomorphism of the $\mathscr{Z}$-module $\mathscr{T}$ onto the set of all bounded module homomorphisms of I_a into $\mathscr{Z}$ [9, §4].

THEOREM 5.3. *Let $\mathscr{A}$ be a type* I *von Neumann algebra. Let I_a be the closed two-sided ideal of $\mathscr{A}$ generated by the abelian projections of $\mathscr{A}$. The vector state space $V(\mathscr{A})$ of the module $\mathscr{A}$ consists of the set of all states of the module $\mathscr{A}$ of the form*

$$C\phi + (1-C)\tau_E$$

where C is a central element of $\mathscr{A}$ with $0 \leqq C \leqq 1$, ψ is a state of the module $\mathscr{A}$ such that $C\psi$ vanishes on I_a and E is a maximal abelian projection of $\mathscr{A}$.

Proof. First let ϕ be an element of $V(\mathscr{A})$; set $\phi \mid I_a = \theta_1$. There is a positive element B in the trace class of $\mathscr{A}$ such that $\theta_1(A) = \mathrm{Tr}\,(AB)$ for every A in I_a. Let $\theta(A) = \mathrm{Tr}\,(AB)$ for every A in $\mathscr{A}$. We show that the functional $\phi - \theta$ is positive. Let $A \in \mathscr{A}^+$. There is a monotonely increasing net $\{A_n\}$ in I_a^+ which converges strongly to A [1, I, §3, Theorem 2, Corollary 5] because I_a is strongly dense in $\mathscr{A}$. Let x be a vector in the Hilbert space of $\mathscr{A}$. We have that

$$(\phi(A)x, x) - (\theta(A_n)x, x) \geqq (\phi(A_n)x, x) - (\theta(A_n)x, x) = 0$$

for every A_n. Thus

$$(\phi(A)x, x) - (\theta(A)x, x) = \lim_n ((\phi(A)x, x) - (\theta(A_n)x, x)) \geqq 0.$$

This proves $\phi-\theta$ is a positive functional of the module $\mathscr{A}$. We also have that $\phi(A)-\theta(A)=0$ for every $A\in I_a$.

Now let $B=\sum B_iE_i$ be a spectral decomposition for B. Here $\{E_i\}$ is a sequence of orthogonal abelian projections with $E_1\succ E_2\succ\cdots$; $\{B_i\}$ is a decreasing sequence of positive central elements with $\lim B_n=0$ (uniformly); and the support of each B_i is equal to the central support of E_i. There is a set of mutually orthogonal central projections $\{P_n\}$ of sum 1 such that for each P_n the series $\sum\{P_nB_i \mid i=1,2,\ldots\}$ converges uniformly [9, Theorem 4.1]. Let n be fixed and let X_n be the set of ζ in the spectrum Z of the center of $\mathscr{A}$ such that $P_n^\frown(\zeta)=1$. For $\zeta\in X_n$ let Ψ_ζ be an irreducible representation of $\mathscr{A}$ with kernel $[\zeta]$ on a Hilbert space $H(\zeta)$. Let ϕ_ζ be the positive functional on $\Psi_\zeta(\mathscr{A})=\mathscr{A}(\zeta)$ given by $\phi_\zeta(A(\zeta))=\phi(A)^\frown(\zeta)$. Here $\Psi_\zeta(A)=A(\zeta)$. Since every functional f having the form $f(A(\zeta))=\tau_F(A)^\frown(\zeta)$, where F is an abelian projection of $\mathscr{A}$ of central support 1, is a vector state of $\mathscr{A}(\zeta)$, the functional ϕ_ζ is in the vector state space of $\mathscr{A}(\zeta)$. By Glimm's theorem [3, Theorem 2], there is an α_ζ in the interval [0, 1], a state g_ζ of $\mathscr{A}(\zeta)$ vanishing on the completely continuous operators of $H(\zeta)$, and a unit vector x_ζ in $H(\zeta)$ such that

$$\phi_\zeta = \alpha_\zeta g_\zeta+(1-\alpha_\zeta)w_{x_\zeta}.$$

Now we have that

$$\theta(A)^\frown(\zeta) = \Big(\sum B_i\tau_{E_i}(A)\Big)^\frown(\zeta) = \sum B_i^\frown(\zeta)\tau_{E_i}(A)^\frown(\zeta)$$

by the uniform convergence of $\sum_i B_iP_n$. Since $\Psi_\zeta(I_a)$ is precisely the ideal of completely continuous operators on $H(\zeta)$, we must have that

$$(1-\alpha_\zeta)w_{x_\zeta}(A(\zeta)) = \sum B_i^\frown(\zeta)\tau_{E_i}(A)^\frown(\zeta)$$

for each A in I_a. For each E_i there is a unit vector y_i in $H(\zeta)$ such that

$$B_i^\frown(\zeta)\tau_{E_i}(A)^\frown(\zeta) = B_i^\frown(\zeta)(A(\zeta)y_i, y_i).$$

