CATEGORICAL HOMOTOPY AND FIBRATIONS

BY

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Let $\mathcal{X}$ be an arbitrary category, and let $\mathcal{M}$ be any family of its morphisms. It is known [3] that there is a category $\mathcal{X}/\mathcal{M}$ (the Gabriel-Zisman "category of fractions of $\mathcal{X}$ by $\mathcal{M}$") having the same objects as $\mathcal{X}$, and a covariant functor $\eta: \mathcal{X} \to \mathcal{X}/\mathcal{M}$ which is the identity on objects, such that $\eta(f)$ is invertible in $\mathcal{X}/\mathcal{M}$ for each $f \in \mathcal{M}$.

We will use each class $\mathcal{M}$ to determine a notion of homotopy in $\mathcal{X}$, by defining two morphisms $f, g$ to be $\mathcal{M}$-homotopic if $\eta(f) = \eta(g)$. This notion has the usual properties expected of a homotopy notion; moreover, in the category Top of topological spaces and continuous maps, suitable choices of $\mathcal{M}$ (for example, $\mathcal{M} =$ all homotopy equivalences) reveal $\mathcal{M}$-homotopy equivalent to the usual notion of homotopy.

Each class $\mathcal{M}$ determines a notion of fibration in $\mathcal{X}$. In the category Top, a suitable choice of $\mathcal{M}$ determines both the usual homotopy and the Hurewicz fibrations; however, different classes $\mathcal{M}$ may yield the usual notion of homotopy and distinct notions of fibration. More generally, in an arbitrary category $\mathcal{X}$, it is the given class $\mathcal{M}$ itself, rather than the homotopy notion induced by $\mathcal{M}$, that determines the concept of fibration; from this viewpoint, it turns out, surprisingly, that the notion of a Hurewicz fibration is not a homotopy notion. By "reversing arrows," $\mathcal{M}$ determines also a concept of cofibration; and again there is a splitting: in fact, two classes may determine the same notion of homotopy but distinct notions of cofibration.

In the last section, we introduce the concept of a weak $\mathcal{M}$-fibration. This notion does not, in general, possess all the advantages of the previous one; however, it reduces to the previous notion under suitable restrictions on the class $\mathcal{M}$. Moreover, there are classes, $\mathcal{M}, \mathcal{N}$ in the category Top such that \{weak $\mathcal{M}$-fibrations\} = \{Hurewicz fibrations\} and \{weak $\mathcal{N}$-fibrations\} = \{Dold fibrations\}.

Each covariant functor $\Phi: \mathcal{X} \to \mathcal{L}$ determines a $\Phi$-homotopy in $\mathcal{X}$, by choosing $\mathcal{M} = \{f | \Phi(f) \text{ is invertible}\}$. Most of the homotopy notions encountered in various categories turn out to stem from this general construction. For example: if $\mathcal{G}$ is the category of graded groups and degree zero homomorphisms, and if $\pi(X)$ is the total homotopy of a space $X$, then in the category $\mathcal{X}$ of spaces dominated by CW-complexes the functor $\pi: \mathcal{X} \to \mathcal{G}$ determines the usual notion of homotopy;

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if $\mathcal{K}$ is the category of Kan-complexes, then $\pi$-homotopy is the same as Kan-homotopy; and if $H: \mathcal{K} \to \mathcal{C}$ is the total homology functor on the category of chain-complexes, then $H$-homotopy is the usual chain-homotopy.

In the Appendix, we give a construction of $\mathcal{K}/\mathcal{M}$ that differs slightly from that of Gabriel-Zisman, and contains slightly more detail; one advantage of this procedure is that some results in [3] are seen to hold without the additional requirement that "$\mathcal{M}$ admits a calculus of left fractions".

1. **The quotient category $\mathcal{K}/\mathcal{M}$.** Let $\mathcal{K}$ be any category, and let $\mathcal{M}$ be any family of its morphisms. By a quotient category we shall mean a pair $(\mathcal{K}/\mathcal{M}, \eta)$, where $\mathcal{K}/\mathcal{M}$ is a category with the same objects as $\mathcal{K}$ and $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is a covariant functor that preserves objects, having the following two properties:

- **C1.** If $\alpha \in \mathcal{M}$, then $\eta(\alpha)$ is invertible in $\mathcal{K}/\mathcal{M}$.
- **C2.** [Universality]: If $F: \mathcal{K} \to \mathcal{L}$ is any covariant functor to any category $\mathcal{L}$ such that $F(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$, then there exists a unique covariant functor $A: \mathcal{K}/\mathcal{M} \to \mathcal{L}$ such that $F = A \circ \eta$.

1.1. **Theorem.** Let $\mathcal{K}$ be any category, and let $\mathcal{M}$ be any family of its morphisms. Then a quotient category $(\mathcal{K}/\mathcal{M}, \eta)$ exists.

This theorem is mentioned in [3]. However, no detailed construction of $(\mathcal{K}/\mathcal{M}, \eta)$ having the generality we require appears in the literature, so we will give a proof of 1.1 in the Appendix. It follows directly from our construction, and we will need this in the sequel, that

1.2. **Proposition.** Each morphism $G$ in $\mathcal{K}/\mathcal{M}$ has a factorization $G = f_n \circ \xi_n \circ \cdots \circ f_1 \circ \xi_1$, where $f_i = \eta(f_i)$ for some $f_i \in \mathcal{K}$, and $\xi_i$ is the inverse in $\mathcal{K}/\mathcal{M}$ for some $\eta(\alpha_i)$, $\alpha_i \in \mathcal{M}$.

It is immediate that

1.3. If $\mathcal{M}$ is the class of all invertible morphisms in $\mathcal{K}$, or if $\mathcal{M}$ is the class of all identities in $\mathcal{K}$, then $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is an equivalence.

**Proof.** Since $1: \mathcal{K} \to \mathcal{K}$ sends each $\alpha \in \mathcal{M}$ to an invertible morphism, there is, by C2, a factorization $1 = \Delta \circ \eta$ where $\Delta: \mathcal{K}/\mathcal{M} \to \mathcal{K}$. This implies $\eta = \eta \circ (\Delta \circ \eta) = (\eta \circ \Delta) \circ \eta$ and since the factorization through $\eta$ is unique, we must have $\eta \circ \Delta = 1$. Thus $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is an equivalence.

Whenever $\mathcal{M} \subset \mathcal{M}'$, the canonical projection $\eta': \mathcal{K} \to \mathcal{K}/\mathcal{M}'$ sends each $\alpha \in \mathcal{M}$ to an invertible morphism, so there is a unique covariant functor $\Delta: \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}'$ such that $\eta' = \Delta \circ \eta$.

(2) The unique factorization property characterizes $(\eta, \mathcal{K}/\mathcal{M})$ up to an equivalence; for the same technique can be used to prove: Let $\mathcal{A}$ be any category, and let $\lambda: \mathcal{K} \to \mathcal{A}$ be a covariant functor such that $\lambda(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$. Assume that for each category $\mathcal{L}$, every covariant functor $T: \mathcal{K} \to \mathcal{L}$ such that $T(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$ factors uniquely through $\lambda$. Then there is a unique equivalence $\Delta: \mathcal{K}/\mathcal{M} \approx \mathcal{A}$ with $\lambda = \Delta \circ \eta$. 

