RECURSION THEORY AND DEDEKIND CUTS(1)

BY
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Considerable attention has been paid by mathematicians to the subject of recursive analysis, and in particular to the recursive real numbers, i.e. the class of Dedekind cuts which can be defined by an effective algorithm. From the point of view of recursion theory, however, it is more natural to consider certain nonrecursive Dedekind cuts, especially those which are recursively enumerable (r.e.) because most results in recursion theory are trivial at the level of recursive sets. In addition to generalizing well-known properties of recursive real numbers, the study of r.e. Dedekind cuts has applications to ordinary recursion theory. The dense linear ordering imposed by the rationals enables simplification of some ordinary proofs, and gives rise to several new order analogues of standard properties. Although our results were first derived for cuts, most are easily generalized to Jockusch's semirecursive sets [5], which may be viewed as generalized "cuts" in some recursive linear ordering of N.

We begin by establishing the reducibility relationships between the standard definitions of a real number. We prove that there are no creative or quasicreative Dedekind cuts, although there is a Dedekind cut in every truth table degree. We use a priority argument to construct a r.e. Dedekind cut which is not a cylinder. This has as a corollary the result of P. R. Young [26] that there are pseudo-creative sets which are not splinters. We develop the cylinder properties of cuts, and disprove the order analogue of Myhill's theorem. Finally, we generalize a theorem of Yates [23] by constructing a semicreative Dedekind cut of every Turing degree, and as a corollary we generalize Jockusch's theorem [4] that in every Turing degree there is an m-degree consisting of a single 1-degree.

The dense linear ordering apparently prevents the classification of r.e. Dedekind cuts by the standard division of r.e. sets into categories such as creative sets or simple sets. However, some of our main results may be summarized as attempts to partially classify r.e. (lower) Dedekind cuts by certain classes of fixed point free maps which preserve them. Let $R^L(R^U)$ denote the class of real numbers in the

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interval [0, 2] whose lower (upper) Dedekind cut is r.e., Q the set of rationals in [0, 2], and define,

$$F(R^U, Q) = \{f \mid f \text{ a recursive function from } R^U \text{ to } Q \text{ and } (\alpha)[\alpha \in R^U \Rightarrow f(\alpha) \neq \alpha]\},$$
$$F^-(R^U, Q) = \{f \mid f \in (R^U, Q) \ \& \ (\alpha)[0 \neq \alpha \in R^U \Rightarrow f(\alpha) < \alpha]\}.$$

In similar fashion, define $F^-(R^U, R^U)$, $F(Q, Q)$ and $F^-(Q, Q)$. A recursive function $f \in F(\cdot, \cdot)$ is said to *preserve* the real number α if

$$(\beta)[\beta \in \text{domain } f \Rightarrow [[\beta > \alpha \Rightarrow f(\beta) > \alpha] \ \& \ [\beta \leqq \alpha \Rightarrow f(\beta) \leqq \alpha]]].$$

We now uniformly summarize some main results:

(1) For all nontrivial $\alpha \in R^L$ and $f \in F^-(R^U, Q)$, f fails to preserve α (Corollary 1.11).

(2) In every Turing degree there exists some $\alpha \in R^L$, and $f \in F^-(R^U, R^U)$ such that f preserves α (Theorem 4.4).

(3) There is an $\alpha \in R^L$ not preserved under any $f \in F(Q, Q)$ (Theorem 3.5). (Since the functions in $F(Q, Q)$ are much weaker than those whose domain is R^U, it is trivial to find elements $\alpha \in R^L$ preserved under some $f \in F^-(Q, Q)$, but more interesting to find elements *not* preserved.)

Background material on recursion theory may be found in Kleene [6], and Rogers [17]. We use the standard enumeration of r.e. sets, $W_0, W_1, \ldots$, that is obtained by setting $W_e = \{x \mid (\exists y)T_1(e, x, y)\}$ for each e; and we set

$$W_e^z = \{x \mid (\exists y)_{<z} T_1(e, x, y)\}$$

for each e and z. For natural numbers $x < y$, $I[x, y]$ will denote the finite set $\{x, x+1, x+2, \ldots, y\}$. We will also use the standard effective indexing of the finite sets, $\{D_x\}$. Namely, if $x_1, x_2, \ldots, x_n$ are distinct natural numbers, and $x = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$, then D_x denotes $\{x_1, x_2, \ldots, x_n\}$, and D_0 denotes the empty set $\varnothing$. The standard pairing function is $j(x, y) = x + (\frac{1}{2})(x+y)(x+y+1)$, and $k(x)$, $l(x)$ are recursive functions such that $x = j(k(x), l(x))$ for all x.

Lower case Greek letters denote real numbers and $[\alpha, \beta]$ (respectively (α, β), $(\alpha, \beta]$) denotes the closed (respectively open, half-open) real interval with endpoints α and β, as in ordinary real analysis. We will refer to the standard reducibilities, such as many-one ($m : 1$), truth table (tt), and Turing (T) reducibilities, which are defined by Post [14]. If R is a reducibility, and A, B are sets of integers, $A \leqq_R B$ will denote "A is R-reducible to B." An *R-degree* is the collection of all sets which are R-equivalent to some set, and a *degree* is a Turing degree. A set A is *R-complete* if it is r.e. and each r.e. set is R-reducible to it. If $A \subseteq N$, A' denotes the complement $N - A$.

1. **Creativity and many-one reducibility on Dedekind cuts.** With each subset $A \subseteq N$, there is naturally associated a real number in the interval [0, 2], namely $\Phi(A) = \sum_{n \in A} 2^{-n}$ and $\Phi(\varnothing) = 0$. Let Q denote the set of rationals in the interval

[0, 2], and fix a one-one effective map from N onto Q, denoting the image of n under this map by the **bold** face $\boldsymbol{n}$. Similarly, for any function f on N, the rational image of the natural number $f(n)$ is denoted by $\boldsymbol{f(n)}$, and the image of the set $A \subseteq N$ by $\boldsymbol{A}$. (The map serves merely to effectively index the rationals, and it will be clear that all results are independent of the particular map chosen.) Identifying each n with its image $\boldsymbol{n}$, the lower Dedekind cut, $L(A)$, associated with the set A is defined to be

$$L(A) = \{n \mid \boldsymbol{n} \leqq \Phi(A)\}.$$

(From now on "cut" will always mean Dedekind cut.) It is well known in recursive analysis, and easy to prove (see [15]):

LEMMA 1.1. *A is recursive if and only if $L(A)$ is recursive.*

Furthermore, $\Phi(A)$ is defined to be a *recursive real number* if $L(A)$ is a recursive set. Now in generalizing to the r.e. sets, we find that the equivalence in Lemma 1.1 no longer holds if "recursive" is replaced by "recursively enumerable." Instead we have: A r.e. implies $L(A)$ r.e., but not conversely. (The positive direction is obvious. At the end of §3 we give a counterexample to the other direction.) Nevertheless, our first theorem establishes a generalization of Lemma 1.1 for arbitrary $A \subseteq N$, namely that $A \equiv_T L(A)$.

THEOREM 1.2. *For any $A \subseteq N$, $[A \leqq_{tt} L(A)\ \&\ L(A) \leqq_T A]$.*

Proof. Assume that $\Phi(A)$ is not a rational, else A and $L(A)$ are both recursive, and the conclusion is immediate. To prove that $L(A) \leqq_T A$, observe that since $\Phi(A)$ is nonrational,

$$n \in L(A) \Leftrightarrow (\exists k)[\boldsymbol{n} < \Phi(A \cap I[0, k])]$$

and

$$n \in (L(A))' \Leftrightarrow (\exists k)[\boldsymbol{n} > \Phi(A \cap I[0, k])].$$

Hence, by Post's theorem (Kleene [6, p. 293]), $L(A) \leqq_T A$. We omit the straightforward induction which uses the fact that $\Phi(A)$ is nonrational to establish that $A \leqq_{tt} L(A)$. We merely observe that,

$$0 \in A \Leftrightarrow \Phi(\{0\}) \in \boldsymbol{L}(A),$$

and that

$$n \in A \Leftrightarrow \Phi((A \cap I[0, n-1]) \cup \{n\}) \in \boldsymbol{L}(A).$$

The reducibility relations established in Theorem 1.2 are in general the strongest standard reducibilities which hold between A and $L(A)$, because:

(i) In §2 we construct a set A such that $L(A) \nleqq_{tt} A$ (Theorem 2.3);

(ii) If M is a maximal set, then $M \nleqq_{btt} L(M)$ Jockusch [5, Corollary 4.6].

C. G. Jockusch [5] has introduced a new notion which is relevant to the study of cuts.

DEFINITION 1.3 (JOCKUSCH). A set $A \subseteq N$ is *semirecursive* if there is a recursive function f of two variables such that for all x and y,

(i) $f(x, y) = x$ or $f(x, y) = y$ and

(ii) $[x \in A$ or $y \in A] \Rightarrow f(x, y) \in A$.

It can be shown (see the construction of McLaughlin and Appel in [5, Theorem 4.1]) that a set A is semirecursive if and only if A is an initial segment of some recursive linear ordering of N. Since any recursive linear ordering of N can be effectively embedded in the ordering of the rationals we immediately have

THEOREM 1.4. *A set $A \subseteq N$ is semirecursive if and only if there is a recursive one-one map ψ_A from N into Q, and a cut $L(B)$, such that $\psi_A(A) \subseteq \boldsymbol{L}(\boldsymbol{B})$ and $\psi_A(A') \subseteq (\boldsymbol{L}(\boldsymbol{B}))'$.*

Note that every cut is a semirecursive set. In fact, the cuts are those semirecursive sets for which ψ_A may be chosen to be *onto* Q. For each semirecursive set A, we choose some ψ_A according to the theorem and regard it as fixed from now on. The following results were first derived for cuts, but can be established for the more general case of semirecursive sets by virtually the same proofs.

