MODULAR REPRESENTATIONS OF SPLIT BN PAIRS

BY
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Introduction. In 1964 C. W. Curtis classified the absolutely irreducible modular representations of a large class of groups, the so called finite groups of Lie type. This was done by finding in each irreducible module an element, called a weight element, which is an eigenvector for certain elements in the modular group algebra; proving that two irreducible modules are isomorphic if and only if the corresponding eigenvalues (the collection of such being called a weight) are equal; and finally by determining which weights are associated with irreducible modules, i.e. which weights actually occur. This paper is a continuation of Curtis' work. We construct the absolutely irreducible modular representations of a finite group of Lie type by finding weight elements in the modular group algebra which generate a full set of nonisomorphic minimal left ideals.

In some work of Steinberg and Curtis (see [14] and [6]) these representations were constructed for the covering groups of the Chevalley groups using the representations of the associated modular Lie algebras. The discussion given here avoids the Lie algebras altogether. Instead we construct each irreducible submodule of certain induced modules by finding the required weight elements. We prove at the same time that these induced modules have multiplicity free socles. Some remarks are made about the related questions of degrees and block structure of these representations.

The finite groups of Lie type whose representations were classified by Curtis in [7] were defined by a large number of axioms. These apparently consisted of basic properties possessed by all known examples of such groups, the Chevalley groups and variations thereon as defined by Steinberg [13], for example. A simplified axiom scheme is used here, the heart of which is the axioms for groups with BN pairs (Tits [16]). The groups discussed in [7] satisfy the simplified axioms.

The disadvantage of introducing new axioms is that the classification described in the first paragraph of this introduction must be redone. The proofs in this reworking are always similar to Curtis' but rarely identical, for the new axioms are enough weaker that some of the properties Curtis used do not follow from them. The advantages of reworking the classification theorem are twofold. First, the theorem is proved in greater generality. Secondly, the commutator relation is never assumed. It is usually replaced by arguments involving lengths of words in the Weyl group. The effect is to clarify the role of the root system in the classification theorem.

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Many of the preliminary results of Chapter II, whose aim is to imbed a root system in a group with a *BN* pair, are undoubtedly known in some form or other to many people. The idea for them was given to the author by Curtis who in turn obtained it from Tits. According to Benson [1] some of the preliminary lemmas were given in a course by Feit at Harvard University in 1963–64 and some of these are recorded in [1]. So far as the author knows, the main results of Chapter II have never been published. However, one should see Tits' paper [17] (which should appear soon) for some more general theorems.

After this paper was submitted, the author became acquainted with Robert Steinberg's *Lectures on Chevalley groups* [15]. In §§12–14 of these lectures he deals with the representation theory of the Chevalley groups and twisted types, and §13 overlaps Chapter III of this paper a great deal. Most of his arguments carry over to the axiom scheme used here. (See the remarks following 3.9 and 3.17.)

The author owes a great deal of thanks to Professor Curtis who suggested the problems dealt with here and who provided help and encouragement during the preparation of this paper. Thanks are also due to Professors Clark Benson and James Humphreys for their many helpful conversations with the author. The referee must be thanked for a careful reading and for observing a redundancy in the hypothesis of 3.17.

Notation and assumptions. The abstract group given by generators $x_1, \ldots, x_n$ and relations $r_i(x_1, \ldots, x_n)$ each of which is some word in the x_j's for $1 \leqq i \leqq m$ (alternatively a group with presentation $G = \langle x_1, \ldots, x_n : r_i(x_1, \ldots, x_n), 1 \leqq i \leqq m\rangle$) is F/N where F is the free group on the x's and N is the smallest normal subgroup of F which contains the r's. If G' is any group generated by $y_1, \ldots, y_n$ such that $r_i(y_1, \ldots, y_n) = 1$ in G', then $x_j \to y_j$ extends to a homomorphism $G \to G'$. $l(g)$ is the unique integer such that g may be expressed as a word in the x_i's of length $l(g)$ but not of a word in the x_i's of any shorter length. If $g \in G$, a reduced expression for g is a word in the x_j's of length $l(g)$.

We assume familiarity with representation theory background for which may be found in [9]. From Clifford's Theorem 49.7 [9], it follows that the kernel of an irreducible representation of a group in a field of characteristic p contains every normal p-subgroup of that group. For G a group, K a field, $H \leqq G$, M and N KH and KG modules respectively, M^G denotes the KG module induced from M and $N|_H$ denotes the KH module obtained from N by restriction. For P a p-group and K an algebraically closed field of characteristic p, KP is a local algebra, $KP \cong_K K \oplus \operatorname{rad} KP$ and

$$\operatorname{rad} KP = \langle 1-p : p \in P\rangle = \left\{\sum_{p\in P} \alpha_p p : \sum \alpha_p = 0\right\}.$$

For G a group, K a field X a subset of G, $\bar{X} = \sum_{x \in X} x \in KG$ and $X^g = g^{-1}Xg$.

I. *BN* pairs and root systems: Background.

DEFINITION 1.1 (TITS [16]). A group G has a *BN pair* if there exist subgroups B and N of G such that $\langle B, N\rangle = G$, $H=(B\cap N)\lhd N$, $N/H=W$ is generated by involutions $s_1,\ldots,s_n$ and

(a) $s_iBw\subseteq BwB\cup Bs_iwB$, $1\leqq i\leqq n$, $w\in W$,

(b) $s_iBs_i\neq B$, $1\leqq i\leqq n$.

W is called the *Weyl group* of the *BN* pair and n is called its *rank*.

Notice that the symbols s_i and w in 1.1 stand for cosets of H in N, but since $H\leqq B$ left, right and double cosets of B whose coset representatives are indicated by s_i or w are well defined. This convention in notation is well established, and we will not need to choose coset representatives for cosets of H in N until Chapter III.

BRUHAT THEOREM 1.2 (TITS [16]). *If G has a BN pair, then*

(a) $G=\bigcup_{w\in W} BwB$,

(b) $BwB=Bw'B$ *for* $w, w'\in W$ *implies* $w=w'$,

(c) $l(s_iw)>l(w)$ *for* $1\leqq i\leqq n$ *and* $w\in W$ *implies* $s_iBw\subseteq Bs_iwB$.

The definition of *BN* pair was probably motivated by a close examination of Chevalley's paper [3] in which he associated with every complex simple Lie algebra an infinite family of simple groups. Each group so constructed has a *BN* pair. With each complex simple Lie algebra we can associate a unique root system (which is a geometric object and will be defined below); and conversely, from every simple root system which satisfies a certain crystallographic condition we can construct a complex simple Lie algebra by generators and relations. Thus the source of the Chevalley groups is ultimately the root system, and the dependence on the root system is further intensified by the fact that in proving most of the properties of the Chevalley groups, one prefers to refer to the root system rather than the Lie algebra. For example, the argument which establishes 1.1(a) for the Chevalley groups (Lemma 10 in [3]) is very geometric. This chapter and the next show that the axioms for a *BN* pair are strong enough to bury a root system inside a group with a *BN* pair.

We now discuss root systems and finite groups generated by reflections briefly. Proofs may be found in [2], *Séminaire Chevalley*, Exposé 14 or in the appendix of [15].

Let V be an n-dimensional real Euclidean space whose inner product is (,). An orthogonal linear transformation s of V is called a reflection if $s\neq 1$ and s fixes some hyperplane pointwise. It follows that if r is a vector perpendicular to that hyperplane then s is given by

$$s_r\colon v\to v-\frac{2(v,r)}{(r,r)}\,r,\quad \text{for all } v\in V.$$

Note that $s_r^2=1$ and that $s_r(r)=-r$.

If W is a finite group generated by reflections let Δ be the set of all unit vectors which are perpendicular to the hyperplane fixed by some reflection in W. We can show that if r, $r' \in \Delta$, then $s_r(r') \in \Delta$ by observing that $s_r(r')$ is a unit vector perpendicular to the hyperplane associated with the reflection $s_r s_{r'} s_r^{-1}$, which is then $s_{s_r(r')}$ of course. It follows that (W, Δ) is a permutation group such that (1) Δ is finite and generates V (if Δ did not generate V then we could consider V' the space generated by Δ and show that W fixed the space orthogonal to V' pointwise and so we would lose nothing by replacing V with V'), (2) r and $tr \in \Delta$ for some real number t implies that $t = \pm 1$, and (3) $r \in \Delta$ implies that there exists the reflection $s_r \in W$, and the s_r's generate W. Any set Δ of vectors which satisfies (1)–(3) is called a *root system*, and the group W generated by the s_r's, $r \in \Delta$, is called a *Weyl group*.

If Δ is a root system we call a subset $\Pi \subseteq \Delta$ a *base* for if (1) Π is a vector space basis for V and (2) if every $r' \in \Delta$ has an expression

$$r' = \sum_{r \in \Pi} t_r r$$

with all the t_r's either nonnegative or nonpositive. Every root system contains a base and its Weyl group is generated by the s_r's for $r \in \Pi$. The length $l(w)$ of an element w in the Weyl group is always with respect to these generators. The roots in a base are enumerated $r_1, \ldots, r_n$ and are called *fundamental roots*, and the corresponding reflections, s_i written for s_{r_i}, are also fundamental. Every root is the image of some fundamental root by an element in W. For $r_i \in \pi \subseteq \Pi$, we often write $s_i \in \pi$ or just $i \in \pi$.

For a given base we define the sets of *positive* and *negative roots* in the obvious way. Namely

$$\Delta^+ = \left\{ \sum_{i=1}^{n} t_i r_i \in \Delta : t_i \geqq 0, 1 \leqq i \leqq n \right\}$$

and

$$\Delta^- = \left\{ \sum_{i=1}^{n} t_i r_i \in \Delta : t_i \leqq 0, 1 \leqq i \leqq n \right\}.$$

If $r \in \Delta^+ (\Delta^-)$ we often write $r > 0$ (< 0). For $w \in W$ define

$$\Delta_w^+ = \{r \in \Delta^+ : w(r) > 0\}, \qquad \Delta_w^- = \{r \in \Delta^+ : w(r) < 0\}$$

and $n(w) = |\Delta_w^-|$. Thus $\Delta = \Delta^+ \cup \Delta^-$ and $\Delta^+ = \Delta_w^+ \cup \Delta_w^-$ for $w \in W$.

LEMMA 1.3 (SOLOMON [12]). *For $w \in W$ and s_i a fundamental reflection*

(i) $\Delta_{ws_i}^- = \{r_i\} \cup s_i(\Delta_w^-)$ *and* $n(ws_i) = n(w) + 1$ *if* $w(r_i) > 0$,

(ii) $\Delta_w^- = \{r_i\} \cup s_i(\Delta_{ws_i}^-)$ *and* $n(ws_i) = n(w) - 1$ *if* $w(r_i) < 0$,

(iii) $n(w) = l(w)$, *and*

(iv) *w has a reduced expression $s_{i_1} \cdots s_{i_k}$ ending in s_{i_k} if and only if $w(r_{i_k}) < 0$.*

The following corollary is also mentioned in Solomon's paper.

