n-APOSYNDETIC CONTINUA AND CUTTING THEOREMS

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1. **Introduction.** In the study of compact, metric continua certain separation and cutting theorems have been established using the notions of semi-locally-connectedness and aposyndesis. The purpose of this paper is to introduce the concepts of 2-aposyndesis and 2-semi-locally-connectedness in order to further characterize compact, metric continua and to generalize some of the earlier results.

In 1928, G. T. Whyburn and W. L. Ayres [1] proved that a necessary and sufficient condition for a continuum to be locally connected is that each pair of disjoint closed sets can be separated from each other by the sum of a finite number of subcontinua. Disjoint closed sets can be replaced by disjoint continua and even by a point and a disjoint continuum without altering the result. One of the theorems of this paper characterizes a 2-aposyndetic continuum as one in which each pair of distinct points can be separated by the sum of a finite number of subcontinua. In 1925, R. L. Moore showed that for the plane this condition is equivalent to local connectedness but a more general characterization was not possible until the concept of aposyndesis due to F. B. Jones was introduced in the early 1940's [3].

A second result of this paper shows that an aposyndetic continuum that is not 2-semi-locally-connected at any point contains a dense set each point of which is one of a pair of points whose sum cuts the continuum. If the continuum is not aposyndetic however, there may not be such a dense set of cutting pairs of points (in fact there may not be even one pair of points whose sum cuts). The examples in §4 show this as well as the fact that a continuum M that is not 2-aposyndetic at any point need have no more than one pair of points whose sum cuts M. One such pair must exist, of course [2].

The reader should observe throughout that most of the definitions, theorems and examples can be generalized and stated for n-aposyndesis and n-semi-locally-connectedness where n is any positive integer greater than 1. For the statements of these generalizations see [7].

2. **Definitions and preliminary theorems.** A continuum M is a compact, metric, closed and connected point set. However, for the theorems of this section and the

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next section, M need be only Hausdorff instead of metric. A subset S *separates* M if $M-S$ is not connected and *cuts* M if $M-S$ is not continuum-wise connected, i.e., there exist two points in $M-S$ such that every subcontinuum of M containing both of these points intersects S.

The continuum M is *2-aposyndetic at* x if for every pair of points p and q lying in $M-x$, there is an open set U and subcontinuum H so that $x \in U \subset H \subset M-(p+q)$ and is *2-aposyndetic* if it is 2-aposyndetic at every one of its points.

If $x \in M$ and for every pair of points p and q lying in $M-x$, there is an open set U and subcontinuum H so that $p \in U \subset H \subset M-(q+x)$, then M is *2-semi-locally-connected at* x (2-slc at x), and if M is 2-slc at all of its points, then M is *2-semi-locally-connected* (2-slc).

For other terms used but not defined, the reader is referred to Jones [4] and Moore [6].

THEOREM 1. *The continuum M is 2-aposyndetic if and only if M is 2-slc.*

THEOREM 2. *The continuum M is 2-slc if and only if for every pair of points p and q and open set U containing $p+q$, there is an open set V so that $p+q \subset V \subset U$ and $M-V$ has a finite number of components.*

Theorem 2 is an easy consequence of the Heine-Borel Theorem.

In [4] F. B. Jones has shown for compact continua that if M is not aposyndetic at x with respect to y, then the set N consisting of y plus all the points of M at which M is not aposyndetic with respect to y is a continuum. However, if M is not 2-aposyndetic at x with respect to $p+q$, then the set N consisting of $p+q$ and all points at which M is not 2-aposyndetic with respect to $p+q$ is not necessarily a continuum although it does have at most two components. But if M were aposyndetic at x with respect to p and with respect to q, then N would be a continuum. These facts are consequences of the next theorem. First we need to generalize the notion of a continuum being 2-aposyndetic at a point.

DEFINITION. The continuum M is *j-aposyndetic* at x with respect to the j points $x_1, x_2, \ldots, x_j$ if there is an open set U and continuum H such that $x \in U \subset H \subset M-\sum_1^j x_i$. By convention, any continuum is 0-aposyndetic at each point with respect to the empty set.

