CATEGORICAL HOMOTOPY AND FIBRATIONS

BY
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Let $\mathscr{K}$ be an arbitrary category, and let $\mathfrak{M}$ be any family of its morphisms. It is known [3] that there is a category $\mathscr{K}/\mathfrak{M}$ (the Gabriel-Zisman "category of fractions of $\mathscr{K}$ by $\mathfrak{M}$") having the same objects as $\mathscr{K}$, and a covariant functor $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ which is the identity on objects, such that $\eta(f)$ is invertible in $\mathscr{K}/\mathfrak{M}$ for each $f \in \mathfrak{M}$.

We will use each class $\mathfrak{M}$ to determine a notion of homotopy in $\mathscr{K}$, by defining two morphisms f, g to be $\mathfrak{M}$-homotopic if $\eta(f) = \eta(g)$. This notion has the usual properties expected of a homotopy notion; moreover, in the category Top of topological spaces and continuous maps, suitable choices of $\mathfrak{M}$ (for example, $\mathfrak{M}$ = all homotopy equivalences) reveal $\mathfrak{M}$-homotopy equivalent to the usual notion of homotopy.

Each class $\mathfrak{M}$ determines a notion of fibration in $\mathscr{K}$. In the category Top, a suitable choice of $\mathfrak{M}$ determines both the usual homotopy and the Hurewicz fibrations; however, different classes $\mathfrak{M}$ may yield the usual notion of homotopy and distinct notions of fibration. More generally, in an arbitrary category $\mathscr{K}$, it is the given class $\mathfrak{M}$ itself, rather than the homotopy notion induced by $\mathfrak{M}$, that determines the concept of fibration; from this viewpoint, it turns out, surprisingly, that the notion of a Hurewicz fibration is *not* a homotopy notion. By "reversing arrows," $\mathfrak{M}$ determines also a concept of cofibration; and again there is a splitting: in fact, two classes may determine the same notion of homotopy but distinct notions of cofibration.

In the last section, we introduce the concept of a weak $\mathfrak{M}$-fibration. This notion does not, in general, possess all the advantages of the previous one; however, it reduces to the previous notion under suitable restrictions on the class $\mathfrak{M}$. Moreover, there are classes, $\mathfrak{M}$, $\mathfrak{N}$ in the category Top such that {weak $\mathfrak{M}$-fibrations} = {Hurewicz fibrations} and {weak $\mathfrak{N}$-fibrations} = {Dold fibrations}.

Each covariant functor $\Phi: \mathscr{K} \to \mathscr{L}$ determines a Φ-homotopy in $\mathscr{K}$, by choosing $\mathfrak{M} = \{f \mid \Phi(f) \text{ is invertible}\}$. Most of the homotopy notions encountered in various categories turn out to stem from this general construction. For example: if $\mathscr{G}^Z$ is the category of graded groups and degree zero homomorphisms, and if $\pi(X)$ is the total homotopy of a space X, then in the category $\mathscr{K}$ of spaces dominated by CW-complexes the functor $\pi: \mathscr{K} \to \mathscr{G}^Z$ determines the usual notion of homotopy;

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if $\mathscr{K}$ is the category of Kan-complexes, then π-homotopy is the same as Kan-homotopy; and if $H: \mathscr{K} \to \mathscr{G}^Z$ is the total homology functor on the category of chain-complexes, then H-homotopy is the usual chain-homotopy.

In the Appendix, we give a construction of $\mathscr{K}/\mathfrak{M}$ that differs slightly from that of Gabriel-Zisman, and contains slightly more detail; one advantage of this procedure is that some results in [3] are seen to hold without the additional requirement that "$\mathfrak{M}$ admits a calculus of left fractions".

1. **The quotient category $\mathscr{K}/\mathfrak{M}$.** Let $\mathscr{K}$ be any category, and let $\mathfrak{M}$ be any family of its morphisms. By a quotient category we shall mean a pair $(\mathscr{K}/\mathfrak{M}, \eta)$, where $\mathscr{K}/\mathfrak{M}$ is a category with the same objects as $\mathscr{K}$ and $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ is a covariant functor that preserves objects, having the following two properties:

C1. If $\alpha \in \mathfrak{M}$, then $\eta(\alpha)$ is invertible in $\mathscr{K}/\mathfrak{M}$.

C2. [Universality]: If $T: \mathscr{K} \to \mathscr{L}$ is any covariant functor to any category $\mathscr{L}$ such that $T(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$, then there exists a unique covariant functor $\Delta: \mathscr{K}/\mathfrak{M} \to \mathscr{L}$ such that $T = \Delta \circ \eta$.

1.1. THEOREM. *Let $\mathscr{K}$ be any category, and let $\mathfrak{M}$ be any family of its morphisms. Then a quotient category $(\mathscr{K}/\mathfrak{M}, \eta)$ exists.*

This theorem is mentioned in [3]. However, no detailed construction of $(\mathscr{K}/\mathfrak{M}, \eta)$ having the generality we require appears in the literature, so we will give a proof of 1.1 in the Appendix. It follows directly from our construction, and we will need this in the sequel, that

1.2. PROPOSITION. *Each morphism G in $\mathscr{K}/\mathfrak{M}$ has a factorization $G = \hat{f}_n \circ \hat{\alpha}_n \circ \cdots \circ \hat{f}_1 \circ \hat{\alpha}_1$, where $\hat{f}_i = \eta(f_i)$ for some f_i in $\mathscr{K}$, and $\hat{\alpha}_i$ is the inverse in $\mathscr{K}/\mathfrak{M}$ for some $\eta(\alpha_i)$, $\alpha_i \in \mathfrak{M}$.*

It is immediate that

1.3. If $\mathfrak{M}$ is the class of all invertible morphisms in $\mathscr{K}$, or if $\mathfrak{M}$ is the class of all identities in $\mathscr{K}$, then $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ is an equivalence.

Proof. Since $1: \mathscr{K} \to \mathscr{K}$ sends each $\alpha \in \mathfrak{M}$ to an invertible morphism, there is, by C2, a factorization $1 = \Delta \circ \eta$ where $\Delta: \mathscr{K}/\mathfrak{M} \to \mathscr{K}$. This implies $\eta = \eta \circ (\Delta \circ \eta) = (\eta \circ \Delta) \circ \eta$ and since the factorization through η is unique, we must have $\eta \circ \Delta = 1$. Thus $\eta: \mathscr{K} \approx \mathscr{K}/\mathfrak{M}$ is an equivalence(2).

Whenever $\mathfrak{M} \subset \mathfrak{M}'$, the canonical projection $\eta': \mathscr{K} \to \mathscr{K}/\mathfrak{M}'$ sends each $\alpha \in \mathfrak{M}$ to an invertible morphism, so there is a unique covariant functor $\Delta: \mathscr{K}/\mathfrak{M} \to \mathscr{K}/\mathfrak{M}'$ such that $\eta' = \Delta \circ \eta$.

(2) The unique factorization property characterizes $(\eta, \mathscr{K}/\mathfrak{M})$ up to an equivalence; for the same technique can be used to prove: Let $\mathscr{B}$ be any category, and let $\lambda: \mathscr{K} \to \mathscr{B}$ be a covariant functor such that $\lambda(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$. Assume that for each category $\mathscr{L}$, every covariant functor $T: \mathscr{K} \to \mathscr{L}$ such that $T(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$ factors uniquely through λ. Then there is a unique equivalence $\Delta: \mathscr{K}/\mathfrak{M} \approx \mathscr{B}$ with $\lambda = \Delta \circ \eta$.

Proposition 1.3 is a special case of

1.4. THEOREM. *Let* $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ *be a canonical projection, and let* $\overline{\mathfrak{M}} = \{f \mid \eta(f)$ *is invertible*$\}$. *Then there exists a unique equivalence* $\Delta: \mathscr{K}/\mathfrak{M} \approx \mathscr{K}/\overline{\mathfrak{M}}$ *satisfying* $\Delta \circ \eta = \bar{\eta}$. *Moreover,* $\overline{\mathfrak{M}}$ *is the largest class containing* $\mathfrak{M}$ *for which such an equivalence holds.*

Proof. Since $\mathfrak{M} \subset \overline{\mathfrak{M}}$, we have a unique $\Delta: \mathscr{K}/\mathfrak{M} \to \mathscr{K}/\overline{\mathfrak{M}}$ with $\bar{\eta} = \Delta \circ \eta$. Since $\eta(f)$ is invertible for each $f \in \overline{\mathfrak{M}}$, there is also a $\overline{\Delta}: \mathscr{K}/\overline{\mathfrak{M}} \to \mathscr{K}/\mathfrak{M}$ such that $\eta = \overline{\Delta} \circ \bar{\eta}$. Thus, $\eta = \Delta \circ \overline{\Delta} \circ \bar{\eta}$ and $\eta = \overline{\Delta} \circ \Delta \circ \eta$, so by the unique factorization property we find $\Delta \circ \overline{\Delta} = 1$, $\overline{\Delta} \circ \Delta = 1$, therefore Δ is an equivalence. To see that $\overline{\mathfrak{M}}$ is maximal, note that if $\mathfrak{M}_0 \supset \overline{\mathfrak{M}}$, there is only one Δ' such that $\eta_0 = \Delta' \circ \bar{\eta}$, and if $f \in \mathfrak{M}_0 - \overline{\mathfrak{M}}$, then $\eta_0(f)$ is invertible whereas $\bar{\eta}(f)$ is not, so that Δ' cannot be an equivalence.