COROLLARY 1.4. *There exists a unique element $w_0 \in W$ with maximal length. $w_0(\Delta^+) = \Delta^-$ and $w_0^2 = 1$.*

We have just seen how a root system is canonically attached to a finite group generated by reflections. The Weyl group of a BN pair is isomorphic to a group generated by reflections if it is finite, and the proof of this fact proceeds in three steps.

In [11] Matsumoto proves the following two theorems, the second of which he attributes to Iwahori.

THEOREM 1.5. *If $W=\langle s_1,\ldots,s_n\rangle$ is the Weyl group of a BN pair, then the generators of W satisfy the following condition called* C.

C. *If $s_{i_1}\cdots s_{i_k}$ is a reduced word and if s_{i_0} is such that $s_{i_0}s_{i_1}\cdots s_{i_k}$ is not reduced, then there exists an m, $1\leqq m\leqq k$, such that $s_{i_1}\cdots s_{i_m}=s_{i_0}\cdots s_{i_{m-1}}$.*

THEOREM 1.6. *If W is a group generated by involutions $s_1,\ldots,s_n$ which satisfy condition* C, *then W has a presentation*

$$W = \langle s_1,\ldots,s_n : (s_is_j)^{n_{ij}}\rangle$$

where $n_{ii}=1$ and $n_{ij}\geqq 2$ for all i and j.

THEOREM 1.7 (COXETER, §9.3 of [5]). *If W is a finite group with the presentation given in Theorem* 1.6, *then W is isomorphic to a finite group generated by reflections in n-dimensional Euclidean space. If Δ is the root system constructed from W, then a base Π of Δ can be chosen so that the s_i are fundamental reflections.*

We see that the Weyl group of a BN pair is isomorphic to the Weyl group of a root system in a very strong sense. Namely, the root system is given canonically, and the involutions which generate the Weyl group of the BN pair correspond under the isomorphism to fundamental reflections with respect to some base of the root system.

This strong isomorphism, Solomon's Lemma 1.3, and the Bruhat theorem will permit the construction of a set Σ of subgroups of G, a group with a BN pair, permuted by the Weyl group W and such that $(W,\Sigma)\cong(W,\Delta)$. Thus the root system associated with the Weyl group of a BN pair is firmly implanted in the group.

An immediate consequence of 1.3 is that if $w=s_{i_1}\cdots s_{i_k}$ and $l(w)=k$, we may unpeel Δ_w^- as

$$\Delta_w^- = \{r_{i_k}, s_{i_k}(r_{i_{k-1}}),\ldots, s_{i_k}\cdots s_{i_2}(r_{i_1})\}.$$

An analogue of 1.3 and of this unpeeling will be proved for (W,Σ) in 2.9 and 2.10.

While the geometrical properties of Weyl groups of BN pairs are used continually in this paper, knowledge of the presentation of a Weyl group is also important. For at one place in Chapter III we must appeal to the universal properties of groups defined by generators and relations. Proposition 1 in Matsumoto's paper [11] asserts that if $W=\langle s_i : i\in\Pi\rangle$ satisfies the hypotheses of 1.6 and if $\pi\subseteq\Pi$, then W_π (which is the subset of W whose elements have a reduced expression involving only the $s_i \in \pi$) is a group whose generators, the $s_i \in \pi$, satisfy condition C. Thus by 1.6 we have

LEMMA 1.8. *Let $\pi \subseteq \Pi$ and W_π be the subgroup of W generated by the $s_i \in \pi$. Then W_π is a group with presentation*

$$W_\pi = \langle s_i : (s_i s_j)^{n_{ij}}, i, j \in \pi \rangle.$$

(*Notation is the same as in* 1.6.)

II. **Finding a root system in a BN pair.**

The following lemma is well known.

LEMMA 2.1 (3.7 IN [1]). *If G has a BN pair (B, N) then G also has a BN pair (B, N') where $N' = \langle H', N \rangle$ and $H' = \bigcap_{n \in N} B^n$. Moreover $N'/H' = N/H$.*

Proof. Both N and H' normalize H' and so $H' \lhd N'$. Since $H' \subseteq B$, $H' \subseteq B \cap N' = B \cap NH' = (B \cap N)H' = HH' = H'$. $G = \langle B, N \rangle \subseteq \langle B, N' \rangle \subseteq G$. $N \cap H' = H$ and so

$$W = N/H = N/N \cap H' \cong NH'/H' = N'/H'.$$

Thus $W = N'/H'$ is still generated by involutions $s_1, \ldots, s_n$, and 1.1(a) and (b) follow because we may use the same coset representatives for W in both N/H and N'/H'.

Throughout this chapter we let G have a BN pair whose Weyl group $W = \langle s_1, \ldots, s_n \rangle$ is finite. We let Δ and Π be the set of roots and a base for Δ respectively which correspond to W as described in Chapter I. Recall that s_i is identified with the reflection which corresponds to $r_i \in \Pi$. Furthermore we assume that $H = \bigcap_{n \in N} B^n$ which does not alter the geometry in view of 2.1.

As mentioned in the introduction, some of the following results are probably well known. I learned 2.5 from Clark Benson. We begin with two technical lemmas.

LEMMA 2.2. *Suppose $l(s_i w) > l(w)$. Then $s_i B s_i \cap BwBw^{-1} \subseteq B$.*

Proof. By 1.1(a) $s_i B s_i \subseteq B \cup B s_i B$. Thus if the lemma is not true, $B s_i B \cap BwBw^{-1} \neq \varnothing$. Hence

$$\varnothing \neq BwB \cap Bs_iBw \subseteq BwB \cap Bs_iwB$$

by 1.2(c). By 1.2(a) $w = s_i w$ a contradiction.

LEMMA 2.3 (3.8 IN [1]). *Suppose $l(s_i w) > l(w)$. Then $B \cap B^{s_i w} \subseteq B \cap B^w$.*

Proof. $B \cap B^{s_i w} \subseteq B$.

$$w(B \cap B^{s_i w})w^{-1} = wBw^{-1} \cap s_i B s_i \subseteq B, \quad \text{by 2.2.}$$

Thus $B \cap B^{s_i w} \subseteq B^w$.

Notice that since $H^w = H$ for all $w \in W$, we have $H \subseteq B \cap B^w$.

LEMMA 2.4. *$B \cap B^{w_0} = H$ (where $w_0(\Delta^+) = \Delta^-$).*

Proof. Let $w \in W$ and define a sequence $w_1, \ldots, w_k \in W$ as follows. Let $w_1 = w$ and if $w_i = w_0$ let $k = i$. If not then by 1.3 there exists $r_j \in \Pi$ such that $w_i(r_j) > 0$. Let $w_{i+1} = w_i s_j$. Then by 1.3

$$l(w_{i+1}) = l(w_i s_j) = n(w_i s_j) = n(w_i)+1 = l(w_i)+1.$$

By 1.4 this sequence terminates with $w_k = w_0$.

By 2.3 and the fact that $l(w) = l(w^{-1})$

$$B \cap B^{w_0} = B \cap B^{w_k^{-1}} \subseteq B \cap B^{w_{k-1}^{-1}} \subseteq \cdots \subseteq B \cap B^{w_1^{-1}} = B \cap B^{w^{-1}}.$$

Thus $H \subseteq B \cap B^{w_0} \subseteq \bigcap_{w \in W} (B \cap B^{w^{-1}}) = \bigcap_{n \in N} B^n = H$.

Lemma 2.5 (3.4 in [1]). *If $l(ws_i) > l(w)$ then*

$$B = (B \cap B^w)(B \cap B^{s_i}) = (B \cap B^{s_i})(B \cap B^w).$$

Proof. Since $l(ws_i) > l(w)$, $l(s_i w^{-1}) > l(w^{-1})$ and 1.2(c) implies

$$B \subseteq (s_i B s_i)(w^{-1} B w).$$

For $b \in B$, write $b = b'b''$ as in the above factorization. Then $b' = bb''^{-1} \in B^{s_i} \cap Bw^{-1}Bw \subseteq B$ by 2.2. Hence $b'' = b'^{-1}b \in B$ also and the lemma follows.

Corollary 2.6. $B = (B \cap B^{s_i})(B \cap B^{w_0 s_i})$ *and* $B \cap B^{w_0 s_i} \neq H$, $1 \leq i \leq n$.

Proof. By 1.4 $l(w_0 s_i s_i) = l(w_0) > l(w_0 s_i)$ for all i, and so the first assertion is just 2.5. If the second assertion were false then $H = B \cap B^{w_0 s_i}$ for some i. Thus by 2.5 we would have $B = B \cap B^{s_i}$. This would contradict 1.1(b), the fact that $B \neq B^{s_i}$.

It is now convenient to define the analogues in G of the fundamental roots, Δ_w^+ and Δ_w^-.

Definition 2.7. Let $B_i = B \cap B^{w_0 s_i}$, $1 \leq i \leq n$. Let $B_w = B \cap B^w$ and $B_w^- = B \cap B^{w_0 w}$, $w \in W$.

Thus $B_i = B_{s_i}^-$ and $B_{w_0 w} = B_w^-$, which is reasonable in view of $\Delta_{s_i}^- = \{r_i\}$ and $\Delta_{w_0 w}^+ = \Delta_w^-$.

Lemma 2.8. *If $w \in W$ and $r_i \in \Pi$, then*

(a) $r_i \in \Delta_w^+$ *implies* $B_i \subseteq B_w$,

(b) $r_i \in \Delta_w^-$ *implies* $B_i \cap B_w = H$ *and* $B_i \nsubseteq B_w$.

Proof. First observe that for all $w \in W$, $l(w_0 w) = l(w_0) - l(w)$. (This follows by 1.3 and taking cardinalities in the following set equations: $\Delta_{w_0 w}^- = \Delta^+ - \Delta_w^-$ and $\Delta_w^- \subseteq \Delta^+$.) If $r_i \in \Delta_w^+$ for some fixed $w \in W$, 1.3 gives $l(ws_i) = l(w)+1$. The opening remark of the proof gives $l(w_0 s_i) = l(w_0) - 1$ and $l(w_0 s_i w^{-1}) = l(w_0) - l(s_i w^{-1}) = l(w_0) - l(w) - 1 = k$ say. Therefore if $s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w_0 s_i w^{-1}$, and if we let $w_j = s_{i_j} \cdots s_{i_k} w$, we must have $l(w_k) = 1 + l(w)$ and $l(w_j) = 1 + l(w_{j+1})$, $1 \leq j \leq k-1$ since otherwise

$$l(w_0) - 1 = l(w_0 s_i) = l(w_1) < l(w) + k = l(w) + l(w_0) - l(w) - 1,$$

a contradiction. 2.3 applied k times gives $B \cap B^{w_k} \subseteq B \cap B^{w}$ and $B \cap B^{w_j} \subseteq B \cap B^{w_{j+1}}$, $1 \leqq j \leqq k-1$. Thus we have $B_i = B \cap B^{w_1} \subseteq B \cap B^{w} = B_w$. For (b) we have

$$\begin{aligned} B_i \cap B_w &= B \cap B^{w_0 s_i} \cap B^{w} = (B^{s_i} \cap B^{w_0} \cap B^{ws_i})^{s_i} \\ &\subseteq (B \cap B^{w_0})^{s_i} = H \end{aligned}$$

where the inclusion follows from 2.2 and the fact that $l(s_i w^{-1}) = l(ws_i) = l(ws_i s_i) - 1$. $B_i \nsubseteq B_w$ since $B_i \neq H$, (2.6).