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Proposition 1.3 is a special case of

1.4. Theorem. Let $\eta: \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be a canonical projection, and let $\mathcal{M} = \{ f | \eta(f) \text{ is invertible} \}$. Then there exists a unique equivalence $\Delta: \mathcal{X}/\mathcal{M} \to \mathcal{X}/\overline{\mathcal{M}}$ satisfying $\Delta \circ \eta = \bar{\eta}$. Moreover, $\mathcal{M}$ is the largest class containing $\mathcal{M}$ for which such an equivalence holds.

Proof. Since $\mathcal{M} \subseteq \overline{\mathcal{M}}$, we have a unique $\Delta: \mathcal{X}/\mathcal{M} \to \mathcal{X}/\overline{\mathcal{M}}$ with $\bar{\eta} = \Delta \circ \eta$. Since $\eta(f)$ is invertible for each $f \in \mathcal{M}$, there is also a $\Delta: \mathcal{X}/\mathcal{M} \to \mathcal{X}/\overline{\mathcal{M}}$ such that $\eta = \Delta \circ \eta$. Thus, $\eta = \Delta \circ \bar{\eta}$ and $\eta = \Delta \circ \eta$, so by the unique factorization property we find $\Delta \circ \Delta = 1$, $\Delta \circ \Delta = 1$, therefore $\Delta$ is an equivalence. To see that $\mathcal{M}$ is maximal, note that if $\mathcal{M}_0 \supset \mathcal{M}$, there is only one $\Delta'$ such that $\eta_0 = \Delta' \circ \eta$, and if $f \in \mathcal{M}_0 - \mathcal{M}$, then $\eta_0(f)$ is invertible whereas $\bar{\eta}(f)$ is not, so that $\Delta'$ cannot be an equivalence.

2. Homotopy. Let $\mathcal{M}$ be an arbitrary class of morphisms in an arbitrary category $\mathcal{X}$, and let $\eta: \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the canonical projection.

2.1. Definition. Two morphisms $f, g: X \to Y$ in $\mathcal{X}$ are called $\mathcal{M}$-homotopic (written $f \simeq g \text{ mod } \mathcal{M}$) if $\eta(f) = \eta(g)$.

This has the standard properties required of a homotopy notion:

2.2. Theorem. (a) $\mathcal{M}$-homotopy is an equivalence relation in each $\mathcal{X}(X, Y)$.
(b) Let $f_0, f_1: X \to Y$ be $\mathcal{M}$-homotopic. If $g: W \to X$ and $h: Y \to Z$ are any two morphisms, then $f_0 \circ g \simeq f_1 \circ g \text{ mod } \mathcal{M}$ and $h \circ f_0 \simeq h \circ f_1 \text{ mod } \mathcal{M}$.
(c) Each invertible morphism in $\mathcal{X}$ is an $\mathcal{M}$-homotopy equivalence.
(d) If $f, g: X \to Y$ have an equalizer (or a coequalizer) $c \in \mathcal{M}$, then $f \simeq g \text{ mod } \mathcal{M}$.

Proof. (a)–(c) are trivial, since $\eta$ is a covariant functor. For (d): Assume, say, $c \circ f = c \circ g$; then $\eta(c) \eta(f) = \eta(c) \eta(g)$ and, since $c \in \mathcal{M}$ so that $\eta(c)$ is invertible, we conclude that $\eta(f) = \eta(g)$.

Denote the $\mathcal{M}$-homotopy class of $f \in \mathcal{X}(X, Y)$ by $[f]_\mathcal{M}$ and the set of all $\mathcal{M}$-homotopy classes in $\mathcal{X}(X, Y)$ by $[X, Y]_\mathcal{M}$. From 2.2 it follows in the usual way that (1) there is a category $\mathcal{X}(\mathcal{M})$ whose objects are those of $\mathcal{X}$ and whose morphisms are the $\mathcal{M}$-homotopy classes, and (2) the function $p: \mathcal{X} \to \mathcal{X}(\mathcal{M})$ which is the identity on objects and sends each $f$ to $[f]_\mathcal{M}$, is a covariant functor.

This method of defining a homotopy notion in a category is reasonable, in that by a suitable choice (in fact, many choices) of $\mathcal{M}$, one gets the usual notion of homotopy in the category $\text{Top}$ of all topological spaces and continuous maps. This will follow from the general

(3) Precisely, $f: X \to Y$ is an $\mathcal{M}$-homotopy equivalence if there exists a $g: Y \to X$ such that $f \circ g \simeq 1 \text{ mod } \mathcal{M}$ and $g \circ f \simeq 1 \text{ mod } \mathcal{M}$.

(4) In particular, if $c: A \to A$ is such that $c \circ f = c \circ g$ for every $f, g: Z \to A$, then whenever $c \in \mathcal{M}$, the set $[Z, A]_\mathcal{M}$ consists of a single element.
2.3. **Theorem.** Let \( \mathcal{K} \) be a category and \( \sim \) an equivalence relation in each \( \mathcal{K}(A, B) \) such that the transformation \( h: \mathcal{K} \to \mathcal{K}_h \) taking each object \( A \) to itself and each \( f \) to its \( \sim \)-class is a covariant functor.

Assume that, if \( f_0 \sim f_1: X \to Y \) then there exists an object \( I_X \in \mathcal{K} \), and morphisms, \( r: I_X \to X, i_0, i_1: X \to I_X, F: I_X \to Y, \) such that \( r \circ i_j = 1 \) and \( f_j = F \circ i_j \) \((j=0, 1)\).

Finally, let \( \mathcal{M} \) be a class of morphisms in \( \mathcal{K} \) and \( \eta: \mathcal{K} \to \mathcal{K}/\mathcal{M} \) the projection. Then

(a) If \( \eta(r) \) is invertible in \( \mathcal{K}/\mathcal{M} \), then \( f_0 \sim f_1 \) implies \( f_0 \equiv f_1 \mod \mathcal{M} \).

(b) If \( h(\alpha) \) is invertible in \( \mathcal{K}_h \) for each \( \alpha \in \mathcal{M} \), then \( f_0 \equiv f_1 \mod \mathcal{M} \) implies \( f_0 \sim f_1 \).

**Proof.** (a) Assume \( f_0 \sim f_1 \). Since \( \eta(r) \circ \eta(i_j) = 1 \) \((j=0, 1)\) and \( \eta(r) \) is invertible, we find that each \( \eta(i_j) \) is invertible, and that \( \eta(i_0) = \eta(i_1) \) \( = \eta(F) \eta(i_0) \eta(F) \eta(i_1) = \eta(f_0) \eta(f_1) \). So \( f_0 \sim f_1 \) and \( f_0 \equiv f_1 \mod \mathcal{M} \).

(b) Because of C2, the functor \( h \) has a factorization \( h = \Delta \circ \eta \); therefore \( f_0 \equiv f_1 \) implies \( h(f_0) = h(f_1) \), i.e., that \( f_0 \sim f_1 \). This completes the proof.