THEOREM 3. *If the continuum M is not n-aposyndetic at x with respect to the sum of the n points $x_1, x_2, \ldots, x_n$ but is m-aposyndetic at x with respect to any m of these points, $0 \leqq m < n$, then the set K consisting of $\sum_1^n x_i$ plus all the points at which M is not n-aposyndetic with respect to $\sum_1^n x_i$ is a closed set with at most $n-m$ components.*

Proof. To show that K is closed let us show that $M-K$ is open. If $q \in M-K$, then there exists an open set U and continuum H such that $q \in U \subset H \subset M-\sum_1^n x_i$. Since $U \subset M-K$, $M-K$ is open.

The proof will be complete if we show that each component of the set K consists of exactly one of the points $x_1, x_2, \ldots, x_n$ or contains at least $m+1$ of them. In order for this to happen K could have at most $n-m$ components since $K \neq \sum_1^n x_i$.

Suppose C is a nondegenerate component of K containing exactly p of the n points $x_1, x_2, \ldots, x_n$ where $p \leqq m$. No geneıality is lost in assuming the p points to be $x_1, x_2, \ldots, x_p$. Now K is closed so there is a separation of K, $K = A + B$, such that $C \subset A$ and $\sum_{p+1}^n x_i \subset B$. Let U be an open set in M so that $A \subset U$ and $B \subset M - \bar{U}$. For each point y of the boundary of U, Fr (U), there is a continuum P such that $y \in P^0 \subset P \subset M - \sum_1^n x_i$. Since Fr (U) is compact, there is a finite number of these continua $P_1, P_2, \ldots, P_k$ such that Fr $(U) \subset \sum P_i^0 \subset \sum P_i \subset M - \sum_1^n x_i$. Let z be some point of C not in $\sum P_i$ and not one of the points $x_1, x_2, \ldots, x_p$. There exists a continuum Q such that $z \in Q^0 \subset Q \subset M - \sum_1^p x_i$ since $p \leqq m$ and M is m-aposyndetic at x with respect to any m of the points $x_1, x_2, \ldots, x_n$. If $Q \cdot \sum_{p+1}^n x_i = \varnothing$ then M is n-aposyndetic at z with respect to $\sum_1^n x_i$ contrary to hypothesis. Hence $Q \cdot \sum_{p+1}^n x_i \neq \varnothing$ and there is a separation of $(Q + \sum P_i) - \sum P_i$ into sets E and F such that $z \in E \subset U$ and $F \subset M - U$. Now $E + \sum P_i$ has a finite number of components since $Q + \sum P_i$ does and consequently z is in tl.ɔ interior of the component of $E + \sum P_i$ containing it. But this component has no point in common with $\sum_1^n x_i$ which makes M n-aposyndetic at z with respect to $\sum_1^n x_i$. This is impossible and the proof is complete.

COROLLARY. *If the continuum M is not n-aposyndetic at x with respect to the sum of the n points $x_1, x_2, \ldots, x_n$ but is $(n-1)$-aposyndetic at x with respect to any $n-1$ of the points where $n \geqq 1$, then the set K consisting of $\sum_1^n x_i$ plus all the points at which M is not n-aposyndetic with respect to $\sum_1^n x_i$ is a continuum.*

As shown with simple examples where $n=2$, the converse of Theorem 3 is not true.

The next theorem shows that the cartesian product of two continua is always "more aposyndetic" than either factor. In this theorem the continua do not need to be compact. The proof is easy and is left to the reader.

THEOREM 4. *If H and K are continua, H being p-aposyndetic and K being q-aposyndetic where $p \geqq 0$ and $q \geqq 0$, then $H \times K$ is a $(p+q+1)$-aposyndetic continuum*(2).

3. **A characterization of 2-aposyndesis.** The next theorem establishes the characterization of a 2-aposyndetic continuum referred to in the introduction.