Indeed, $E_i(\zeta)$ is a projection on $H(\zeta)$ of dimension not exceeding 1. Therefore, we have that

$$(1-\alpha_\zeta)w_{x_\zeta}(A(\zeta)) = B_1^\frown(\zeta)\tau_{E_1}(A)^\frown(\zeta)$$

for every A in I_a. Then $B_2^\frown(\zeta), B_3^\frown(\zeta),\ldots$ vanish. Because ζ in X_n is arbitrary, we conclude that $0=B_2P_n=B_3P_n=\cdots$ and thus that $BP_n=(B_1E_1)P_n$. Because P_n is arbitrary, we find that $B_2, B_3,\ldots$ vanish. Thus $B=B_1E_1$ and $\theta(A)=B_1\tau_{E_1}(A)$ for every A in $\mathscr{A}$. Since the support of $B_1^\frown$ is equal to that of E_1, we may assume $E=E_1$ is a maximal abelian projection and still retain the formula $B_1\tau_E(A)=\theta(A)$.

There is a sequence $\{Q_n\}$ of orthogonal central projections of sum equal to the support Q of $C=\phi(1)-\theta(1)$ such that for each Q_n there is a positive central element D_n with $D_nQ_n=D_n$ and $D_nC=Q_n$. The sequence $\{\|D_n(\phi(A)-\theta(A))\|\}$ is bounded above by $\|A\|$ for each A in $\mathscr{A}$ since $\phi-\theta$ is a positive functional of the module $\mathscr{A}$. Set $\psi_1(A)=\sum_n D_n(\phi(A)-\theta(A))$ for each A in $\mathscr{A}$. Then ψ is a positive functional of the module $\mathscr{A}$, with the property $\psi_1(1)=Q$. We extend ψ_1 to a state ψ on the module $\mathscr{A}$ by setting $\psi=\psi_1+\psi_2$ where ψ_2 is a positive functional of the module $\mathscr{A}$ with $\psi_2(1)=1-Q$.

We show that $C\psi+B_1\tau_E=\phi$. For each Q_n we have that

$$Q_n(C\psi(A)+B_1\tau_E(A)) = Q_n(\phi(A)-\theta(A)+\theta(A)) = Q_n\phi(A)$$

for every A in $\mathscr{A}$. Also

$$(1-Q)(C\psi(A)+B_1\tau_E(A)) = (1-Q)\theta(A) = (1-Q)\phi(A).$$

So $C\psi+B_1\tau_E=\phi$. Since both ψ and τ_E are states, we have that $C+B_1=1$. This completes the first part of the proof.

Conversely, let ϕ be a state of the module $\mathscr{A}$ of the form

$$\phi = C\psi+(1-C)\tau_E,$$

where C is a central element of $\mathscr{A}$ with $0\leqq C\leqq 1$, ψ is a state of the module $\mathscr{A}$ such that $C\psi$ vanishes on I_a, and E is an abelian projection of central support 1. Let $A_1, A_2,\ldots, A_n$ be elements of $\mathscr{A}$. Let ζ be a maximal ideal of the center of $\mathscr{A}$ and let Ψ_ζ be an irreducible representation with kernel $[\zeta]$ of $\mathscr{A}$ on the Hilbert space $H(\zeta)$. Let $\Psi_\zeta(\mathscr{A})=\mathscr{A}(\zeta)$ and $\Psi_\zeta(A)=A(\zeta)$. The relation

$$\phi_\zeta(A(\zeta)) = \phi(A)^\wedge(\zeta)$$

defines a functional in the vector state space of $\mathscr{A}(\zeta)$ [3, Theorem 2] since $\Psi_\zeta(I_a)$ is the ideal of completely continuous operators on $H(\zeta)$. There is a unit vector x_ζ in $H(\zeta)$ such that

$$|\phi_\zeta(A_j(\zeta))-(A_j(\zeta)x_\zeta, x_\zeta)| < 1$$

for $j=1, 2,\ldots, n$. But there is an abelian projection E_ζ in A such that

$$(A(\zeta)x_\zeta, x_\zeta) = \tau_{E_\zeta}(A)^\wedge(\zeta)$$

for every A in $\mathscr{A}$. By the same procedure as employed in Theorem 5.2, we obtain an abelian projection F of central support 1 in $\mathscr{A}$ such that

$$|\phi_\zeta(A_j(\zeta))-\tau_F(A_j)^\wedge(\zeta)| < 1$$

for every $j=1, 2,\ldots, n$ and every maximal ideal ζ. So

$$\|\phi(A_j)-\tau_F(A_j)\| < 1$$

for $j=1, 2,\ldots, n$. Thus ϕ is in the vector state space of the module $\mathscr{A}$. Q.E.D.

In a type I algebra every state is the pointwise limit of σ-weakly continuous states.

