SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 7

BY

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1. **Introduction.** One of the famous congruences of the unrestricted partition function $p(n)$ due to Ramanujan [4], [5] is

$$p(7n+5) \equiv 0 \pmod 7. \tag{1}$$

Others of a similar character but involving powers of 7 are also known. But there is not much of a variety in the type of congruences that we come across for the modulus 7. The author is not aware, for example, of any congruence relation for $p(n)$ of the type

$$\tau(n) \equiv 0 \pmod 7 \tag{2}$$

for "almost all" values of n, or of the type

$$\tau(7n+i) \equiv 0 \pmod 7 \tag{3}$$

for all values of n, if i is a quadratic nonresidue of 7, [4], where $\tau(n)$ is Ramanujan's function defined by

$$x\left[\prod_{n=1}^{\infty}(1-x^n)\right]^{24} = \sum_{n=1}^{\infty} \tau(n)x^n. \tag{4}$$

Nor does he know of any congruence relation between $p(n)$ and some simple arithmetical functions analogous to

$$\tau(n) \equiv n\sigma_3(n) \pmod 7, \tag{5}$$

also due to Ramanujan [9].

This paper deals with partition functions of a restricted nature for which a variety of congruences including those of the above types exist. It is obviously necessary to explain the nature of the above restrictions on the partitioning of numbers. We merely impose the condition that no number of the forms αn or $\alpha n \pm r$, where α and r are specified numbers, shall be a part (necessarily positive integral) of the relevant partitions. In other words, in order to determine the value of ${}^{\alpha}_{r}p(n)$, the partition function restricted in this way, one should count all the unrestricted partitions of n excepting those which contain a number of any of the above forms as a part. We shall assume ${}^{\alpha}_{r}p(n)$ and $p(n)$ to be unity when $n=0$, and

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vanishing when n is negative. In this paper we shall deal mainly with the seven partition functions corresponding to $\alpha=147$ and $r=7$, 14, 28, 35, 49, 56 and 70.

The restricted partition function ${}^{3\beta}_{\beta}p(n)$ has a somewhat simpler interpretation. It is easily seen that this function requires the count of all the unrestricted partitions of n excepting those which contain β or any multiple thereof as a part. We have considered it desirable to use the simpler notation

$$ {}^{\beta}p(n) = {}^{\beta}_{0}p(n) = {}^{3\beta}_{\beta}p(n), \tag{6} $$

in this case, in order to emphasize the simpler interpretation.

The restricted partition functions ${}^{\alpha}_{r}p(n)$ are not new in number theory literature. These functions appear in the arithmetical interpretations of the Rogers–Ramanujan identities [4], [5] and in their combinatorial generalization due to Gordon [3], [1]. His theorem which gives another interpretation to these partition functions is of interest in this connection. He shows that ${}^{2d+1}_{t}p(n)$, where $1\leqq t\leqq d$, is equal to the number of partitions of the form $n=n_1+\cdots+n_k$, where $n_i\geqq n_{i+1}$, $n_i\geqq n_{i+d-1}+2$, and $n_{k-t+1}\geqq 2$.

2. **The principal results.** We are now in a position to state our main results which are listed below.

THEOREM 1. *For almost all values of n*

$$ \begin{aligned} {}^{49}p(n) &\equiv 0 \pmod 7, \\ {}^{147}_{70}p(n) &\equiv -{}^{147}_{28}p(n-7) \pmod 7, \\ {}^{147}_{56}p(n) &\equiv -{}^{147}_{7}p(n-14) \pmod 7, \\ {}^{147}_{35}p(n) &\equiv {}^{147}_{14}p(n-7) \pmod 7. \end{aligned} $$

THEOREM 2. *For all values of $n\geqq 0$,*

$$ {}^{49}p(7n+5) \equiv 0 \pmod 7, $$

and more generally with $0\leqq\rho\leqq 10$

$$ {}^{147}_{7\rho}p(7n+5) \equiv 0 \pmod 7. $$

THEOREM 3. *The following congruences modulo 7 hold for all values of $n\geqq 0$.*