THEOREM 5. *In order for a continuum M to be 2-aposyndetic (2-slc) it is necessary and sufficient that every distinct pair of its points can be separated in M by the sum of a finite number of subcontinua of M.*

Proof. First we prove the necessity. Let x and y be distinct points of M and let U be an open subset such that $x \in U \subset \bar{U} \subset M - y$. For each $p \in$ Fr (U), there exists a continuum Hp such that $p \in Hp^0 \subset Hp \subset M - (x+y)$ and by compactness there

(2) This theorem in essence was brought to my attention by Clifford Arnquist and Lynn Gref in a topology seminar.

exists a finite number of the Hp's, $Hp_1, Hp_2, \ldots, Hp_n$, such that $\mathrm{Fr}(U) \subset \sum Hp_i$. This sum of continua separates x from y in M since $\mathrm{Fr}(U)$ does.

Now let us prove the sufficiency. First we need to show that M is aposyndetic. Let x and y be distinct points of M and let us show that M is aposyndetic at x with respect to y. By hypothesis there are continua, $H_1, H_2, \ldots, H_n$, such that $M - \sum H_i = A + B$, which is a separation such that $x \in A$, $y \in B$. Then $A + \sum H_i$ has at most n components; so x is in the interior of the component of $A + \sum H_i$ containing it. Since y is not in this component, M is aposyndetic at x with respect to y.

Now suppose that M is not 2-aposyndetic at x with respect to $y+z$ and let $H_1, H_2, \ldots, H_n$ be the continua of the hypothesis such that $\sum H_i$ separates y from z. The point x may not be in $\sum H_i$, but if it is, the set K defined in Theorem 3 is a continuum containing both y and z, so that we can find a point p such that (1) $p \notin \sum H_i$ and (2) M is not 2-aposyndetic at p with respect to $y+z$.

Since $\sum H_i$ separates y from z, we have a separation $M - \sum H_i = C + D$ where $y \in C$ and $z \in D$. We can assume that $p \in C$. Now M is aposyndetic at p with respect to y, so there is a continuum Q such that $p \in Q^0 \subset Q \subset M - y$. By assumption z belongs to Q. The set $Q + \sum H_i$ is separated by $\sum H_i$ into sets E and F such that $p \in E$ and $z \in F$. But $E + \sum H_i$ has a finite number of components and thus p lies in the interior of the component of $E + \sum H_i$ containing it. This component does not contain either y or z and so M is 2-aposyndetic at p with respect to $y+z$. This contradiction completes the proof.

REMARK. The condition that every pair of distinct points in M can be separated from each other in M by the sum of a finite number of degenerate continua (finite point set) characterizes a regular curve. This induces a very strong type of local connectedness on M in which not only M but each of its subcontinua is locally connected.

By the use of Moore's result mentioned in the introduction, we have the following corollary to Theorem 5.

COROLLARY. *Let M be a continuum lying in the plane. Then M is locally connected if and only if M is 2-aposyndetic.*

4. **Cuttings of M by pairs of points.** In [4] Jones has shown that a continuum M that is not semi-locally-connected at any point contains a dense set of points each of which cuts M. However, if the continuum M is not 2-semi-locally-connected at any point it need not have even one pair of points whose sum cuts M. If we add the requirement though that M be aposyndetic, then M will contain a dense set for which each point is one of a pair of points whose sum does cut M. This is the aim of the following three lemmas.

LEMMA 1. *Let M be an aposyndetic (slc) continuum and let U be an open set in M. Suppose that M is not 2-slc at any point of U. Then U contains an open set T and M contains closed sets H and K such that $\overline{T}$, H, K are mutually disjoint and for each* *point x in T, there are points y in H and z in K such that M is not 2-aposyndetic at y with respect to $x+z$.*