2. **Homotopy.** Let $\mathfrak{M}$ be an arbitrary class of morphisms in an arbitrary category $\mathscr{K}$, and let $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ be the canonical projection.

2.1. DEFINITION. Two morphisms $f, g: X \to Y$ in $\mathscr{K}$ are called $\mathfrak{M}$-homotopic (written $f \simeq g \bmod \mathfrak{M}$) if $\eta(f) = \eta(g)$.

This has the standard properties required of a homotopy notion:

2.2. THEOREM. (a) *$\mathfrak{M}$-homotopy is an equivalence relation in each* $\mathscr{K}(X, Y)$.

(b) *Let* $f_0, f_1: X \to Y$ *be $\mathfrak{M}$-homotopic. If* $g: W \to X$ *and* $h: Y \to Z$ *are any two morphisms, then* $f_0 \circ g \simeq f_1 \circ g \bmod \mathfrak{M}$ *and* $h \circ f_0 \simeq h \circ f_1 \bmod \mathfrak{M}$.

(c) *Each invertible morphism in* $\mathscr{K}$ *is an $\mathfrak{M}$-homotopy equivalence*(3).

(d) *If* $f, g: X \to Y$ *have an equalizer (or a coequalizer)* $c \in \mathfrak{M}$, *then* $f \simeq g \bmod \mathfrak{M}$.

Proof. (a)–(c) are trivial, since η is a covariant functor. For (d): Assume, say, $c \circ f = c \circ g$; then $\eta(c)\eta(f) = \eta(c)\eta(g)$ and, since $c \in \mathfrak{M}$ so that $\eta(c)$ is invertible, we conclude that $\eta(f) = \eta(g)$.

Denote the $\mathfrak{M}$-homotopy class of $f \in \mathscr{K}(X, Y)$ by $[f]_{\mathfrak{M}}$ and the set of all $\mathfrak{M}$-homotopy classes in $\mathscr{K}(X, Y)$ by $[X, Y]_{\mathfrak{M}}$. From 2.2 it follows in the usual way that (1) there is a category $\mathscr{K}(\mathfrak{M})$ whose objects are those of $\mathscr{K}$ and whose morphisms are the $\mathfrak{M}$-homotopy classes, and (2) the function $p: \mathscr{K} \to \mathscr{K}(\mathfrak{M})$ which is the identity on objects and sends each f to $[f]_{\mathfrak{M}}$, is a covariant functor(4).

This method of defining a homotopy notion in a category is reasonable, in that by a suitable choice (in fact, many choices) of $\mathfrak{M}$, one gets the usual notion of homotopy in the category Top of all topological spaces and continuous maps. This will follow from the general

(3) Precisely, $f: X \to Y$ is an $\mathfrak{M}$-homotopy equivalence if there exists a $g: Y \to X$ such that $f \circ g \simeq 1 \bmod \mathfrak{M}$ and $g \circ f \simeq 1 \bmod \mathfrak{M}$.

(4) In particular, if $c: A \to A$ is such that $c \circ f = c \circ g$ for every $f, g: Z \to A$, then whenever $c \in \mathfrak{M}$, the set $[Z, A]_{\mathfrak{M}}$ consists of a single element.

2.3. THEOREM. *Let $\mathscr{K}$ be a category and $\sim$ an equivalence relation in each $\mathscr{K}(A, B)$ such that the transformation $h: \mathscr{K} \to \mathscr{K}_H$ taking each object A to itself and each f to its $\sim$-class is a covariant functor.*

Assume that, if $f_0 \sim f_1: X \to Y$ then there exists an object $I_X \in \mathscr{K}$, and morphisms, $r: I_X \to X$, $i_0, i_1: X \to I_X$, $F: I_X \to Y$, such that $r \circ i_j = 1$ and $f_j = F \circ i_j$ $(j=0, 1)$.

Finally, let $\mathfrak{M}$ be a class of morphisms in $\mathscr{K}$ and $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ the projection. Then

(a) *If $\eta(r)$ is invertible in $\mathscr{K}/\mathfrak{M}$, then $f_0 \sim f_1$ implies $f_0 \simeq f_1$* mod $\mathfrak{M}$.

(b) *If $h(\alpha)$ is invertible in $\mathscr{K}_H$ for each $\alpha \in \mathfrak{M}$, then $f_0 \simeq f_1$* mod $\mathfrak{M}$ *implies $f_0 \sim f_1$.*

Proof. (a) Assume $f_0 \sim f_1$. Since $\eta(r) \circ \eta(i_j) = 1$ $(j=0, 1)$ and $\eta(r)$ is invertible, we find that each $\eta(i_j)$ is invertible, and that $\eta(i_0) = \eta(r)^{-1} = \eta(i_1)$. Thus $\eta(f_0) = \eta(F)\eta(i_0) = \eta(F)\eta(i_1) = \eta(f_1)$ so $f_0 \simeq f_1$ mod $\mathfrak{M}$.

(b) Because of C2, the functor h has a factorization $h = \Delta \circ \eta$; therefore $\eta(f_0) = \eta(f_1)$ implies $h(f_0) = h(f_1)$, i.e., that $f_0 \sim f_1$. This completes the proof.

Let $\mathscr{K} = \text{Top}$ and $\sim$ the usual homotopy in Top; then by taking $\mathfrak{M}$ to be any one of the four classes:

$\mathfrak{U}$ = all homotopy equivalences,

$\mathfrak{B}$ = all maps $r: X \times I \to X$, where $r(x, t) = x$,

$\mathfrak{J}$ = all maps $i: X \to X \times I$, where $i(x) = (x, 0)$,

$\mathfrak{X}$ = all inclusion maps $j: W \to X$, where $j(W)$ is a zero-set([5]) and a strong deformation retract of X, it follows immediately from 2.3 that $\mathfrak{M}$-homotopy is exactly the same as the usual homotopy.

We shall say that two classes $\mathfrak{M}$, $\mathfrak{N}$ in a category $\mathscr{K}$ determine the same notion of homotopy, and write $[\mathfrak{M}] = [\mathfrak{N}]$, whenever $f \simeq g$ mod $\mathfrak{M}$ if and only if $f \simeq g$ mod $\mathfrak{N}$. As the examples in Top show, distinct classes may yield the same homotopy notion. We now examine briefly some questions that arise, such as: to find conditions assuring that the $\mathfrak{M}$-homotopy equivalences determine the same homotopy notion as $\mathfrak{M}$ itself.

Let $\mathfrak{M}$ be fixed, let $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ be the canonical map, and let

$$\begin{aligned} \overline{\mathfrak{M}} &= \{f \mid \eta(f) \text{ is invertible}\}, \\ E(\mathfrak{M}) &= \{f \mid \eta(f) \text{ is invertible and has some } \eta(g) \text{ as inverse}\} \\ &= \{f \mid f \text{ is an } \mathfrak{M}\text{-homotopy equivalence}\}. \end{aligned}$$

It is clear that, if $\mathfrak{M} \subset \mathfrak{N}$, then $\overline{\mathfrak{M}} \subset \overline{\mathfrak{N}}$ and $E(\mathfrak{M}) \subset E(\mathfrak{N})$. The general relation of these concepts is

2.4. (a) $[\mathfrak{M}] = [\overline{\mathfrak{M}}]$ and $E(\mathfrak{M}) = E(\overline{\mathfrak{M}}) \subset \overline{\mathfrak{M}}$; (b) $\overline{\mathfrak{M}} = \overline{\mathfrak{N}} \Rightarrow [\mathfrak{M}] = [\mathfrak{N}] \Rightarrow E(\mathfrak{M}) = E(\mathfrak{N})$.