$$ \begin{aligned} {}^{147}_{70}p(7n+4)+{}^{147}_{28}p(7n-3) &\equiv -{}^{49}p(7n+2), \\ {}^{147}_{56}p(7n+1)+{}^{147}_{7}p(7n-13) &\equiv {}^{49}p(7n), \\ {}^{147}_{35}p(7n+3)-{}^{147}_{14}p(7n-4) &\equiv -{}^{49}p(7n+6); \\ {}^{147}_{70}p(7n+6)+{}^{147}_{28}p(7n-1) &\equiv -2\cdot{}^{49}p(7n+4), \\ {}^{147}_{56}p(7n+2)+{}^{147}_{7}p(7n-12) &\equiv 2\cdot{}^{49}p(7n+1), \\ {}^{147}_{35}p(7n)-{}^{147}_{14}p(7n-7) &\equiv -2\cdot{}^{49}p(7n+3); \\ {}^{147}_{70}p(7n+3)+{}^{147}_{28}p(7n-4) &\equiv 3\cdot{}^{49}p(7n+1), \\ {}^{147}_{56}p(7n+4)+{}^{147}_{7}p(7n-10) &\equiv -3\cdot{}^{49}p(7n+3), \\ {}^{147}_{35}p(7n+1)-{}^{147}_{14}p(7n-6) &\equiv 3\cdot{}^{49}p(7n+4). \end{aligned} $$

THEOREM 4. *The following congruences modulo* 7 *hold for all values of* $n \geqq 0$.

$$
\begin{aligned}
{}^{147}_{70}p(49n+49)-{}^{147}_{70}p(7n+7) &\equiv -{}^{147}_{28}p(49n+42)+{}^{147}_{28}p(7n),\\
{}^{147}_{56}p(49n+48)-{}^{147}_{56}p(7n+6) &\equiv -{}^{147}_{7}p(49n+34)+{}^{147}_{7}p(7n-8),\\
{}^{147}_{35}p(49n+44)-{}^{147}_{35}p(7n+2) &\equiv {}^{147}_{14}p(49n+37)-{}^{147}_{14}p(7n-5).
\end{aligned}
$$

3. **Notations and conventions.** We shall use m to denote an integer, positive, zero or negative, but n is reserved to denote a nonnegative integer only.

We define u_r by the expression,

$$u_r = \sum_{n=0}^{\infty} n^r a_n x^n \cdot \sum_{n=0}^{\infty} p(n)x^n, \tag{7}$$

where a_n is defined by the well-known "pentagonal number" theorem of Euler,

$$f(x) = \prod_{n=1}^{\infty} (1-x^n) = \sum_{-\infty}^{+\infty} (-1)^m x^{m(3m+1)/2} = \sum_{n=0}^{\infty} a_n x^n, \tag{8}$$

and $p(n)$ is the number of unrestricted partitions of n given by the expansion,

$$[f(x)]^{-1} = \left[\prod_{n=1}^{\infty} (1-x^n)\right]^{-1} = \sum_{n=0}^{\infty} p(n)x^n. \tag{9}$$

When $n=r=0$, n^r is to be interpreted as unity so that

$$u_0 = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} p(n)x^n = 1. \tag{10}$$

We shall use v to denote the pentagonal numbers,

$$v = \tfrac{1}{2}m(3m+1), \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{11}$$

and with each v there corresponds an "associated" sign, viz., $(-1)^m$. We shall come across sums of the type $\sum_v [\mp V(v)]$, where it is understood that the sign to be prefixed is the associated one, which would thus be (a) negative if v is 1, 2, 12, 15, 35, ..., that is, when it is of the form $(2m+1)(3m+1)$, and (b) positive if it is 0, 5, 7, 22, 26, ..., that is, when it is of the form $m(6m+1)$. It is clear that with the above summation notation,

$$u_r = \sum_v (\mp v^r x^v)/f(x), \tag{12}$$

$$\sum_v (\mp x^v)/f(x) = 1. \tag{13}$$

We shall require the functions U_i, $i=0, 1, 2$ and 5 which are defined in terms of u_r's, $r=0, 1, 2$ and 3, as

$$
\begin{aligned}
U_0 &= 2u_3-2u_2-u_1+u_0,\\
U_1 &= 2u_3 - u_1,\\
U_2 &= u_3+u_2-2u_1,\\
U_5 &= 2u_3+u_2-3u_1.
\end{aligned}
\tag{14}
$$

We also need certain polynomials $P_i(v)$ in v, $i=0$, 1, 2 and 5 which can be obtained by replacing U_i by $P_i(v)$, and u_r by v^r in the above relations (14), so that

$$\begin{aligned} P_0(v) &= 2v^3-2v^2-\ v+1, \\ P_1(v) &= 2v^3 \qquad\ \ - v, \\ P_2(v) &= \ \ v^3+\ v^2-2v, \\ P_5(v) &= 2v^3+\ v^2-3v. \end{aligned} \tag{15}$$

4. **Some lemmas.** Remembering that the pentagonal numbers fall only in the residue classes $i=0$, 1, 2 and 5 modulo 7, the truth of the following lemma can be easily verified.