Proof. Let W be an open set such that $W\subset \overline{W}\subset U$. For each positive integer i, let K_i be the set of all points x of $\overline{W}$ such that there are points y and z of M with M not 2-aposyndetic at y with respect to $x+z$ and $d(x, z)\geqq 1/i$. Let us show that K_i is closed. Take x in $\overline{K}_i$ and let $x_1, x_2, \ldots$ be a sequence of points of K_i converging to x. Now for each x_j there are points y_j and z_j such that $d(x_j, z_j)\geqq 1/i$ and M is not 2-aposyndetic at y_j with respect to x_j+z_j. Taking subsequences, if necessary, let $z_1, z_2, \ldots$ converge to a point z where, of course, $d(x, z)\geqq 1/i$. Let U and V be open sets such that $x\in U$, $z\in V$ and $\overline{U}\cdot\overline{V}=\varnothing$. Now for j sufficiently large, $x_j\in U$, $z_j\in V$ and by virtue of the corollary of Theorem 3 the point y_j can be chosen in $M-(U+V)$. Let y be a limit point of $y_1, y_2, \ldots$. Since y is in $M-(U+V)$, y is not equal to x or z and it follows easily that M is not 2-aposyndetic at y with respect to $x+z$. This proves that x belongs to K_i and that K_i is closed.

Now $\overline{W}=\sum K_i$ so there is an integer n such that K_n contains an open set. Let V be an open set in K_n with a diameter less than $1/n$ and let V_1 and V_2 be open sets such that $\overline{V}_2\subset V_1\subset \overline{V}_1\subset V$. For every x in V_2 there is a point z in $M-V$ and again by the corollary of Theorem 3 a point y in $\operatorname{Fr}(V_1)$ such that M is not 2-aposyndetic at y with respect to $x+z$. The sets V_2, $\operatorname{Fr}(V_1)$, $M-V$ are the required sets T, H, K respectively in the conclusion of Lemma 1.

LEMMA 2. *Let M be an aposyndetic continuum, U an open subset of M, H and K closed subsets of M such that U, H, K are mutually disjoint. Suppose that for each point $x\in U$, there is a point $y\in H$, and a point $z\in K$ such that M is not 2-aposyndetic at y with respect to $x+z$. Then U contains an open set W and H and K contain closed subsets H_1 and K_1 respectively, such that*: (1) *For every point $x\in W$ there exist points $y\in H_1$ and $z\in K_1$ such that M is not 2-aposyndetic at y with respect to $x+z$.* (2) *K_1 lies in an open set D, H_1 lies in an open set B such that $\overline{B}$, $\overline{D}$, W are mutually disjoint and $\overline{B}+W$ is a set contained in the interior of a component C of $M-D$.*

Proof. For each $y\in H$, $z\in K$, there is a continuum P and an open set O such that $y\in P^0\subset P$, $z\in O$ and $\overline{O}\cdot P=\varnothing$. Hold y fixed in H. Because K is compact there is a finite number of continua $P_1, P_2, \ldots, P_n$ and a finite number of open sets $O_1, O_2, \ldots, O_n$ such that $\overline{O}_i\cdot P_i=\varnothing$ and $y\in P_i^0$, $1\leqq i\leqq n$, and for every $z\in K$ there is an integer i, $1\leqq i\leqq n$, such that $z\in O_i$. Let U_y be an open set so that $y\in U_y\subset \overline{U}_y\subset\prod P_i^0$. Now set $K_y^i=\overline{O}_i\cdot K$ for $1\leqq i\leqq n$ and set $H_y=\overline{U}_y\cdot H$. This can be done for each $y\in H$ and so because H is compact there is a finite number of these closed sets $H_{y_1}, H_{y_2}, \ldots, H_{y_m}$ such that $H=\sum H_{y_i}$.

Let V be an open set such that $V\subset \overline{V}\subset U$ and denote by A_{ij} the set of all points p of $\overline{V}$ so that there is $q\in H_{y_i}$ and $r\in K_{y_i}^j$ with M not 2-aposyndetic at q with respect to $p+r$. Again A_{ij} is closed and $\overline{V}=\sum A_{ij}$ is closed; so because $\overline{V}$ is the sum of a finite number of closed sets one of them, say A_{IJ}, contains an open set W. Now, as seen above, H_{y_I} is in the interior of a continuum P such that $P\cdot K^J_{y_I}=\varnothing$. Thus $W\subset P$ for otherwise there would be a point $p\in W-P$ such that for every $q\in H_{y_I}$ and $r\in K^J_{y_I}$, M is 2-aposyndetic at q with respect to $p+r$. Let D be an open set such that $K^J_{y_I}\subset D\subset M-P$, let B be an open set such that $H_{y_I}\subset B\subset \bar{B}\subset P^0-W$, and let C be the component of $M-D$ containing P. Relabel $K^J_{y_I}$ by K_1 and H_{y_I} by H_1. These sets together with W are the required sets in Lemma 2.