Here follows the analogue of 1.3. The formulas are suggested by (4.17) and (4.18) of [7].

THEOREM 2.9. *If $w \in W$ and s_i is a fundamental generator then*

(a) *$w(r_i) > 0$, equivalently $l(ws_i) > l(w)$, implies $B^-_{ws_i} = B_i (B^-_w)^{s_i}$ and $B_i \cap (B^-_w)^{s_i} = H$;*

(b) *$w(r_i) < 0$, equivalently $l(w) > l(ws_i)$, implies $B^-_w = B_i (B^-_{ws_i})^{s_i}$ and $B_i \cap (B^-_{ws_i})^{s_i} = H$.*

Proof. (b) is immediate from (a). For (a), $w(r_i) > 0$ and so 2.8 gives $B_i \subseteq B_{w_0 w s_i} = B^-_{ws_i}$. Using the fact that for subgroups A, B, C with $A \subseteq B$, $A(B \cap C) = B \cap AC$ we find

$$\begin{aligned} B^-_{ws_i} &= B^-_{ws_i} \cap B = B^-_{ws_i} \cap (B_i B_{s_i}) \quad \text{by 2.6} \\ &= B_i (B^-_{ws_i} \cap B_{s_i}) = B_i (B^{s_i} \cap B^{w_0 w} \cap B)^{s_i}. \end{aligned}$$

We analyze the last factor. From 2.2 it follows that $B^{s_i} \cap B^{w_0 w s_i} \subseteq B$ which implies $B \cap B^{w_0 w} \subseteq B \cap B^{s_i}$. Thus

$$(B^{s_i} \cap B^{w_0 w} \cap B) = B \cap B^{w_0 w} = B^-_w$$

which proves the first assertion.

For the second,

$$\begin{aligned} H \subseteq B_i \cap (B^-_w)^{s_i} &= B \cap B^{w_0 s_i} \cap B^{s_i} \cap B^{w_0 w s_i} \\ &\subseteq (B^{w_0} \cap B)^{s_i} = H^{s_i} = H. \end{aligned}$$

THEOREM 2.10. *For $w \in W$, $B = B^-_w B_w$.*

Proof. Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression for w. Then $l(w_0 s_{i_k}) < l(w_0)$ and

$$\begin{aligned} l(w_0 s_{i_k} \cdots s_{i_j}) &= l(w_0) - l(s_{i_k} \cdots s_{i_j}) \\ &< l(w_0) - l(s_{i_k} \cdots s_{i_{j+1}}) \\ &= l(w_0 s_{i_k} \cdots s_{i_{j+1}}), \qquad 1 \leqq j \leqq k-1. \end{aligned}$$

We let $w_j = s_{i_j} \cdots s_{i_k}$, $1 \leqq j \leqq k$. Then applying 2.9(b) k times

$$\begin{aligned} B &= B^-_{w_0} \\ &= B_{i_k} (B^-_{w_0 w_k})^{w_k} \\ &= B_{i_k} B^{w_k}_{i_{k-1}} (B^-_{w_0 w_{k-1}^{-1}})^{w_{k-1}} \\ &= B_{i_k} \cdots B^{w_2}_{i_1} (B_{w_0 w_1^{-1}})^{w_1}. \end{aligned}$$

The last factor is just $(B^{-}_{w_0w^{-1}})^w = (B\cap B^{w^{-1}})^w = B_w$. We may collapse the first k factors with 2.9(a) to find that they equal B^{-}_{w}.

COROLLARY 2.11. $G = \bigcup_{w\in W} B^{-}_{w} w^{-1}B$.

Proof.

$$\begin{aligned} G &= \bigcup_{w\in W} Bw^{-1}B && \text{by 1.2(a)} \\ &= \bigcup B^{-}_{w}B_w w^{-1}B && \text{by 2.10} \\ &= \bigcup B^{-}_{w} w^{-1}B \end{aligned}$$

since $w(B\cap B^w)w^{-1} \subseteq B$.

To pin down the geometrical significance of the *BN* axioms we will finish this chapter by showing that the root system lives inside G.

We define $\Sigma = \{wB_iw^{-1} : 1\leqq i\leqq n,\ w\in W\}$. Since $H\subseteq B\cap B^{w_0s_i} = B_i$, this set is well defined and for the same reason W permutes it via

$$w\colon w'B_iw'^{-1} \to ww'B_iw'^{-1}w^{-1}.$$

This permutation representation of W is faithful since for each $1\neq w\in W$ there exists an $r_i\in \Pi$ such that $w(r_i)<0$. By 2.9(b) $wB_iw^{-1}\neq B_i$. Thus (W,Σ) is a permutation group.

THEOREM 2.12. $(W,\Sigma)\cong(W,\Delta)$.

Proof. $r_i\to B_i$, $1\leqq i\leqq n$ is a well-defined map. It suffices to show that $w(r_i) \to wB_iw^{-1}$ defines a one-to-one map of Δ onto Σ.

To show that it is a map we recall from Chapter I that every root is of the form $w(r_i)$ and then observe that it suffices to show that $w(r_i)=r_j$ implies $wB_iw^{-1}=B_j$. But $w(r_i)=r_j$ and $w_0s_jw(r_i)>0$ by 1.3 and so 2.8(a) used twice implies $B_i\subseteq B_w$ and $B_i\subseteq B_{w_0s_jw}$. These in turn imply

$$wB_iw^{-1} \subseteq B\cap B^{w^{-1}}$$

and

$$wB_iw^{-1} \subseteq B^{w^{-1}} \cap B^{w_0s_j}.$$

Thus $wB_iw^{-1}\subseteq B_j$. The same argument applied to the equation $w^{-1}(r_j)=r_i$ shows that $w^{-1}B_jw\subseteq B_i$. Hence $B_j = wB_iw^{-1}$.

The map is onto by definition of Σ. To see that it is one-to-one it suffices to show that $wB_iw^{-1}=B_j$ implies $w(r_i)=r_j$. If $wB_iw^{-1}=B_j=B\cap B^{w_0s_j}$ then $B_i\subseteq B\cap B^w$ and by 2.8 $w(r_i)>0$. Moreover $B_i\subseteq B\cap B^{w_0s_jw}$ and again 2.8 gives $w_0s_jw(r_i)>0$ which implies $s_jw(r_i)<0$. By 1.3 $\{r_j\}=\Delta^{-}_{s_j}$ and so $w(r_i)=r_j$.

III. Modular representations of split *BN* pairs.

DEFINITION 3.1. A finite group G is said to have a *split BN pair of rank n at characteristic p* if G has a BN of rank n, $\bigcap_{n\in N} B^n = H$, and $B=XH$, $X\lhd B$ is a p-group and H is an abelian p'-group.

Throughout this chapter G will usually denote a finite group with a split BN pair of rank n at characteristic p and K will usually be an algebraically closed field of characteristic p. As in the last chapter we will identify $N/H=W$ with a finite group generated by reflections whose root system will again be called Δ. The base Π of Δ will again be chosen so that the reflections s_r, $r\in\Pi$ correspond to the generators of N/H.

We will need to modify the construction of Σ, which was the root system transplanted to the group G, and this will be done by translating some of the results of Chapter II into our new setting. To accomplish this translation we will need to choose coset representatives of H in N because symbols such as wX, $w\in W$ do not make sense. Thus for each $w\in W$, we let $(w)\in N$ be such that $(w)H=w$. For the time being $\{(w) : w\in W\}$ is a fixed but arbitrary set of coset representatives. Whenever it is correct to do so, we will omit the parentheses from (w). For example let $w\in W$, $h\in H$. Since H normalizes X, $X^{(w)}=X^{h(w)}$. Thus we write X^w. Arguments such as this which permit omission of parentheses will often be left to the reader.

Definition 3.2. Let $X_i=X\cap X^{w_0 s_i}$, $1\leqq i\leqq n$. Let $X_w=X\cap X^w$ and $X_w^-=X\cap X^{w_0 w}$, $w\in W$. Let $Y_i=X_i^{s_i}$ and $Y=X^{w_0}$.

As remarked after 2.7, $X_i=X_{s_i}^-$ and $X_w^-=X_{w_0 w}$. We introduce the two new symbols since we will frequently make use of the negative of a fundamental root and the set of negative roots.

Theorem 3.3. *If* $w\in W$ *and* $r_i\in\Pi$ *then*

(a) $w(r_i)>0$, *equivalently* $l(ws_i)>l(w)$, *implies* $X_{ws_i}^-=X_i(X_w^-)^{s_i}$ *and* $X_i\cap(X_w^-)^s=\{1\}$,

(b) $w(r_i)<0$, *equivalently* $l(ws_i)<l(w)$, *implies* $X_w^-=X_i(X_{ws_i}^-)^{s_i}$ *and* $X_i\cap(X_{ws_i}^-)^{s_i}=\{1\}$.

Proof. (b) is immediate from (a). For any $w\in W$, $B^w=H^wX^w=HX^w$. Thus

$$B\cap B^w = HX\cap HX^w \supseteq H(X\cap X^w).$$

Conversely if $g\in HX\cap HX^w$, there exist $h,h'\in H$, $x,x'\in X$ such that $g=h'x'=hx^{(w)}$. Thus $x^{(w)}=h^{-1}h'x'$ is a p element in B and X contains all p elements of B. Thus $x'=x^{(w)}$ and $g\in H(X\cap X^w)$. We have shown $B_w=HX_w$.

Moreover H normalizes each intersection X_w.

These remarks and 2.9 allow the following computations.

$$\begin{aligned} HX_{ws_i}^- &= B_{ws_i}^- = B_i(B_w^-)^{s_i} \\ &= HX_iH(X_w^-)^{s_i} = HX_i(X_w^-)^{s_i}. \\ H &= B_i\cap(B_w^-)^{s_i} \\ &= HX_i\cap H(X_w^-)^{s_i} \supseteq H(X_i\cap(X_w^-)^{s_i}). \end{aligned}$$

3.3 follows.

Theorem 3.4. *For* $w\in W$, $X=X_w^-X_w$ *and* $X_w\cap X_w^-=\{1\}$.

Proof. Either translate the corresponding theorem from Chapter II 2.10 as we did above or copy the proof of 2.10 appealing to 3.3 instead of 2.9.