LEMMA 1.

$$\begin{aligned} P_i(v) &\equiv 1 \pmod 7, \quad \textit{if } v \equiv i \pmod 7 \\ &\equiv 0 \pmod 7, \quad \textit{if } v \not\equiv i \pmod 7. \end{aligned}$$

If we replace the u_r's appearing in the expressions (14) for U_i by

$$\sum_v (\mp v^r x^v)/f(x),$$

which is justified according to (12) we obtain,

$$U_i = \sum_v [\mp P_i(v)x^v]/f(x); \tag{16}$$

and then Lemma 1 leads to Lemma 2 given below.

LEMMA 2.

$$U_i \equiv \sum_{v\equiv i} (\mp x^v)/f(x) \pmod 7,$$

the summation being extended over all pentagonal numbers $v\equiv i \pmod 7$.

The truth of the following lemma can be verified without much difficulty by writing $7m+j$ with $j=0$, 2; -1, 3; 1; -2, -3; respectively in place of m in the expression $\frac{1}{2}m(3m+1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It may be remembered (when j is negative, say $-j'$) that the same set of numbers is represented by $\frac{1}{2}(7m-j')(21m-3j'+1)$ and $\frac{1}{2}(7m+j')(21m+3j'-1)$.

LEMMA 3. *With respect to the modulus* 7 *the pentagonal numbers* v *fall in the four residue classes* $i=0$, 1, 2 *and* 5; *the solutions of* $v\equiv i \pmod 7$ *and the corresponding associated signs are as given below.*

i	*solutions* (1*st set*) :	*sign*	*solutions* (2*nd set*) :	*sign*
0	$\frac{1}{2}(147m^2+7m)$	$(-1)^m$	$\frac{1}{2}(147m^2+91m)+7$	$(-1)^m$
1	$\frac{1}{2}(147m^2+35m)+1$	$(-1)^{m+1}$	$\frac{1}{2}(147m^2+133m)+15$	$(-1)^{m+1}$
2	$\frac{1}{2}(147m^2+49m)+2$	$(-1)^{m+1}$	———	———
5	$\frac{1}{2}(147m^2+77m)+5$	$(-1)^m$	$\frac{1}{2}(147m^2+119m)+12$	$(-1)^{m+1}$

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [5, p. 283], viz.,

$$\prod_{n=0}^{\infty} [(1-x^{2kn+k-l})(1-x^{2kn+k+l})(1-x^{2kn+2k})] = \sum_{-\infty}^{+\infty} (-1)^m x^{km^2+lm}. \tag{17}$$

In establishing this lemma k and l are given values which are in conformity with the quadratic expressions in m given in Lemma 3. As an illustration we have the initial steps for $\sum_{v\equiv 1}(\mp x^v)$ as

$$\begin{aligned}\sum_{v\equiv 1}(\mp x^v) &= \sum_{-\infty}^{+\infty}(-1)^{m+1}x^{(147m^2+35m)/2+1} + \sum_{-\infty}^{+\infty}(-1)^{m+1}x^{(147m^2+133m)/2+15}\\ &= -x\sum_{-\infty}^{+\infty}(-1)^m x^{(147m^2+35m)/2} - x^{15}\sum_{-\infty}^{+\infty}(-1)^m x^{(147m^2+133m)/2}.\end{aligned} \tag{18}$$

Lemma 4. *Writing $v\equiv i$ simply for $v\equiv i$* (mod 7)

$$\begin{aligned}\sum_{v\equiv 0}(\mp x^v) &= \prod_{n=0}^{\infty}[(1-x^{147n+70})(1-x^{147n+77})(1-x^{147n+147})]\\ &\quad + x^7\prod_{n=0}^{\infty}[(1-x^{147n+28})(1-x^{147n+119})(1-x^{147n+147})],\\ \sum_{v\equiv 1}(\mp x^v) &= -x\prod_{n=0}^{\infty}[(1-x^{147n+56})(1-x^{147n+91})(1-x^{147n+147})]\\ &\quad - x^{15}\prod_{n=0}^{\infty}[(1-x^{147n+7})(1-x^{147n+140})(1-x^{147n+147})],\\ \sum_{v\equiv 2}(\mp x^v) &= -x^2\prod_{n=0}^{\infty}(1-x^{49n+49}),\\ \sum_{v\equiv 5}(\mp x^v) &= x^5\prod_{n=0}^{\infty}[(1-x^{147+35})(1-x^{147n+112})(1-x^{147n+147})]\\ &\quad - x^{12}\prod_{n=0}^{\infty}[(1-x^{147n+14})(1-x^{147n+133})(1-x^{147n+147})].\end{aligned}$$