Let \( \mathcal{K} = \text{Top} \) and \( \sim \) the usual homotopy in \( \text{Top} \); then by taking \( \mathcal{M} \) to be any one of the four classes:

- \( \mathcal{M} = \text{all homotopy equivalences} \),
- \( \mathcal{M} = \text{all maps } r: X \times I \to X, \text{ where } r(x, t) = x, \)
- \( \mathcal{M} = \text{all maps } i: X \to X \times I, \text{ where } i(x) = (x, 0), \)
- \( \mathcal{M} = \text{all inclusion maps } j: W \to X, \text{ where } j(W) \text{ is a zero-set(5) and a strong deformation retract of } X, \) it follows immediately from 2.3 that \( \mathcal{M} \)-homotopy is exactly the same as the usual homotopy.

We shall say that two classes \( \mathcal{M}, \mathcal{N} \) in a category \( \mathcal{K} \) determine the same notion of homotopy, and write \( [\mathcal{M}] = [\mathcal{N}] \), whenever \( f \equiv g \mod \mathcal{M} \) if and only if \( f \equiv g \mod \mathcal{N} \). As the examples in Top show, distinct classes may yield the same homotopy notion. We now examine briefly some questions that arise, such as: to find conditions assuring that the \( \mathcal{M} \)-homotopy equivalences determine the same homotopy notion as \( \mathcal{M} \) itself.

Let \( \mathcal{M} \) be fixed, let \( \eta: \mathcal{K} \to \mathcal{K}/\mathcal{M} \) be the canonical map, and let

- \( \overline{\mathcal{M}} = \{ f | \eta(f) \text{ is invertible} \}, \)
- \( E(\mathcal{M}) = \{ f | \eta(f) \text{ is invertible and has some } \eta(g) \text{ as inverse} \} = \{ f | f \text{ is an } \mathcal{M} \text{-homotopy equivalence} \}. \)

It is clear that, if \( \mathcal{M} \subset \mathcal{N} \), then \( \overline{\mathcal{M}} \subset \overline{\mathcal{N}} \) and \( E(\mathcal{M}) \subset E(\mathcal{N}) \). The general relation of these concepts is

2.4. (a) \([\mathcal{M}] = [\overline{\mathcal{M}}]\) and \( E(\mathcal{M}) = E(\overline{\mathcal{M}}) \subset \mathcal{M} \); (b) \( \mathcal{M} = \overline{\mathcal{M}} \Rightarrow [\mathcal{M}] = [\overline{\mathcal{M}}] \Rightarrow E(\mathcal{M}) = E(\overline{\mathcal{M}}) \).

**Proof.** (a) According to 1.4, there are equivalences \( \Delta: \mathcal{K}/\mathcal{M} \approx \mathcal{K}/\overline{\mathcal{M}}, \overline{\Delta}: \mathcal{K}/\overline{\mathcal{M}} \approx \mathcal{K}/\mathcal{M} \) such that \( \overline{\eta} = \Delta \circ \eta, \eta = \Delta \circ \overline{\eta} \). This implies that \( \eta(f) = \eta(g) \) if and only if

(5) I.e., there exists a continuous \( \phi: X \to I \) with \( \phi^{-1}(0) = j(W) \).
\(\eta(f) = \eta(g)\), so that \([\mathcal{W}] = [\mathcal{W}]\). It also implies that \(E(\mathcal{W}) = E(\mathcal{W})\); and the definition shows \(E(\mathcal{W}) < \mathcal{W}\).

(b) If \(\mathcal{W} = \mathcal{W}\) then obviously \([\mathcal{W}] = [\mathcal{W}]\) and, from (a), also \([\mathcal{W}] = [\mathcal{W}]\). Now let \(f \in E(\mathcal{W})\); then there is a \(g\) with \(f \circ g \equiv 1 \mod \mathcal{W}\), \(g \circ f \equiv 1 \mod \mathcal{W}\); since \([\mathcal{W}] = [\mathcal{W}]\), we find \(f \circ g \equiv 1 \mod \mathcal{W}\), \(g \circ f \equiv 1 \mod \mathcal{W}\) and so \(f \in E(\mathcal{W})\). Similarly, \(f \in E(\mathcal{W})\) implies \(f \in E(\mathcal{W})\) and the proof is complete.

In general, none of the implications in 2.4(b) are reversible.

(A) Let \(\mathcal{X}\) consist of two objects \(A, B\) together with the identity morphisms and a single morphism \(f: A \to B\). Let \(\mathcal{W} = \{f, 1_A, 1_B\}\) and \(\mathcal{W} = \{1_A, 1_B\}; then \([\mathcal{W}] = [\mathcal{W}]\) yet \(\mathcal{W} \neq \mathcal{W}\) (see 1.3).

(B) Let \(\mathcal{X}\) consist of three intervals \(A_r = [0, r], r = 1, 2, 3\), with the identity maps, the inclusion maps \(j_{rs}: A_r \to A_s, r < s\), and the map \(h: A_2 \to A_3\) given by \(h(x) = x, 0 \leq x \leq 1, h(x) = 1, x > 1\), so that \(j_{13} = j_{23} \circ j_{12} = h \circ j_{12}\). Let \(\mathcal{W} = \{1, 1_2, 1_3, j_{12}\}\) and \(\mathcal{W} = \{1, 1_2, 1_3\}\); then \([\mathcal{W}] \neq [\mathcal{W}]\) since \(h\) is \(\mathcal{W}\)-homotopic to \(j_{23}\); however \(E(\mathcal{W}) = E(\mathcal{W})\) since there is no map in \(\mathcal{X}\) going backwards. Note that this shows also that \([\mathcal{W}] \neq [\mathcal{W}]\) (see 1.3).

2.5. Definition. The class \(\mathcal{W}\) is called stable if \(E(\mathcal{W}) = \mathcal{W}\); i.e., if the class of \(\mathcal{W}\)-homotopy equivalences is exactly \(\mathcal{W}\).

If \(\mathcal{W}\) is stable, then \(E(\mathcal{W})\) and \(\mathcal{W}\) determine the same notion of homotopy, since \([E(\mathcal{W})] = [\mathcal{W}] = [\mathcal{W}]\). Furthermore, distinct stable classes determine distinct notions of homotopy, because, from 2.4(b) and stability, we have

2.6. If \(\mathcal{W}, \mathcal{W}\) are stable classes, then \(\mathcal{W} = \mathcal{W} \iff \mathcal{W} = [\mathcal{W}] = [\mathcal{W}] = E(\mathcal{W}) = E(\mathcal{W})\).

The stable classes are characterized by

2.7. Theorem. A class \(\mathcal{W}\) in \(\mathcal{X}\) is stable if and only if \(\eta: \mathcal{X} \to \mathcal{X}/\mathcal{W}\) is epic.

Proof. “If”: We need show only that \(\mathcal{W} \subset E(\mathcal{W})\). Let \(f \in \mathcal{W}\); then \(\eta(f)\) has an inverse \(G\) in \(\mathcal{X}/\mathcal{W}\); since \(\eta\) is epic, \(G = \eta(g)\) for some \(g \in \mathcal{X}\), so \(f \in E(\mathcal{W})\). “Only if”: We have seen in 1.2 that each given \(G\) in \(\mathcal{X}/\mathcal{W}\) has a factorization as \(G = f_1 \circ \alpha_1 \circ \cdots \circ f_1 \circ \alpha_1\), where each \(f_1 = \eta(f)\) for some \(f_1\) in \(\mathcal{X}\), and each \(\alpha_1\) is the inverse of some \(\eta(\alpha_1)\). In particular, each \(\alpha_1 \in \mathcal{W} = E(\mathcal{W})\), so \(\eta(\alpha_1)\) has some \(\eta(\beta_1)\) for inverse and, since inverses are unique, \(\alpha_1 = \eta(\beta_1)\). Thus, \(G = \eta[f_1 \circ \beta_1 \circ \cdots \circ f_1 \circ \beta_1]\) and \(\eta\) is epic.