LEMMA 1.5. *If A and B are semirecursive and $A \leqq_m B$, then $A \leqq_m B$ via a recursive function f such that*

$$(x)(y)[\psi_A(x) < \psi_A(y) \Rightarrow \psi_B(f(x)) \leqq \psi_B(f(y))].$$

Proof. To avoid notational complications we will prove only the case where A and B are cuts, and where $\psi_A(x) = \psi_B(x) = \boldsymbol{x}$ for all x. It will be clear how to modify the proof for the general case. With these assumptions let $A \leqq_m B$ via a recursive function g. If $\boldsymbol{g}$ (the recursive function on Q induced by g) is not already order preserving, say $(\exists x)(\exists y)[\boldsymbol{x} > \boldsymbol{y}\ \&\ \boldsymbol{g}(\boldsymbol{x}) < \boldsymbol{g}(\boldsymbol{y})]$, then either $x, y \in A$ or $x, y \in A'$ because $\boldsymbol{g}(\boldsymbol{A}) \subseteq \boldsymbol{B}$. Hence, the following recursive function $\boldsymbol{f}$ is clearly an $m{:}1$ reduction of A to B such that $\boldsymbol{f}$ is order preserving. Define f by induction. For $x \geqq 0$ set

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}) &= \max\{\boldsymbol{f}(\boldsymbol{y}) \mid 0 \leqq y < x\ \&\ \boldsymbol{y} < \boldsymbol{x}\} \text{ if } (\exists y)_{<x}[\boldsymbol{y} < \boldsymbol{x}\ \&\ \boldsymbol{f}(\boldsymbol{y}) \geqq \boldsymbol{g}(\boldsymbol{x})] \\ &= \min\{\boldsymbol{f}(\boldsymbol{y}) \mid 0 \leqq y < x\ \&\ \boldsymbol{y} > \boldsymbol{x}\} \text{ if } (\exists y)_{<x}[\boldsymbol{y} > \boldsymbol{x}\ \&\ \boldsymbol{f}(\boldsymbol{y}) \leqq \boldsymbol{g}(\boldsymbol{x})] \\ &= \boldsymbol{g}(\boldsymbol{x}) \quad \text{otherwise.} \end{aligned}$$

DEFINITION 1.6. A subset C of a semirecursive set A is *cofinal in A* (denoted $C \underset{\to}{\subseteq} A$) if

$$(n)[n \in A \Rightarrow (\exists m)[m \in C\ \&\ \psi_A(m) \geqq \psi_A(n)]].$$

Note that every r.e. semirecursive set A has a recursive subset cofinal in A, and that any semirecursive set containing an r.e. cofinal subset is itself r.e.

THEOREM 1.7. *If A and B are semirecursive, A is nonrecursive, $A \leqq_m B$ via recursive function f, and $C \underset{\to}{\subseteq} A$, then $f(C) \underset{\to}{\subseteq} B$.*

Proof. We may assume that f satisfies the conclusion of Theorem 1.5. Now if $C \subsetneqq A$ but $f(C) \not\subsetneqq B$, then there exists $y \in B$ such that

$$(x)[x \in A \Leftrightarrow \psi_B(f(x)) < \psi_B(y)].$$

Hence, since ψ_B and f are recursive functions, A is recursive contrary to hypothesis.

This gives rise to an amusing property of semirecursive r.e. sets not shared by ordinary r.e. sets.

COROLLARY 1.8. *If A and B are semirecursive, A is nonrecursive, and $A \leqq_m B$, then A is r.e. if and only if B is r.e.*

Proof. Let $A \leqq_m B$ via recursive function f. By Theorem 1.7, $f(A) \subsetneqq B$, and $f^{-1}(B) \subsetneqq A$.

It can be shown (see [5, Theorem 4.2]) that if $A \leqq_m S$, where S is semirecursive, then A is semirecursive also. Thus, after showing that there are r.e. sets which are not semirecursive, Jockusch concludes that there is no r.e. semirecursive set (hence, no r.e. cut) which is m:1 complete. We can strengthen this result as a corollary of the following theorem, which further illustrates the difference between semirecursive sets and ordinary sets.

THEOREM 1.9. *If S is semirecursive, A is nonrecursive, and $A \leqq_m S$, then $S \leqq_T A$.*

Proof. By Jockusch's result A must also be semirecursive. Since A is nonrecursive $f(A) \subsetneqq S$ and $f(A') \subsetneqq S'$. Clearly, then S' is r.e. in A' and S is r.e. in A, and hence $S \leqq_T A$. (Theorem 2.8 will demonstrate that in general this conclusion cannot be strengthened to "$S \leqq_{tt} A$".)

(Rogers [17] defines a set A to be *norm*:1 *reducible* to a set B if there is a recursive function f such that $(x)[x \in A \Leftrightarrow f(x) \in B \text{ join } B']$, i.e. if and only if $A \leqq_{btt} B$ with a single column in each truth table. By the appropriate modification of Theorem 1.7, we can strengthen Theorem 1.9 to the following: If S is semirecursive, A is nonrecursive and A is norm:1 reducible to S, then $S \leqq_T A$.)

By Theorem 1.2 there is a cut (and thus a semirecursive set) of every Turing degree. In particular, there are cuts which are not Turing complete, and thus we have,

COROLLARY 1.10. *There is no semirecursive set (or cut) which is m:1 complete even with respect to the class of semirecursive r.e. sets (respectively r.e. cuts). (That is, there is no r.e. semirecursive set S such that $(A)[A$ r.e. and semirecursive $\Rightarrow A \leqq_m S]$.)*

A set P is *productive* if there is a partial recursive function f such that for all e, if $W_e \subset P$, then $f(e)$ is defined and $f(e) \in P - W_e$. An r.e. set C is creative if C' is productive. Myhill [12] proved that any creative set is m:1 complete. (In fact, he proved that if P is productive, and A is r.e., then $A \leqq_m P'$.)

COROLLARY 1.11 (JOCKUSCH). *There is no semirecursive set which is a creative* (*or even productive*) *set.*

Shoenfield defined a set P to be *quasiproductive* if there is a partial recursive function f such that for all e, if $W_e \subset P$, then $f(e)$ is defined, $D_{f(e)} \subset P$, and $D_{f(e)} \not\subset W_e$. An r.e. set A is *quasicreative* if A' is quasiproductive. Clearly, every creative set is quasicreative. The converse fails for ordinary sets, but on semirecursive sets we have,

LEMMA 1.12. *For any semirecursive set A, if A is quasiproductive, then A is productive.*

Proof. Let A be quasiproductive via the partial recursive function f. Define the recursive function h as follows:

$$W_{h(e)} = \{x \mid (\exists y)[y \in W_e \ \& \ \psi_A(y) \geqq \psi_A(x)]\}.$$

Now A is easily seen to be productive via the partial recursive function g defined as follows:

$$g(x) = y, \quad \text{where } y \text{ is such that } \psi_A(y) = \max \{\psi_A(z) \mid z \in D_{f(h(x))}\}.$$

COROLLARY 1.13. *There is no quasicreative* (*quasiproductive*) *semirecursive set.*

In view of Corollaries 1.11 and 1.13, it is now natural to ask whether the recursive linear ordering of the rationals will permit even weak forms of creativity such as semicreativity (defined by Dekker [1].) This question is answered affirmatively in §4.

T. G. McLaughlin has observed another interesting corollary of Theorem 1.9. If C is a creative set, then for every r.e. set A, $A \leqq_m C$. However, the deficiency set of C, defined in [2, p. 365], denoted D_C, is semirecursive (Jockusch [5, Theorem 3.2]). Thus by Theorem 1.9, if $A \leqq_m D_C$, then A has degree either $\mathbf{0}$ or $\mathbf{0}'$. Thus passing to the deficiency set "filters out" most m:1 reducibility, although it preserves Turing degree.

2. **Dedekind cuts and truth table reducibility.** In this section we justify our parenthetical remark following Theorem 1.2 by constructing a set A such that $L(A)$ is not tt-reducible to A (Theorem 2.3). After constructing a cut of every tt-degree, we show how tt-reducibility *on cuts* can be conveniently characterized in terms of conditions on certain half open real intervals (Corollary 2.7). We then use this characterization (in Theorem 2.8) to construct nonrecursive r.e. cuts, $L(A)$ and $L(B)$, such that $L(A) \leqq_m L(B)$, but $L(B) \not\leqq_{tt} L(A)$ thus establishing Theorem 1.9 as the best (standard) reducibility result. It will be convenient to have in mind the following definition of truth table reducibility.

DEFINITION 2.1. $B \leqq_{tt} A$ if there are recursive functions $f(x)$, $g(x)$ such that for all x,

(1) $(y)[y \in D_{f(x)} \Rightarrow D_y \subseteq D_{g(x)}]$ and

(2) $x \in B \Leftrightarrow (\exists y)[y \in D_{f(x)} \ \& \ D_y \subseteq A \ \& \ (D_{g(x)} - D_y) \subseteq A']$.

The conditions on A can be rephrased as conditions on the real number $\Phi(A)$ by using the initial binary expansions of real numbers which lie in certain open real intervals. In the following lemma we exclude the case where $\Phi(A)$ is rational so as to avoid certain rationals with two distinct binary expansions. The lemma is used only to facilitate the proof of Theorem 2.3.

LEMMA 2.2. *If $\Phi(A)$ is not rational, then $B \leqq_{tt} A$ if and only if there are recursive functions $f(x)$, $m(x)$ such that for all x,*

(1) $(y)[y \in D_{f(x)} \Rightarrow D_y \subseteq I[0, m(x)]]$ *and*

(2) $x \in B \Leftrightarrow (\exists y)[y \in D_{f(x)}\ \&\ \Phi(A) \in (\Phi(D_y), \Phi(D_y)+2^{-m(x)})]$.

Proof. Let $B \leqq_{tt} A$. Given $g(x)$ as in Definition 2.1 define $m(x) = \max\{z \mid z \in D_{g(x)}\}$. If $\Phi(A)$ is nonrational then for $D_y \subseteq I[0, m(x)]$,

$$\Phi(D_y) < \Phi(A) < \Phi(D_y)+2^{-m(x)} \Leftrightarrow A \cap I[0, m(x)] = D_y.$$

The converse is clear.

Let Q_F denote the subset of Q consisting of those rationals which can be given by a *finitely nonzero* binary expansion. Namely,

$$Q_F = \{\boldsymbol{x} \mid (\exists y)[\boldsymbol{x} = \Phi(D_y)]\}. \tag{2.1}$$

(Note that for any x, $\boldsymbol{x} \in Q$ is effectively presented as a quotient of natural numbers, so we can effectively determine whether $\boldsymbol{x} \in Q_F$, or $\boldsymbol{x} \in Q - Q_F$.)

(Theorem 2.6 which will be proved for semirecursive sets in general appears to imply Lemma 2.2, but notice that in Lemma 2.2 the rational endpoints are in Q_F not just in Q. This is essential for the proof of Theorem 2.3.)

THEOREM 2.3. *There is a r.e. cut $L(A)$ such that $L(A) \nleqq_{tt} A$.*

Proof. Effectively enumerate all potential truth table reductions (tt-reductions) by enumerating all partial recursive functions of one variable, $\{h_e(x)\}$, and letting the potential tt-reduction of index e correspond as defined in Lemma 2.2 to the pair of functions, $\{f_e(x), m_e(x)\}$, where $f_e(x) = k(h_e(x))$, and $m_e(x) = l(h_e(x))$. (This will be a tt-reduction if the latter functions are defined for all x.)