The next lemma is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v\equiv i}(\mp x^v)$ as given in Lemma 4. The following fact is to be used in addition.

$$\begin{aligned}&\prod_{n=0}^{\infty}[(1-x^{147n+r})(1-x^{147n+147-r})(1-x^{147n+147})]/f(x)\\ &\qquad = \frac{\prod_{n=0}^{\infty}[(1-x^{147n+r})(1-x^{147n+147-r})(1-x^{147n+147})]}{[(1-x)(1-x^2)(1-x^3)\cdots]}\\ &\qquad = \sum_{n=0}^{\infty} {}^{147}_{\;\;r}p(n)x^n.\end{aligned} \tag{19}$$

LEMMA 5. *With respect to the modulus* 7,

$$U_0 \equiv \sum_{n=0}^{\infty} \tfrac{147}{70}p(n)x^n + \sum_{n=0}^{\infty} \tfrac{147}{28}p(n-7)x^n,$$

$$U_1 \equiv -\sum_{n=0}^{\infty} \tfrac{147}{56}p(n-1)x^n - \sum_{n=0}^{\infty} \tfrac{147}{7}p(n-15)x^n,$$

$$U_2 \equiv -\sum_{n=0}^{\infty} {}^{49}p(n-2)x^n,$$

$$U_5 \equiv \sum_{n=0}^{\infty} \tfrac{147}{35}p(n-5)x^n - \sum_{n=0}^{\infty} \tfrac{147}{14}p(n-12)x^n.$$

We require another set of congruences which are derivable from the following identities.

$$p(n-1)+2p(n-2)-5p(n-5)-7p(n-7)+\cdots = \sigma(n); \tag{20}$$

$$\begin{aligned} p(n-1)+2^2p(n-2)-5^2p(n-5)-7^2p(n-7)+\cdots \\ = -\tfrac{1}{12}[5\sigma_3(n)-(18n-1)\sigma(n)]; \end{aligned} \tag{21}$$

$$\begin{aligned} p(n-1)+2^3p(n-2)-5^3p(n-5)-7^3p(n-7)+\cdots \\ = \tfrac{1}{192}[7\sigma_5(n)-10(15n-1)\sigma_3(n)+(360n^2-36n+1)\sigma(n)]. \end{aligned} \tag{22}$$

Here, $\sigma_k(n)$, as usual, stands for the sum of the kth powers of the divisors of n. The first identity (20) is a classical one due to Catalan [2, p. 290], and the second one is due to Glaisher [2, p. 312]. The third one has been proved by the author [6], where the other two are also proved. These three identities can be rewritten according to our present notation as,

$$\sum_v [\mp vp(n-v)] = -\sigma(n), \tag{23}$$

$$12\sum_v [\mp v^2p(n-v)] = 5\sigma_3(n)-(18n-1)\sigma(n), \tag{24}$$

$$192\sum_v [\mp v^3p(n-v)] = -7\sigma_5(n)+10(15n-1)\sigma_3(n)-(360n^2-36n+1)\sigma(n). \tag{25}$$

Now from (12) we have

$$\begin{aligned} u_r &= \sum_v (\mp v^r x^v)/f(x) = \sum_v (\mp v^r x^v)\cdot \sum_{n=0}^{\infty} p(n)x^n \\ &= \sum_{n=1}^{\infty} \left\{\sum_v [\mp v^r p(n-v)]\right\}x^n, \qquad r > 0. \end{aligned} \tag{26}$$

It is now easy to establish the validity of the following lemma from the above four relations (23), (24), (25) and (26).

LEMMA 6.

$$u_1 = -\sum_{n=1}^{\infty} \sigma(n)x^n,$$

$$u_2 \equiv \sum_{n=1}^{\infty} [\sigma_3(n)+(2n+3)\sigma(n)]x^n \pmod 7,$$

$$u_3 \equiv \sum_{n=1}^{\infty} [(n-1)\sigma_3(n)-(n^2+2n-2)\sigma(n)]x^n \pmod 7.$$

The following lemma can be easily obtained by the application of Lemma 6 to the expressions (14) for U_i's in terms of u_r's.