Proof. (a) According to 1.4, there are equivalences $\Delta: \mathscr{K}/\mathfrak{M} \approx \mathscr{K}/\overline{\mathfrak{M}}$, $\overline{\Delta}: \mathscr{K}/\overline{\mathfrak{M}} \approx \mathscr{K}/\mathfrak{M}$ such that $\bar{\eta} = \Delta \circ \eta$, $\eta = \overline{\Delta} \circ \bar{\eta}$. This implies that $\eta(f) = \eta(g)$ if and only if

([5]) I.e., there exists a continuous $\phi: X \to I$ with $\phi^{-1}(0) = j(W)$.

$\bar{\eta}(f)=\bar{\eta}(g)$, so that $[\mathfrak{M}]=[\bar{\mathfrak{M}}]$. It also implies that $E(\mathfrak{M})=E(\bar{\mathfrak{M}})$; and the definition shows $E(\mathfrak{M})\subset\bar{\mathfrak{M}}$.

(b) If $\bar{\mathfrak{M}}=\bar{\mathfrak{N}}$ then obviously $[\bar{\mathfrak{M}}]=[\bar{\mathfrak{N}}]$ and, from (a), also $[\mathfrak{M}]=[\mathfrak{N}]$. Now let $f\in E(\mathfrak{M})$; then there is a g with $f\circ g\simeq 1 \bmod \mathfrak{M}$, $g\circ f\simeq 1 \bmod \mathfrak{M}$; since $[\mathfrak{M}]=[\mathfrak{N}]$, we find $f\circ g\simeq 1 \bmod \mathfrak{N}$, $g\circ f\simeq 1 \bmod \mathfrak{N}$ and so $f\in E(\mathfrak{N})$. Similarly, $f\in E(\mathfrak{N})$ implies $f\in E(\mathfrak{M})$ and the proof is complete.

In general, none of the implications in 2.4(b) are reversible.

(A) Let $\mathscr{K}$ consist of two objects A, B together with the identity morphisms and a single morphism $f\colon A\to B$. Let $\mathfrak{M}=\{f, 1_A, 1_B\}$ and $\mathfrak{N}=\{1_A, 1_B\}$; then $[\mathfrak{M}]=[\mathfrak{N}]$ yet $\bar{\mathfrak{M}}\neq\bar{\mathfrak{N}}$ (see 1.3).

(B) Let $\mathscr{K}$ consist of three intervals $A_i=[0, r]$, $r=1, 2, 3$, with the identity maps, the inclusion maps $j_{rs}\colon A_r\to A_s$, $r<s$, and the map $h\colon A_2\to A_3$ given by $h(x)=x$, $0\leqq x\leqq 1$, $h(x)=1$, $x>1$, so that $j_{13}=j_{23}\circ j_{12}=h\circ j_{12}$. Let $\mathfrak{M}=\{1_1, 1_2, 1_3, j_{12}\}$ and $\mathfrak{N}=\{1_1, 1_2, 1_3\}$; then $[\mathfrak{M}]\neq[\mathfrak{N}]$ since h is $\mathfrak{M}$-homotopic to j_{23}; however $E(\mathfrak{M})=E(\mathfrak{N})=\mathfrak{N}$ since there is no map in $\mathscr{K}$ going backwards. Note that this shows also that $[\mathfrak{M}]\neq[\mathfrak{N}]=[E(\mathfrak{M})]$.

2.5. Definition. The class $\mathfrak{M}$ is called stable if $E(\mathfrak{M})=\bar{\mathfrak{M}}$; i.e., if the class of $\bar{\mathfrak{M}}$-homotopy equivalences is exactly $\bar{\mathfrak{M}}$.

If $\mathfrak{M}$ is stable, then $E(\mathfrak{M})$ and $\mathfrak{M}$ determine the same notion of homotopy, since $[E(\mathfrak{M})]=[\bar{\mathfrak{M}}]=[\mathfrak{M}]$. Furthermore, distinct stable classes determine distinct notions of homotopy, because, from 2.4(b) and stability, we have

2.6. If $\mathfrak{M}$, $\mathfrak{N}$ are stable classes, then $\bar{\mathfrak{M}}=\bar{\mathfrak{N}} \Leftrightarrow [\mathfrak{M}]=[\mathfrak{N}] \Leftrightarrow E(\mathfrak{M})=E(\mathfrak{N})$.

The stable classes are characterized by

2.7. Theorem. *A class $\mathfrak{M}$ in $\mathscr{K}$ is stable if and only if $\eta\colon \mathscr{K}\to\mathscr{K}/\mathfrak{M}$ is epic*[6].

Proof. "If": We need show only that $\bar{\mathfrak{M}}\subset E(\mathfrak{M})$. Let $f\in\bar{\mathfrak{M}}$; then $\eta(f)$ has an inverse G in $\mathscr{K}/\mathfrak{M}$; since η is epic, $G=\eta(g)$ for some g in $\mathscr{K}$, so $f\in E(\mathfrak{M})$. "Only if": We have seen in 1.2 that each given G in $\mathscr{K}/\mathfrak{M}$ has a factorization as $G=\hat{f}_n\circ\hat{\alpha}_n\circ\cdots\circ\hat{f}_1\circ\hat{\alpha}_1$, where each $\hat{f}_i=\eta(f_i)$ for some f_i in $\mathscr{K}$, and each $\hat{\alpha}_i$ is the inverse of some $\eta(\alpha_i)$. In particular, each $\alpha_i\in\bar{\mathfrak{M}}=E(\mathfrak{M})$, so $\eta(\alpha_i)$ has some $\eta(\beta_i)$ for inverse and, since inverses are unique, $\hat{\alpha}_i=\eta(\beta_i)$. Thus, $G=\eta[f_n\circ\beta_n\circ\cdots\circ f_1\circ\beta_1]$ and η is epic.

2.8. Corollary. *Let $\mathfrak{M}$ be any class. Then there exists a unique maximal stable class $\hat{\mathfrak{M}}\subset\mathfrak{M}$ with the following property: if $\mathfrak{N}\subset\mathfrak{M}$ is stable, then $\mathfrak{N}\subset\hat{\mathfrak{M}}$.*

Proof. In view of our constructions, the empty class $\varnothing\subset\mathfrak{M}$ is (1.3) clearly stable. It is therefore enough to show that the union of any family of stable classes is stable. Let $\{\mathfrak{M}_\beta \mid \beta\in B\}$ be the family of all stable classes in $\mathfrak{M}$, so that each $\eta_\beta\colon\mathscr{K}\to\mathscr{K}/\mathfrak{M}_\beta$ is epic; letting $\hat{\mathfrak{M}}=\bigcup\mathfrak{M}_\beta$, we will prove that $\hat{\eta}\colon\mathscr{K}\to\mathscr{K}/\hat{\mathfrak{M}}$ is also epic. Each morphism G in $\mathscr{K}/\hat{\mathfrak{M}}$ has a factorization $G=\hat{f}_n\circ\hat{\alpha}_n\circ\cdots\circ\hat{f}_1\circ\hat{\alpha}_1$,

[6] I.e., for each A, B and $\hat{f}\in$ hom (A, B), there exists an $f\colon A\to B$ with $\eta(f)=\hat{f}$. Note that the stability of $\mathfrak{M}$ implies that each hom (A, B) is in fact a set, cf. footnote 14, 15.

where each $\hat{f}_i = \eta(f_i)$ for some f_i in $\mathscr{K}$. Each α_i lies in some $\mathfrak{M}_i$ so there is a β_i in $\mathscr{K}$ with $\eta(\beta_i) = \alpha_i$ in $\mathscr{K}/\mathfrak{M}_i$. Since (1.4) there is an $\hat{\eta}_i: \mathscr{K}/\mathfrak{M}_i \to \mathscr{K}/\hat{\mathfrak{M}}$ with $\eta = \hat{\eta}_i \circ \eta_i$, it follows that $\eta(\beta_i) = \hat{\alpha}_i$ in $\mathscr{K}/\hat{\mathfrak{M}}$ and therefore that $\eta(f_n \circ \beta_n \circ \cdots \circ f_1 \circ \beta_1) = G$. This completes the proof.

The maximal stable class $\hat{\mathfrak{M}}$ may not give the same notion of homotopy as $\mathfrak{M}$. In fact, there are notions of $\mathfrak{M}$-homotopy that cannot be obtained from any stable class: in example (B) above, $\hat{\mathfrak{M}} = \mathfrak{N}$ and, indeed, $\mathfrak{M}$-homotopy cannot be obtained from a stable class. Moreover, $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$ in general (cf. Appendix A5).

The machinery developed so far can be applied to any covariant functor $\Phi: \mathscr{K} \to \mathscr{L}$ to introduce into $\mathscr{K}$ a notion of homotopy induced (or adapted to) the functor Φ.