THEOREM 3.5. *$G=\bigcup_{w\in W} X_w^- w^{-1}B$ and $\bigcup_{w\in W} X_w^-(w)^{-1}$ is a set of coset representatives of B in G. Also $X^{w_0}\cap B=\{1\}$.*

Proof. By 2.11 $G=\bigcup_{w\in W} B_w^- w^{-1}B=\bigcup X_w^- Hw^{-1}B=\bigcup X_w^- w^{-1}B$. Suppose $x\in X_w^-$, $x'\in X_{w'}^-$ and

$$x(w)^{-1}B = x'(w')^{-1}B.$$

Then by the Bruhat Theorem 1.2 $w=w'$ and so

$$(w)x^{-1}x'(w)^{-1}\in B\cap (X_w^-)^{w^{-1}} = B\cap X^{w^{-1}}\cap X^{w_0}.$$

But the p-group $B\cap X^{w_0}\subseteq B\cap B^{w_0}=H$ which is a p'-group. Thus $B\cap X^{w_0}=\{1\}$. Hence $x=x'$.

THEOREM 3.6. *Let Σ be the set of W conjugates of the X_i's $1\leqq i\leqq n$. Then (W,Σ) is a permutation group under*

$$w\colon w'X_iw'^{-1}\to ww'X_iw'^{-1}w^{-1}$$

and $X_i^{w^{-1}}\to w(r_i)$ defines an isomorphism $(W,\Sigma)\cong(W,\Delta)$.

This is translated from 2.12 just as the preceding results were.

It is immediate from 1.1(a) that $B\cup Bs_iB$ is a subgroup for each $s_i\in\Pi$. Moreover from 3.5 we see that $\{1\}\cup X_i(s_i)$ is a set of coset representatives of B in $B\cup Bs_iB$. Now if $1\neq x\in X_i$, then $(s_i)^{-1}x(s_i)\in Y_i\subseteq X^{w_0}$. $X^{w_0}\cap B=\{1\}$ and so $(s_i)^{-1}x(s_i)\subseteq Bs_iB=X_iH(s_i)X$. Hence we write

$$(s_i)^{-1}x(s_i) = f_i(x)h_i(x)(s_i)g_i(x),\quad \text{for } 1\neq x\in X_i$$

where $f_i(x)\in X_i$, $h_i(x)\in H$ and $g_i(x)\in X$.

f_i is a function from $X_i-\{1\}$ into itself since $f_i(x)(s_i)B$ is just the coset of B to which $x^{(s_i)}$ belongs, and if $f_i(x)=1$ then $x^{(s_i)}\in H(s_i)X$. This would imply that $x(s_i)\in(s_i)H(s_i)X\subseteq B$ contrary to the Bruhat Theorem 1.2. h_i is a function for if $h, h'\in H$ are such that

$$x^{(s_i)} = f_i(x)h(s_i)g_i(x) = f_i(x)h'(s_i)g_i(x)$$

then $s_i^{-1}h^{-1}h's_i$ is a p'-element in X and so $h=h'$. Similarly g_i is a function.

Finally f_i is a permutation of $X_i-\{1\}$. For if $f_i(x)=f_i(x')$, $x, x'\in X_i-\{1\}$,

$$(s_i)^{-1}x^{-1}x'(s_i) = g_i(x)^{-1}(s_i)^{-1}h_i(x)^{-1}h_i(x')(s_i)g_i(x')$$

is in $B\cap Y_i\subseteq B\cap X^{w_0}=\{1\}$. This means that $x=x'$. We can now make the following

DEFINITION 3.7. The *ith structural equation* $1\leqq i\leqq n$ is

$$(s_i)^{-1}x(s_i) = f_i(x)h_i(x)(s_i)g_i(x),\qquad 1\neq x\in X_i$$

where f_i is a permutation of $X_i-\{1\}$, and h_i and g_i are functions from $X_i-\{1\}$ to H and X respectively.

Let $S_i=\langle X, Y_i\rangle$ and notice that $h_i(x)(s_i)\in S_i$ for all $x\in X_i-\{1\}$. Thus we may choose a coset representative (s_i) of s_i to lie in S_i. We will do this and at the same time we will let $(w_0s_i)=(w_0)(s_i)$.

From the structural equation and the root system, the representation theory will now blossom forth. The entire weight element approach is suggested in [7]. Background in representation theory may be found in [9].

DEFINITION 3.8. Let G have a split BN pair of rank n at characteristic p, and let K be an algebraically closed field of characteristic p. A nonzero element m in a KG module M is called a *weight element* if there exist a linear K representation χ of B and scalars μ_i, $1\leqq i\leqq n$ such that

$$bm = \chi(b)m, \qquad b\in B$$
$$\bar{X}_i(s_i)^{-1}m = \mu_i m, \qquad 1\leqq i\leqq m,$$

where $\bar{X}_i=\sum_{w\in W_i} x$. The $(n+1)$-tuple $(\chi, \mu_1, \ldots, \mu_n)$ is called a *weight*.

THEOREM 3.9. *Let G have a split BN pair of rank n at characteristic p; let K be an algebraically closed field of characteristic p. Then*

(a) *every KG module M contains a weight element,*

(b) *if $m\in M$ is a weight element, then $KGm=KYm$ (where $Y=X^{w_0}$),*

(c) *if M and M' are irreducible KG modules with weight elements m, m' of weight $(\chi, \mu_1, \ldots, \mu_n)$, $(\chi', \mu_1', \ldots, \mu_n')$ respectively, then $\chi=\chi'$ and $\mu_i=\mu_i'$, $1\leqq i\leqq n$ implies that $M\cong M'$,*

(d) *if M is an irreducible KG module with weight element m, then the subspace of invariants of $M|_X$ is Km. Thus M has a unique weight element up to scalar multiple.*

Parts (a)–(c) of this theorem assert the same facts as 4.1(a)–(c) of [7]. Part (d) was taken from p. 239 of Steinberg's *Lectures* [15], and it replaced a weaker result in the original manuscript. The proof of each part of 3.9 follows its respective source in outline, and where the details of our proof differ from those in [7] or [15] it is because the commutator relation is not available to us. We break the proof down into a series of lemmas.

Recall that since X is a normal p-subgroup of B, it is contained in the kernel of every irreducible K representation of B. Thus if χ is a linear K representation of B, so is $xh\to\chi(xh^{(w)^{-1}})$, $w\in W$. Moreover since H is abelian h^w for $w\in W$ is well defined. Thus we define $\chi^w\colon xh\to\chi(xh^{w^{-1}})$ a linear representation of B.

LEMMA 3.10. *Let M be a KG module and $0\neq m\in M$ afford a linear representation χ of B. Then for $w\in W$, $\bar{X}_w^-(w)^{-1}m$ affords χ^w (hence affords the trivial representation of X).*

Proof. Let $h\in H$. Since H normalizes X_w^- we have

$$\begin{aligned} h\bar{X}_w^-(w)^{-1}m &= \bar{X}_w^- h(w)^{-1}m \\ &= \bar{X}_w^-(w)^{-1}h^{w^{-1}}m \\ &= \chi(h^{w^{-1}})\bar{X}_w^-(w)^{-1}m. \end{aligned}$$

Since $B=XH$, it suffices to show that $\bar{X}_w^-(w)^{-1}m$ affords the trivial representation of X. Let $x_0 \in X$. By 3.4 $X=X_w^- X_w$, $X_w^- \cap X_w=1$ and so for each $x \in X_w^-$ there exist unique $x' \in X_w^-$, $x'' \in X_w$ such that $x_0 x=x'x''$. We claim that $x \to x'$ is a permutation of X_w^-; for if $x_1, x_2 \in X_w^-$ are such that $x_0x_1=x_1'x_1''$ and $x_0x_2=x_2'x_2''$ and $x_1'=x_2'$ then

$$x_2''^{-1}x_1'' = x_2^{-1}x_1 \in X_w \cap X_w^- = \{1\}.$$

Thus $x_1=x_2$. With this permutation we may compute

$$\begin{aligned} x_0\bar{X}_w^-(w)^{-1}m &= \sum_{x\in X_w^-} x_0x(w)^{-1}m \\ &= \sum_{x\in X_w^-} x'x''(w)^{-1}m \\ &= \sum x'(w)^{-1}m \quad \text{since } (w)X_w(w)^{-1} \subseteq X \\ &= \bar{X}_w^-(w)^{-1}m \end{aligned}$$

since $x \to x'$ is a permutation of X_w^-.

LEMMA 3.11. *Every KG module M contains a weight element.*

Proof. Since X is in the kernel of every irreducible B-submodule of M and since $B/X \cong H$ is abelian, there exists $0 \neq m \in M$ which affords some linear representation χ on B. For all $w \in W$ consider elements of the form $\bar{X}_w^-(w)^{-1}m$, and among those that are nonzero choose one for which $l(w)$ is maximal. We show that this $\bar{X}_w^-(w)^{-1}m$ is a weight element.

By 3.10 $\bar{X}_w^-(w)^{-1}m$ affords χ^w on B. If $r_i \in \Delta_w^+$ for some $r_i \in \Pi$, then

$$\begin{aligned} \bar{X}_i(s_i)^{-1}\bar{X}_w^-(w)^{-1}m &= \bar{X}_i(s_i)^{-1}\bar{X}_w^-(s_i)(s_i)^{-1}(w)^{-1}m \\ &= \bar{X}_{ws_i}^-(s_i)^{-1}(w)^{-1}m \quad \text{by 3.3(a)} \\ &= \bar{X}_{ws_i}^-(ws_i)^{-1}hm \\ &= \chi(h)\bar{X}_{ws_i}(ws_i)^{-1}m \end{aligned} \tag{3.12}$$

where $h=(ws_i)(s_i)^{-1}(w)^{-1} \in H$. But $l(ws_i)>l(w)$ and so maximality of $l(w)$ among the nonzero $\bar{X}_w^-(w)^{-1}m$ forces

$$\bar{X}_i(s_i)^{-1}\bar{X}_w^-(w)^{-1}m = 0.$$

If $r_i \in \Delta_w^-$ for some $r_i \in \Pi$, then

$$\begin{aligned} \bar{X}_i(s_i)^{-1}\bar{X}_w^-(w)^{-1}m &= \bar{X}_i(s_i)^{-1}\bar{X}_i(s_i)\bar{X}_{ws_i}^-(s_i)^{-1}(w)^{-1}m \quad \text{by 3.3(b)} \\ &= \bar{X}_i\Big[1+\sum_{\substack{1\neq x\\ x\in X_i}} f_i(x)h_i(x)(s_i)g_i(x)\Big]\bar{X}_{ws_i}^-(s_i)^{-1}(w)^{-1}m \quad \text{by 3.7} \\ &= \bar{X}_i\Big[\sum_{1\neq x;x\in X_i} h_i(x)\Big](s_i)\bar{X}_{ws_i}^-(s_i)^{-1}(w)^{-1}m \quad \text{by 3.10 and 3.7} \\ &= \Big[\sum \chi^w(h_i(x))\Big]\bar{X}_i(\bar{X}_{ws_i}^-)^{s_i}(w)^{-1}m \quad \text{by 3.10} \\ &= \Big[\sum_{1\neq x;x\in X_i} \chi^w(h_i(x))\Big]\bar{X}_w^-(w)^{-1}m \quad \text{by 3.3(b).} \end{aligned} \tag{3.13}$$

Thus $\bar{X}_w^-(w)^{-1}m$ is a weight element.