LEMMA 7.

$$U_0-1 \equiv \sum_{n=1}^{\infty} [(2n+3)\sigma_3(n)-(2n^2+n+1)\sigma(n)]x^n \pmod 7,$$

$$U_1 \equiv \sum_{n=1}^{\infty} [(2n-2)\sigma_3(n)-(2n^3-3n+2)\sigma(n)]x^n \pmod 7,$$

$$U_2 \equiv \sum_{n=1}^{\infty} [n\cdot\sigma_3(n)-n^2\cdot\sigma(n)]x^n \pmod 7,$$

$$U_5 \equiv \sum_{n=1}^{\infty} [(2n-1)\sigma_3(n)-(2n^2+2n-3)\sigma(n)]x^n \pmod 7.$$

Another useful lemma is given below. The first congruence belonging to it is already known [8], [7], and the second one is easily established from first principles.

LEMMA 8. $\sigma_3(7n+j) \equiv 0 \pmod 7$, *if j is a quadratic nonresidue of* 7; *i.e. if* $j=3$, 5 *or* 6.

$$\sigma_k(7n) \equiv \sigma_k(n) \pmod 7, \qquad k > 0.$$

5. **Proof of the theorems.** By comparing the coefficients of like powers of x in the two (right-hand) expressions for U_i modulo 7 given in Lemmas 5 and 7 we obtain the following congruences for $n > 0$.

$$(27) \qquad {}^{147}_{70}p(n) + {}^{147}_{28}p(n-7) \equiv (2n+3)\sigma_3(n)-(2n^2+n+1)\sigma(n) \pmod 7,$$

$$(28) \qquad -{}^{147}_{56}p(n-1) - {}^{147}_{7}p(n-15) \equiv (2n-2)\sigma_3(n)-(2n^2-3n+2)\sigma(n) \pmod 7,$$

$$(29) \qquad -\ {}^{49}p(n-2) \equiv n\cdot\sigma_3(n)-n^2\cdot\sigma(n) \pmod 7,$$

$$(30) \qquad {}^{147}_{35}p(n-5) - {}^{147}_{14}p(n-12) \equiv (2n-1)\sigma_3(n)-(2n^2+2n-3)\sigma(n) \pmod 7.$$

These are our basic relations from which our final conclusions are drawn.

Remembering the well-known congruence [10], [4, p. 167]

$$(31) \qquad \sigma_s(n) \equiv 0 \pmod k \quad \text{for almost all values of } n$$

for arbitrarily fixed k and odd s, it is a straightforward matter to deduce Theorem 1. In relation (31) and in Theorem 1 "for almost all values of n" means that the number of integers $n \leqq N$ for which the specified congruence does not hold is $o(N)$.

The first relation of Theorem 2 is obtained immediately by writing $7n+7$ for n in (29). The general result enunciated in this theorem actually emanates from the first part, and the process of derivation has two stages. In the first one Ramanujan's congruence (1) is derived from the first relation and then this derived relation is used in the second stage to establish the general proposition. It easily follows from (19), (17) and (9) that ${}^{147}_{7\rho}p(n)$ can be expressed in the (really finite) form,

$$(32) \qquad {}^{147}_{7\rho}p(n) = p(n) + \sum_{n'=1}^{\infty} \varepsilon(n')p(n-7n')$$

where $\varepsilon(n')=0$ or ± 1. For the special case corresponding to $\rho=7$ we have the fully specified expression,

$$(33) \qquad {}^{49}p(n) = \sum_{v} [\mp p(n-49v)].$$

Keeping in mind the first relation of Theorem 2, viz.,

$$(34) \qquad {}^{49}p(7n+5) \equiv 0 \pmod 7,$$

Ramanujan's congruence (1) is seen to be valid by putting successively $n=5$, 12, 19, 26, ... in (33). And to derive the general proposition we merely write $7n+5$ for n in (32) and make use of Ramanujan's congruence.

The two remaining results, Theorem 3 and Theorem 4, are not so simple. In Theorem 4 we are still concerned with relations between two differently restricted partition functions as in Theorem 1, but there are as many as four terms in the congruence relations. In the other there is no doubt a smaller number of terms, but there are as many as three differently restricted partition functions.