We will construct a real number, $\Phi(A)$, by a sequence of nested intervals, $\{[\boldsymbol{x}_e, \boldsymbol{y}_e]\}$, such that $\Phi(A) \in [\boldsymbol{x}_e, \boldsymbol{y}_e]$ will imply that the potential tt-reduction of index e does not reduce $L(A)$ to A. These intervals are constructed by defining at each stage s, the intervals $\{[\boldsymbol{x}_e^s, \boldsymbol{y}_e^s]\}$ for every e such that

$$(s)(e)[[\boldsymbol{x}_{e+1}^s, \boldsymbol{y}_{e+1}^s] \subseteq [\boldsymbol{x}_e^s, \boldsymbol{y}_e^s]], \tag{2.2}$$

$$(s)(e)[0 < \boldsymbol{y}_e^s - \boldsymbol{x}_e^s \leqq 2^{-e}], \tag{2.3}$$

$$(s)(e)[\boldsymbol{x}_e^s \leqq \boldsymbol{x}_e^{s+1}], \tag{2.4}$$

$$(e)[\lim_s \boldsymbol{x}_e^s = \boldsymbol{x}_e \quad \text{and} \quad \lim_s \boldsymbol{y}_e^s = \boldsymbol{y}_e]. \tag{2.5}$$

Stage $s=0$. For every i, set $x_i^0=0$ and $y_i^0=2^{-i}$. For convenience in treating the case $e=0$, define $x_{-1}^s=0$, and $y_{-1}^s=2$ for all $s\geqq 0$.

Stage $s>0$. Let $e=k(s)$, and define

$$z_e^s = \mu s[z\in (y_e^{s-1}, y_{e-1}^{s-1})\cap (Q-Q_F)].$$

Case I. Either $x_e^{s-1}>x_{e-1}^{s-1}$, or $h_e(z_e^s)$ is not defined by stage s. For all i, set $x_i^s=x_i^{s-1}$, and $y_i^s=y_i^{s-1}$. (It will be clear from the construction as a whole that if $x_e^{s-1}>x_{s-1}^{e-1}$, then, in Sack's terminology [17], the eth requirement has been met at an earlier stage and not injured at any subsequent stage.)

Case II. Otherwise, $x_e^{s-1}=x_{e-1}^{s-1}$, and $f_e(z_e^s)$ and $m_e(z_e^s)$ are both defined.

Let $u'=\mu u[D_u\subseteq I[0, m]\ \&\ z_e^s\in (\Phi(D_u), \Phi(D_u)+2^{-m})]$, where $m=m_e(z_e^s)$.

Define $[x_e^s, y_e^s]=[k(v'), l(v')]$, where v' is the least v such that:

$$k(v), l(v)\in Q_F, \tag{2.6}$$

$$0 < l(v)-k(v) < 2^{-e}, \tag{2.7}$$

$$[k(v), l(v)] \subseteq (y_e^{s-1}, y_{e-1}^{s-1}), \tag{2.8}$$

$$(i)_{\leqq e}[i\notin [k(v), l(v)], \tag{2.9}$$

$$\begin{aligned}[k(v), l(v)] &\subseteq (\Phi(D_{u'}), z_e^s) && \text{if } u'\in D_{f_e}(z_e^s),\\ &\subseteq (z_e^s, \Phi(D_{u'})+2^{-m}) && \text{otherwise.}\end{aligned} \tag{2.10}$$

(Note that because $z_e^s\in Q-Q_F$, while $\Phi(D_{u'})$, $\Phi(D_{u'})+2^{-m}$, y_e^{s-1}, and $y_{e-1}^{s-1}\in Q^s$, z_e^s is never equal to any of these endpoints.)

For all $i<e$, set $x_i^s=x_i^{s-1}$, and $y_i^s=y_i^{s-1}$.

For all $i>e$, set $x_i^s=x_e^s$, and

$$y_i^s = \mu y[y\in Q_F\ \&\ y < y_{i-1}^s\ \&\ 0 < y-x_i^s < 2^{-i}].$$

We omit the straightforward induction on e which proves simultaneously that $\lim_s x_e^s$ and $\lim_s y_e^s$ exist. By (2.7) and (2.8), $\bigcap_e [x_e, y_e]$ contains a unique real number, say α. The associated lower cut, $L_\alpha=\{x \mid x\leqq\alpha\}$ is r.e. because by our construction (2.4) holds and the r.e. set $\bigcup_{s,e}\{x_e^s\}$ is cofinal in L_α. Furthermore, α is nonrational because by (2.9) each rational is excluded from at least one interval. Being nonrational, α has a unique binary expansion so there is a unique set A such that $\Phi(A)=\alpha$, and $L(A)=L_\alpha$.

Since $\Phi(A)$ is nonrational, we can apply Lemma 2.2 to prove that $L(A)\nleqq_{tt}A$. Suppose to the contrary that $L(A)\leqq_{tt}A$ by a tt-reduction of index e. Then $h_e(x)$ must be (total) recursive, and thus Case II in the definition of $[x_e^s, y_e^s]$ holds at some stage s such that $e=k(s)$ and $(i)_{\leqq e}[x_i=x_i^s\ \&\ y_i=y_i^s]$. Now suppose that the first clause of (2.10) holds, namely that $[x_e^s, y_e^s]\subseteq(\Phi(D_{u'}), z_e^s)$ and $u'\in D_{f_e(z_e^s)}$. Then as in Lemma 2.2, the tt-reduction of index e asserts that because $u'\in D_{f_e(z_e^s)}$, we must have $z_e^s\in L(A)$. However, $\Phi(A)<z_e^s$ implies $z_e^s\notin L(A)$, a contradiction. A similar contradiction arises when the other clause of (2.10) holds, because there is only

one candidate in $D_{f_e(z_e^s)}$ (namely u') for a tt-reduction. This completes the proof of the theorem.

By Theorem 2.3, it is false in general that $L(A) \equiv_{tt} A$. Nevertheless, it is easy to show that there is a cut of every tt-degree. (The following result is similar to Theorem 3.6 in [5] where Jockusch proves that there is a semirecursive set of every tt-degree. In fact, he proves that $A \equiv_{tt} L_F(A)$, where $L_F(A) = L(A) \cap Q_F$. For genuine Dedekind cuts, however, we have seen that in general $L(A) \not\leq_{tt} A$. We will show that in every tt-degree there is a set A such that $L(A) \equiv_{tt} A$.)

THEOREM 2.4. *For every set B, there is an A such that $B \equiv_m A \equiv_{tt} L(A)$.*

Proof. Given B, if B is recursive, let $A = B$. Otherwise, $A = \{2^y \mid y \in B\}$. Clearly $B \equiv_m A$. Since A is nonrecursive, in particular A is nonrational. By the well-known properties of binary expansions of rationals, for any rational $\boldsymbol{x} \in Q$, either $\boldsymbol{x} \in Q_F$ (see (2.1)), or $\boldsymbol{x} \in Q - Q_F$, and $\boldsymbol{x}$ has an (eventually) repeating binary expansion. In the latter case, define $[\boldsymbol{x}]_i = 0$ or 1 according as the ith place in the (unique) binary expansion if $\boldsymbol{x}$ is 0 or 1. We can define the recursive function,

$$
\begin{aligned}
m(x) &= 1 && \text{if } \boldsymbol{x} = 2\\
&= 1 + \max\{z \mid z \in D_y\} && \text{if } \boldsymbol{x} \in Q_F \;\&\; \boldsymbol{x} = \Phi(D_y)\\
&= \mu i[[\boldsymbol{x}]_i = 1 \;\&\; i \notin \{2^y \mid y \in N\}] && \text{if } \boldsymbol{x} \in Q - Q_F.
\end{aligned}
$$

Since $\Phi(A)$ is nonrational, it is immediate from the definition of $m(x)$ that for every x, $\boldsymbol{x} \neq \Phi(A \cap I[0, m(x)])$, and hence that $x \in L(A) \Leftrightarrow [\boldsymbol{x} < \Phi(A \cap I[0, m(x)])]$. This procedure can easily be formulated as a tt-reduction in the form of Definition 2.1.

THEOREM 2.5. *There is a tt-complete cut.*

Proof. If A is a tt-complete set, then $L(A)$ is r.e., and $L(A)$ is tt-complete since $A \leqq_{tt} L(A)$ by Theorem 1.2.

The tt-reduction of Definition 2.1 assumes an especially simple form if A is semi-recursive.

THEOREM 2.6. *Let S be semirecursive but neither $\varnothing$ nor N. Then $B \leqq_{tt} S$ if and only if there is a recursive function $g'(x)$ such that*

$$x \in B \Leftrightarrow (\exists y)[y \in D_{g'(x)} \;\&\; k(y) \in S \;\&\; l(y) \in S'].$$

Proof. Let $B \leqq_{tt} S$ via recursive functions f and g as in Definition 2.1. We may assume that for all x, $D_{g(x)}$ intersects S and S'. Let $D_{g(x)} = \{q_i\}_{i=1}^{C(x)}$ such that $\psi_S(q_1) < \psi_S(q_2) < \cdots < \psi_S(q_{C(x)})$, where $C(x) = \text{card } D_{g(x)}$. Now we may eliminate any elements of $D_{f(x)}$ superfluous to the tt-reduction by defining $D_{p(x)} \subseteq D_{g(x)}$ as follows,

$$D_{p(x)} = \{q_i \mid 1 \leqq i < C(x) \;\&\; (\exists y)[D_y = \{q_1, q_2, \ldots, q_i\} \;\&\; y \in D_{f(x)}]\}.$$

If we define $D_{g'(x)} = \{j(q_i, q_{i+1}) \mid q_i \in D_{p(x)}\}$ the assertion follows. The converse is clear.

We will use the following special case of Theorem 2.6 along with this notation in our proof of Theorem 2.8.

COROLLARY 2.7. *If $L(A)$ is neither $\varnothing$ nor N, then $B \leqq_{tt} L(A)$ if and only if there are recursive functions $p(x)$ and $g(x)$ such that for all x,*

(1) $D_{p(x)} \subseteq D_{g(x)} = \{q_i\}_{i=1}^{C(x)}$, *where* $C(x) = \text{card } D_{g(x)}$,

(2) $q_1 < q_2 < \cdots < q_{C(x)}$,

(3) $x \in B \Leftrightarrow (\exists i)_{<C(x)}[q_i \in D_{p(x)} \,\&\, \Phi(A) \in [q_i, q_{i+1})]$.

Theorem 1.9 established that for nonrecursive cuts $L(A)$ and $L(B)$, if $L(A) \leqq_m L(B)$, then $L(B) \leqq_T L(A)$. This, however, leaves open the question of whether there is a stronger result, such as: for nonrecursive cuts $L(A)$ and $L(B)$, does $L(A) \leqq_m L(B)$ imply that $L(B) \leqq_m L(A)$? Of course, such an assertion cannot hold for *ordinary* sets even if they are of the same degree, because for example, no creative set is m:1 reducible to a simple set. An analagous proof for cuts fails, however, because the linear ordering of the rationals imposes such great uniformity on cuts that no cut can be a simple or creative set. The following theorem settles these questions.