2.9. DEFINITION. Let $\Phi: \mathscr{K} \to \mathscr{L}$ be any covariant functor. Let

$$\mathfrak{M}(\Phi) = \{f \text{ in } \mathscr{K} \mid \Phi(f) \text{ is invertible}\}.$$

The homotopy in $\mathscr{K}$ determined by the class $\mathfrak{M}(\Phi)$ is called Φ-homotopy.

The category $\mathscr{K}/\mathfrak{M}(\Phi)$ is written $\mathscr{K}^{\Phi}$ and is essentially the category first studied by Bauer in [1]. Observe that, because of C2, the functor Φ has a unique factorization $\Phi = \Delta \circ \eta$ through $\mathscr{K}^{\Phi}$. It follows from this that $\mathfrak{M}(\Phi) = [\mathfrak{M}(\Phi)]^-$ always: for, if $f \in [\mathfrak{M}(\Phi)]^-$, then $\eta(f)$ is invertible, therefore so also is $\Delta(\eta(f)) = \Phi(f)$, consequently $f \in \mathfrak{M}(\Phi)$; thus, $[\mathfrak{M}(\Phi)]^- \subset \mathfrak{M}(\Phi)$ and the opposite inclusion is trivial.

We give some applications of Φ-homotopy.

(1) Let Φ: Top $\to$ Ens be the forgetful functor. Then f is Φ-homotopic to g if and only if $f = g$. For, consider the factorization $\Phi = \Delta \circ \eta$ through $\mathscr{K}^{\Phi}$. If $\eta(f) = \eta(g)$, then $\Phi(f) = \Phi(g)$ so that $f = g$ as maps of sets; and the converse is trivially true. The class $\mathfrak{M} = \mathfrak{M}(\Phi)$ consists of all bijective continuous maps, whereas $E(\mathfrak{M})$ is the class of all bijective bicontinuous maps (i.e., homeomorphisms).

(2) Let $\mathscr{K}$ be the category of spaces dominated by CW-complexes, let $\mathscr{G}^Z$ be the category of graded groups and degree zero homomorphisms, and let $\pi: \mathscr{K} \to \mathscr{G}^Z$ be the total homotopy functor. Then two maps are π-homotopic if and only if they are homotopic. For, we apply 2.3 because (a) each projection $r: X \times I \to X$ is such that $\pi(r)$ is invertible, therefore $r \in \mathfrak{M}(\pi)$, consequently $\eta(r)$ is invertible, and (b) if $f \in \mathfrak{M}(\pi)$, i.e., if $\pi(f)$ is invertible, then, by Whitehead's theorem [6] f is a homotopy equivalence and therefore $h(f)$ is invertible. Note that, again using Whitehead's theorem, the class $\mathfrak{M}(\pi)$ is stable, so that $\eta: \mathscr{K} \to \mathscr{K}^{\pi}$ is epic. Similarly, if $\mathscr{K}$ is the category of Kan-complexes, then π-homotopy in $\mathscr{K}$ is exactly Kan-homotopy.

(3) Let $\mathscr{K}$ be the category of all simply connected CW-complexes, and $H: \mathscr{K} \to \mathscr{G}^Z$ the total homology functor. Exactly the same considerations as in (2) reveal H-homotopy to be the usual homotopy. Using $H: \mathscr{K} \to \mathscr{G}^Z$ on the category of css complexes yields a homotopy notion extending the usual one for Kan-complexes. And on the category $\mathscr{K}$ of chain-complexes, H-homotopy yields the usual notion of chain-homotopy.

Thus, the various notions of homotopy encountered, and usually defined differently, in various categories turn out to be essentially the homotopy relation determined in the above manner by a suitable covariant functor. For arbitrary contravariant functors, one turns to the dual category.

3. **Fibrations.** Each class $\mathfrak{M}$ of morphisms in a category $\mathscr{K}$ determines a concept of fibration in $\mathscr{K}$:

3.1. DEFINITION. A morphism $p: E \to B$ in $\mathscr{K}$ is called an $\mathfrak{M}$-fibration if for each diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\ & & & \searrow^{f} & \downarrow p \\ & & & & B \end{array}$$

in which $\mu \in \mathfrak{M}$ and $p \circ g \circ \mu = f \circ \mu$, there exists a $g': X \to E$ in $\mathscr{K}$ with $g \circ \mu = g' \circ \mu$ and $p \circ g' = f$.

We are requiring simply that every triangle involving p which can be equalized by a morphism belonging to $\mathfrak{M}$, can be made itself commutative. Note that if $\mathfrak{M}$-homotopy is used in $\mathscr{K}$, the conditions $p \circ g \circ \mu = f \circ \mu$ and $g \circ \mu = g' \circ \mu$ imply, by 2.2(d), that $p \circ g \simeq f \bmod \mathfrak{M}$ and $g \simeq g' \bmod \mathfrak{M}$. The relations between $\mathfrak{M}$-homotopy and $\mathfrak{M}$-fibration will be considered after we verify that the class of $\mathfrak{M}$-fibrations has the properties usually required of fibrations.

3.2. THEOREM. *Let $\mathfrak{M}$ be a fixed class of morphisms in $\mathscr{K}$. Then*:

(a) *If $p: E \to B$ is invertible, then p is an $\mathfrak{M}$-fibration.*

(b) *If $p_1: E_1 \to E_0$ and $p_0: E_0 \to B$ are $\mathfrak{M}$-fibrations, so also is $p_0 \circ p_1: E_1 \to B$.*

(c) *If $p: E \to B$ is an $\mathfrak{M}$-fibration, and if*

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{h}} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[h]{} & B \end{array}$$

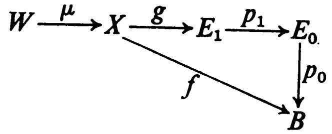
is a cartesian diagram(7), *then p' is also an $\mathfrak{M}$-fibration*; *and h can be lifted into E if and only if p' has a section*(8).

Proof. (a) is trivial.

(7) In the terminology of Mitchell [4] the indicated diagram would be called a pullback of p, h.

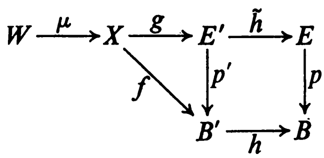
(8) I.e., an $s: B' \to E'$ such that $p' \circ s' = 1$.

(b) Given



with $p_0 \circ p_1 \circ g \circ \mu = f \circ \mu$ and $\mu \in \mathfrak{M}$, there exists first a $G\colon X \to E_0$ with $p_0 \circ G = f$, $G \circ \mu = p_1 \circ g \circ \mu$; and then a $g'\colon X \to E_1$ with $p_1 \circ g' = G$, $g' \circ \mu = g \circ \mu$; since $p_0 \circ p_1 \circ g = p_0 \circ G = f$, the required morphism is g'.

(c) We are given

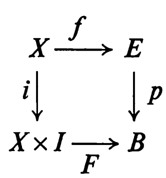


where the square is cartesian and $f \circ \mu = p' \circ g \circ \mu$. Thus, $h \circ f \circ \mu = p \circ \tilde{h} \circ g \circ \mu$ so, because p is an $\mathfrak{M}$-fibration, there is a $G\colon X \to E$ such that $G \circ \mu = \tilde{h} \circ g \circ \mu$ and $p \circ G = h \circ f$. Because the square is cartesian and $p \circ G = h \circ f$, there exists a unique $g'\colon X \to E'$ such that $\tilde{h} \circ g' = G$ and $p' \circ g' = f$. Since also $\tilde{h} \circ (g' \circ \mu) = G \circ \mu = \tilde{h} \circ (g \circ \mu)$ and $p' \circ (g \circ \mu) = f \circ \mu = p' \circ (g' \circ \mu)$, the uniqueness of a morphism $W \to E'$ satisfying these two conditions in a cartesian diagram shows $g' \circ \mu = g \circ \mu$ and therefore we find that p' is an $\mathfrak{M}$-fibration. The second part is proved in a similar manner.