2.8. Corollary. Let \(\mathcal{W}\) be any class. Then there exists a unique maximal stable class \(\mathcal{W} \subset \mathcal{W}\) with the following property: if \(\mathcal{W} \subset \mathcal{W}\) is stable, then \(\mathcal{W} \subset \mathcal{W}\).

Proof. In view of our constructions, the empty class \(\emptyset \subset \mathcal{W}\) is (1.3) clearly stable. It is therefore enough to show that the union of any family of stable classes is stable. Let \(\{\mathcal{W}_\beta | \beta \in B\}\) be the family of all stable classes in \(\mathcal{W}\), so that each \(\eta_\beta: \mathcal{X} \to \mathcal{X}/\mathcal{W}_\beta\) is epic; letting \(\mathcal{W} = \bigcup \mathcal{W}_\beta\), we will prove that \(\eta: \mathcal{X} \to \mathcal{X}/\mathcal{W}\) is also epic. Each morphism \(G\) in \(\mathcal{X}/\mathcal{W}\) has a factorization \(G = f_1 \circ \alpha_1 \circ \cdots \circ f_1 \circ \alpha_1\), 

(*) I.e., for each \(A, B \text{ and } f \in \text{hom}(A, B)\), there exists an \(f: A \to B\) with \(\eta(f) = f\). Note that the stability of \(\mathcal{W}\) implies that each \(\text{hom}(A, B)\) is in fact a set, cf. footnote 14, 15.

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where each \( f_i = \eta(f_i) \) for some \( f_i \) in \( \mathcal{K} \). Each \( a_i \) lies in some \( \mathcal{M}_i \) so there is a \( \beta_i \) in \( \mathcal{K} \) with \( \eta(\beta_i) = a_i \) in \( \mathcal{K}/\mathcal{M}_i \). Since (1.4) there is an \( \eta_i : \mathcal{K}/\mathcal{M}_i \to \mathcal{K}/\mathcal{M} \) with \( \eta = \eta_i \circ \eta_i \), it follows that \( \eta(\beta_i) = \tilde{a}_i \) in \( \mathcal{K}/\mathcal{M} \) and therefore that \( \eta(f_n \circ \beta_n \circ \cdots \circ f_1 \circ \beta_1) = G \). This completes the proof.

The maximal stable class \( \mathcal{M} \) may not give the same notion of homotopy as \( \mathcal{M} \). In fact, there are notions of \( \mathcal{M} \)-homotopy that cannot be obtained from any stable class: in example (B) above, \( \mathcal{M} = \mathcal{M} \) and, indeed, \( \mathcal{M} \)-homotopy cannot be obtained from a stable class. Moreover, \( E(\mathcal{M}) \neq E(\mathcal{M}) \) in general (cf. Appendix A5).

The machinery developed so far can be applied to any covariant functor \( \Phi : \mathcal{K} \to \mathcal{L} \) to introduce into \( \mathcal{K} \) a notion of homotopy induced (or adapted to) the functor \( \Phi \).

2.9. Definition. Let \( \Phi : \mathcal{K} \to \mathcal{L} \) be any covariant functor. Let

\[ \mathcal{M}(\Phi) = \{ f \in \mathcal{K} \mid \Phi(f) \text{ is invertible} \}. \]

The homotopy in \( \mathcal{K} \) determined by the class \( \mathcal{M}(\Phi) \) is called \( \Phi \)-homotopy.

The category \( \mathcal{K}/\mathcal{M}(\Phi) \) is written \( \mathcal{K}^\Phi \) and is essentially the category first studied by Bauer in [1]. Observe that, because of C2, the functor \( \Phi \) has a unique factorization \( \Phi = \Delta \circ \eta \) through \( \mathcal{K}^\Phi \). It follows from this that \( \mathcal{M}(\Phi) = [\mathcal{M}(\Phi)]^- \) always: for, if \( f \in [\mathcal{M}(\Phi)]^- \), then \( \eta(f) \) is invertible, therefore so also is \( \Delta(\eta(f)) = \Phi(f) \), consequently \( f \in \mathcal{M}(\Phi) \); thus, \( [\mathcal{M}(\Phi)]^- \subseteq \mathcal{M}(\Phi) \) and the opposite inclusion is trivial.

We give some applications of \( \Phi \)-homotopy.

(1) Let \( \Phi : \text{Top} \to \text{Ens} \) be the forgetful functor. Then \( f \) is \( \Phi \)-homotopic to \( g \) if and only if \( f = g \). For, consider the factorization \( \Phi = \Delta \circ \eta \) through \( \mathcal{K}^\Phi \). If \( \eta(f) = \eta(g) \), then \( \Phi(f) = \Phi(g) \) so that \( f = g \) as maps of sets; and the converse is trivially true. The class \( \mathcal{M} = \mathcal{M}(\Phi) \) consists of all bijective continuous maps, whereas \( E(\mathcal{M}) \) is the class of all bijective bicontinuous maps (i.e., homeomorphisms).

(2) Let \( \mathcal{K} \) be the category of spaces dominated by CW-complexes, let \( \mathcal{G}^\pi \) be the category of graded groups and degree zero homomorphisms, and let \( \pi : \mathcal{K} \to \mathcal{G}^\pi \) be the total homotopy functor. Then two maps are \( \pi \)-homotopic if and only if they are homotopic. For, we apply 2.3 because (a) each projection \( r : X \times I \to X \) is such that \( \pi(r) \) is invertible, therefore \( r \in \mathcal{M}(\pi) \), consequently \( \eta(r) \) is invertible, and (b) if \( f \in \mathcal{M}(\pi) \), i.e., if \( \pi(f) \) is invertible, then, by Whitehead’s theorem [6], \( f \) is a homotopy equivalence and therefore \( h(f) \) is invertible. Note that, again using Whitehead’s theorem, the class \( \mathcal{M}(\pi) \) is stable, so that \( \eta : \mathcal{K} \to \mathcal{K}^\pi \) is epic. Similarly, if \( \mathcal{K} \) is the category of Kan-complexes, then \( \pi \)-homotopy in \( \mathcal{K} \) is exactly Kan-homotopy.

(3) Let \( \mathcal{K} \) be the category of all simply connected CW-complexes, and \( H : \mathcal{K} \to \mathcal{G}^\pi \) the total homology functor. Exactly the same considerations as in (2) reveal \( H \)-homotopy to be the usual homotopy. Using \( H : \mathcal{K} \to \mathcal{G}^\pi \) on the category of css complexes yields a homotopy notion extending the usual one for Kan-complexes. And on the category \( \mathcal{K} \) of chain-complexes, \( H \)-homotopy yields the usual notion of chain-homotopy.