THEOREM 2.8. *There are nonrecursive r.e. cuts $L(A)$ and $L(B)$ such that $L(A) \leqq_m L(B)$, and $L(B) \nleqq_{tt} L(A)$.*

Proof. The proof depends upon a more complicated version of the nested interval priority construction used in proving Theorem 2.3. Here we must simultaneously construct a recursive map f, as well as two sequences of nested intervals: $\{[x_e, y_e]\}$ converging to $\Phi(A)$; and $\{[\hat{x}_e, \hat{y}_e]\}$ converging to $\Phi(B)$. These intervals are constructed by defining at each stage s, the intervals $\{[x_e^s, y_e^s]\}$ and $\{[\hat{x}_e^s, \hat{y}_e^s]\}$ for every e, which will satisfy (2.2), (2.3), (2.4) and (2.5) of the proof of Theorem 2.3.

Effectively enumerate all potential tt-reductions *on cuts* by enumerating all partial recursive functions, $\{h_e(x)\}$, and letting the eth tt-reduction correspond as described in Lemma 2.6 to the pair of functions $\langle p_e(x), g_e(x)\rangle$, where $p_e(x) = k(h_e(x))$, and $g_e(x) = l(h_e(x))$, (whenever the right-hand sides are defined).

Stage $s=0$. For every i, set $x_i^0 = \hat{x}_i^0 = 0$, and $y_i^0 = \hat{y}_i^0 = 2^{-i}$. Also define $f^0(x_i^0) = \hat{x}_i^0$, and $f^0(y_i^0) = \hat{y}_i^0$ for all i. For convenience in treating the case $i=0$, we define $x_{-1}^s = \hat{x}_{-1}^s = 0$, and $y_{-1}^s = \hat{y}_{-1}^s = 2$, for all $s \geqq 0$.

Stage $s>0$. Let $e = k(s)$, and define

$$\hat{z}_e^s = \mu z[z \in (\hat{y}_e^{s-1}, \hat{y}_{e-1}^{s-1})].$$

Case I. Either $x_e^{s-1} > x_{e-1}^{s-1}$, or $h_e(\hat{z}_e^s)$ is not defined by stage s. For all i, set $x_i^s = x_i^{s-1}$, $y_i^s = y_i^{s-1}$, $\hat{x}_i^s = \hat{x}_i^{s-1}$, and $\hat{y}_i^s = \hat{y}_i^{s-1}$.

Case II. Otherwise $x_e^{s-1} = x_{e-1}^{s-1}$, and $p_e(\hat{z}_e^s)$ and $g_e(\hat{z}_e^s)$ are both defined.

Define

$$z_e^s = \mu z[z < y_{e-1}^{s-1} \,\&\, (w)[w \in [z, y_{e-1}^{s-1}) \Rightarrow w \notin \text{dom} f^{s-1}]].$$

(Note that $z_e^s > y_e^{s-1}$, since $y_e^{s-1} \in \text{dom} f^{s-1}$.)

Let $D_{g_e(z_e^s)}=\{q_1, q_2, \ldots, q_{c(\lambda)}\}$ as described in Lemma 2.6, and let

$$i' = \mu i[[q_i, q_{i+1}) \cap (z_e^s, y_{e-1}^{s-1}) \neq \varnothing].$$

Define $[x_e^s, y_e^s]=[k(u'), l(u')]$, where u' is the least u such that: $0<l(u)-k(u)<2^{-e}$, and

$$[k(u), l(u)] \subseteq [q_{i'}, q_{i'}+1) \cap (z_e^s, y_{e-1}^{s-1}).$$

Define $[\hat{x}_e^s, \hat{y}_e^s]=[k(v'), l(v')]$, where v' is the least v such that $0<l(v)-k(v)<2^{-e}$, and

$$\begin{aligned}[k(v), l(v)] &\subseteq (\hat{y}_e^{s-1}, \hat{z}_e^s) \quad \text{if } q_{i'} \in D_{p_e(\hat{z}_e^s)}\\ &\subseteq (\hat{z}_e^s, \hat{y}_{e-1}^{s-1}) \quad \text{otherwise.}\end{aligned}$$

For $i<e$, set $x_i^s=x_i^{s-1}$, $y_i^s=y_i^{s-1}$, $\hat{x}_i^s=\hat{x}_i^{s-1}$, and $\hat{y}_i^s=\hat{y}_i^{s-1}$.

For $i>e$, set $x_i^s=x_e^s$, $\hat{x}_i^s=\hat{x}_e^s$,

$$y_i^s = \mu y[y < y_{i-1}^s \;\&\; 0 < y-x_i^s < 2^{-i}],$$

and

$$\hat{y}_i^s = \mu y[y < \hat{y}_{i-1}^s \;\&\; 0 < y-\hat{x}_i^s < 2^{-i}].$$

Define the partial recursive function f^s:

$$\begin{aligned}f^s(w) &= f^{s-1}(w) && \text{if } w \in \text{domain } f^{s-1}\\ &= \hat{x}_i^s && \text{if } w = x_i^s\\ &= \hat{y}_i^s && \text{if } w = y_i^s\\ &= \hat{x}_i^s && \text{if } w = \mu v[v \in (x_{i-1}^s, x_i^s) \;\&\; v \notin \text{dom } f^{s-1}]\\ &= \hat{y}_i^s && \text{if } w = \mu v[v \in (y_i^s, y_{i-1}^s) \;\&\; v \notin \text{dom } f^{s-1}].\end{aligned}$$

We omit the straightforward proof by induction on e that for every e, the sequences $\{x_e^s\}$, $\{y_e^s\}$, $\{\hat{x}_e^s\}$, and $\{\hat{y}_e^s\}$ all attain limits as s increases. As in Theorem 2.3, $\bigcap_e [x_e, y_e]$ and $\bigcap_e [\hat{x}_e, \hat{y}_e]$ define unique real numbers, say α and β respectively. The associated lower cuts $L_\alpha=\{x \mid x \leqq \alpha\}$, and $L_\beta=\{x \mid x \leqq \beta\}$, are r.e. as in proof of Theorem 2.3, because (2.4) holds and the r.e. set $\bigcup_{s,e} \{x_e^s\}$ is cofinal in L_α, and $\bigcup_{s,e} \{\hat{x}_e^s\}$ is cofinal in L_β.

By the first clause in the definition of f^s, it is clear that for all s, f^{s+1} extends f^s. Let $f=\lim_s f^s$. Then f is *total* recursive by a trivial induction because of the fourth and fifth clauses (in the definition of f^s). Furthermore, by the second and third clauses,

$$(s)(e)[f(x_e^s) = \hat{x}_e^s \;\&\; f(y_e^s) = \hat{y}_e^s].$$

Hence, $f(L_\alpha) \subsetneqq L_\beta$ and $f(L'_\alpha) \subsetneqq L'_\beta$. Thus by proof of Theorem 1.8, $L_\alpha \equiv_T L_\beta$. Once we prove that $L_\beta \nleqq_{tt} L_\alpha$, we can conclude that neither L_α nor L_β is recursive.

Suppose that $L_\beta \leqq_{tt} L_\alpha$ by the *tt*-reduction of index e. Then $h_e(x)$ is total recursive, and thus Case II (of our construction) holds at some stage t such that $e=k(t)$, and

$$(i)_{\leqq e}[x_i^t = x_i \;\&\; y_i^t = y_i \;\&\; \hat{x}_i^t = \hat{x}_i \;\&\; \hat{y}_i^t = \hat{y}_i].$$

Now suppose that $[\hat{x}_e^t, \hat{y}_e^t] \subseteq (\hat{y}_e^{t-1}, \hat{z}_e^t)$ and hence that $q_{i'} \in D_{p_e(z_e^s)}$. Then according to Lemma 2.6, since $\alpha \in [x_e^t, y_e^t] \subseteq [q_i, q_{i+1})$, the *tt*-reduction of index e asserts that $\hat{z}_e^t \in L_\beta$. However, $\beta \in [\hat{x}_e^t, \hat{y}_e^t] \subseteq (\hat{y}_e^{t-1}, \hat{z}_e^t)$ implies that $\beta < \hat{z}_e^t$, and hence that $\hat{z}_e^t \notin L_\beta$, a contradiction. A similar contradiction arises in the other case, namely where $[\hat{x}_e^t, \hat{y}_e^t] \subseteq (\hat{z}_e^t, \hat{y}_{e-1}^{t-1})$. We conclude that $L_\beta \nleqq_{tt} L_\alpha$.

Thus L_α and L_β are not recursive sets. In particular, α and β are not rational numbers, and thus have unique binary expansions. There are unique sets A and B such that $\Phi(A) = \alpha$ and $\Phi(B) = \beta$ so that $L(A) = L_\alpha$ and $L(B) = L_\beta$.

The characterization in Lemma 2.6 of *tt*-reducibility on semirecursive sets has another consequence, which is the *tt*-reducibility analogue of the cofinality (Theorem 1.7) associated with *m*-reducibility. Suppose that A and B are semi-recursive and that $A \leqq_{tt} B$ via recursive functions $p(x)$ and $g(x)$ as defined in the proof of Lemma 2.6. For any fixed x, letting $D_{g(x)} = \{q_i\}_{i=1}^{c(x)}$, we can find (although not effectively) the unique i' such that $q_{i'} \in B$ and $q_{i'+1} \in B'$. Let $u_x = q_{i'}$, $v_x = q_{i'+1}$, and let $U^A = \{u_x \mid x \in A\}$, $V^A = \{v_x \mid x \in A\}$. By definition $U^A \subseteq B$ and $V^A \subseteq B'$. We will sketch the easy proof that if A is nonrecursive, then

(1) $A \leqq_{tt} B \Rightarrow [U^A \underset{\rightarrow}{\subsetneq} B$ *or* $V^A \underset{\rightarrow}{\subsetneq} B']$, and

(2) $A \leqq_{tt} B$ and $B \nleqq_T A \Rightarrow [U^A \underset{\rightarrow}{\subsetneq} B$ *and* $V^A \underset{\rightarrow}{\subsetneq} B']$.

Now (1) follows because otherwise A and A' are both easily seen to be r.e. In (2), suppose that $U^A \underset{\rightarrow}{\subsetneq} B$ but $V^A \not\underset{\rightarrow}{\subsetneq} B'$, then it is easy to show that $A \leqq_m B$ and hence $B \leqq_T A$ by Theorem 1.9. Finally, if $V^A \underset{\rightarrow}{\subsetneq} B'$ but $U^A \not\underset{\rightarrow}{\subsetneq} B$ then $A \leqq_m B'$ and again $B \leqq_T A$.

In §1, Theorem 1.7 led immediately to the reflexiveness associated with *m*-reducibility (Theorem 1.9). No such reflexiveness holds here because u_x and v_x cannot be determined *effectively* from x.

3. **Cylinders and order isomorphism types.** In attempting to classify r.e. sets, Myhill [13] introduced the notion of (r.e.) cylinder. Rogers [17] extended the concept to sets in general, and derived several equivalent characterizations.