THEOREM 5.4. *Let $\mathscr{A}$ be a type* I *von Neumann algebra. Every state of the module $\mathscr{A}$ is the pointwise limit of normal states of the module $\mathscr{A}$.*

Proof. Let ϕ be a state of $\mathscr{A}$. Let θ_1 be the restriction of ϕ to I_a. There is a positive element B of the trace class of $\mathscr{A}$ such that $\theta_1(A)=\mathrm{Tr}\,(BA)$ for every A in I_a. Let

$\theta(A) = \mathrm{Tr}\,(BA)$ for every A in $\mathscr{A}$. Then $\phi - \theta = \psi_1$ is a positive functional on the module $\mathscr{A}$ which vanishes on I_a (cf. proof of Theorem 5.3). Let the central projection Q be the support of $C = \psi_1(1)$. There is a positive functional ψ of the module $\mathscr{A}$ such that $\psi(1) = Q$ and such that $C\psi = \psi_1$. Now let $A_1, A_2, \ldots, A_n$ be elements of $\mathscr{A}$. The restriction of ψ to the $\mathscr{Z}Q$-module $\mathscr{A}Q$ vanishes on the closed two-sided ideal I_aQ generated by the abelian projections of $\mathscr{A}Q$. There is an abelian projection E in $\mathscr{A}Q$ with central support Q such that

$$\|\psi(AQ) - \tau_E(A_jQ)\| < (\|C\| + 1)^{-1}$$

for $j = 1, 2, \ldots, n$ (Theorem 5.3). This means that

$$\|\psi_1(A_j) - C\tau_E(A_j)\| < 1$$

for $j = 1, 2, \ldots, n$. The functional

$$A \to C\tau_E(A) + \mathrm{Tr}\,(BA)$$

is a σ-weakly continuous positive functional of the module $\mathscr{A}$. We have that

$$C\tau_E(1) + \mathrm{Tr}\,(B) = \phi(1) - \theta(1) + \theta(1) = \phi(1) = 1.$$

Also

$$\|\phi(A_j) - C\tau_E(A_j) - \mathrm{Tr}\,(BA_j)\| < 1$$

for $j = 1, 2, \ldots, n$. Thus the state ϕ is the pointwise limit of positive σ-weakly continuous states. Q.E.D.

Bibliography

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
2. ———, *Les C*-algèbres et leur représentations*, Gauthier-Villars, Paris, 1964.
3. J. Glimm, *A Stone-Weierstrass theorem for C*-algebras*, Ann. of Math. (2) **72** (1960), 216–244.
4. ———, *Type* I *C*-algebras*, Ann. of Math. (2) **73** (1961), 572–612.
5. R. Godement, *Sur la théorie des représentations unitaires*, Ann. of Math. (2) **53** (1951), 68–124.
6. M. Goldman, *Structure of AW*-algebras of type* I, Duke Math. J. **23** (1956), 23–34.
7. H. Halpern, *The maximal GCR ideal of an AW*-algebra*, Proc. Amer. Math. Soc. **17** (1966), 906–914.
8. ———, *An integral representation of a normal functional on a von Neumann algebra*, Trans. Amer. Math. Soc. **125** (1966), 32–46.
9. ———, *A spectral decomposition for self-adjoint elements in the maximal GCR ideal of a von Neumann algebra with applications to noncommutative integration theory*, Trans. Amer. Math. Soc. **133** (1968), 281–306.
10. ———, *Commutators in properly infinite von Neumann algebra*, Trans. Amer. Math. Soc. **139** (1969), 55–73.
11. ———, *Proper values for the elements of the maximal GCR ideal of a von Neumann algebra*, (to appear).
12. ———, *Module homomorphisms of a von Neumann algebra into its center*, Trans. Amer. Math. Soc. **140** (1969), 183–193.

13. R. V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276.

14. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

15. ———, *A theorem on rings of operators*, Pacific J. Math. **1** (1951), 227–232.

16. ———, *Algebras of type* I, Ann. of Math. (2) **56** (1952), 460–472.

17. ———, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.

18. J. Kelly and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.

19. M. Nakai, *Some expectations in AW*-algebras*, Proc. Japan Acad. Sci. **34** (1958), 411–416.

20. J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. (2) **50** (1949), 401–485.

21. C. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, N. J., 1960.

22. S. Sakai, *Theory of W*-algebras* (mimeographed notes), Yale Univ., New Haven, Conn., 1962.

23. I. E. Segal, *Decomposition of operator algebras*. I, II, Mem. Amer. Math. Soc. No. 9 (1951), 67 pp. and 66 pp.

24. M. Takesaki, *On the Hahn-Banach type theorem and the Jordan decomposition of module linear mappings over some operator algebras*, Kōdai Math. Sem. Rep. **12** (1960), 1–10.

25. J. Taylor, *The Tomita decomposition of rings of operators*, Trans. Amer. Math. Soc. **113** (1964), 30–39.

26. M. Tomita, *Representation of operator algebras*, Math. J. Okayama Univ. **3** (1954), 142–173.

27. ———, *Spectral theory of operator algebras*. I, Math. J. Okayama Univ. **9** (1959), 63–98.

28. H. Widom, *Embedding in algebras of type* I, Duke Math. J. **23** (1956), 309–324.

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