This proves 3.9(a). 3.12 and 3.13 are computations which are used again. The next lemma proves part (b).

LEMMA 3.14. *Let* $(\chi, \mu_1, \ldots, \mu_n)$ *be a weight (associated with some weight element). Then there exists a function* $\mathcal{Y}: G \to KY$, *depending only on the weight itself, such that whenever m is a weight element of weight* $(\chi, \mu_1, \ldots, \mu_n)$

$$gm = \mathcal{Y}(g)m, \quad \text{for } g \in G.$$

Proof. It suffices to define $\mathcal{Y}$ on $\{(w) : w \in W\}$ for if this is done, we may use the decomposition $G = \bigcup_{w \in W} Y(w)B$ and define $\mathcal{Y}$ as follows. For all $g \in G$ fix an expression $g = y(w)b$, $y \in Y$, $b \in B$. Define $\mathcal{Y}(g) = \chi(b)y\mathcal{Y}((w))$. Clearly $\mathcal{Y}$ fulfills the requirements of the theorem.

We define $\mathcal{Y}$ on $\{(w)\}$ by induction on $l(w)$. If $l(w)=0$, $w=1$ and $\mathcal{Y}(1)=1$. For $l(w)=1$, $w=s_i$ for some i. The ith structural equation 3.7 gives

$$(s_i)(s_i)^{-1}\bar{X}_i(s_i) = (s_i) + \sum_{1 \neq x; x \in X_i} (s_i)f_i(x)h_i(x)(s_i)g_i(x).$$

If m is any weight element of weight $(\chi, \mu_1, \ldots, \mu_n)$ then

$$\begin{aligned} \chi((s_i)^2)\mu_i m &= (s_i)(s_i)^{-1}\bar{X}_i(s_i)m \\ &= (s_i)m + \chi((s_i)^2) \sum_{1 \neq x; x \in X_i} \chi^{s_i}(h_i(x))(f_i(x))^{(s_i)^{-1}}m. \end{aligned}$$

Thus we define

$$\mathcal{Y}((s_i)) = \chi((s_i)^2)\left[\mu_i - \sum_{1 \neq x; x \in X_i} \chi^{s_i}(h_i(x))(f_i(x))^{(s_i)^{-1}}\right]$$

which is in $KY_i \subset KY$.

Suppose $w \in W$ and that $\mathcal{Y}$ has been defined on all elements in W of strictly shorter length. Since $l(w) > 1$ there is a fundamental root $r_i \in \Delta_w^-$. Then $l(ws_i) < l(w)$.

$$\begin{aligned} (w)m &= (w)(s_i)^{-1}(s_i)m \\ &= (w)(s_i)^{-1}\mathcal{Y}(s_i)m \\ &= \mathcal{Y}(s_i)^{(s_i)(w)^{-1}}(w)(s_i)^{-1}m \\ &= \chi(h)\mathcal{Y}(s_i)^{(s_i)(w)^{-1}}\mathcal{Y}(ws_i)m \end{aligned}$$

where $h = (ws_i)^{-1}(w)(s_i)^{-1}$. Since $\mathcal{Y}(s_i) \in KY_i$ and since $Y_i^{s_i w^{-1}} = X_i^{w^{-1}} \subseteq Y$ we may define

$$\mathcal{Y}((w)) = \chi(h)\mathcal{Y}(s_i)^{(s_i)(w)^{-1}}\mathcal{Y}((ws_i)).$$

COROLLARY 3.15. *If two irreducible KG modules M, M′ contain weight elements m, m′ of the same weight* $(\chi, \mu_1, \ldots, \mu_n)$, *then they are isomorphic.*

Proof. Since $KY \cong_K K \oplus \text{rad } KY$, we may follow the $\mathcal{Y}$ defined in 3.14 by the projection onto K to get a function $f: G \to K$ which depends only on the weight itself and has the property that for any weight element m of weight $(\chi, \mu_1, \ldots, \mu_n)$

$$gm = f(g)m \ (\text{mod rad } KYm), \qquad g \in G.$$

For all $g \in G$ define $\phi(gm) = gm'$. Let

$$U = \left\{\sum_g \xi_g gm : \sum_g \xi_g gm' = 0\right\}.$$

U is a submodule of M. Irreducibility implies $U=0$ or M. If the latter occurs then there exist scalars $\xi_g \in K$ such that

$$m = \sum \xi_g gm \quad \text{and} \quad 0 = \sum \xi_g gm'.$$

Since $KY \cong K \oplus \text{rad } KY$,

$$1 = \sum \xi_g f(g) \quad \text{and} \quad 0 = \sum \xi_g f(g)$$

a contradiction. Thus $U=0$ and ϕ extends to a well-defined map $M \to M'$. Knowing this it is obviously a nonzero homomorphism, and irreducibility implies that it is an isomorphism.

Proof of 3.9(d). Let M be an irreducible KG module with weight element m. Then by 3.9(b) $M = KYm = Km \oplus (\text{rad } KY)m$. Following Steinberg we first show that $X_i(s_i)^{-1}(\text{rad } KY)m \subseteq (\text{rad } KY)m$, for $i \in \Pi$.

Since $X = X_w^- X_w$ with $w = w_0 s_i w_0$ we have

$$Y = X^{w_0} = X_i^{s_i}(X^{w_0} \cap X^{w_0 s_i}).$$

For $y \in Y$ write $y = y_i y_i'$ according to this factorization of Y. Then

$$\begin{aligned} \bar{X}_i(s_i)^{-1}y &= \bar{X}_i(s_i)^{-1}y_i y_i' \\ &= \bar{X}_i(s_i)^{-1}y_i' && \text{since } y_i^{(s_i)} \in X_i \\ &= \bar{X}_i y_i''(s_i)^{-1} && \text{with } y_i'' = y_i'^{(s_i)} \in X^{w_0} \cap X^{w_0 s_i} \\ &= \sum_{x \in X_i} y(x)x(s_i)^{-1} && \text{where } y(x) \in X^{w_0} \cap X^{w_0 s_i} \end{aligned}$$

and where the last equality follows since the map $x \to x'$ defined by $xy_i'' = y(x)x'$, $y(x) \in X^{w_0 s_i} \cap X^{w_0}$ and $x' \in X_i$, is a permutation of X_i. (See proof of 3.10.) Thus

$$\bar{X}_i(s_i)^{-1}(1-y)m = \sum_{x \in X_i} (1-y(x))x(s_i)^{-1}m \in (\text{rad } KY)m$$

proving that $(\text{rad } KY)m$ is stable under multiplication by $\bar{X}_i(s_i)^{-1}$.

Now suppose some $m' \neq m$ affords the 1 representation of X. Then as $M = Km \oplus (\text{rad } KY)m$ we may assume that $m' \in (\text{rad } KY)m$. Since H is abelian and normalizes X we may assume that m' affords a linear representation of B. By multiplying m' by suitable $X_i(s_i)^{-1}$ as in Lemma 3.11 we can construct a weight element m'' which must be in $(\text{rad } KY)m$. Using irreducibility of M and 3.9(b) we see that $M = KYm'' \subseteq (\text{rad } KY)m$, a contradiction.

The last sentence of 3.9(d) is now immediate from the definition of weight element. This completes the proof of 3.9.

We are now concerned with finding the possible weights and weight elements. The next lemma tells us where to look.

LEMMA 3.16. *If G is any finite group, H any subgroup and K any field, then every irreducible KG module is contained in a module induced from an irreducible KH module.*

Proof. Let M be an irreducible KG module and let N be the top factor module in some composition series for $M|_H$. Then $\mathrm{Hom}_{KH}(M|_H, N)\neq 0$. We claim that

$$\mathrm{Hom}_{KH}(M|_H, N) \underset{K}{\cong} \mathrm{Hom}_{KG}(M, N^G).$$

The map ϕ: $\mathrm{Hom}_{KH}(M|_H, N) \to \mathrm{Hom}_{KG}(M, N^G)$ defined by

$$\phi(f)(m) = \sum g_i \otimes f(g_i^{-1}m)$$

for $f \in \mathrm{Hom}(M|_H, N)$, $m \in M$ and $\{g_i\}$ a set of left coset representatives of H in G, and the map π_*: $\mathrm{Hom}_{KG}(M, N^G) \to \mathrm{Hom}_{KH}(M|_H, N)$ defined by

$$\pi_*(f)\colon m \overset{f}{\to} \sum g_i \otimes m_i \to g_1 m_1, \qquad f \in \mathrm{Hom}_{KG}(M, N^G)$$

are K linear maps which are inverses of each other. Thus $\mathrm{Hom}_{KG}(M, N^G)\neq 0$. Since M is irreducible any nonzero homomorphism f: $M \to N^G$ embeds M in N^G.

In light of 3.16 we will confine our search for weights and weight elements to KG modules which are induced from irreducible (hence linear) representations of B. This is the main theorem.

THEOREM 3.17. *Let G be a finite group with a split BN pair of rank n at characteristic p, and let K be an algebraically closed field of characteristic p. Let* $S_i = \langle X, Y_i\rangle$ *for* $i \in \Pi$.

(a) *Let m afford a linear representation* χ *of B and let* $\pi_1 = \{i \in \Pi : \chi^{w_0}|_{H_i} = 1\}$ *where* $H_i = H \cap S_i$. *Every weight element in* $(Km_B)^G$ *determines a subset* $\pi \subseteq \pi_1$, *and distinct (i.e., nonproportional) weight elements determine distinct subsets of* π_1. *If a weight element* $m_\pi \in (Km_B)^G$ *determines* $\pi \subseteq \pi_1$, *then it has weight*

$$(\chi^{w_0}, \mu_1, \ldots, \mu_n) \quad \textit{where } \mu_i = \begin{cases} -1 & \textit{if } i \in \pi_1 - \pi \\ 0 & \textit{otherwise} \end{cases}$$

and has the form

$$m_\pi = \sum_{w \in W_\pi} \alpha_{w_0 w} \bar{X}_{w_0 w}^{-}(w_0 w)^{-1} \otimes m$$

where the α'*s are nonzero scalars.*

(b) *Every weight element in* $(Km_B)^G$ *generates an irreducible submodule. Thus each* $(Km_B)^G$ *has a multiplicity free socle.*

(c) *Suppose that* (s_i) *can be chosen in* $X_i Y_i X_i$, $1 \leq i \leq n$. *Let m,* χ *and* π_1 *be as in* (a).