Thus, the various notions of homotopy encountered, and usually defined
differently, in various categories turn out to be essentially the homotopy relation
determined in the above manner by a suitable covariant functor. For arbitrary
contravariant functors, one turns to the dual category.

3. Fibrations. Each class \( \mathcal{M} \) of morphisms in a category \( \mathcal{X} \)
determines a concept of fibration in \( \mathcal{X} \):

3.1. Definition. A morphism \( p: E \rightarrow B \) in \( \mathcal{X} \) is called an \( \mathcal{M} \)-fibration if for
each diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow{f} & & \downarrow{p} \\
B & \xrightarrow{g} & E
\end{array}
\]

in which \( \mu \in \mathcal{M} \) and \( p \circ g \circ \mu = f \circ \mu \), there exists a \( g': X \rightarrow E \) in \( \mathcal{X} \) with \( g \circ \mu = g' \circ \mu \) and \( p \circ g' = f \).

We are requiring simply that every triangle involving \( p \) which can be equalized
by a morphism belonging to \( \mathcal{M} \), can be made itself commutative. Note that if
\( \mathcal{M} \)-homotopy is used in \( \mathcal{X} \), the conditions \( p \circ g \circ \mu = f \circ \mu \) and \( g \circ \mu = g' \circ \mu \) imply,
by 2.2(d), that \( p \circ g \equiv f \text{ mod } \mathcal{M} \) and \( g \equiv g' \text{ mod } \mathcal{M} \). The relations between \( \mathcal{M} \)-
homotopy and \( \mathcal{M} \)-fibration will be considered after we verify that the class of
\( \mathcal{M} \)-fibrations has the properties usually required of fibrations.

3.2. Theorem. Let \( \mathcal{M} \) be a fixed class of morphisms in \( \mathcal{X} \). Then:

(a) If \( p: E \rightarrow B \) is invertible, then \( p \) is an \( \mathcal{M} \)-fibration.
(b) If \( p_1: E_1 \rightarrow E_0 \) and \( p_0: E_0 \rightarrow B \) are \( \mathcal{M} \)-fibrations, so also is \( p_0 \circ p_1: E_1 \rightarrow B \).
(c) If \( p: E \rightarrow B \) is an \( \mathcal{M} \)-fibration, and if

\[
\begin{array}{ccc}
E' & \xrightarrow{\bar{h}} & E \\
\downarrow{p'} & & \downarrow{p} \\
B' & \xrightarrow{h} & B
\end{array}
\]

is a cartesian diagram\(^7\), then \( p' \) is also an \( \mathcal{M} \)-fibration; and \( h \) can be lifted into \( E \) if
and only if \( p' \) has a section\(^8\).

Proof. (a) is trivial.

---

\(^7\) In the terminology of Mitchell [4] the indicated diagram would be called a pullback of
\( p, h \).

\(^8\) I.e., an \( s: B' \rightarrow E' \) such that \( p' \circ s' = 1 \).
(b) Given

\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow{f} & & \downarrow{p} \\
B & & E_0
\end{array}
\]

with \( p_0 \circ p_1 \circ g \circ \mu = f \circ \mu \) and \( \mu \in \mathcal{W} \), there exists first a \( G: X \to E_0 \) with \( p_0 \circ G = f \), \( G \circ \mu = p_1 \circ g \circ \mu \); and then a \( g': X \to E_1 \) with \( p_1 \circ g' = G \), \( g' \circ \mu = g \circ \mu \); since \( p_0 \circ p_1 \circ g = p_0 \circ G = f \), the required morphism is \( g' \).

(c) We are given

\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow{f} & & \downarrow{h} \\
B' & & E
\end{array}
\]

where the square is cartesian and \( f \circ \mu = p' \circ g \circ \mu \). Thus, \( h \circ f \circ \mu = p \circ h \circ g \circ \mu \) so, because \( p \) is an \( \mathcal{W} \)-fibration, there is a \( G: X \to E \) such that \( G \circ \mu = h \circ g \circ \mu \) and \( p \circ G = h \circ f \). Because the square is cartesian and \( p \circ G = h \circ f \), there exists a unique \( g': X \to E' \) such that \( h \circ g' = G \) and \( p' \circ g' = f \). Since also \( h \circ (g' \circ \mu) = G \circ \mu = h \circ (g \circ \mu) \) and \( p' \circ (g \circ \mu) = f \circ \mu = p' \circ (g' \circ \mu) \), the uniqueness of a morphism \( W \to E' \) satisfying these two conditions in a cartesian diagram shows \( g' \circ \mu = g \circ \mu \) and therefore we find that \( p' \) is an \( \mathcal{W} \)-fibration. The second part is proved in a similar manner.

We now examine the notion of \( \mathcal{W} \)-fibration in \( \text{Top} \), using the classes, and the symbols for those classes, that are listed following Theorem 2.3. We have

3.3 Theorem. Let \( \mathcal{W} = \mathcal{A} \) in \( \text{Top} \). Then \( \mathcal{A} \)-homotopy is the classical notion, and \( p: E \to B \) is a \( \mathcal{A} \)-fibration if and only if it is a Hurewicz fibration.

Proof. The first part was established in 2.3. Let now \( p: E \to B \) be a \( \mathcal{A} \)-fibration; we show it has the covering homotopy property, i.e., that it is a Hurewicz fibration. We start with the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
X \times I & \xrightarrow{F} & B
\end{array}
\]

where \( i(x) = (x, 0) \). Let \( r: X \times I \to X \) be the map \( r(x, t) = x \); we then have

\[
\begin{array}{ccc}
X & \xleftarrow{i} & X \times I \\
\downarrow{F} & & \downarrow{p} \\
B & & E
\end{array}
\]
where $F \circ i = p \circ f = p \circ f \circ r \circ i$. Since $p$ is a $\mathfrak{S}$-fibration, there is an $\tilde{F}: X \times I \to E$ with $p \circ \tilde{F} = F$ and $\tilde{F} \circ i = f \circ r \circ i = f$, so that $\tilde{F}$ is a homotopy of $f$ covering $F$, and therefore $p$ is a Hurewicz fibration.

Conversely, let $p: E \to B$ be a Hurewicz fibration. The morphisms in $\mathfrak{S}$ being simply the maps $i: X \to X \times I$, we consider any diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X \times I \\
\downarrow{F} & & \downarrow{p} \\
\phantom{X} & \phantom{i} & \phantom{X \times I} \\
\phantom{p} & \phantom{F} & \phantom{B}
\end{array}
$$

in which $p \circ f \circ i = F \circ i$; since $p$ is a Hurewicz fibration, there is a homotopy $\tilde{F}$ of $f \circ i$ covering $F$, so that $p \circ \tilde{F} = F$ and $f \circ i = \tilde{F} \circ i$. Thus, $p$ is a $\mathfrak{W}$-fibration, and the proof is complete.

If $\mathfrak{W} \subseteq \mathfrak{S}$, it is clear(9) that $\{\mathfrak{W}\text{-fibrations}\} \subseteq \{\mathfrak{S}\text{-fibrations}\}$. Moreover,

(a) Two classes may determine the same notion of homotopy, but distinct notions of fibration.