LEMMA 3. *Let U be an open set and H, K closed sets of an aposyndetic continuum M such that*: (1) *H and K are contained in open sets B and D respectively such that U, $\bar{B}$, $\bar{D}$ are mutually disjoint and $\bar{B}+U$ is contained in the interior of a component C of $M-D$.* (2) *For each point x in U, there is a point y in H and z in K such that M is not* 2*-aposyndetic at y with respect to $x+z$. Then there exists a point p in U and a point q in K such that $p+q$ cuts M.*

Proof. Let W_1 and V be open sets with $\overline{W}_1\subset V\subset \bar{V}\subset U$. Let us make use of the following result from [7, p. 32]. Suppose that T is an open set, R and S disjoint closed sets lying in $M-T$ such that for each point x in T there are points y in R and z in S with M not 2-aposyndetic at x with respect to $y+z$. Then there exist points p and q in R and S respectively such that $p+q$ cuts M. Since $\bar{V}-W_1$ and K are disjoint closed sets, it follows from this result that if for each point $x\in W_1$, there are points y in $\bar{V}-W_1$ and z in K such that M is not 2-aposyndetic at x with respect to $y+z$, then there are points $p\in \bar{V}-W_1$ and $q\in K$ such that $p+q$ cuts M. Because $\bar{V}\subset U$, then $p\in U$ and the conclusion of the lemma is satisfied. So let us assume that there is a point $x_1\in W_1$ such that for any $y\in \bar{V}-W_1$ and $z\in K$, M is 2-aposyndetic at x_1 with respect to $y+z$.

Because $\bar{V}-W_1$ and K are disjoint and compact, by an argument similar to the one in Lemma 2, there is a finite number of open sets $O_1, O_2, \ldots, O_n$ and for every i, $1\leqq i\leqq n$, a finite number of continua $P^i_1, P^i_2, \ldots, P^i_{n_i}$, and a finite number of closed subsets $K^i_1, K^i_2, \ldots, K^i_{n_i}$ of K such that if $y\in \bar{V}-W_1$, $z\in K$, there exist integers i, j with $y\in O_i$, $z\in K^i_j$, $x_1\in (P^i_j)^0\subset P^i_j\subset M-(O_i+K^i_j)$. No generality is lost by taking each K^i_j to be of diameter less than 1. Now let $T_1=W_1\cdot\prod_{ij}(P^i_j)^0$. The set B cannot be in the same component of $C-T_1$, for otherwise M would be 2-aposyndetic at any point y of H with respect to $p+q$ where p and q are arbitrary points of T_1 and K, respectively. Hence $C-T_1$ has a separation, $C-T_1=E_1+F_1$ such that $E_1\cdot B\neq\varnothing\neq F_1\cdot B$. There is a natural number I such that $O_I\cdot(V-\overline{W}_1)\cdot E_1\neq\varnothing$. Let O be an open set such that $O\subset\bar{O}\subset O_I\cdot(V-\overline{W}_1)\cdot E_1$. If Q_j is the set of all points x of $\bar{O}$ so that there is a $y\in H$ and $z\in K^I_j$ such that M is not 2-aposyndetic at y with respect to $x+z$, then as before Q_j is closed, $\bar{O}=\sum Q_j$ and there is an integer J such that Q_J contains an open set V_1 whose diameter can be chosen less than 1. It follows that $T_1\subset P^I_J\subset M-(V_1+K^I_J)$.

Relabel K^I_J by K_1, P^I_J by P_1 and take D_1 to be an open set containing K_1, lying in $D-P_1$ such that the distance from any point of D_1 to K_1 is less than 1. Denote by C_1 the component of $M-D_1$ containing C.