Then for every $\pi \subseteq \pi_1$ *there exists a weight element in* $(Km_B)^G$ *of weight*

$$(\chi^{w_0}, \mu_1, \ldots, \mu_n), \quad \mu_i = \begin{cases} -1 & \text{if } i \in \pi_1 - \pi \\ 0 & \text{otherwise} \end{cases}$$

and if the (w) *are suitably chosen that weight element is*

$$m_\pi = \sum_{w \in W} \bar{X}^-_{w_0 w}(w_0 w)^{-1} \otimes m.$$

REMARKS. If we could show that each $X_i H \cup X_i H s_i X_i$ was a group, then by using a structural equation argument (3.7) we could choose the $(s_i) \in X_i Y_i X_i$ making the additional hypothesis of 3.17(c) superfluous. The most we can conclude from 3.7 as it stands is that the (s_i) may be chosen in $X_i Y_i X$. Both Curtis and Steinberg use this mobility of the (s_i) to prove the existence of weight elements (see 4.1(d) of [7] and p. 238 of [15] respectively).

3.17(b) is new information about the groups considered in [7]. However for the Chevalley groups, Steinberg states that the weight elements of 3.17(c) (viewed as part of KG) generate irreducible modules and he gives a proof in the rank one case.

Note that 3.17(a) says that if $(\lambda, \mu_1, \ldots, \mu_n)$ is a weight then (a) $\lambda|_{H_i} \neq 1$ implies that $\mu_i = 0$; and (b) $\mu_i \neq 0$ implies $\mu_i = -1$. Given the added hypothesis of 3.17(c), 3.17(c) says that given any $(n+1)$-tuple $(\lambda, \mu_1, \ldots, \mu_n)$ that satisfies these two conditions there exists a weight element of that weight.

The proof of 3.17(a), which will be dissected into numerous lemmas, involves a close examination of the induced module $M = (Km_B)^G$. Since $\bigcup_{w \in W} X^-_w (w)^{-1}$ is a set of coset representatives of B in G (3.5), M has a K basis

$$\bigcup_{w \in W} \{x(w)^{-1} \otimes m : x \in X^-_w\}.$$

We choose $(s_i) \in S_i$ and $(w_0 s_i) = (w_0)(s_i)$ which is justified by the discussion following 3.7.

LEMMA 3.18. *The subspace of invariants of* $M|_X$ *is the linear span of* $\bar{X}^-_w (w)^{-1} \otimes m$, $w \in W$.

Proof. First observe that

$$M|_X = \oplus \sum_{w \in W} \{\text{linear span of } x(w)^{-1} \otimes m, x \in X^-_w\}$$

due to the linear independence of the $x(w)^{-1} \otimes m$'s and due to the fact that each summand is an X module. This follows from $X = X^-_w X_w$ (3.4) and $(w)X_w(w)^{-1} \subseteq X$. Thus it suffices to show that the invariant subspace of $\langle x(w)^{-1} \otimes m, x \in X^-_w \rangle$ is just $K\bar{X}^-_w (w)^{-1} \otimes m$. But if

$$m_0 = \sum_{x \in X^-_w} \alpha_x x(w)^{-1} \otimes m$$

is such that $xm_0 = m_0$ for all $x \in X$ then

$$\sum \alpha_x x' x(w)^{-1} \otimes m = \sum \alpha_x x(w)^{-1} \otimes m$$

for all $x' \in X_w^-$. Thus $\alpha_{x'^{-1}x} = \alpha_x$ for all $x, x' \in X_w^-$ which shows that all of the α_x's are equal. Thus m_0 is scalar multiple of $\bar{X}_w^-(w)^{-1} \otimes m$. $\bar{X}_w^-(w)^{-1} \otimes m$ is an X invariant by 3.10.

LEMMA 3.19. *If $m_0 \in M$ affords a linear representation λ of B, then*

$$m_0 = \sum_{w \in W} \alpha_w \bar{X}_w^-(w)^{-1} \otimes m$$

and

$$\alpha_w(\chi^w(h) - \lambda(h)) = 0, \qquad w \in W, h \in H.$$

(*Recall m affords χ.*)

Proof. X is in the kernel of any linear representation of B; thus the first assertion follows from 3.18. The last assertion follows by comparing coefficients in the following equation.

$$\begin{aligned}\sum_{w \in W} \lambda(h)\alpha_w \bar{X}_w(w)^{-1} \otimes m &= \lambda(h) m_0 = h m_0 \\ &= \sum_{w \in W} \alpha_w \bar{X}_w^-(w)^{-1} \otimes h^{w^{-1}} m \\ &= \sum_{w \in W} \alpha_w \chi^w(h) \bar{X}_w^-(w)^{-1} \otimes m.\end{aligned}$$

LEMMA 3.20. *If the m_0 of 3.19 is a weight element of weight $(\lambda, \mu_1, \ldots, \mu_n)$, and $r_i \in \Pi$, then*

(a) $r_i \in \Delta_w^+$ *implies* $\alpha_w \mu_i = 0$.

(b) $r_i \in \Delta_w^-$ *implies*

$$\alpha_w \mu_i = \alpha_w \left[\sum_{1 \neq x; x \in X_i} \chi^w(h_i(x)) \right] + \alpha_{ws_i} K_{iw}$$

where $K_{iw} = \chi((ws_i)(s_i)^{-1}(w)^{-1}) \neq 0$.

Proof. Again we compare coefficients in

$$\begin{aligned}\sum_{w \in W} \alpha_w \mu_i \bar{X}_w^-(w)^{-1} \otimes m &= \mu_i m_0 = \bar{X}_i(s_i)^{-1} m_0 \\ &= \sum_{w; w(r_i) > 0} \alpha_w \bar{X}(s_i)^{-1} \bar{X}_w^-(w)^{-1} \otimes m \\ &\quad + \sum_{w; w(r_i) < 0} \alpha_w \bar{X}_i(s_i)^{-1} \bar{X}_w^-(w)^{-1} \otimes m \\ &= \sum \alpha_w K_{iw} \bar{X}_{ws_i}^-(ws_i)^{-1} \otimes m \\ &\quad + \sum \alpha_w \left[\sum_{1 \neq x; x \in X_i} \chi^w(h_i(x)) \right] \bar{X}_w^-(w)^{-1} \otimes m \quad \text{by 3.12 and 3.13} \\ &= \sum_{w; w(r_i) < 0} \left[\alpha_{ws_i} K_{iw} + \alpha_w \left[\sum_{1 \neq x; x \in X_i} \chi^w(h_i(x)) \right] \right] \bar{X}_w^-(w)^{-1} \otimes m\end{aligned}$$

by a change of variable in the first sum.

Before determining the α_w's we prove two useful results about the sums $\sum \chi^w(h_i(x))$.

LEMMA 3.21. *Let χ be any linear K representation of B and let $w \in W$, $r_i \in \Pi$ be such that $w(r_i) < 0$. Then either*

$$\chi^w = \chi^{ws_i} \quad \text{or} \quad \sum_{1 \neq x; x \in X_i} \chi^w(h_i(x)) = 0.$$

Proof. We work inside of $M = (Km_B)^G$. Lemma 3.10 implies that $\bar{X}_w^-(w)^{-1} \otimes m$ affords χ^w on B and $\bar{X}_i(s_i)^{-1}\bar{X}_w^-(w)^{-1} \otimes m$ affords χ^{ws_i}. But 3.13 implies

$$\bar{X}_i(s_i)^{-1}\bar{X}_w^-(w)^{-1} \otimes m = \left[\sum_{1 \neq x; x \in X_i} \chi^w(h_i(x))\right] \bar{X}_w^-(w)^{-1} \otimes m$$

and the result follows.

COROLLARY 3.22. *Let $H_i = S_i \cap H$ and suppose $H_i \subseteq \ker \chi$ for all $r_i \in \pi \subseteq \Pi$. Then for all $w \in W_\pi$, $\chi^w = \chi$ and in particular $w^{-1}H_i w \subseteq \ker \chi$, $i \in \pi$.*

Proof. Induction on $l(w)$. Suppose the lemma is true for all words in W_π which are shorter than $w \in W_\pi$. By 1.8 and 1.3 there exists $r_i \in \pi$ such that $w(r_i) < 0$ and $l(ws_i) < l(w)$. By induction $H_i^{s_i} = H_i \subseteq \ker \chi^{ws_i}$. Thus

$$-1 = \sum_{x \neq 1; x \in X_i} \chi^{ws_i}(h_i(x)^{s_i}) = \sum \chi^w(h_i(x)).$$

By 3.21, $\chi^w = \chi^{ws_i}$ which equals χ by induction.

LEMMA 3.23. *Let $S_i = \langle X, Y_i \rangle$ and $H_i = S_i \cap H$. If χ is a linear K-representation of B such that $\chi|_{H_i} \neq 1$, then*

$$\sum_{1 \neq x; x \in X_i} \chi(h_i(x)) = 0.$$

This result is due to Curtis; one or two details in the proof differ due to the different axiom scheme.

Proof. Let $G_i = B \cup Bs_iB$, and let χ^{s_i} afford a KB module Km. Since 3.5 implies that $X_i(s_i)^{-1} \cup \{1\}$ is a set of left coset representative of B in G_i, the same analysis as that used in 3.18 and 3.19 shows that the only possible G_i weight elements in $(Km)^{G_i}$ are of the form

$$\alpha 1 \otimes m + \beta \bar{X}_i(s_i)^{-1} \otimes m, \qquad \alpha, \beta \in K.$$

(Weight element here means a nonzero element which affords a linear representation of B and is an eigenvector for $\bar{X}_i(s_i)^{-1}$.)

If $\chi^{s_i} \neq \chi$ then 3.21 shows that $\sum_{1 \neq x; x \in X_i} \chi^{s_i}(h_i(x)) = 0$. Thus $\bar{X}_i(s_i)^{-1} \otimes m$ affords χ on B and 3.13 and 3.21 show that $\bar{X}_i(s_i)^{-1}\bar{X}_i(s_i)^{-1} \otimes m = 0$. On the other hand if $\chi^{s_i} = \chi$ then $\alpha 1 \otimes m + \beta \bar{X}_i(s_i)^{-1} \otimes m$ affords χ on B and

$$\bar{X}_i(s_i)^{-1}(\alpha 1 \otimes m + \beta \bar{X}_i(s_i)^{-1} \otimes m) = \left[\alpha + \beta \sum_{1 \neq x; x \in X_i} \chi^{s_i}(h_i(x))\right] \bar{X}_i(s_i)^{-1} \otimes m$$

by 3.13. Thus α and β can be chosen so that this last term is zero. Hence there exists a G_i weight element of weight $(\chi, 0)$. Hence there exists an irreducible KG_i module of weight $(\chi, 0)$ (which may be constructed by factoring out a submodule of $(Km)^{G_i}$ which is maximal with respect to not containing the weight element).