Example in Top: We have seen that $\mathfrak{S}$ determines the same homotopy notion as $\mathfrak{S}$, that is, the usual notion of homotopy. However, every continuous map is a $\mathfrak{S}$-fibration: for, clearly, $\mathfrak{S} \subseteq \{\text{all continuous surjections}\}$, and, if $\mu$ is surjective, the condition $p \circ g \circ \mu = f \circ \mu$ of Definition 3.1 implies that $p \circ g = f$; thus, every continuous $p: E \to B$ belongs to $\{\text{surjective map fibrations}\} \subseteq \{\mathfrak{S}\text{-fibrations}\}$.

(b) Two classes may determine the same notion of fibration but distinct notions of homotopy.

Example in Top: Let $\mathfrak{W} = \text{class of all identity maps} 1_A: A \to A$. Then every continuous map is an $\mathfrak{W}$- and also a $\mathfrak{S}$-fibration; yet $\mathfrak{W}$-homotopy is the usual notion of homotopy, whereas we have seen that $\mathfrak{W}$-homotopy is simply equality.

From this viewpoint, the notions of homotopy and of fibration are independent. In the category Top, the Hurewicz fibrations appear as a concept dependent on a particular class of morphisms that happens to yield the usual homotopy notion, rather than as a concept dependent on the homotopy notion itself(10).

Returning to 3.1, we shall establish a simple criterion for two classes to determine the same notion of fibration. This is based on

3.4. Theorem. Let $\mathfrak{W}, \mathfrak{Q}$, be two classes of morphisms in $\mathfrak{X}$. Assume that for each $\mu: W \to X$, $\mu \in \mathfrak{W}$, there exist a $\lambda: Y \to Z$, $\lambda \in \mathfrak{Q}$ and morphisms in $\mathfrak{X}$ such that

9) We denote the class of $\mathfrak{W}$-fibrations by $\{\mathfrak{W}\text{-fibrations}\}$.

10) The class $\mathfrak{H}$ of Hurewicz fibrations does not determine the usual homotopy: in fact, since every constant map $f: E \to e$ is a Hurewicz fibration, it follows (cf. footnote 4) that for each $X, Y$, all $f, g: X \to Y$ are $\mathfrak{H}$-homotopic.
is commutative and $k \circ j = 1$. Then $\{\mathcal{W}\text{-fibrations}\} \subseteq \{\mathcal{M}\text{-fibrations}\}$.

**Proof.** Let $p: E \to B$ be an $\mathcal{W}$-fibration. Given a diagram

$$
\begin{array}{ccc}
W & \xrightarrow{i} & Y \\
\mu \downarrow & & \downarrow \lambda \\
X & \xrightarrow{j} & Z \\
\downarrow & & \downarrow k \\
 & & X
\end{array}
$$

with $p = g \circ f$, we find that $\mu \circ \lambda = \lambda \circ \mu = s \circ f \circ \mu = s = f \circ k \circ \lambda$, so since $p$ is an $\mathcal{W}$-fibration, there is a $\Gamma: Z \to E$ with $p \circ \Gamma = f \circ k$ and $\Gamma \circ \lambda = g \circ k \circ \lambda$. Letting $G = \Gamma \circ j$, we find $p \circ G = p \circ \Gamma \circ j = f \circ k \circ j = f$, and $G \circ \mu = \Gamma \circ j \circ \mu = g \circ k \circ \lambda \circ i = g \circ k \circ j \circ \mu = g \circ \mu$, so that $p$ is an $\mathcal{M}$-fibration.

As an application,

3.5. In $\text{Top}$, $\{\mathcal{X}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}$.

**Proof.** Clearly, $\mathcal{X} \subseteq \mathcal{X}$, so $\{\mathcal{X}\text{-fibrations}\} \subseteq \{\mathcal{X}\text{-fibrations}\}$. For the converse, let $\mu: W \to X$, $\mu \in \mathcal{X}$, be given; we identify $W$ with a subset of $X$ in order to cut down excessive notation, and let $\Phi: r \simeq 1$ be a strong deformation retraction. Choose $\phi: X \to I$ vanishing exactly on $W$, and define $\Phi: X \times I \to X$ by

$$
\Phi(x, t) = \begin{cases} 
\Phi(x, t/\phi(x)), & x \notin W, \\
\Phi(x, 1), & x \in W,
\end{cases}
$$

which is easily verified to be continuous. Then the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\mu \downarrow & & \downarrow \lambda \\
X & \xrightarrow{j} & X \times I \\
\downarrow & & \downarrow \Phi \\
 & & X
\end{array}
$$

where $r$ = retraction, $\mu$ = inclusion, $\lambda(x) = (x, 0)$ and $j(x) = (x, \phi(x))$ is easily seen to be commutative. Thus, $\{\mathcal{X}\text{-fibrations}\} \subseteq \{\mathcal{X}\text{-fibrations}\}$ and, with 3.3, the theorem is proved.

We recall that, in $\text{Top}$, a fibration is called regular whenever a covering homotopy can always be chosen stationary at all the points where the given homotopy is stationary\(^{(1)}\). Let $\mathcal{F}$ be the family of all inclusions $\mu: A \to X$, where $A$ is a

\(^{(1)}\) An example of a nonregular Hurewicz fibration, as well as a general condition on a topological space $B$ that assures the regularity of every Hurewicz fibration over $B$, is given in [5].
strong deformation retract of $X$, and let $\mathfrak{B}$ be the class of all maps $\lambda_B: X \to X_B = (X \times I) \cup_{r:B \times 1} B$, $r \in B$, where $B \subseteq X$ is arbitrary, and $\lambda_B = \rho_B \circ i$ with $i \in \mathfrak{S}$ and $\rho_B: X \times I \to X_B$ the identification. Clearly, $\mathfrak{K} \subseteq \mathfrak{B}$ and $\mathfrak{S} \subseteq \mathfrak{B} \subseteq \mathfrak{K}$, so that every $\mathfrak{B}$- and every $\mathfrak{K}$-fibration is a Hurewicz fibration.

3.7. Theorem\(^{(12)}\). In $\text{Top} \{\mathfrak{B}$-fibrations$\} = \{\mathfrak{K}$-fibrations$\} = \{\text{regular Hurewicz fibrations}\}$.\

Proof. The proof that $\{\mathfrak{B}$-fibrations$\} = \{\text{regular Hurewicz fibrations}\}$ is a repetition of that for 3.3, with $X \times I$ replaced by suitable $X_B$ and $\mathfrak{S}$ replaced by $\mathfrak{B}$. Because $\mathfrak{B} \subseteq \mathfrak{K}$, we have $\{\mathfrak{B}$-fibrations$\} \subseteq \{\mathfrak{K}$-fibrations$\}$. To prove the converse inclusion, let $\mu: A \to X$, $\mu \in \mathfrak{K}$, be given, let $\Phi: r \simeq 1$, be a strong deformation retraction of $X$ onto $A$; then a continuous $\Phi: X_A \to X$ such that $\Phi \circ \rho_A = \Phi$ exists. Let $i_1: X \to X \times I$ be the map $x \to (x, 1)$; then the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow \lambda_A \\
X & \xrightarrow{\mu} & X_A \\
\downarrow j & \Phi & \\
& & X
\end{array}
$$

with $j = \rho_A \circ i_1$, is commutative and $\Phi \circ j = 1$. Thus, by 3.5, the proof is complete.