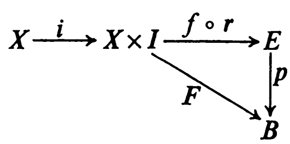
We now examine the notion of $\mathfrak{M}$-fibration in Top, using the classes, and the symbols for those classes, that are listed following Theorem 2.3. We have

3.3 THEOREM. *Let* $\mathfrak{M} = \mathfrak{I}$ *in* Top. *Then* $\mathfrak{I}$*-homotopy is the classical notion, and* $p\colon E \to B$ *is a* $\mathfrak{I}$*-fibration if and only if it is a Hurewicz fibration.*

Proof. The first part was established in 2.3. Let now $p\colon E \to B$ be a $\mathfrak{I}$-fibration; we show it has the covering homotopy property, i.e., that it is a Hurewicz fibration. We start with the commutative diagram

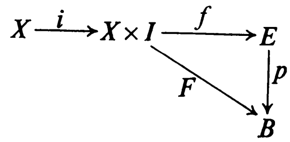


where $i(x) = (x, 0)$. Let $r\colon X \times I \to X$ be the map $r(x, t) = x$; we then have



where $F \circ i = p \circ f = p \circ f \circ r \circ i$. Since p is a $\mathfrak{I}$-fibration, there is an $\bar{F}\colon X \times I \to E$ with $p \circ \bar{F} = F$ and $\bar{F} \circ i = f \circ r \circ i = f$, so that $\bar{F}$ is a homotopy of f covering F, and therefore p is a Hurewicz fibration.

Conversely, let $p\colon E \to B$ be a Hurewicz fibration. The morphisms in $\mathfrak{I}$ being simply the maps $i\colon X \to X \times I$, we consider any diagram



in which $p \circ f \circ i = F \circ i$; since p is a Hurewicz fibration, there is a homotopy $\bar{F}$ of $f \circ i$ covering F, so that $p \circ \bar{F} = F$ and $f \circ i = \bar{F} \circ i$. Thus, p is a $\mathfrak{W}$-fibration, and the proof is complete.

If $\mathfrak{M} \subset \mathfrak{N}$, it is clear([9]) that $\{\mathfrak{N}\text{-fibrations}\} \subset \{\mathfrak{M}\text{-fibrations}\}$. Moreover,

(a) Two classes may determine the same notion of homotopy, but distinct notions of fibration.

Example in Top: We have seen that $\mathfrak{B}$ determines the same homotopy notion as $\mathfrak{I}$, that is, the usual notion of homotopy. However, every continuous map is a $\mathfrak{B}$-fibration: for, clearly, $\mathfrak{B} \subset$(all continuous surjections), and, if μ is surjective, the condition $p \circ g \circ \mu = f \circ \mu$ of Definition 3.1 implies that $p \circ g = f$; thus, every continuous $p\colon E \to B$ belongs to {surjective map fibrations} $\subset \{\mathfrak{B}\text{-fibrations}\}$.

(b) Two classes may determine the same notion of fibration but distinct notions of homotopy.

Example in Top: Let $\mathfrak{M}$ = class of all identity maps $1_A\colon A \to A$. Then every continuous map is an $\mathfrak{M}$- and also a $\mathfrak{B}$-fibration; yet $\mathfrak{B}$-homotopy is the usual notion of homotopy, whereas we have seen that $\mathfrak{M}$-homotopy is simply equality.

From this viewpoint, the notions of homotopy and of fibration are independent. In the category Top, the Hurewicz fibrations appear as a concept dependent on a particular class of morphisms that happens to yield the usual homotopy notion, rather than as a concept dependent on the homotopy notion itself([10]).

Returning to 3.1, we shall establish a simple criterion for two classes to determine the same notion of fibration. This is based on

3.4. THEOREM. *Let $\mathfrak{M}$, $\mathfrak{L}$, be two classes of morphisms in $\mathscr{K}$. Assume that for each $\mu\colon W \to X$, $\mu \in \mathfrak{M}$, there exist a $\lambda\colon Y \to Z$, $\lambda \in \mathfrak{L}$ and morphisms in $\mathscr{K}$ such that*

([9]) We denote the class of $\mathfrak{M}$-fibrations by $\{\mathfrak{M}\text{-fibrations}\}$.

([10]) The class $\mathscr{H}$ of Hurewicz fibrations does not determine the usual homotopy: in fact, since every constant map $f\colon E \to e$ is a Hurewicz fibration, it follows (cf. footnote 4) that for each X, Y, all $f, g\colon X \to Y$ are $\mathscr{H}$-homotopic.

$$\begin{array}{ccccc} W & \xrightarrow{i} & Y & \xrightarrow{s} & W \\ \mu\downarrow & & \lambda\downarrow & & \downarrow\mu \\ X & \xrightarrow[j]{} & Z & \xrightarrow[k]{} & X \end{array}$$

is commutative and $k \circ j = 1$. *Then* $\{\mathfrak{L}\text{-fibrations}\} \subset \{\mathfrak{M}\text{-fibrations}\}$.

Proof. Let $p: E \to B$ be an $\mathfrak{L}$-fibration. Given a diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\ & & & \searrow^{f} & \downarrow p \\ & & & & B \end{array}$$

with $p \circ g \circ \mu = f \circ \mu$ and $\mu \in \mathfrak{M}$, we find that $p \circ g \circ k \circ \lambda = p \circ g \circ \mu \circ s = f \circ \mu \circ s = f \circ k \circ \lambda$ so, since p is an $\mathfrak{L}$-fibration, there is a $\Gamma: Z \to E$ with $p \circ \Gamma = f \circ k$ and $\Gamma \circ \lambda = g \circ k \circ \lambda$. Letting $G = \Gamma \circ j$, we find $p \circ G = p \circ \Gamma \circ j = f \circ k \circ j = f$, and $G \circ \mu = \Gamma \circ j \circ \mu = g \circ k \circ \lambda \circ i = g \circ k \circ j \circ \mu = g \circ \mu$, so that p is an $\mathfrak{M}$-fibration.

As an application,

3.5. In Top, $\{\mathfrak{X}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}$.

Proof. Clearly, $\mathfrak{J} \subset \mathfrak{X}$, so $\{\mathfrak{X}\text{-fibrations}\} \subset \{\mathfrak{J}\text{-fibrations}\}$. For the converse, let $\mu: W \to X$, $\mu \in \mathfrak{X}$, be given; we identify W with a subset of X in order to cut down excessive notation, and let $\Phi: r \simeq 1$ be a strong deformation retraction. Choose $\phi: X \to I$ vanishing exactly on W, and define $\bar{\Phi}: X \times I \to X$ by

$$\begin{aligned} \bar{\Phi}(x, t) &= \Phi(x, t/\phi(x)), \quad x \notin W, \\ &= \Phi(x, 1), \qquad\quad x \in W, \end{aligned}$$

which is easily verified to be continuous. Then the diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{r} & W \\ \mu\downarrow & & \downarrow\lambda & & \downarrow\mu \\ X & \xrightarrow[j]{} & X \times I & \xrightarrow[\bar{\Phi}]{} & X \end{array}$$

where r = retraction, μ = inclusion, $\lambda(x) = (x, 0)$ and $j(x) = (x, \phi(x))$ is easily seen to be commutative. Thus, $\{\mathfrak{J}\text{-fibrations}\} \subset \{\mathfrak{X}\text{-fibrations}\}$ and, with 3.3, the theorem is proved.

We recall that, in Top, a fibration is called regular whenever a covering homotopy can always be chosen stationary at all the points where the given homotopy is stationary[11]. Let $\tilde{\mathfrak{X}}$ be the family of all inclusions $\mu: A \to X$, where A is a

(11) An example of a nonregular Hurewicz fibration, as well as a general condition on a topological space B that assures the regularity of every Hurewicz fibration over B, is given in [5].

strong deformation retract of X, and let $\tilde{\mathfrak{W}}$ be the class of all maps $\lambda_B\colon X \to X_B = (X\times I) \bigcup_{r\mid B\times I} B$, $r \in \mathfrak{B}$, where $B\subset X$ is arbitrary, and $\lambda_B=\rho_B \circ i$ with $i\in \mathfrak{I}$ and $\rho_B\colon X\times I \to X_B$ the identification. Clearly, $\mathfrak{X}\subset\tilde{\mathfrak{X}}$ and $\mathfrak{I}\subset\tilde{\mathfrak{W}}\subset\tilde{\mathfrak{X}}$, so that every $\tilde{\mathfrak{X}}$- and every $\tilde{\mathfrak{W}}$-fibration is a Hurewicz fibration.

3.7. Theorem(12). *In* Top $\{\tilde{\mathfrak{X}}$*-fibrations*$\}=\{\tilde{\mathfrak{W}}$*-fibrations*$\}=\{$*regular Hurewicz fibrations*$\}$.