DEFINITION 3.1. A set A is a *cylinder* if there is some set B such that $A \equiv_1 B \times N$.

THEOREM 3.2 (ROGERS). *A is a cylinder if and only if one of the following conditions holds*:

(1) $A \equiv_1 A \times N$.

(2) *For all B*, $[B \leqq_m A \Rightarrow B \leqq_1 A]$.

(3) *There is a recursive function f, such that for all x*

(i) $\varnothing \neq D_x \subseteq A \Rightarrow f(x) \in A - D_x$,

(ii) $\varnothing \neq D_x \subseteq A' \Rightarrow f(x) \in A' - D_x$.

(For a proof see Rogers [17].) Condition (2) illustrates the connection between cylinders and m:1 reducibility. Condition (3) characterizes cylinders by a "doubly productive" property with respect to finite sets, and gives rise to a generalization due to P. R. Young [25].

DEFINITION 3.3 (YOUNG). A set A is a *semicylinder* if there is a (total) recursive function f such that

(1) $(x)[f(x) \neq x]$,

(2) $f(A) \subseteq A$ and $f(A') \subseteq A'$.

Clearly every cylinder is a semicylinder, but Young has shown [25] that the converse is false. On cuts, however, these conditions are equivalent.

LEMMA 3.4. *A cut $L(A)$ is a cylinder if and only if $L(A)$ is a semicylinder.*

Proof. Suppose $L(A)$ is a semicylinder with recursive function $f(x)$. We define a recursive function $g(x)$ which satisfies condition (3) of Theorem 3.2. For every x such that $D_x \neq \varnothing$, define $m_x = \mu z[z \in D_x]$, and define the infinite recursive set

$$\begin{aligned} B_x &= \{y \mid m_x < y < f(m_x)\} && \text{if } m_x < f(m_x) \\ &= \{y \mid f(m_x) < y < m_x\} && \text{if } f(m_x) < m_x. \end{aligned}$$

Define

$$\begin{aligned} g(x) &= \mu y[y \in B_x - D_x] && \text{if } D_x \neq \varnothing \\ &= 0 && \text{otherwise.} \end{aligned}$$

THEOREM 3.5. *There is an r.e. cut $L(A)$ which is not a cylinder.*

Proof. Simultaneously enumerate all partial recursive functions of one variable, letting $f_e(x)$ denote the eth such function. We construct a real number $\Phi(A)$ by a sequence of nested intervals $\{[x_e, y_e]\}$ such that for each e, $\Phi(A) \in [x_e, y_e]$ will imply that $L(A)$ is not a semicylinder by the function f_e. These intervals are constructed by defining at each stage s and for every e the intervals $[x_e^s, y_e^s]$ satisfying (2.2), (2.3), (2.4) and (2.5) as in the proof of Theorem 2.3.

Stage $s=0$. For every i, set $x_i^0 = 0$, and $y_i^0 = 2^{-i}$. Again for convenience, define $x_{-1}^s = 0$, and $y_{-1}^s = 2$ for all $s \geqq 0$.

Stage $s>0$. Let $e = k(s)$, and define

$$z_e^s = \mu z[z \in (y_e^{s-1}, y_{e-1}^{s-1})].$$

Case I. Either (1) $f_e(z_e^s)$ is not defined by stage s, or (2) $f_e(z_e^s) = z_e^s$, or (3) $x_e^{s-1} > x_{e-1}^{s-1}$. (If (2) holds then f_e, having a fixed point, cannot be a semicylinder function.)

For all i, set $x_i^s = x_i^{s-1}$ and $y_i^s = y_i^{s-1}$.

Case II. Otherwise define $[x_e^s, y_e^s] = [k(u'), l(u')]$ where u' is the least u such that

(1) $0 < l(u) - k(u) < 2^{-e}$,

(2) $[k(u), l(u)] \subseteq (y_e^{s-1}, y_{e-1}^{s-1})$,

(3)
$$\begin{aligned} [k(u), l(u)] &\subseteq (z_e^s, f_e(z_e^s)) && \text{if } z_e^s < f_e(z_e^s) \\ &\subseteq (f_e(z_e^s), z_e^s) && \text{if } f_e(z_e^s) < z_e^s. \end{aligned}$$

For all $i < e$, set $x_i^s = x_i^{s-1}$, and $y_i^s = y_i^{s-1}$.

For all $i > e$, set $x_i^s = x_e^s$, and

$$y_i^s = \mu y[y < y_{i-1}^s \;\&\; 0 < y - x_i^s < 2^{-i}].$$

We again omit the straightforward proof by induction on e that for every e, the sequences $\{x_e^s\}$ and $\{y_e^s\}$ attain limits, say x_e and y_e, as s increases. As in the proof of Theorems 2.3 and 2.7, $\bigcap_e [x_e, y_e]$ defines a unique real number, say α, whose (lower) Dedekind cut L_α is r.e. Now if L_α is a semicylinder by the function f_e, then f_e is a total recursive function and $(x)[f_e(x) \neq x]$. Hence, Case II holds at some stage t at which $e = k(t)$ and $(i)_{\leq e}[x_i^t = x_i \;\&\; y_i^t = y_i]$. But then by the definition of $[x_e^t, y_e^t]$, either $z_e^s \in L_\alpha$ and $f_e(z_e^s) \in L'_\alpha$ or vice versa. In either case f_e cannot be a semicylinder function for L_α.

In particular L_α is not recursive (since every nontrivial recursive cut is a cylinder). Since α is nonrational it has a unique binary expansion. We complete the proof by letting A be the unique set such that $\Phi(A) = \alpha$ and $L(A) = L_\alpha$.

Two results of P. R. Young are immediate corollaries of Theorem 3.5. A non-creative r.e. set A is *pseudo-creative* [24] if for every r.e. set $B \subset A'$, there is an infinite r.e. set $C \subset A' \cap B'$. Young showed [24] that not all pseudo-creative sets are cylinders. This result follows from Theorem 3.5 because *every nonrecursive r.e. cut is psuedo-creative*, since no cut is a creative, simple or pseudo-simple set.

A set A is defined to be a *splinter* if there exist $x_0 \in N$ and a recursive function f, such that $A = \{f^i(x_0) \mid i \in N\}$. Young's main motivation in [26] was to prove that not all pseudo-creative sets are splinters, a result which he points out is interesting because "classification of recursively enumerable (r.e.) sets is traditionally by the recursive structure in their complements; thus when one has a class of r.e. sets which is not defined by reference to the complements, one wants to know whether the class may also be defined by the more traditional methods." (Young's remark in substance appears in Myhill [13, p. 215].) Young's result is also a corollary of Theorem 3.5 because *any nontrivial cut which is a splinter is a cylinder*, although this is false for sets in general.

LEMMA 3.6. *An r.e. cut $L(A)$ is a cylinder if and only if $L(A)$ is a splinter.*

Proof. We may assume $L(A)$ nonrecursive since every nontrivial recursive cut is both a cylinder and a splinter. It is easy to prove (see Myhill [13]) that every non-empty r.e. set which is a cylinder is a splinter. Conversely, suppose that $L(A)$ is a splinter via the recursive function f. Define the r.e. set

$$X = \{x \mid f(x) = x \vee (\exists y)[x > y \;\&\; f(x) = f(y)]\}.$$

Now $X \subseteq (L(A))'$ because $L(A)$ is a splinter. Since X is r.e., X is not cofinal in $(L(A))'$ else $(L(A))'$ is r.e. and hence recursive. Thus there exists $x_0 \in (L(A))'$ such that $(y)[y \in X \Rightarrow y > x_0]$. But then $L(A)$ is a semicylinder via the recursive function,

$$\begin{aligned} g(x) &= f(x) \quad \text{if } x \leq x_0 \\ &= x_0 \quad\;\;\, \text{if } x > x_0. \end{aligned}$$

Thus $L(A)$ is a cylinder by Lemma 3.4.

COROLLARY 3.7 (YOUNG). *There is a pseudo-creative set which is not a splinter.*

(Young [26] does in fact prove somewhat more, namely that any creative set has a subset which is pseudo-creative, not a splinter, and *btt*-complete. This result neither implies nor is implied by Theorem 3.5 and corollaries since in particular $L(A)$ cannot be *btt*-complete.)

The ordering imposed by the rationals on cuts suggests an order analogue of many-one (one-one) reducibility which in turn gives rise to a stronger version for cuts of Theorem 3.2.

DEFINITION 3.8. If a recursive linear ordering, denoted by $\preccurlyeq$, is imposed on N, and if $A, B \subseteq N$, then

(1) *A is many-one order preserving reducible* to B (denoted $A \leqq_m^{\text{ord}} B$) if there is a recursive function $f(x)$ such that

(i) $f(A) \subseteq B$ and $f(A') \subseteq B'$,

(ii) $(x)(y)[[x \prec y \Rightarrow f(x) \preccurlyeq f(y)]$ & $[x \succ y \Rightarrow f(x) \succcurlyeq f(y)]]$.

(2) If in addition f is one-one, then A is *one-one order preserving reducible* to B ($A \leqq_1^{\text{ord}} B$).

(3) If f is one-one and onto then A is *recursively order isomorphic* to B ($A \simeq^{\text{ord}} B$).

By Lemma 1.5 it follows that for any semirecursive sets A and B (each with recursive linear ordering induced by ψ_A and ψ_B respectively) if $A \leqq_m B$, then $A \leqq_m^{\text{ord}} B$.

DEFINITION 3.9. If a recursive linear ordering, denoted by $\preccurlyeq$, is imposed on N, $A \otimes B$ is the cartesian product $A \times B$ with lexicographic recursive ordering (i.e. $\langle x, y\rangle \prec \langle u, v\rangle$ holds if $x \prec u$ or $x = u$ and $y \prec v$).

By identifying each n with $\boldsymbol{n}$ (and thus N with Q) we abuse notation and sometimes write $\boldsymbol{L(A)} \otimes \boldsymbol{L(B)}$ and $\boldsymbol{L(A)} \leqq_m^{\text{ord}} \boldsymbol{L(B)}$ instead of $L(A) \otimes L(B)$ and $L(A) \leqq_{\text{ord}}^m L(B)$. The recursive linear ordering for cuts will always be that induced by the fixed map, $x \rightarrow \boldsymbol{x}$. The following elementary theorem is the order analogue for cuts of Theorem 3.2.