As in the previous section, a covariant functor $\Phi: \mathcal{X} \to \mathcal{L}$ gives rise to a notion of $\Phi$-fibration, i.e., $\mathfrak{B}(\Phi)$-fibration. Observe that, if $\pi: \mathcal{X} \to \mathcal{L}^2$ is the total homotopy functor on the category of spaces dominated by CW-complexes, then the corresponding notion of $\Phi$-fibration is not that of Hurewicz fibration, but of a certain subset: for, $\mathfrak{B}(\Phi) =$ set of all homotopy equivalences and $\mathfrak{S} \subseteq \mathfrak{B}(\Phi)$ so that $\{\mathfrak{B}(\Phi)$-fibrations$\} \subseteq \{\mathfrak{S}$-fibrations$\} = \{\text{Hurewicz fibrations}\}$.


4.1. Definition. Let $\mathfrak{M}$ be a class of morphisms in a category $\mathcal{X}$. A morphism $j: B \to E$ in $\mathcal{X}$ is called an $\mathfrak{M}$-cofibration if for each diagram

$$
\begin{array}{ccc}
E & \xrightarrow{g} & X \\
\downarrow j & & \downarrow \mu \\
B & \xrightarrow{f} & W
\end{array}
$$

in which $\mu \in \mathfrak{M}$ and $\mu \circ g \circ j = \mu \circ f$, there exists a $g': E \to X$ with $g' \circ j = f$ and $\mu \circ g' = \mu \circ g$.

This concept depends on the class $\mathfrak{M}$ itself rather than on the homotopy notion that $\mathfrak{M}$ determines. In fact, the corresponding homotopy notions may be the same, and the associated cofibrations different, since

4.2. Theorem. In the category $\text{Top}$,

\(^{(12)}\) For $\mathfrak{E}$-fibrations, this theorem was suggested by the referee.
(a) $\{\mathfrak{9}$-cofibrations$\} = \{\mathfrak{8}$-cofibrations$\} = \text{all morphisms in } \text{Top}$;
(b) $\{\mathfrak{8}$-cofibrations$\} = \text{class of all } j : B \to E \text{ such that every continuous } \phi : B \to I \text{ has a factorization } \phi = \psi \circ j \text{ where } \psi : E \to I$.

Proof. (a) is trivial. (b) Assume that $j : B \to E$ is a $\mathfrak{9}$-cofibration. Let $\phi : B \to I$ be given and consider

$$
\begin{array}{ccc}
E & \xrightarrow{g} & x_0 \times I \\
\downarrow{j} & & \downarrow{r} \\
B & \xrightarrow{f} & x_0
\end{array}
$$

where $f(b) = (x_0, \phi(b))$ and $g(e) = (x_0, 0)$. By hypothesis, there exists a $g' : E \to x_0 \times I$ such that $r \circ g' = r \circ g$ and $g' \circ j = f$, i.e., $g'(e) = (x_0, \psi(e))$ and $\phi \circ j = \phi$.

Conversely, assume that the property is satisfied for $j : B \to E$, and consider any diagram

$$
\begin{array}{ccc}
E & \xrightarrow{g} & X \times I \\
\downarrow{j} & & \downarrow{r} \\
B & \xrightarrow{f} & X
\end{array}
$$

where $r \circ f = r \circ g \circ j$. Using the projections on each factor, write $f(b) = (f_x(b), f_I(b))$ and $g(e) = (g_x(e), g_I(e))$; the commutativity requirement assures $f_x(b) = g_x(jb)$; but $f_I(b) = \psi j(b)$ for suitable $\psi : E \to I$, so that $g'(e) = (g_x(e), \phi(e))$ is the desired map.

In particular, if $j : B \to E$ is inclusion onto a retract of $E$, then $j$ is a $\mathfrak{8}$-cofibration. Moreover,

4.3. If $B$ is a functional Hausdorff space$^{(13)}$, then each $\mathfrak{8}$-cofibration $j : B \to E$ is injective.

Proof. Let $j : B \to E$ be a map such that $j(b_0) = j(b_1)$ for some $b_0 \neq b_1$. There exists a map $\phi : B \to I$ with $\phi(b_0) = 0$, $\phi(b_1) = 1$; and $\phi$ cannot factor as $\phi = \psi \circ j$.

It is interesting to note that, within this framework, $\{\mathfrak{9}$-fibrations$\} \equiv \{\text{Hurewicz fibrations}$ has $\{\mathfrak{9}$-cofibrations$\} \equiv \{\text{all morphisms}$ for dual; and that $\{\mathfrak{8}$-fibrations$\} \equiv \{\text{all morphisms}$ has $\{\mathfrak{9}$-cofibrations$\} \equiv \{\text{a restricted class of morphisms}$ for dual.

For a covariant functor $\Phi : \mathcal{X} \to \mathcal{L}$ the notion of $\Phi$-cofibration is defined, in the customary manner, to be that of $\mathfrak{M}(\Phi)$-cofibration. For contravariant functors, one uses the dual category, in which fibrations are exchanged against cofibrations.

5. Weak $\mathfrak{M}$-fibrations. We recall that, in the category Top, a map $p : E \to B$ is called a Dold fibration (= a map with the WCHP in [2]) if for each commutative diagram

$^{(13)}$ I.e., for each pair $b_0 \neq b_1$ of points in $B$, there exists a continuous $\phi : B \to I$ such that $\phi(b_0) = 0$ and $\phi(b_1) = 1$. 

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there exists a map \( F: X \times I \to E \) such that \( p \circ F = F \) and \( F \circ i_0 \) is fiber-homotopic to \( f \).

For any \( X \in \text{Top} \), let \( \rho = \rho_X: X \times I \to X \times I \) denote the map
\[
\rho(x, t) = (x, 0), \quad 0 \leq t \leq \frac{1}{2},
\]
\[
= (x, 2t - 1), \quad \frac{1}{2} \leq t \leq 1,
\]
and let \( \mathcal{R} = \{\rho_X \mid X \in \text{Top}\} \). Then

5.1. **Lemma.** A map \( p: E \to B \) is a Dold fibration if and only if for each diagram
\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & X \times I \\
\downarrow & & \downarrow p \\
X \times I & \xrightarrow{F} & E
\end{array}
\]
with \( p \circ g \circ i_0 = F \circ i_0 \), there exists a \( g': X \times I \to E \) such that \( g \circ i_0 = g' \circ i_0 \) and \( p \circ g' = F \circ \rho \).

Because the proof of 5.1 is entirely analogous to that of 3.3, it is omitted.

Let \( \mathcal{C} \) be an arbitrary category and \( \mathcal{R} \) be any family of its morphisms. With 5.1 in mind, we make the following

5.2. **Definition.** A morphism \( f: E \to B \) in \( \mathcal{C} \) is called a weak \( \mathcal{R} \)-fibration if for each diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow g \\
& \xrightarrow{f} & E
\end{array}
\]
in which \( \mu \in \mathcal{R} \) and \( p \circ g \circ \mu = f \circ \mu \), there exists a \( g': X \to E \) in \( \mathcal{C} \), and an \( r: X \to X \) in \( \mathcal{R} \) such that \( r \circ \mu = \mu \), \( g \circ \mu = g' \circ \mu \), and \( p \circ g' = f \circ r \).