To establish Theorem 3 we first eliminate $\sigma(n)$ between the three pairs of relations obtained by coupling (29) respectively with (27), (28) and (30), and then in order to eliminate the term involving $\sigma_3(n)$ either (a) replace n by $7n+i$, i being so chosen as to make the coefficient of $\sigma_3(n)$ divisible by 7, or (b) replace n by $7n+j$ where j is a quadratic nonresidue of 7, so as to make it divisible by 7 according to the first part of Lemma 8. As an illustration we shall establish the relations involving the functions

$$ {}^{147}_{35}p(n), \qquad {}^{147}_{14}p(n) \quad \text{and} \quad {}^{49}p(n).$$

Eliminating $\sigma(n)$ between (29) and (30) we have for $n>0$,

$$(35) \qquad \begin{aligned} n^2[{}^{147}_{35}p(n-5) - {}^{147}_{14}p(n-12)] + (2n^2+2n-3)\cdot {}^{49}p(n-2) \\ \equiv -3n(n-1)\sigma_3(n) \pmod 7. \end{aligned}$$

Now to derive the third relation of the theorem we replace n by $7n+i$ with $i=8$ ($i=7$ leads to the first relation of Theorem 2). Again if we replace n by $7n+j$ with

$j=5$ and 6 we get respectively the sixth and the last relation of the theorem ($j=10$ leads to a result which is a trivial deduction from Theorem 2).

Theorem 4 requires the help of the second congruence of Lemma 8. We shall indicate the method followed by establishing the last congruence of the theorem. Writing $7n$ in place of n in (30) and making use of the lemma, we have,

$$ {}^{147}_{35}p(7n-5)-{}^{147}_{14}p(7n-12) \equiv -\sigma_3(n)+3\sigma(n) \pmod 7. \tag{36}$$

Subtracting (30) from (36) we have,

$$\begin{aligned} {}^{147}_{35}p(7n-5)-{}^{147}_{14}p(7n-12)-{}^{147}_{35}p(n-5)+{}^{147}_{14}p(n-12) \\ \equiv -2n\sigma_3(n)+(2n^2+2n)\sigma(n) \pmod 7. \end{aligned} \tag{37}$$

Now putting $7n+7$ in place of n we have the desired result.

Finally, it might be of interest to note that the seven differently restricted partition functions appearing in our discussions are connected by the identical relation,

$$\begin{aligned} {}^{147}_{70}p(n+2)-{}^{147}_{56}p(n+1)+{}^{147}_{35}p(n-3)+{}^{147}_{28}p(n-5) \\ -{}^{147}_{14}p(n-10)-{}^{147}_{7}p(n-13) = {}^{49}p(n). \end{aligned} \tag{38}$$

This can be easily derived from (13) which can be put in the form

$$\sum_i \sum_{v\equiv i} (\mp x^v)/f(x) = 1, \tag{39}$$

where $\sum_i$ denotes summation over the values $i=0$, 1, 2 and 5. All that we need do now is to apply Lemma 4 and the relation (19) before equating to zero the coefficients of x^n, $n>0$, on the left-hand side. If we assume this identical relation to be given then some of the congruences shown in the theorem become derivable from the remaining.

6. **Concluding observations.** This paper deals with the modulus 7 only. Congruences for restricted partition functions of the same type exist for other moduli also. For the moduli 2, 3, 5, 7, 11 and 13 we have rather simple relations. A selection from among those congruences which hold for almost all values of n are given below.

For almost all values of n,

$$ {}^{48}_{14}p(n) \equiv {}^{48}_{2}p(n-4) \pmod 2; \tag{40}$$

$$ {}^{192}_{68}p(n) \equiv {}^{192}_{4}p(n-20) \pmod 2; \tag{41}$$

$$ {}^{27}_{3}p(n) \equiv 0 \pmod 3; \tag{42}$$

$$ -{}^{243}_{99}p(n) \equiv {}^{243}_{63}p(n-6) \equiv {}^{243}_{18}p(n-21) \pmod 3; \tag{43}$$

$$ {}^{75}_{20}p(n) \equiv {}^{75}_{5}p(n-5) \pmod 5; \tag{44}$$

(45) $$\tfrac{363}{110}p(n) \equiv {}^{363}_{11}p(n-33) \pmod{11};$$

(46) $$^{169}p(n) \equiv 3\tau(n+7) \pmod{13}.$$

where $\tau(n)$ is Ramanujan's function defined at the beginning of the paper.

The treatments for the different moduli are not exactly identical, and there exist different types of results for which special methods are to be used. It is therefore proposed to publish separate papers on these moduli.

Finally, I must express my indebtedness to the referee for drawing my attention to Gordon's theorem and also for his useful suggestions.

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