THEOREM 3.10. *A cut $L(A)$ is a cylinder if and only if $L(A)$ satisfies one of the following*:

(1) *$L(A)$ is a semicylinder.*

(2) *$\boldsymbol{L(A)} \simeq_m^{\text{ord}} \boldsymbol{L(A)} \otimes Q$.*

(3) *For all B, $[\boldsymbol{L(B)} \leqq^{\text{ord}} \boldsymbol{L(A)} \Rightarrow \boldsymbol{L(B)} \leqq_1^{\text{ord}} \boldsymbol{L(A)}]$.*

Proof. In Lemma 3.4 we proved (1). Now (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1) are trivial analogues of the standard proofs in Rogers [17]. To prove (1) $\Rightarrow$ (2), assume that $L(A)$ is a semicylinder via the recursive function f. Let J: $Q \otimes Q \rightarrow Q$ be a 1:1 recursive, order preserving map from $Q \otimes Q$ onto Q. For any $\boldsymbol{x}, \boldsymbol{y} \in Q$, let $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ denote the image $J(\boldsymbol{x}, \boldsymbol{y})$. To prove (2) it suffices to construct a 1:1 recursive, order preserving map $\boldsymbol{h}$ from $\boldsymbol{L(A)}$ onto $J(\boldsymbol{L(A)} \otimes Q)$. For any x define the rational intervals,

$$
\begin{aligned}
I_x &= [\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})] \quad \text{if } \boldsymbol{x} < \boldsymbol{f}(\boldsymbol{x}) \\
&= [\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}] \quad \text{if } \boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{x}. \\
\hat{I}_x &= [\langle \boldsymbol{x}, 0\rangle, \langle \boldsymbol{f}(\boldsymbol{x}), 2\rangle] \quad \text{if } \boldsymbol{x} < \boldsymbol{f}(\boldsymbol{x}) \\
&= [\langle \boldsymbol{f}(\boldsymbol{x}), 0\rangle, \langle \boldsymbol{x}, 2\rangle] \quad \text{if } \boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{x}.
\end{aligned}
$$

Note that for each x, $I_x \subset \boldsymbol{L}(A)$ or $I_x \subset (\boldsymbol{L}(A))'$, similarly for $\hat{I}_x$, and that $I_x \subset \boldsymbol{L}(A)$ if and only if $\hat{I}_x \subseteq J(\boldsymbol{L}(A) \otimes Q)$.

At each stage s we define $\boldsymbol{h}$ on the interval I_s such that $\boldsymbol{h}$ is 1:1, recursive, order preserving, and is onto $\hat{I}_s$. (Of course, $\boldsymbol{h}$ may be already defined on some subintervals of I_s, in which case we need define $\boldsymbol{h}$ only on the new subintervals of I_s.) Clearly, $\boldsymbol{L}(A) \cong^{\text{ord}} J(\boldsymbol{L}(A) \otimes Q)$ via $\boldsymbol{h}$.

Myhill's Theorem [12] asserts that if two sets A and B are of the same one-one degree ($A \equiv_1 B$), then they are recursively isomorphic ($A \simeq B$). To see that the order analogue of Myhill's theorem fails for cuts, we define certain "monotone" cylinders. These notions are suggested by the interpretation of a cut $L(A)$ as a cylinder if both $L(A)$ and $(L(A))'$ are preserved under some fixed point free recursive function (Lemma 3.4).

Definition 3.11. A cut $L(A)$ is an *increasing* (*decreasing*) *cylinder* if there is a recursive function f such that

(1) $f(L(A)) \subseteq L(A)$ and $f(L(A))' \subset (L(A))'$, and

(2) $(x)[\boldsymbol{x} \neq 2 \ \&\ \boldsymbol{x} \neq \Phi(A) \Rightarrow \boldsymbol{f}(\boldsymbol{x}) > \boldsymbol{x}]$ (respectively, $(x)[\boldsymbol{x} \neq 0 \Rightarrow \boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{x}]$).

Furthermore, $L(A)$ is a *two-way cylinder* ($\updownarrow$-cylinder) if $L(A)$ is both an increasing cylinder ($\uparrow$-cylinder) and a decreasing cylinder ($\downarrow$-cylinder).

In the terminology of Lacombe [9], if $L(A)$ is a $\uparrow$-cylinder ($\downarrow$-cylinder), then $L(A)$ ($(L(A))'$) is a "recursively open" set, as can be seen more clearly by the characterization in Theorem 3.12.

These new cylinders can be neatly characterized by analogy with condition (2) of Theorem 3.10. We omit the straightforward proofs.

Theorem 3.12. *The cut $L(A)$ is a*:

(1) *cylinder* $\Leftrightarrow \boldsymbol{L}(A) \cong^{\text{ord}} \boldsymbol{L}(A) \otimes Q$.

(2) $\uparrow$*-cylinder* $\Leftrightarrow L(A) \cong^{\text{ord}} L(A) \otimes Q \cap [0, 2)$.

(3) $\downarrow$*-cylinder* $\Leftrightarrow L(A) \cong^{\text{ord}} L(A) \otimes Q \cap (0, 2]$.

(4) $\updownarrow$*-cylinder* $\Leftrightarrow L(A) \cong^{\text{ord}} L(A) \otimes Q \cap (0, 2)$.

One speaks of "cylindrifying" a set A by taking the image $j(A \times N)$, of the cartesian product $A \times N$ under the recursive isomorphism j from $N \times N$ onto N. Analogously, Theorem 3.12 allows us to cylindrify a nontrivial *cut* $\boldsymbol{L}(A)$ in any of four distinct ways, $J_1(L(A) \otimes Q)$, $J_2(L(A) \otimes Q \cap [0, 2))$, $J_3(L(A) \otimes Q \cap (0, 2])$, or $J_4(L(A) \otimes Q \cap (0, 2))$, where J_1 (respectively J_2, J_3, J_4) is a recursive order isomorphism from $Q \otimes Q$ (respectively $Q \otimes Q \cap [0, 2)$, $Q \otimes Q \cap (0, 2]$, $Q \otimes Q \cap (0, 2)$) onto Q (respectively $Q \cap [0, 2)$, $Q \cap (0, 2]$, or $Q \cap (0, 2)$), and where for all i, $1 \leq i \leq 4$, $J_i(0, 0) = 0$ and $J_i(2, 2) = 2$. Furthermore, it is easy to prove that such cylindrification introduces no extraneous cylinder properties. (For example, $J_2(\boldsymbol{L}(A) \otimes Q \cap [0, 2))$ is a $\downarrow$-cylinder if and only if $L(A)$ is a $\downarrow$-cylinder, etc.) We can now easily disprove the order analogue of Myhill's theorem.

Theorem 3.13. *There are cuts $\boldsymbol{L}(A)$ and $\boldsymbol{L}(B)$ such that $\boldsymbol{L}(A) \equiv_1^{\text{ord}} \boldsymbol{L}(B)$ but $\boldsymbol{L}(A) \not\cong^{\text{ord}} \boldsymbol{L}(B)$.*

Proof. Let $\boldsymbol{L}(\boldsymbol{C})$ be an r.e. lower cut which is not a cylinder. (By Theorem 3.5 $\boldsymbol{L}(\boldsymbol{C})$ exists.) Define $\boldsymbol{L}(A)=J_2(\boldsymbol{L}(\boldsymbol{C})\otimes Q\cap[0,2))$, and $\boldsymbol{L}(\boldsymbol{B})=J_3(\boldsymbol{L}(\boldsymbol{C})\otimes Q\cap(0,2])$, where J_2 and J_3 are the recursive order isomorphisms described above. Define a one-one order preserving recursive map $\boldsymbol{h}\colon Q\to Q\cap[1/2,3/2]$ by $\boldsymbol{h}(\boldsymbol{n})=(1/2)(\boldsymbol{n})+1/2$. Now $\boldsymbol{L}(A)\leqq_1^{\text{ord}}\boldsymbol{L}(\boldsymbol{B})$ via the recursive map $\boldsymbol{g}$, defined by $\boldsymbol{g}(\langle\boldsymbol{n},\boldsymbol{m}\rangle)=\langle\boldsymbol{n},\boldsymbol{h}(\boldsymbol{m})\rangle$. Similarly $\boldsymbol{L}(\boldsymbol{B})\leqq_1^{\text{ord}}\boldsymbol{L}(A)$. However, if $\boldsymbol{L}(A)\simeq^{\text{ord}}\boldsymbol{L}(\boldsymbol{B})$, then clearly $\boldsymbol{L}(A)$ is a $\downarrow$-cylinder if and only if $\boldsymbol{L}(\boldsymbol{B})$ is a $\downarrow$-cylinder. By our earlier remark, $\boldsymbol{L}(A)$ cannot be a $\downarrow$-cylinder since $\boldsymbol{L}(\boldsymbol{C})$ is not. Since $\boldsymbol{L}(\boldsymbol{B})$ is a $\downarrow$-cylinder, $\boldsymbol{L}(A)\not\simeq^{\text{ord}}\boldsymbol{L}(\boldsymbol{B})$.

There is an interesting theorem observed by R. W. Robinson which demonstrates that if $\boldsymbol{L}(A)$ is a $\uparrow$-cylinder, then A cannot be too "sparse".

THEOREM 3.14 (R. W. ROBINSON). *If $L(A)$ is a $\uparrow$-cylinder, then A is not hyperimmune.*

Proof. Let $\boldsymbol{L}(A)$ be a $\uparrow$-cylinder via the recursive function $\boldsymbol{f}$ (defined on Q). We will construct a recursive function g (defined on N) such that

$$(n)(\exists a)[a\in A\ \&\ g(n)<a\leqq g(n+1)].$$

Thus the strong array of finite sets, $\{B_n\}$, will witness that A is not hyperimmune, where $B_n=\{m\mid g(n)<a\leqq g(n+1)\}$. Given any rational $\boldsymbol{y}$, define $[\boldsymbol{y}]_i$ to be 0 or 1 according as the ith place in the dyadic expansion of $\boldsymbol{y}$ is 0 or 1. (Since each $\boldsymbol{y}$ is effectively presented as a ratio of natural numbers, we can always eliminate any ambiguous expansions by always favoring the expansion ...1000... instead of ...0111....) Define recursive functions m and g,

$$\begin{aligned} m(x,y) &= (\mu i)_{>y}[[\boldsymbol{f}(\Phi(D_x))]_i=1] && \text{if such exists}\\ &=0 && \text{otherwise.}\\ g(0) &= \mu i[[\boldsymbol{f}(0)]_i=1],\\ g(n+1) &= \max\{m(x,g(n))\mid \text{all } x \text{ such that } D_x\subseteq I(0,g(n))\}.\end{aligned}$$

Now if $A\cap I(0,g(n))=D_z$, then $\Phi(D_z)\in\boldsymbol{L}(A)$, and hence $\boldsymbol{f}(\Phi(D_z))\in\boldsymbol{L}(A)$. But $\boldsymbol{f}(\Phi(D_z))>\Phi(D_z)$ so $(\exists i)_{>g(n)}[[\boldsymbol{f}(\Phi(D_z))]_i=1]$ and hence $m(z,g(n))>g(n)$. Now $\boldsymbol{f}(\Phi(D_z))\in\boldsymbol{L}(A)$ implies that $(\exists a)[a\in A\ \&\ g(n)<a\leqq g(n+1)]$.