For the second step of this process, let W_2 be an open set whose closure is contained in V_1. As before we can assume that W_2 contains a point x_2 such that M is 2-aposyndetic at x_2 with respect to $y+z$, y being any point of $\bar{V}_1 - W_2$ and z being any point of K_1. Using the same notation and procedure as in the first step of this construction, there is a finite number of open sets $O_1, O_2, \ldots, O_n$ and for each i, a finite number of closed subsets of K_1, $K_1^i, K_2^i, \ldots, K_{n_i}^i$, and a finite number of continua $P_1^i, P_2^i, \ldots, P_{n_i}^i$, such that (1) dia $K_j^i < 1/2$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$ and (2) for each $y \in \bar{V}_1 - W_2$ and $z \in K_1$ there are integers i, j such that $y \in O_i$, $z \in K_j^i$ and $x_2 \in (P_j^i)^0 \subset P_j^i \subset M-(O_i+K_j^i)$. Let $T_2 = W_2 \cdot \prod_{ij} (P_j^i)^0$. As before, $C_1 - T_2$ has a separation, $C_1 - T_2 = E_2 + F_2$ such that $E_2 \cdot B \neq \varnothing \neq F_2 \cdot B$. Since $P_1 + F_1$ is a continuum lying in $C_1 - T_2$, let us assume that $P_1 + F_1 \subset F_2$. There is an integer I, $1 \leqq I \leqq n$, such that $O_I \cdot (V_1 - \bar{W}_2) \cdot E_2 \neq \varnothing$. Let O be an open set with $\bar{O} \subset O_I \cdot (V_1 - \bar{W}_2) \cdot E_2$. There is an integer J and an open set V_2 of diameter less than $1/2$ such that $\bar{V}_2 \subset O$ and for every $x \in \bar{V}_2$ there are points $y \in H$ and $z \in K_J^I$ such that M is not 2-aposyndetic at y with respect to $x+z$. Again $T_2 \subset P_J^I \subset M-(V_2+K_J^I)$.

Relabel K_J^I by K_2, P_J^I by P_2 and let D_2 be an open set such that $K_2 \subset D_2 \subset D_1 - P_2$ and every point of D_2 is at a distance less than $1/2$ from K_2. Let C_2 be the component of $M - D_2$ containing C_1.

Continuing in this way, let $p = \prod V_i = \prod \bar{V}_i$, $q = \prod K_i = \prod D_i$, $x \in \prod (E_i \cdot \bar{B})$ and $y \in F_1 \cdot \bar{B}$. Now $p+q$ cuts x from y, for if there were a continuum Q such that $x+y \subset Q \subset M-(p+q)$, there would be an integer n such that $Q \cdot (V_n + D_n) = \varnothing$. Then $Q \subset C_n - V_n$, hence $Q \subset C_n - T_{n+1} = E_{n+1} + F_{n+1}$ with $x \in Q \cdot E_{n+1}$, $y \in Q \cdot F_{n+1}$. This is impossible and the proof is complete.

As a direct application of these three lemmas we have

THEOREM 6. *Let M be an aposyndetic* (*slc*) *continuum and let U be an open set in M. Suppose that M is not* 2-*slc at any point of U. Then U contains a point p and M contains a point q such that $p+q$ cuts M.*

As an immediate consequence of Theorem 6 we have

THEOREM 7. *Let M be a semi-locally-connected continuum that is* 2 *semi-locally-connected at no point. Then M contains a dense set of points S with the property that for each point $p \in S$ there is a point $q \in S$ such that $p+q$ cuts M.*

EXAMPLE. Erect vertical intervals of length one at each point of the Cantor set of the interval [0, 1] of the x-axis. At the bottom collapse to a point the end points of the vertical intervals and at the top do likewise to the other end points of the intervals (i.e., the two-point suspension of the Cantor set).

This continuum is semi-locally-connected at every point but is 2-semi-locally-connected at no point. In this continuum, if U is any open set, not only p but q as well can be found in U such that $p+q$ cuts. Theorem 6 guarantees only that p can be found in U. Under these conditions whether a point q "close to p" can always be found so that $p+q$ cuts is not known.