Let M' be this irreducible KG_i module with weight element m' of weight $(\chi, 0)$. We claim that the space of invariants of $M'|_X$ has dimension 1. It is at least 1 since m' affords the one representation of X. If it were greater than 1, then by embedding M' in some $(Km_B)^{G_i}$ (3.16) and analyzing the invariants of $(Km_B)^{G_i}|_X$ just as in 3.18 we would see that the space of X invariants was just $K(1 \otimes m) + K(\bar{X}_i(s_i)^{-1} \otimes m)$ which would mean that $1 \otimes m \in M'$. This would imply that $M' = (Km_B)^{G_i}$ which contradicts the fact that the dimension of the induced module is $|X_i| + 1$ while the dimension of M' is less than or equal to $|X_i|$ (see proof of 3.14). Thus the space of elements in M' which afford the one representation of $Y_i = X_i^{s_i}$ is one dimensional also.

We claim that $(s_i)^{-1}m' \in (\text{rad } KY_i)m'$. $(s_i)^{-1}m'$ affords the one representation of Y_i. The proof of 3.14 shows that $KG_i m' = KY_i m'$ and so since $Y_i \leqq S_i \leqq G_i$, $KS_i m' = KY_i m'$. Since $KY_i = K \oplus \text{rad } KY_i$, either $(\text{rad } KY_i)m' \neq 0$ or else $KS_i m' = KY_i m'$ is one dimensional. If the latter alternative occured, the fact that $H_i \leqq S_i$ and $\chi|_{H_i} \neq 1$ would imply that S_i had a linear K-representation which was not the one representation. This cannot happen as S_i is generated by its p-elements and as 1 is the only p^nth root of 1 in K. Hence $(\text{rad } KY_i)m' \neq 0$ and there exists $v \in \text{rad } KY_i$ such that $vm' \neq 0$ and $(1-y)vm' = 0$ for all $y \in Y'$. Thus vm affords the one representation of Y_i. By the last sentence of the preceding paragraph and the second sentence of this paragraph we have the claim.

Since m' is of weight $(\chi, 0)$ we have

$$\begin{aligned} 0 &= (s_i)^{-1}(s_i)\bar{X}_i(s_i)^{-1}m' \\ &= (s_i)^{-1}m' + \sum_{x \in X_i; 1 \neq x} \chi^{s_i}(h_i(x))(f_i(x))^{(s_i)^{-1}}m' \end{aligned}$$

by 3.7. Thus

$$\sum_{1 \neq x; x \in X_i} \chi^{s_i}(h_i(x))(f_i(x))^{(s_i)^{-1}}m' = -(s_i)^{-1}m' \in (\text{rad } KY_i)m'.$$

Since Y_i is a p-group $\sum \chi^{s_i}(h_i(x)) = 0$ and the lemma follows.

We can now determine the weight elements in $(Km)^G$. Recall that m afforded χ on B and that if m_0 was a weight element in $(Km)^G$, it was of the form

$$m_0 = \sum_{w \in W} \alpha_w \bar{X}_w^-(x)^{-1} \otimes m.$$

We now assume this m_0 is a fixed weight element in $(Km)^G$ of weight $(\lambda, \mu_1, \ldots, \mu_n)$.

3.24. $\alpha_{w_0} \neq 0$ and hence m_0 affords χ^{w_0}, i.e. $\lambda = \chi^{w_0}$.

Proof. If $\alpha_{w_0} = 0$ then 3.20(b) shows that $\alpha_{w_0 s_i} = 0$ for all i. Using 3.20(b) again and again in this way, we conclude that $\alpha_w = 0$ for all w and hence $m_0 = 0$, a contradiction. The second assertion follows by 3.19.

Let m_0 be normalized by setting $\alpha_{w_0}=1$ and let

$$\pi_0 = \{i \in \Pi : \chi^{w_0}|_{H_i} \neq 1\},$$
$$\pi_1 = \{i \in \Pi : \chi^{w_0}|_{H_i} = 1\}.$$

3.25. $\mu_i \neq 0$ implies $\mu_i = -1$, $\chi^{w_0}|_{H_i}=1$ and $\alpha_{w_0 s_i}=0$.

Proof. $w_0 s_i(r_i)>0$ implies $\alpha_{w_0 s_i}=0$ by 3.20(a). $w_0(r_i)<0$ and $\alpha_{w_0 s_i}=0$ and 3.20(b) imply

$$\alpha_{w_0}\mu_i = \alpha_{w_0} \sum_{1 \neq x; x \in X_i} \chi^{w_0}(h_i(x))+0.$$

Since $\mu_i \neq 0$, 3.23 implies that $\chi^{w_0}|_{H_i}=1$ which in turn implies that $\mu_i=-1$ by the same equation.

Thus $\mu_i=0$ for all $i \in \pi_0$. For $i \in \pi_1$ 3.20(b) shows that

$$\mu_i \alpha_{w_0} = \alpha_{w_0} \sum \chi^{w_0}(h_i(x)) + \alpha_{w_0 s_i} K_{i w_0}.$$

That is, $\mu_i + 1 = \alpha_{w_0 s_i}$. (Recall $K_{i w_0}=1$ since we choose $(w_0 s_i)=(w_0)(s_i)$.) Thus for $i \in \pi_1$, $\mu_i = 0$ (-1) if and only if $\alpha_{w_0 s_i}=1$ (0) by 3.25.

We define

$$\pi = \{i \in \pi_1 : \alpha_{w_0 s_i} = 1\} = \{i \in \pi_1 : \mu_i = 0\}.$$

Summarizing we have

3.26. $i \in \pi_0$ implies $\mu_i = 0$.

$i \in \pi$ implies $\mu_i=0$ and $\alpha_{w_0 s_i}=1$.

$i \in \pi_1 - \pi$ implies $\mu_i=-1$, $\alpha_{w_0 s_i}=0$ and $\chi^{w_0}|_{H_i}=1$.

We have shown so far that the weight element m_0 determines a subset π of π_1 and that this subset determines the weight. 3.27 finishes the proof of 3.17(a) by showing that the scalars α_w which give m_0 are defined in terms of nothing but π.

3.27. (a) $\alpha_{w_0 w}$ is a fixed nonzero scalar for $w \in W_\pi$, which is defined only in terms of π and the K_{iw}'s. (See 3.20 for definition.)

(b) $\alpha_{w_0 w}=0$ if $w \notin W_\pi$.

Proof of (a). Induction on the length of w. For words of length 0 or 1, 3.27(a) is true by 3.24 and 3.26 respectively. Assume $w \in W_\pi$ has length l and assume 3.27(a) true for all shorter words in W_π. By 1.8 and 1.3 there exists $r_i \in \pi$ such that $w(r_i)<0$. Thus $l(ws_i)<l(w)$ and $w_0 w s_i(r_i)<0$. By induction $\alpha_{w_0 w s_i} \neq 0$. 3.19 implies $\chi^{w_0 w s_i}=\lambda =\chi^{w_0}$ which gives $\chi^{w_0 w s_i}|_{H_i}=1$. By 3.20(b)

$$\mu_i \alpha_{w_0 w s_i} = \alpha_{w_0 w s_i}\left[\sum_{x \neq 1; x \in X_i} \chi^{w_0 w s_i}(h_i(x))\right] + \alpha_{w_0 w} K_{i w_0 w s_i}.$$

$\mu_i=0$ and the sum is just -1. Thus $\alpha_{w_0 w}=(1/K_{i w_0 w s_i})\alpha_{w_0 w s_i}$.

Proof of (b). We show $\alpha_{w_0 w}=0$ if $w \notin W_\pi$ by induction on the length of w. For $l(w)=1$ then $w=s_i$ for $i \in$ either $\pi_1 - \pi$ or π_0. In the former case $\alpha_{w_0 s_i}=0$ by 3.26 and in the latter, 3.20(b), $\mu_i=0$ and $\sum \chi^{w_0}(h_i(x))=0$ combine to show $\alpha_{w_0 s_i}=0$.

Now suppose $l(w)>1$, $w \notin W_\pi$ and that $\alpha_{w_0w}=0$ for all $w \in W-W_\pi$ of shorter length. By 1.3 there exists $r_i \in \Pi$ such that $w(r_i)<0$.

(i) If $r_i \in \pi_1-\pi$, then $\mu_i=-1$, $w_0w(r_i)>0$ and 3.20 combine to give $\alpha_{w_0w}=0$.

(ii) If $r_i \in \pi_0$ we use 3.20(b) which asserts

$$(*) \qquad \alpha_{w_0ws_i}\left[\mu_i - \sum_{x\neq 1; x\in X_i} \chi^{w_0ws_i}(h_i(x))\right] = \alpha_{w_0w}K_{iw_0ws_i}.$$

If $\alpha_{w_0ws_i}=0$ then $\alpha_{w_0w}=0$ automatically by (*). If $\alpha_{w_0ws_i}\neq 0$ then 3.19 implies $\chi^{w_0ws_i}=\chi^{w_0}$ and so the sum $\sum \chi^{w_0ws_i}(h_i(x))=0$ by 3.23 and the definition of π_0. Thus $\alpha_{w_0w}=0$ by (*).

(iii) If $r_i \in \pi$ then $ws_i \notin W_\pi$ since $w \notin W_\pi$. Moreover $l(ws_i)<l(w)$ by 1.3. This with (*) implies that $\alpha_{w_0w}=0$. The proof of 3.27 is complete.

We now prove 3.17(b). Let $M=(Km)^G$ where m affords a linear representation χ of B and let $m_\pi \in M$ be some weight element of weight $(\chi^{w_0}, \mu_1, \dots, \mu_n)$. By factoring out a submodule of KGm_π maximal with respect to noncontainment of m_π we see that there exists an irreducible module M' of weight $(\chi^{w_0}, \mu_1, \dots, \mu_n)$. By 3.16 we can embed M' in a KG module induced from a linear B module, say $(Km')^G$ where m affords λ on B. From 3.17(a) all weight elements in $(Km')^G$ afford λ^{w_0} on B so that we may assume $\lambda=\chi$ and $m=m'$. Also by 3.17(a) no two linearly independent weight elements in $M=(Km)^G$ have the same weight so that m_π must belong to the irreducible submodule of M which is isomorphic to M'. Thus m_π generates that irreducible module.

That the socle of $(Km)^G$ is multiplicity free is now immediate from 3.17(a), 3.9(d) and what has been proved of (b). This completes the proof of 3.17(b).

To prove 3.17(c) we need an auxiliary result due to Curtis (see p. 198 of [7]).