Proof. The proof that $\{\tilde{\mathfrak{W}}$-fibrations$\}=\{$regular Hurewicz fibrations$\}$ is a repetition of that for 3.3, with $X\times I$ replaced by suitable X_B and $\mathfrak{I}$ replaced by $\tilde{\mathfrak{W}}$. Because $\tilde{\mathfrak{W}}\subset\tilde{\mathfrak{X}}$, we have $\{\tilde{\mathfrak{X}}$-fibrations$\}\subset\{\tilde{\mathfrak{W}}$-fibrations$\}$. To prove the converse inclusion, let $\mu\colon A\to X$, $\mu\in\tilde{\mathfrak{X}}$, be given, let $\Phi\colon r\simeq 1$, be a strong deformation retraction of X onto A; then a continuous $\bar{\Phi}\colon X_A\to X$ such that $\bar{\Phi}\circ\rho_A=\Phi$ exists. Let $i_1\colon X\to X\times I$ be the map $x\to(x,1)$; then the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mu} & X & \xrightarrow{r} & A \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\lambda_A} & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow[j]{} & X_A & \xrightarrow[\bar{\Phi}]{} & X \end{array}$$

with $j=\rho_A\circ i_1$, is commutative and $\bar{\Phi}\circ j=1$. Thus, by 3.5, the proof is complete.

As in the previous section, a covariant functor $\Phi\colon\mathscr{K}\to\mathscr{L}$ gives rise to a notion of Φ-fibration, i.e., $\mathfrak{M}(\Phi)$-fibration. Observe that, if $\pi\colon\mathscr{K}\to\mathscr{G}^Z$ is the total homotopy functor on the category of spaces dominated by CW-complexes, then the corresponding notion of Φ-fibration is not that of Hurewicz fibration, but of a certain subset: for, $\mathfrak{M}(\Phi)=$ set of all homotopy equivalences and $\mathfrak{I}\subset\mathfrak{M}(\Phi)$ so that $\{\mathfrak{M}(\Phi)$-fibrations$\}\subset\{\mathfrak{I}$-fibrations$\}=\{$Hurewicz fibrations$\}$.

4. Cofibrations.

4.1. Definition. Let $\mathfrak{M}$ be a class of morphisms in a category $\mathscr{K}$. A morphism $j\colon B\to E$ in $\mathscr{K}$ is called an $\mathfrak{M}$-cofibration if for each diagram

$$\begin{array}{ccccc} E & \xrightarrow{g} & X & \xrightarrow{\mu} & W \\ {\scriptstyle j}\uparrow & \nearrow{\scriptstyle f} & & & \\ B & & & & \end{array}$$

in which $\mu\in\mathfrak{M}$ and $\mu\circ g\circ j=\mu\circ f$, there exists a $g'\colon E\to X$ with $g'\circ j=f$ and $\mu\circ g'=\mu\circ g$.

This concept depends on the class $\mathfrak{M}$ itself rather than on the homotopy notion that $\mathfrak{M}$ determines. In fact, the corresponding homotopy notions may be the same, and the associated cofibrations different, since

4.2. Theorem. *In the category* Top,

(12) For $\tilde{\mathfrak{X}}$-fibrations, this theorem was suggested by the referee.

(a) $\{\mathfrak{S}$*-cofibrations*$\}=\{\mathfrak{X}$*-cofibrations*$\}=$ *all morphisms in* Top;

(b) $\{\mathfrak{W}$*-cofibrations*$\}=$ *class of all* $j: B\to E$ *such that every continuous* $\phi: B\to I$ *has a factorization* $\phi=\psi\circ j$ *where* $\psi: E\to I$.

Proof. (a) is trivial. (b) Assume that $j: B\to E$ is a $\mathfrak{W}$-cofibration. Let $\phi: B\to I$ be given and consider

$$\begin{array}{ccccc} E & \xrightarrow{g} & x_0\times I & \xrightarrow{r} & x_0 \\ \uparrow{\scriptstyle j} & \nearrow{\scriptstyle f} & & & \\ B & & & & \end{array}$$

where $f(b)=(x_0, \phi(b))$ and $g(e)=(x_0, 0)$. By hypothesis, there exists a $g': E\to x_0\times I$ such that $r\circ g'=r\circ g$ and $g'\circ j=f$, i.e., $g'(e)=(x_0, \psi(e))$ and $\psi\circ j=\phi$.

Conversely, assume that the property is satisfied for $j: B\to E$, and consider any diagram

$$\begin{array}{ccccc} E & \xrightarrow{g} & X\times I & \xrightarrow{r} & X \\ \uparrow{\scriptstyle j} & \nearrow{\scriptstyle f} & & & \\ B & & & & \end{array}$$

where $r\circ f=r\circ g\circ j$. Using the projections on each factor, write $f(b)=(f_X(b), f_I(b))$ and $g(e)=(g_X(e), g_I(e))$; the commutativity requirement assures $f_X(b)=g_X(jb)$; but $f_I(b)=\psi j(b)$ for suitable $\psi: E\to I$, so that $g'(e)=(g_X(e), \psi(e))$ is the desired map.

In particular, if $j: B\to E$ is inclusion onto a retract of E, then j is a $\mathfrak{W}$-cofibration. Moreover,

4.3. If B is a functional Hausdorff space([13]), then each $\mathfrak{W}$-cofibration $j: B\to E$ is injective.

Proof. Let $j: B\to E$ be a map such that $j(b_0)=j(b_1)$ for some $b_0\neq b_1$. There exists a map $\phi: B\to I$ with $\phi(b_0)=0$, $\phi(b_1)=1$; and ϕ cannot factor as $\phi=\psi\circ j$.

It is interesting to note that, within this framework, $\{\mathfrak{S}$-fibrations$\}\equiv\{$Hurewicz fibrations$\}$ has $\{\mathfrak{S}$-cofibrations$\}\equiv\{$all morphisms$\}$ for dual; and that $\{\mathfrak{W}$-fibrations$\}\equiv\{$all morphisms$\}$ has $\{\mathfrak{W}$-cofibrations$\}\equiv$ (a restricted class of morphisms) for dual.

For a covariant functor $\Phi: \mathscr{K}\to\mathscr{L}$ the notion of Φ-cofibration is defined, in the customary manner, to be that of $\mathfrak{M}(\Phi)$-cofibration. For contravariant functors, one uses the dual category, in which fibrations are exchanged against cofibrations.

5. **Weak $\mathfrak{M}$-fibrations.** We recall that, in the category Top, a map $p: E\to B$ is called a Dold fibration (= a map with the WCHP in [2]) if for each commutative diagram

([13]) I.e., for each pair $b_0\neq b_1$ of points in B, there exists a continuous $\phi: B\to I$ such that $\phi(b_0)=0$ and $\phi(b_1)=1$.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ X\times I & \xrightarrow[F]{} & B \end{array}$$

there exists a map $\tilde{F}\colon X\times I\to E$ such that $p\circ\tilde{F}=F$ and $\tilde{F}\circ i_0$ is fiber-homotopic to $\tilde{f}$.

For any $X\in\text{Top}$, let $\rho=\rho_X\colon X\times I\to X\times I$ denote the map

$$\begin{aligned} \rho(x, t) &= (x, 0), & 0 \leqq t \leqq \tfrac{1}{2}, \\ &= (x, 2t-1), & \tfrac{1}{2} \leqq t \leqq 1, \end{aligned}$$

and let $\mathfrak{R}=\{\rho_X \mid X\in\text{Top}\}$. Then

5.1. Lemma. *A map $p\colon E\to B$ is a Dold fibration if and only if for each diagram*

$$\begin{array}{ccccc} X & \xrightarrow{i_0} & X\times I & \xrightarrow{g} & E \\ & & & {\scriptstyle F}\searrow & \downarrow{\scriptstyle p} \\ & & & & B \end{array}$$

with $p\circ g\circ i_0=F\circ i_0$, there exists a $g'\colon X\times I\to E$ such that $g\circ i_0=g'\circ i_0$ and $p\circ g'=F\circ\rho$.

Because the proof of 5.1 is entirely analogous to that of 3.3, it is omitted.

Let $\mathscr{K}$ be an arbitrary category and $\mathfrak{M}$ be any family of its morphisms. With 5.1 in mind, we make the following

5.2. Definition. A morphism $p\colon E\to B$ in $\mathscr{K}$ is called a weak $\mathfrak{M}$-fibration if for each diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\ & & & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & & & B \end{array}$$

in which $\mu\in\mathfrak{M}$ and $p\circ g\circ\mu=f\circ\mu$, there exists a $g'\colon X\to E$ in $\mathscr{K}$, and an $r\colon X\to X$ in $\mathfrak{M}$ such that $r\circ\mu=\mu$, $g\circ\mu=g'\circ\mu$, and $p\circ g'=f\circ r$.

Remarks. (1) If $\mathfrak{M}$ is such that $\mathfrak{M}\cap\mathscr{K}(X, X)=\{1_X : X\to X\}$ for each $X\in\mathscr{K}$, then {weak $\mathfrak{M}$-fibrations} = {$\mathfrak{M}$-fibrations}.