Another consequence of Theorem 6 is the next theorem.

THEOREM 8. *If M is a semi-locally-connected continuum and for no pair of points p and q does $p+q$ cut M, then M is 2-semi-locally-connected on a dense G_δ subset of M.*

In [2], E. E. Grace has proved results for which the next two theorems are consequences. Here the continuum M does not need to be aposyndetic.

THEOREM 9. *If M is 2-aposyndetic at none of its points then there exists a pair of points p and q in M such that $p+q$ cuts M.*

THEOREM 10. *If M is a continuum not cut by the sum of any pair of its points, then M is 2-aposyndetic on a dense G_δ subset of M.*

By combining Theorems 8 and 10 we obtain a result for aposyndetic continua analogous to Theorem 18 of Jones' paper [4].

THEOREM 11. *If M is an aposyndetic continuum and for no pair of its points p and q does $p+q$ cut M, then M is both 2-aposyndetic and 2-semi-locally-connected on a dense G_δ subset of M.*

Next we give an example to show that a continuum M that is 2-aposyndetic at none of its points need have no more than one pair of points whose sum cuts M. One such pair is guaranteed by Theorem 9. This example also shows that Theorem 6 fails without the additional requirement that M be semi-locally-connected.

EXAMPLE. Start with a $\sin 1/x$ curve, the limiting interval pq, and at the end point y let two rays branch out and connect at p and at q as shown in Figure 1. Now expand this figure to three dimensions in the following manner:

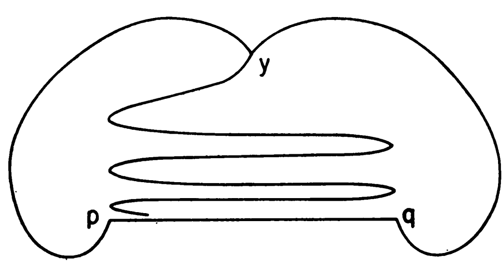

FIGURE 1

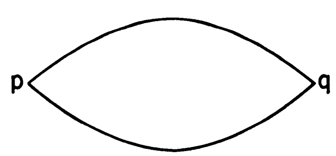

FIGURE 2

(1) Consider that part of the continuum in Figure 1 consisting of the sin $1/x$ curve and the interval pq. Take the cartesian product of this subcontinuum with the unit interval I. Then alter the resulting 3-dimensional continuum by replacing the rectangle $(pq) \times I$ by the plane 2-dimensional continuum in Figure 2 and requiring the sin $1/x$ part to close down to this set instead of $(pq) \times I$.

(2) At y multiply the right "horn" of Figure 1 by the interval [0, 1/4] and let it gradually taper down to q; multiply the left "horn" by the interval [3/4, 1] and let it taper down to p.

(3) From the continuum thus far obtained subtract the limiting set (Figure 2) and take an uncountable number c copies of the resulting connected set. Locate these in 3-space so that: (A) Each pair is disjoint. (B) Each one has the same limiting set in 3-space, that being the set pq (Figure 2), with the left "horns" and right "horns" of all of them being joined at p and q respectively. (C) The sum of the sin $1/x$ portions of the c sets projected into the xy-plane would be the sin $1/x$ portion of the continuum of Figure 1 with a cross section of a Cantor discontinuum instead of a single point. The projection of the sum of the "left horns" into the xy-plane gives the "left horn" of Figure 1 with a cross section also of a Cantor discontinuum instead of a point. A similar statement holds for "right horns."

This continuum M is not semi-locally-connected at p or at q and although it is 2-slc at none of its points the conclusions of Theorems 6 and 7 fail here since the only pair of points whose sum cuts M is p and q. (By a slight alteration this continuum could even fail to have one pair of points whose sum cuts M.) Also M is not 2-aposyndetic at any of its points and thus illustrates that such a continuum need have no more than one pair of points whose sum cuts M.

The continuum M is not even aposyndetic at any point of the limiting set pq of Figure 2. However, it is possible to alter this example in such a way that it *is* aposyndetic at every point but even so is still 2-aposyndetic at no point and has but one pair of points whose sum cuts M.

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