LEMMA 3.28. *Let $\pi \subseteq \Pi$ and suppose that the (s_i) can be chosen in $X_iY_iX_i$. Then the coset representatives (w) of $w \in W_\pi$ can be chosen so that if $w, w' \in W_\pi$*

$$(w)(w')(ww')^{-1} \in H_\pi = \langle H_i^w : r_i \in \pi, w \in W_\pi\rangle.$$

Proof. For this proof only, we use a different notation for the coset representatives corresponding to fundamental reflections. Let $\zeta\colon N \to W$ be the natural map. By hypothesis there exists $\sigma_i \in X_iY_iX_i$ such that $\zeta(\sigma_i)=s_i$. So for $i \in \pi$ we define

$$H_i' = \langle \sigma_i\sigma_i' : s_i = \zeta(\sigma_i) = \zeta(\sigma_i'), \sigma_i, \sigma_i' \in N \cap X_iY_iX_i\rangle.$$

$H_i' \subseteq H \cap S_i$ since $s_i^2=1$ and $X_iY_iX_i \subseteq S_i$. Since H is abelian $(H_i')^w$, $w \in W$ is well defined and so we define

$$H_\pi' = \langle (H_i')^w : i \in \pi, w \in W_\pi\rangle.$$

Note that $H_\pi' \subseteq H_\pi$, (H_π being defined in the statement of 3.28). Notice that $\sigma_i \in N \cap X_iY_iX_i$ such that $\zeta(\sigma_i)=s_i$ normalizes H_π' for all $i \in \pi$. Define

$$N_\pi = \langle \sigma_i, H_\pi' : i \in \pi\rangle.$$

Then $H'_\pi \lhd N_\pi$. Since $N_\pi \subseteq N$, the inclusion map followed by ζ gives $\varphi: N_\pi \to N/H = W$ and $H'_\pi \subseteq \ker \varphi$ so that we have $\varphi^*: N_\pi/H'_\pi \to W$. $\operatorname{Im} \varphi^* = W_\pi$ since $W_\pi = \langle s_i : i \in \pi \rangle$. If we can show that

$$\varphi^*: N_\pi/H'_\pi \to W_\pi$$

is an isomorphism then we can pick our coset representatives for W_π from N_π. It will follow that for $w, w' \in W_\pi$, and $(w), (w'), (ww') \in N_\pi$ mapping onto w, w', ww' respectively under φ^*, we have $(w)(w')(ww')^{-1} \in H'_\pi \subseteq H_\pi$.

To prove φ^* is an isomorphism we use 1.8, the fact that W_π has a presentation

$$W_\pi = \langle s_i : (s_i s_j)^{n_{ij}}, i, j \in \pi \rangle$$

$n_{ii} = 1$ and $n_{ij} \geqq 2$ for $i \neq j$. For each $i \in \pi$ define $\psi(s_i) = \sigma_i H'_\pi$ where $\sigma_i \in N \cap X_i Y_i X_i$ and $\zeta(\sigma_i) = s_i$. This coset is well defined by the definition of H'_i. For all $i \in \pi$,

$$\sigma_i H'_\pi \sigma_i H'_\pi = \sigma_i^2 H'_\pi = H'_\pi.$$

Thus $\psi(s_i)^2 = H'_\pi = \psi(1)$.

Suppose n is an even integer $\geqq 2$. Then the relation $(s_i s_j)^n = 1$ is equivalent to

$$(s_i s_j)^{n/2} s_i (s_i s_j)^{-n/2} = s_i.$$

Thus $(s_i s_j)^{n/2}(r_i) = \pm r_i$ by Chapter I. By 3.6 and the choice of $\sigma_i \in X_i Y_i X_i$ we have

$$(\sigma_i \sigma_j)^{n/2} \sigma_i (\sigma_i \sigma_j)^{-n/2} \in X_i Y_i X_i \text{ or } Y_i X_i Y_i,$$

and, letting this element by σ'_i, $\zeta(\sigma'_i) = s_i$. In any event $\sigma'_i \in N_\pi$ (for if $\sigma'_i \in Y_i X_i Y_i$ then $(\sigma'_i)^{-1} \sigma'_i \sigma'_i = \sigma'_i \in (Y_i X_i Y_i)^{s_i} = X_i Y_i X_i$). The definition of H'_π implies that σ_i and σ'_i differ by something in H'_π and so $(\sigma_i H'_\pi \sigma_j H'_\pi)^n = H'_\pi$. A similar argument reaches the same conclusion if n is odd.

We have shown that the $\psi(s_i)$'s satisfy the relations which define W_π, and thus ψ extends to a homomorphism

$$\psi^*: W_\pi \to N_\pi/H_\pi$$

which happens to be the inverse of φ^*. This completes the proof of 3.28.

Proof of 3.17(c). Let m afford χ on B, $\pi_0 = \{i \in \Pi : \chi^{w_0}|_{H_i} \neq 1\}$ and

$$\pi_1 = \{i \in \Pi : \chi^{w_0}|_{H_i} = 1\} \quad \text{and} \quad \pi \subseteq \pi_1.$$

We choose coset representatives of H in N so that $(s_i) \in S_i$, $(w_0 s_i) = (w_0)(s_i)$ and so that the conclusion of 3.28 is valid. Then we show that

$$m_\pi = \sum_{w \in W_\pi} \bar{X}^-_{w_0 w}[(w_0)(w)]^{-1} \otimes m$$

is a weight element of weight

$$(\chi^{w_0}, \mu_1, \ldots, \mu_n), \quad \mu_i = \begin{cases} -1, & i \in \pi_1 - \pi \\ 0, & \text{otherwise.} \end{cases}$$

$xm_\pi = m_\pi$ by 3.18 and by 3.22 $\chi^{w_0 w} = \chi^{w_0}$ for all $w \in W_\pi$ which means that m_π affords χ^{w_0} on B.

Let $r_i \notin \pi$. $w(r_i) > 0$ for all $w \in W_\pi$ by 1.3. Thus $w_0 w(r_i) < 0$ for all $w \in W_\pi$. With the 3.13 and 3.22 we compute

$$\bar{X}_i(s_i)^{-1} m_\pi = \sum_{\substack{1 \neq x \\ x \in X_i}} \chi^{w_0}(h_i(x)) m_\pi = -m_\pi \quad \text{if } r_i \in \pi_1 - \pi$$
$$= 0 \quad \text{if } r_i \in \pi_0.$$

When $r_i \in \pi$ we copy the computation in the proof of 3.20 making these changes. All a_w's appearing there are 1. In our situation $K_{iw} = \chi^{w_0}((ws_i)(s_i)^{-1}(w)^{-1}) = 1$ by 3.28 and the fact that $H_\pi \subseteq \ker \chi^{w_0}$ (3.22). Again by 3.22 $\sum_{x \neq 1; x \in X_i} \chi^{w_0 w}(h_i(x)) = -1$, $w \in W_\pi$. With this information the computation in 3.20 shows $\bar{X}_i(s_i)^{-1} m_\pi = 0$. This completes the proof of 3.17.

Remark. We will show that the weight elements obtained here are the same as those found in [7]. Let E be an idempotent in KH which affords a linear representation χ of H, and let $\pi \subseteq \pi_1 = \{r_i \in \Pi : \chi|_{H_i} = 1\}$. In [7] it is proved that

$$m_\pi = \sum_{w \in W_\pi} \bar{X}_w E(w^{-1} w_0) \bar{X}$$

is a weight element in KG of weight

$$(\chi, \mu_1, \ldots, \mu_n), \mu_i = \begin{cases} -1 & \text{if } r_i \in \pi_1 - \pi \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\bar{X}_w = \bar{X}^-_{w_0 w}$. By 3.22 $\chi^w = \chi$ for $w \in W_\pi$. Thus $E(w^{-1} w_0) = (w^{-1} w_0) E^{w_0}$. Hence

$$m_\pi = \sum_{w \in W_\pi} \bar{X}^-_{w_0 w}(w^{-1} w_0^{-1}) E^{w_0} \bar{X}.$$

But $E^{w_0} \bar{X} \in KB$ affords χ^{w_0} on B and $m_\pi \in KG \otimes_{KB} KB$; so m_π can be identified with the weight element in 3.17(c).

We close this chapter with some remarks about the degrees of the absolutely irreducible modular representations and about the block structure.

Theorem 3.29. *Let G and K satisfy the hypotheses of* 4.1 *of* [7]. (*These amount to the hypotheses of our* 3.17 *plus a commutator relation plus the existence of subgroups $X_i H \cup X_i H s_i X_i$. The hypotheses of* 3.17 *will suffice for the inequality on the right.*) *Let m be a weight element of weight $(\chi, \mu_1, \ldots, \mu_n)$ and*

$$\pi = \{i \in \Pi : \mu_i = 0 \textit{ and } \chi|_{H_i} = 1\},$$
$$\pi' = \{i \in \Pi : \mu_i = -1 \textit{ and } \chi|_{H_i} = 1\}.$$

Then $|X^-_{w_{\pi'}}| \leqq \dim KGm \leqq |X^-_{w_\pi w_0}|$ where w_π ($w_{\pi'}$) is the word of maximal length in W_π ($W_{\pi'}$).

The orders of the subgroups $X^-_{w_\pi}$ and $X^-_{w_\pi w_0}$ are computable in the case of the Chevalley groups. See Solomon's paper [12].

The right-hand inequality is proved by showing $w_\pi m = m$ (using the definition of weight elements and structural equations), by factoring $Y = (X^-_{w_\pi w_0})^{w_0}(X^-_{w_\pi})^{w_\pi}$ (using 3.4) and then using 3.9(b). The left-hand inequality is proved by finding a subgroup $G_{\pi'}$ of G which satisfies the hypotheses of 4.1 [7], showing that $KG_{\pi'}m \subseteq KGm$ is a $G_{\pi'}$ module of dimension $|X^-_{w_{\pi'}}|$ (by Theorem 4, [8]).

For background on blocks see §§86–88 of [9].

THEOREM 3.30. *If G is a rank one BN pair of characteristic p, then there are exactly* $|C_H(X)|$ *blocks of full defect and exactly one of defect zero. If G is simple then* $C_H(X) = 1$, *and there is exactly one block of full defect.*

Proof. That there is exactly one block of defect zero is 1.5(f) of [7]. By a result of Green [10] defect groups are always Sylow intersections. One easily checks that X is p-Sylow in G and that it is a *TI* set. Thus all uncounted blocks have full defect, and the number of these is just the number of p-regular conjugate classes of full defect. The number of such classes is easily checked to be $|C_H(X)|$.

The last statement follows since $C_H(X) \lhd G$. And this is easily checked by using the structural equation to show that N centralizes $C_H(X)$.

The block structure for split *BN* pairs of characteristic p and rank greater than one appears to be more complicated. According to Curtis' Theorem 4 of [8] there is still exactly one block of defect zero. We quote a result of Conlon [4]. Let G be a finite group, P a p-subgroup and K an algebraically closed field of characteristic p. Then Conlon's Theorem says that if L is a $KN(P)$ module belonging to a $N(P)$ block e, with defect group P, then there exists an indecomposable direct summand of L^G which belongs to the G block E, where E corresponds to e under the Brauer correspondence. Since $X \lhd B$ and H is abelian, all blocks of B have X as a defect group. Hence Conlon's result together with 3.17 implies that for every linear representation χ of B, there exists an irreducible module of weight $(\chi, \mu_1, \ldots, \mu_n)$, for some μ_i's, belonging to a block of full defect.

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