It was announced by mistake in [20] that the converse of Theorem 3.14 also holds. Although this is false for genuine Dedekind cuts, it easily holds if we replace $L(A)$ by $L_F(A)$, where $\boldsymbol{L}_F(A)=\boldsymbol{L}(A)\cap Q_F$. Now using the fact that $L_F(A)$ is a $\uparrow$-cylinder if and only if A is not hyperimmune, we can construct a hyperimmune set A such that $L(A)$ is r.e. We first slightly modify the proof of Theorem 3.5 (with Q_F in place of Q) in order to construct a set A such that $L_F(A)$ is r.e. but not a cylinder. In particular, $L_F(A)$ is not a $\uparrow$-cylinder and hence A is hyperimmune, but $L(A)$ is r.e. because $L_F(A)$ is r.e.

The existence of a hyperimmune set A which yields an r.e. lower cut $L(A)$ suggests that we ask just how "sparse" the set A can be so that $L(A)$ remains r.e.

In [21] we show that one can even find a cohesive set C with r.e. lower cut $L(C)$.

To observe more easily the weaker fact that there exists a non-r.e. set which yields an r.e. lower cut, C. G. Jockusch Jr. has pointed out that if A is any r.e. set, and if $B=A$ join $A'=\{2n \mid n \in A\} \cup \{2n+1 \mid n \in A'\}$, then $L(B)$ is a r.e. but B is not r.e. unless A is recursive.

4. **Semicreative Dedekind cuts.** The existence of semicreative cuts is of interest because: (1) semicreativity is the strongest type of creativity which can exist on cuts, since by Corollary 1.13 there are no semirecursive sets which are creative or quasicreative, and (2) semicreativity on cuts is related to the cylinder properties developed in §3.

DEFINITION 4.1. A set P is *semiproductive* if there is a partial recursive function p such that for all e,

$$W_e \subset P \Rightarrow [p(e) \text{ is defined and } W_e \subsetneqq W_{p(e)} \subset P].$$

An r.e. set A is *semicreative* if A' is semiproductive.

It is easy to show that there exist semicreative cuts because we can prove that every truth table complete cut is semicreative. (There exist tt-complete cuts by Theorem 2.5.) Since there are no quasicreative cuts, this gives another method of answering Shoenfield's question, answered first by Yates [23], of whether every semicreative set is quasicreative. We use the theorem of Friedberg and Rogers ([3], Corollary 5) that an r.e. set S is tt-complete if and only if S' is tt-productive.

DEFINITION 4.2. A set P is *tt-productive* if there is a recursive function g such that for any x, $g(x)$ is an effective index for the sequence of finite sets $G, F_1, F_2, \ldots, F_n$ (where $F_i \subseteq G$ for $i \leqq n$) such that *either*

(1) for some $i \leqq n$, $[F_i \subseteq P$ and $G-F_i \subseteq P']$, or

(2) for some $i \leqq n$, $[F_i \subseteq W_x$ and $G-F_i \subseteq P']$,

but *not both*.

THEOREM 4.3. *If A is semirecursive and tt-complete, then A is semicreative.*

Proof. For notational simplicity we will assume that A is a cut and that ψ_A is our fixed map, $x \rightarrow \boldsymbol{x}$. The general case follows by precisely the same proof. Let A be tt-complete. Then by Friedberg and Rogers [3] A' is tt-productive via, say the recursive function g. Given any $x \in N$, if $W_x \subset A'$, and if $G, F_1, F_2, \ldots, F_n$ is the sequence associated with $g(x)$, then (1) rather than (2) of Definition 4.2 must hold, since if (2), then for some i, $F_i \subset W_x \subset A'$ implying *both* (1) and (2) contrary to hypothesis. Fixing x, we define $y_i = \min\{z \mid z \in F_i\}$, for $i \leqq n$, and the recursive functions, $f^s(x)$ and $p(x)$,

$$\begin{aligned} f^s(x) &= \max\{y_i \mid i \leqq n \;\&\; y_i < \min\{z \mid z \in W_x^s\}\} && \text{if such exists} \\ &= 2 && \text{otherwise.} \end{aligned}$$

$$W_{p(x)} = W_x \cup \{f^s(x) \mid s \in N\}.$$

Now for all x, $W_x \subset A'$ implies $W_x \subsetneqq W_{p(x)} \subset A'$. Furthermore, $p(x)$ is a recursive function because it is recursive in $g(x)$, uniformly in x. Thus A is semicreative.

It is now natural to ask whether there is a semicreative cut which is not (Turing) complete, for if all semicreative cuts are complete, then they are the natural analogue for cuts of creative sets which are $m{:}1$ complete. Yates [23] proved that there is a semicreative set of every degree. Yates' proof cannot be directly carried over to cuts for the same reason that the standard diagonal argument for constructing creative sets fails for cuts. Namely, when one adds a new element to a cut, one simultaneously introduces all lesser (in the rational ordering) elements, and thus one loses the precise coding of information which these proofs require. By a more complicated construction which relies on Yates' basic ideas, we can prove that there is a semicreative *cut* of every nonrecursive r.e. degree. (Our complications arise from the fact that $g(n)$, our bound on the information enumerated in W_e, must be large enough to enable proof of Lemma II below, but small enough to be recursive in A, uniformly in n.)

THEOREM 4.4. *If A is any nonrecursive r.e. set, then there is a cut $L(B)$ of the same (Turing) degree as A such that $L(B)$ is semicreative.*

Proof. We will construct B as the disjoint union of r.e. sets C and D, each recursive in A, and such that $C \subseteq 3N$, and $D = \{3x+2 \mid x \in A\}$. Since A is recursive in D, B will be of the same degree as A.

Define the rational $\rho_e^s = \min\{x \mid x \in W_e^s\}$, and the real number $\rho = \lim_s \rho_e^s$. (Here lim is used in the analytic sense for the first time.) Let β denote $\Phi(B)$, and β_n denote $\Phi(B \cap I[0, n])$. As we construct B, we simultaneously construct for each e, a nonincreasing sequence of rationals, $\{\sigma_e^s\}$, such that if $\sigma_e = \lim_s \sigma_e^s$, then

$$(e)[\rho_e > \beta \Rightarrow \rho_e > \sigma_e > \beta]. \tag{4.1}$$

This will insure that $(L(B))'$ is semiproductive because we can define $W_{p(e)} = \{x \mid x \geqq \sigma_e\}$, thus satisfying Definition 4.1. That $\{\sigma_e^s\}$ is nonincreasing in s for fixed e insures that the upper cut determined by σ_e is r.e. in $\{\rho_e^s\}$ uniformly in e.

Let $a(y)$ be a one-one recursive function whose range is A, and let $A(s) = \{a(y) \mid y \leqq s\}$. Let $m(s) = \max\{a(y) \mid y \leqq s\}$, and note that $m(s)$ is unbounded and nondecreasing as s increases. Define the recursive function f,

$$\begin{aligned} f(s, x) &= \max\{y \mid y \leqq s \ \&\ a(y) \leqq x\} && \text{if } (\exists y)_{\leqq s}[a(y) \leqq x] \\ &= 0 && \text{otherwise.} \end{aligned}$$

It is easy to verify that,

$$f \text{ is nondecreasing in both variables,} \tag{4.2}$$

$$(s)(x)[f(s, m(s)+x) = s], \tag{4.3}$$

$$(s)(x)[f(s, x) \leqq s]. \tag{4.4}$$

Stage $s>0$. Define $D(s)=\{3x+2 \mid x \in A(s)\}$. Let $C(s, i)$ denote $C(s) \cap I[0, i]$, and likewise, $B(s, i)$ and $D(s, i)$. We will first define the finite sets $C(s, 3m)$ in $m(s)+1$ substages as n varies from 0 to $m(s)$, and will then define $C(s)=C(s, 3m(s))$, and $B(s)=C(s) \cup D(s)$. (It will be obvious that for each i, $C(s) \cap I[0, i]=C(s, i)$, and hence that our notation is consistent.)

Substage $n=0$. Set $C(s, 0)=\varnothing$.

Substage $0<n \leqq m(s)$. Assume that $C(s, 3(n-1))$ has been defined. Set $C(s, 3n-1) = C(s, 3n-2)=C(s, 3n-3)$, and $B(s, 3n-1)=C(s, 3n-1) \cup C(s, 3n-1)$, and let β^s_{3n-1} denote $\Phi(B(s, 3n-1))$. Define the recursive functions t and g,

$$\begin{aligned} t(s, n) &= \max\{t \mid t \leqq s \;\&\; B(t, 3n-1) \neq B(t \dot{-} 1, 3n-1)\} \\ &\qquad\qquad\qquad\qquad \text{if } n > 0 \text{ and such a } t \text{ exists} \\ &= 0 \quad \text{if } n = 0 \text{ or no such } t \text{ exists,} \\ g(s, n) &= \max\{f(s, n), f(t, m(t))\}, \quad \text{where } t = t(s, n). \end{aligned}$$

Now by (4.2), (4.3) and (4.4) we have,

(4.5) $\qquad g$ is nondecreasing in both variables,

(4.6) $\qquad (s)(x)[g(s, x) \leqq f(s, m(s))]$.

Define

$$\begin{aligned} C(s, 3n) &= C(s, 3n-1) \cup \{3n\} \quad \text{if } \rho^{g(s,n)}_{k(n)} \leqq \beta^s_{3n-1}+2^{-3n} \\ &= C(s, 3n-1) \qquad\qquad \text{otherwise.} \end{aligned}$$

(Our motivation for the definition of $C(s, 3n)$ is that if the first clause holds, we can satisfy requirement $k(n)$ by adjoining $3n$ to C, since then $3n \in B$ and thus $\beta^s_{3n} =\beta^s_{3n-1}+2^{-3n}$. Now by hypothesis, $\rho^{g(s,n)}_{k(n)} \leqq \beta^s_{3n-1}+2^{-3n}=\beta_{3n}$, and therefore $\rho_{k(n)} \leqq \beta$.)

Define $C(s)=\bigcup\{C(s, i) \mid i \leqq 3m(s)\}$, and $C=\bigcup_s C(s)$.

Note that C is thus r.e. because if the first clause in the definition of $C(s, 3n)$ holds at one stage, then it holds at all later stages since $g(s, n)$ is nondecreasing and thus,

$$(s)(t)_{>s}[\rho^{g(s,n)}_e \geqq \rho^{g(t,n)}_e \;\&\; \beta^s_{3n-1} \leqq \beta^t_{3n-1}].$$

Simultaneously, at stage s we define σ^s_e for each e. To do this we first define,

$$v(s, e) = \max\{n \mid n = 0 \vee [e = k(n) \;\&\; n \leqq m(s)]\},$$

and $h(s, e)=g(s, v(s, e))$. Clearly,

(4.7) $(e)[v(s, e)$ is unbounded and nondecreasing in $s]$, because $m(s)$ is unbounded and nondecreasing. Hence, by (4.5),

(4.8) $(e)[h(s, e)$ is unbounded and nondecreasing in $s]$, and

(4.9) $(s)(e)[h(s, e) \leqq f(s, m(s))]$, by (4.6).