(2) If $\mathfrak{M}$ is any class that contains all the identity morphisms, then each $\mathfrak{M}$-fibration is also a weak $\mathfrak{M}$-fibration.

(3) Unlike the $\mathfrak{M}$-fibrations, it is not in general true that $(\mathfrak{M}\subset\mathfrak{N})\Rightarrow$ ({weak $\mathfrak{N}$-fibrations} $\subset$ {weak $\mathfrak{M}$-fibrations}).

5.3. Definition. For any family $\mathfrak{M}$ of morphisms in Top, we let $\mathfrak{M}_e=\mathfrak{M}\cup$ {all identities} and $\mathfrak{M}^*=\mathfrak{M}_e\cup\mathfrak{R}$. Then

5.4. Theorem. *In the category* Top,

(a) $\{$*weak* $\mathfrak{X}_e$*-fibrations*$\} = \{$*weak* $\mathfrak{I}_e$*-fibrations*$\} = \{$*Hurewicz fibrations*$\}$;
(b) $\{$*weak* $\mathfrak{X}^*$*-fibrations*$\} = \{$*weak* $\mathfrak{I}^*$*-fibrations*$\} = \{$*Dold fibrations*$\}$.

Proof. (a) follows immediately from Remark (1) above and Theorems 3.3, 3.5. To prove (b): Observe that we have

$$\{\text{weak } \mathfrak{X}^*\text{-fibrations}\} \subset \{\text{weak } \mathfrak{I}^*\text{-fibrations}\}$$

because $\mathfrak{I} \subset \mathfrak{X}$ and all morphisms r that can possibly appear in Definition 5.2 are in $\mathfrak{R}_e$. The converse inclusion follows by an argument entirely analogous to that in the proof of 3.5. Finally, $\{$weak $\mathfrak{I}^*$-fibrations$\} = \{$Dold fibrations$\}$ by 5.1. This completes the proof (b).

Everything which has been said for weak $\mathfrak{M}$-fibrations can be dualized in the usual fashion, to yield weak $\mathfrak{M}$-cofibrations.

Appendix. In this section, we will give a proof of Theorem 1.1. We also include an example to show that in general $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$ and $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$.

Let $\mathscr{K}$ be an arbitrary category, and let $\mathfrak{M}$ be any subclass of its morphisms; the case that $\mathfrak{M}$ is the family of all morphisms in $\mathscr{K}$ is not excluded. We denote the elements of $\mathfrak{M}$ by small Greek letters, $\alpha, \beta, \ldots$.

Given $A, B \in \text{ob}(\mathscr{K})$, by an $\mathfrak{M}$-word $m = (A, X_1, \ldots, X_{2n-1}, B)$ from A to B we mean a finite chain of morphisms and objects

$$A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\alpha_2} X_3 \xrightarrow{f_2} \cdots \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

in which the $\leftarrow$ and $\rightarrow$ alternate, and each morphism α_i "in the wrong direction" either belongs to $\mathfrak{M}$ or is an identity morphism. The class of all $\mathfrak{M}$-words from A to B is denoted by $\mathfrak{M}(A, B)$.

We introduce an equivalence relation $\sim$ in $\mathfrak{M}(A, B)$ by

A.1. Definition. Let $m, \hat{m} \in \mathfrak{M}(A, B)$. Declare $m \sim \hat{m}$ if there is a finite sequence $m = m_1, m_2, \ldots, m_s = \hat{m}$ of elements of $\mathfrak{M}(A, B)$ in which each m_i is obtained from m_{i-1} (or from m_{i+1}) by one of the following two operations:

(i) A pushout operation: replacement of a segment

$$\cdot \xrightarrow{f} X \xleftarrow{\alpha} Y \xrightarrow{g} Z \xleftarrow{\beta} \cdot$$

in m_{i-1} (or m_{i+1}) by

$$\cdot \xleftarrow{u \circ f} W \xrightarrow{v \circ \beta} \cdot$$

if there is a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & Y & \xrightarrow{g} & Z \\ & {\scriptstyle u}\searrow & & \swarrow{\scriptstyle v} & \\ & & W & & \end{array}$$

and $v \circ \beta \in \mathfrak{M}$ or is an identity morphism. This operation is denoted by

$$PO(X,\ Y, Z\colon u,\ W, v).$$

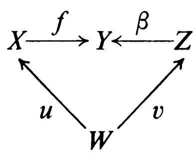
(ii) A pullback operation: replacement of a segment

$$\cdot \xleftarrow{\alpha} X \xrightarrow{f} Y \xleftarrow{\beta} Z \xrightarrow{g} \cdot$$

in m_{i-1} (or m_{i+1}) by

$$\cdot \xleftarrow{\alpha \circ u} W \xrightarrow{g \circ v} \cdot$$

if there is a commutative diagram



and $\alpha \circ u \in \mathfrak{M}$ or is an identity morphism. This operation is denoted by

$$PB(X,\ Y, Z; u,\ W, v).$$

It is clear that $\sim$ is an equivalence relation in $\mathfrak{M}(A, B)$; the equivalence class of m is denoted by $[m]$. Furthermore, the map $\mathfrak{M}(A, B) \otimes \mathfrak{M}(B, C) \to \mathfrak{M}(A, C)$ defined by

$$(A, X_1, \ldots, X_{2n-1}, B) \otimes (B, Y_1, \ldots, C) = (A, X_1, \ldots, X_{2n-1}, B, Y_1, \ldots, C)$$

is associative, and the equivalence class of $m \otimes m'$ depends only on that of m and of m'.

A.2. Definition. The category $\mathscr{K}/\mathfrak{M}$ is that having for objects: $\{A \mid A \in \mathrm{ob}(\mathscr{K})\}$; for morphisms hom (A, B): the equivalence classes(14) in $\mathfrak{M}(A, B)$, with the composition law hom $(B, C) \circ$ hom $(A, B) \to$ hom (A, C) given by $[m'] \circ [m] = [m \otimes m']$.

The identity $[e_A] \in \hom(A, A)$ is the morphism $[A \xleftarrow{1} A \xrightarrow{1} A]$: for, given $[m] \in \hom(B, A)$ we find that $m \otimes e_A$ is a word of form

$$\cdots \longleftarrow X \xrightarrow{f} A \xleftarrow{1} A \xrightarrow{1} A$$

and $PB(X, A, A; 1, X, f)$ shows $m \otimes e_A \sim m$; similarly, $e_A \otimes m' \sim m'$ for $m' \in \mathfrak{M}(A, B)$ so $[e_A]$ is the (unique) identity morphism of A.

A.3. Let each morphism in a word $m \in \mathfrak{M}(A, B)$ be either a member of $\mathfrak{M}$ or an identity. Then $[m]$ is invertible in $\mathscr{K}/\mathfrak{M}$; and its inverse is represented by m written in reverse order.

(14) If $\mathscr{K}$ is not a small category, hom (A, B) need not be a set. However, in all the choices of $\mathscr{K}$ and $\mathfrak{M}$ considered in this paper, the class hom (A, B) can be proved to be a set, so that the construction of this category is legitimate.

Proof. Letting m' be the word in reverse order, we have $m' \in \mathfrak{M}(B, A)$, and

$$m \otimes m' = A \xleftarrow{\alpha} X \longrightarrow \cdots \xleftarrow{\beta} Y \xrightarrow{\gamma} B \xleftarrow{\gamma} Y \xrightarrow{\beta} \cdots \longleftarrow X \xrightarrow{\alpha} A.$$

Starting from the middle with $PB(Y, B, Y; 1, Y, 1)$ and repeating, shows $m \otimes m' \sim A \xleftarrow{\alpha} X \xrightarrow{\alpha} A$; inserting $PO(A, X, A; 1, A, 1)$ we find $m \otimes m' \sim e_A$. Similarly $m' \otimes m \sim e_B$ and the proof is complete.

Observe that any word

$$A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \longleftarrow \cdots \longrightarrow X_{2n-2} \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

is equivalent to

$$(A \xleftarrow{\alpha_1} X_1 \xrightarrow{1} X_1) \otimes (X_1 \xleftarrow{1} X_1 \xrightarrow{f_1} X_2) \otimes \cdots$$
$$\otimes (X_{2n-2} \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{1} X_{2n-1}) \otimes (X_{2n-1} \xleftarrow{1} X_{2n-1} \xrightarrow{f_n} B).$$

Thus, each morphism in $\mathscr{K}/\mathfrak{M}$ can be factored as $\hat{f}_n \circ \hat{\alpha}_n \circ \cdots \circ \hat{f}_1 \circ \hat{\alpha}_1$ where, according to A.3, each $\hat{\alpha}_i$ is invertible.