Define

$$\begin{aligned} \sigma^s_e &= 0 && \text{if } \rho^{h(s,e)}_e \leqq \beta^s \\ &= \rho^{h(s,e)}_e && \text{if } v(s, e) = 0 \;\&\; \rho^{h(s,e)}_e > \beta^s \\ &= \rho^{h(s,e)}_e - 2^{-(3z(s,e)+1)} && \text{if } v(s, e) \neq 0 \;\&\; \rho^{h(s,e)}_e > \beta^s \end{aligned}$$

where if clause 3 holds, we define $z(s, e)$ by

$$z(s, e) = \mu n[e = k(n) \ \&\ \rho_e^{h(s,e)} > \beta^s_{3v(s,e)-1}+2^{-3n}].$$

Note that if $\rho_e > \beta$ then clause 3 always holds for sufficiently large s by (4.7). Now $z(s, e)$ is well defined because if clause 3 above holds then

$$(4.10) \qquad z(s, e) \leqq v(s, e),$$

because by clause 3 and the definition of h (letting $v(s, e)=v$), we have

$$\rho_e^{g(s,v)} = \rho^{h(s,e)} > \beta^s,$$

and therefore,

$$\rho_e^{g(s,v)} > \beta^s_{3v-1}+2^{-3v},$$

else by definition of $C(s, v)$ we would have insured that $\beta^s_{3v}=\beta^s_{3v-1}+2^{-3v}$, and hence that $\rho_e^{g(s,v)} \leqq \beta^s$, contradicting clause 3.

Finally (4.7) and (4.8) imply that for fixed e, $z(s, e)$ is nondecreasing (but not necessarily unbounded) in s, and hence that for fixed e, $\{\sigma_e^s\}$ is nonincreasing. This completes the construction.

By the definition of $C(s, 3n)$ we have immediately,

$$(4.11) \qquad (n)[n \in C \Leftrightarrow \rho_{k(n)}^{g(n)} \leqq \beta_{3n-1}+2^{-3n}].$$

Thus a trivial induction on n proves that C is recursive in A once we show that the function $g(n)=\lim_s g(s, n)$ is recursive in A. (Induction on n is necessary because β_{3n-1} itself depends upon $C \cap I[0, 3n-1]$, as well as upon A.)

Define $u(n)=\mu u[(s)_{\geqq u}(i)_{\leqq n}[i \in A(s) \Leftrightarrow i \in A]]$.

Clearly, $u(n)$ is a function recursive in A. Note that

$$(4.12) \qquad (s)_{\geqq u(n)}(i)_{\leqq n}[f(s, i) = f(i)].$$

LEMMA I. $(n)(s)_{\geqq u(n)}(i)_{\leqq n}[g(s, i)=g(i) \ \&\ t(s, i)=t(i)]$, *and hence* $g(n)$ *and* $t(n)$ *are functions recursive in* A.

Proof. Induction on n.

Case $n=0$. We have $(s)[t(s, 0)=0]$, and $f(0, x)=0$ for all x. Hence, $g(s, 0) =f(s, 0)$, and the conclusion follows by (4.12).

Case $n+1$. By induction, assume for n that

$$(4.13) \qquad (s)_{\geqq u(n)}(i)_{\leqq n}[g(s, i) = g(i) \ \&\ t(s, i) = t(i)].$$

Hence, by (4.11) and (4.13),

$$(s)_{\geqq u(n)}(i)_{\leqq 3n}[i \in C \Leftrightarrow i \in C(s)].$$

But $(i)_{\leqq n}[3i+2 \in B \Leftrightarrow i \in A]$. Hence, since $3n+1$ is never in B,

$$(s)_{\geqq u(n)}(i)_{\leqq 3n+2}[i \in B \Leftrightarrow i \in B(s)],$$

$$(s)_{\geqq u(n)}(i)_{\leqq n+1}[t(s, i) = t(i)], \quad \text{by definition.}$$

But $g(n+1)=\max\{f(n+1), f(t, m(t))\}$, where $t=t(n+1)$. Hence, by (4.12),

$$(s)_{\geqq u(n+1)}(i)_{\leqq n+1}[g(s, i) = g(i)].$$

LEMMA II. $(e)[\rho_e > \beta \Rightarrow \rho_e > \sigma_e > \beta]$.

Proof. Fix e and assume $\rho_e > \beta$. Then by (4.11),

$$(n)(s)[e = k(n) \Rightarrow 3n \notin B(s)]. \tag{4.14}$$

We claim $\sigma_e > \beta$. If not, then $(\exists s)(\exists w)_{>s}[\beta^s < \sigma_e^s \leqq \beta^w]$. Let s be the least stage such that clause 3 holds in the definition of σ_e^s, and let w be the least stage such that $\sigma_e^s \leqq \beta^w$. By clause 3 for σ_e^s and by the definition of $z(s, e)$, we have

$$\rho_e^{h(s,e)} - 2^{-(3z+1)} = \sigma_e^s \leqq \beta^w, \quad \text{where } z = z(s, e), \tag{4.15}$$

and

$$\rho_e^{h(s,e)} - 2^{3z} > \beta_{3v-1}^s, \quad \text{where } v = v(s, e). \tag{4.16}$$

But by (4.14), for any s, $3z \notin B(s)$, and $3z+1$ is never in $B(s)$, hence by the properties of real numbers,

$$\beta^w < \beta_{3z-1}^w + 2^{-(3z+1)}, \tag{4.17}$$

and hence

$$\rho_e^{h(s,e)} < \beta_{3z-1}^w + 2^{-3z}, \tag{4.18}$$

by (4.15) and (4.17), but

$$\rho_e^{h(s,e)} > \beta_{3v-1}^s + 2^{-3z} > \beta_{3z-1}^s + 2^{-3z}, \quad \text{by (4.16) and (4.10).} \tag{4.19}$$

Therefore, $\beta_{3z-1}^s < \beta_{3z-1}^w$ by (4.18) and (4.19).

Now since w was taken to be minimal, we have $\beta_{3z-1}^{w-1} < \beta_{3z-1}^w$, and hence $t(w, z) = w$. Hence,

$$g(w, z) \geqq f(w, m(w)) \geqq f(s, m(s)) \geqq h(s, e) \tag{4.20}$$

by the definition of g, (4.2), and (4.9) respectively. Now by (4.18) and (4.20),

$$\rho_e^{g(w,z)} \leqq \rho_e^{h(s,e)} < \beta_{3z-1}^w + 2^{-3z}.$$

Thus by definition of $C(w, 3z)$, $3z \in B(w)$ contradicting (4.14).

Finally, we must prove $\rho_e > \sigma_e$. If $\rho_e \leqq \sigma_e$, then $(s)(\exists t)_{>s}[\rho_e^t \leqq \sigma_e^s]$. For fixed e, $h(s, e)$ is unbounded in s by (4.8), so (assuming $\rho_e > \beta$) we can define an infinite sequence of stages $s(0), s(1), \ldots$, as follows:

$$s(0) = \mu s[\sigma_e^s < \rho_e^{h(s,e)}],$$
$$s(i+1) = \mu s[\rho_e^{h(s,e)} \leqq \rho_e^{h(s(i),e)} - 2^{-(3z(s(i),e)+1)}].$$

Since $\rho_e^{h(s(i),e)} - \rho_e^{h(s(i+1),e)} \geqq 2^{-(3z(s(i),e)+1)}$, clearly $\lim_s z(s, e)$ cannot exist, but rather $z(s, e)$ must increase without bound as s increases. Hence, by definition of $z(s, e)$,

$$(i)(\exists s)(n)[n < i \;\&\; e = k(n) \Rightarrow \rho_e^{h(s,e)} - \beta_{3n-1}^s \leqq 2^{-3n}].$$

But since $\beta^s \geqq \beta_{3n-1}^s$,

$$(i)(\exists s)[\rho_e^{h(s,e)} - \beta^s \leqq 2^{-3i}],$$

and hence $\rho_e \leqq \beta$, contradicting our original assumption. This completes the proof of Theorem 4.4.

LEMMA 4.5. *If A is semirecursive and semicreative then A is a cylinder.*

Proof. Let A' be semiproductive via $p(x)$. For any x define the r.e. set

$$W_{g(x)} = \{y \mid \psi_A(y) \geqq \psi_A(x)\}.$$

Given any finite set $D_x \neq \varnothing$. Let $m(x)$ be the unique y_0 such that $\psi_A(y_0) = \min\{\psi_A(y) \mid y \in D_x\}$. Begin enumerating A and simultaneously attempt to compute the partial recursive functions,

$$h_1(x) = \mu y[y \in W_{p(g(m(x)))} \;\&\; \psi_A(y) > \psi_A(m(x))],$$
$$h_2(x) = \mu y[y \in A - D_x], \quad \text{if } m(x) \text{ has been enumerated in } A.$$

For $x \neq 0$ define the recursive function $h(x)$ to be $h_1(x)$ or $h_2(x)$ whichever is computed first, and define $h(0) = 0$. (For $x \neq 0$ eventually $h_1(x)$ or $h_2(x)$ is defined because if $m(x) \in A$ then $h_2(x)$ is defined, while if $m(x) \in A'$ then $h_1(x)$ is defined.)

To show that A is a cylinder via h suppose that $\varnothing \neq D_x \subset A$. If $h(x) = h_1(x)$ then $\psi_A(h(x)) < \psi_A(m(x))$ so $h(x) \in A - D_x$, while if $h(x) = h_2(x)$ then by definition $h(x) \in A - D_x$. Now suppose $\varnothing \neq D_x \subset A'$. Necessarily $h(x) = h_1(x)$, and by semiproductivity $W_{p(g(m(x)))} \subset A'$ so $h(x) \in A' - D_x$.

COROLLARY 4.6 (JOCKUSCH [4], UNPUBLISHED). *In every nonrecursive r.e.* (*Turing*) *degree there is an m-degree consisting of a single* 1-*degree.*

Proof. By Theorem 4.4 in every nonrecursive r.e. degree there is a semicreative set S which is semirecursive (in fact is a Dedekind cut). It is well known that if A is semicreative and $A \leqq_m B$ then B' is semiproductive. Thus for any B, if $B \equiv_m S$ then B is semicreative and semirecursive, so B and S are cylinders by Lemma 4.5, and thus $B \equiv_1 S$.

(In [5] Jockusch defines the notion of "positive" reducibility, denoted $A \leqq_p B$. He has pointed out to us that Corollary 4.9 actually proves that "In every r.e. (Turing) degree there is a *p-degree* consisting of a single 1-degree," a result which his own construction [4, Corollary 5.10] does not accomplish.)

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