The transformation $\eta: \mathscr{K} \to \mathscr{K}/\mathfrak{M}$ given by rules

$$\eta(A) = A \quad \text{on the objects,}$$
$$\eta(f) = [A \xleftarrow{1} A \xrightarrow{f} B] \quad \text{for each } f \in \mathscr{K}(A, B)$$

is easily seen to be a covariant functor, called the canonical projection. Because of A.3, $\eta(\alpha)$ is invertible in $\mathscr{K}/\mathfrak{M}$ for each $\alpha \in \mathfrak{M}$; and the known [3] universality property of $(\eta, \mathscr{K}/\mathfrak{M})$, in a formulation adapted for our purposes, is

A.4. THEOREM. *Let $T: \mathscr{K} \to \mathscr{L}$ be any covariant functor to any category $\mathscr{L}$. Then*

(a) *There exists a covariant functor $\Delta: \mathscr{K}/\mathfrak{M} \to \mathscr{L}$ with $T = \Delta \circ \eta$ if and only if $T(\alpha)$ is invertible in $\mathscr{L}$ for each $\alpha \in \mathfrak{M}$.*

(b) *If $T = \Delta \circ \eta$, then Δ is unique.*

Proof. (a) If $T = \Delta \circ \eta$ then, because $\eta(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$ and functors preserve invertible morphisms, T has the required property. Conversely, assume $T(\alpha)$ invertible for each $\alpha \in \mathfrak{M}$. Define Δ on the objects of $\mathscr{K}/\mathfrak{M}$ by $\Delta(A) = T(A)$. For each word

$$m = A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \longleftarrow \cdots \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

in $\mathfrak{M}(A, B)$, let

$$\Delta(m) = T(f_n) \circ T(\alpha_n)^{-1} \circ \cdots \circ T(f_1) \circ T(\alpha_1)^{-1} \in \mathscr{L}(\Delta(A), \Delta(B))$$

where $T(\alpha_i)^{-1}$ is the inverse of $T(\alpha_i)$ in $\mathscr{L}$. If $m \sim m'$, then $\Delta(m) = \Delta(m')$ because the operations in changing m to m' are governed by commutative diagrams, and all morphisms "in the wrong direction" belong to $\mathfrak{M}$. Thus, $[m] \mapsto \Delta(m)$ is a well-defined[15] map $\Delta\colon \hom(A, B) \to \mathscr{L}(\Delta(A), \Delta(B))$ for each $A, B \in \text{ob}(\mathscr{K}/\mathfrak{M})$ which with the above indicated correspondence of the objects, is easily seen to determine a covariant functor $\Delta\colon \mathscr{K}/\mathfrak{M} \to \mathscr{L}$. Clearly, $\Delta \circ \eta(A) = A$ on objects, and $\Delta \circ \eta(f) = \Delta[A \xleftarrow{1} A \xrightarrow{f} B] = T(f)$ for each $f \in \mathscr{K}(A, B)$, so $\Delta \circ \eta = T$ and the proof is complete.

(b) Assume $T = \Delta \circ \eta = \Gamma \circ \eta$; then $\Delta(A) = T(A) = \Gamma(A)$ for each $A \in \text{ob}(\mathscr{K}/\mathfrak{M})$. For each morphism of type $\hat{f} = [A \xleftarrow{1} A \xrightarrow{f} B]$ we have $\Delta(\hat{f}) = \Delta \circ \eta(f) = T(f)$ $= \Gamma \circ \eta(f) = \Gamma(\hat{f})$. Each morphism of type $\hat{\alpha} = [A \xleftarrow{\alpha} B \xrightarrow{1} B]$ is invertible in $\mathscr{K}/\mathfrak{M}$, with inverse $\hat{\alpha}^{-1} = [B \xleftarrow{1} B \xrightarrow{\alpha} A]$. Thus $\Delta(\hat{\alpha})$, $\Gamma(\hat{\alpha})$ are invertible in $\mathscr{L}$ and, from what we have already shown, $\Delta(\hat{\alpha}) = \Delta(\hat{\alpha}^{-1})^{-1} = \Gamma(\hat{\alpha}^{-1})^{-1} = \Gamma(\hat{\alpha})$. Since each morphism in $\mathscr{K}/\mathfrak{M}$ is representable as a composition of morphisms of these two types, we find $\Delta[m] = \Gamma[m]$ for each morphism $[m]$ and the proof is complete.

REMARK. If $\mathfrak{M}$ is the family of all morphisms in $\mathscr{K}$, then all the morphisms of $\mathscr{K}/\mathfrak{M}$ are invertible therefore each $\hom(X, X)$ is a group, $\pi(X)$. Furthermore, for each $\phi \in \hom(X, Y)$ there is a homomorphism $\phi^*\colon \pi(X) \to \pi(Y)$ given by $\phi^*(f)$ $= \phi \circ f \circ \phi^{-1}$, and clearly ϕ^* is an isomorphism. Thus the groups $\pi(X)$ belonging to a "component" of $\mathscr{K}$ are all isomorphic; if $\mathscr{K}$ is "connected" then, as in topology, we can define the fundamental group $\pi(\mathscr{K})$ of a category. From C2 it follows that a covariant functor $\Phi\colon \mathscr{K} \to \mathscr{L}$ on "connected" categories induces a homomorphism $\pi(\mathscr{K}) \to \pi(\mathscr{L})$. It is easy to see that for Top_0, the category of based topological spaces with base-point preserving continuous maps, each word from (X, x_0) to (X, x_0) becomes equivalent to $(X, x_0) \leftarrow (x_0, x_0) \to (X, x_0)$ and therefore that $\pi(\text{Top}_0) = 0$. Similarly, it can be shown that $\pi(\text{Top}) = 0$.

We now show

A.5. PROPOSITION. *There exists a category $\mathscr{K}$ and a class $\mathfrak{M}$ such that $E(E(\mathfrak{M}))$ $\neq E(\mathfrak{M})$ and $E(\hat{\mathfrak{M}}) \neq E(\mathfrak{M})$, where $\hat{\mathfrak{M}}$ is the maximal stable subclass in $\mathfrak{M}$.*

Proof. Consider the following diagram

$$*_A \xrightarrow{h'} S_0^2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S_1^2 \xrightarrow{h} *_B$$

where S_0^2, S_1^2 are two distinct 2-spheres, $*_A$, $*_B$ are distinct points, $h'(*_A) =$ north pole of S_0^2, α is the rotation about the z-axis with angle π, and β is the rotation about the same axis with angle π/u, u an irrational. Let $\mathscr{K}$ be the category with these four distinct objects, and with morphisms generated by h', α, β, h.

Let $\mathfrak{M} = \{h, h'\}$; then it is immediate that $\bar{\mathfrak{M}} =$ all morphisms in $\mathscr{K}$ and $E(\mathfrak{M})$ $= \bar{\mathfrak{M}} - \mathfrak{M}$. Moreover

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([15]) The set of all morphisms $A \to B$ in a category $\mathscr{L}$ will be denoted by $\mathscr{L}(A, B)$; if $\mathscr{L} = \mathscr{K}/\mathfrak{M}$, this set is denoted by $\hom(A, B)$.

(a) $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$. Let $\mathscr{L}$ be the smallest subcategory of Top containing $\mathscr{K}$ and the inverses $\bar{\alpha}$, $\bar{\beta}$ of α, β. Let $\lambda: \mathscr{K} \to \mathscr{L}$ be the inclusion functor. Since λ sends every member of $E(\mathfrak{M})$ to an isomorphism in $\mathscr{L}$, there exists (A.4) a functor $\Delta: \mathscr{K}/E(\mathfrak{M}) \to \mathscr{L}$ such that $\Delta \circ \eta = \lambda$. Now, $\alpha \notin E(E(\mathfrak{M}))$: for if $\eta(\alpha)\eta(g) = 1$ for some g in $\mathscr{K}$, then $\lambda(\alpha \circ g) = 1$ in $\mathscr{L}$, i.e., $\alpha \circ g = 1$ in $\mathscr{L}$; however α does not have a right inverse in $\mathscr{K}$, so such a g cannot exist in $\mathscr{K}$. Thus, $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$. Moreover, since $\alpha \in E(\mathfrak{M}) \subset [E(\mathfrak{M})]^-$, this shows also that $E(\mathfrak{M})$ is not stable; furthermore, $\hat{\mathfrak{M}} = \varnothing$, since h and h' do not have inverses in $\mathscr{K}$, so we also have $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$.

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