**Remarks.** (1) If \( \mathcal{R} \) is such that \( \mathcal{R} \cap \mathcal{C}(X, X) = \{1_X : X \to X\} \) for each \( X \in \mathcal{C} \), then \( \{\text{weak } \mathcal{R}\text{-fibrations}\} = \{\text{\mathcal{R}-fibrations}\} \).

(2) If \( \mathcal{R} \) is any class that contains all the identity morphisms, then each \( \mathcal{R}\)-fibration is also a weak \( \mathcal{R}\)-fibration.

(3) Unlike the \( \mathcal{R}\)-fibrations, it is not in general true that \( (\mathcal{R} \subset \mathcal{C}) = (\{\text{weak } \mathcal{R}\text{-fibrations}\} \subset \{\text{weak } \mathcal{R}\text{-fibrations}\}) \).

5.3. **Definition.** For any family \( \mathcal{R} \) of morphisms in \( \text{Top} \), we let \( \mathcal{R}_\circ = \mathcal{R} \cup \{\text{all identities}\} \) and \( \mathcal{R}^\ast = \mathcal{R} \cup \mathcal{R} \). Then

5.4. **Theorem.** In the category \( \text{Top} \),

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(a) \{weak \mathcal{X}_e\text{-fibrations}\} = \{weak \mathcal{S}_e\text{-fibrations}\} = \{Hurewicz fibrations\};
(b) \{weak \mathcal{X}^*\text{-fibrations}\} = \{weak \mathcal{S}^*\text{-fibrations}\} = \{Dold fibrations\}.

**Proof.** (a) follows immediately from Remark (1) above and Theorems 3.3, 3.5. To prove (b): Observe that we have
\{weak \mathcal{X}^*\text{-fibrations}\} \subseteq \{weak \mathcal{S}^*\text{-fibrations}\}
because \mathcal{S} \subseteq \mathcal{X} and all morphisms \( r \) that can possibly appear in Definition 5.2 are in \( \mathcal{S}_e \). The converse inclusion follows by an argument entirely analogous to that in the proof of 3.5. Finally, \{weak \mathcal{S}^*\text{-fibrations}\} = \{Dold fibrations\} by 5.1. This completes the proof (b).

Everything which has been said for weak \( \mathcal{M} \)-fibrations can be dualized in the usual fashion, to yield weak \( \mathcal{M} \)-cofibrations.

**Appendix.** In this section, we will give a proof of Theorem 1.1. We also include an example to show that in general \( E(E(\mathcal{M})) \neq E(\mathcal{M}) \) and \( E(\mathcal{M}) \neq E(\mathcal{M}) \).

Let \( \mathcal{X} \) be an arbitrary category, and let \( \mathcal{M} \) be any subclass of its morphisms; the case that \( \mathcal{M} \) is the family of all morphisms in \( \mathcal{X} \) is not excluded. We denote the elements of \( \mathcal{M} \) by small Greek letters, \( \alpha, \beta, \ldots \).

Given \( A, B \in \mathrm{ob} (\mathcal{X}) \), by an \( \mathcal{M} \)-word \( m=(A, X_1, \ldots, X_{2n-1}, B) \) from \( A \) to \( B \) we mean a finite chain of morphisms and objects
\[
A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\alpha_2} X_3 \xrightarrow{f_2} \cdots \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B
\]
in which the \( \xleftarrow{\ } \) and \( \xrightarrow{\ } \) alternate, and each morphism \( \alpha_i \) "in the wrong direction" either belongs to \( \mathcal{M} \) or is an identity morphism. The class of all \( \mathcal{M} \)-words from \( A \) to \( B \) is denoted by \( \mathcal{M}(A, B) \).

We introduce an equivalence relation \( \sim \) in \( \mathcal{M}(A, B) \) by

A.1. **Definition.** Let \( m, \hat{m} \in \mathcal{M}(A, B) \). Declare \( m \sim \hat{m} \) if there is a finite sequence \( m=m_1, m_2, \ldots, m_n=\hat{m} \) of elements of \( \mathcal{M}(A, B) \) in which each \( m_i \) is obtained from \( m_{i-1} \) (or from \( m_{i+1} \)) by one of the following two operations:

(i) A pushout operation: replacement of a segment
\[
\xrightarrow{f} X \xleftarrow{\alpha} Y \xrightarrow{g} Z \xleftarrow{\beta}.
\]
in \( m_{i-1} \) (or \( m_{i+1} \)) by
\[
\xrightarrow{u \circ f} W \xleftarrow{v \circ \beta}.
\]
if there is a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \xrightarrow{g} Z \\
& \searrow{u} & \swarrow{v} \\
& W
\end{array}
\]
and \( v \circ \beta \in \mathcal{M} \) or is an identity morphism. This operation is denoted by

\[ \text{PO}(X, Y, Z; u, W, v). \]

(ii) A pullback operation: replacement of a segment

\[ \begin{array}{c}
X \xrightarrow{f} Y \xleftarrow{\beta} Z
\end{array} \]

in \( m_{i-1} \) (or \( m_{i+1} \)) by

\[ \begin{array}{c}
\alpha \circ u \quad g \circ v
\end{array} \]

if there is a commutative diagram

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y & \xleftarrow{\beta} & Z \\
& & \downarrow{u} & \searrow{v} & \\
& & W
\end{array} \]

and \( \alpha \circ u \in \mathcal{M} \) or is an identity morphism. This operation is denoted by

\[ \text{PB}(X, Y, Z; u, W, v). \]

It is clear that \( \sim \) is an equivalence relation in \( \mathcal{M}(A, B) \); the equivalence class of \( m \) is denoted by \([m]\). Furthermore, the map \( \mathcal{M}(A, B) \otimes \mathcal{M}(B, C) \to \mathcal{M}(A, C) \)

defined by

\[ (A, X_1, \ldots, X_{2n-1}, B) \otimes (B, Y_1, \ldots, C) = (A, X_1, \ldots, X_{2n-1}, B, Y_1, \ldots, C) \]

is associative, and the equivalence class of \( m \otimes m' \) depends only on that of \( m \)
and of \( m' \).

A.2. Definition. The category \( \mathcal{X}/\mathcal{M} \) is that having for objects \( \{A \mid A \in \text{ob}(\mathcal{X})\} \);
for morphisms \( \text{hom}(A, B) \): the equivalence classes\(^{(14)} \) in \( \mathcal{M}(A, B) \), with the com-
position law \( \text{hom}(B, C) \circ \text{hom}(A, B) \to \text{hom}(A, C) \) given by

\[ [m'] \circ [m] = [m \otimes m']. \]

The identity \([e_A] \in \text{hom}(A, A)\) is the morphism \([A \xrightarrow{1} A \xleftarrow{1} A]\): for, given
\([m] \in \text{hom}(B, A)\) we find that \( m \otimes e_A \) is a word of form

\[ \cdots \xleftarrow{X} \xrightarrow{f} A \xleftarrow{1} A \xrightarrow{1} A \]

and \( \text{PB}(X, A, A; 1, X, f) \) shows \( m \otimes e_A \sim m \); similarly, \( e_A \otimes m' \sim m' \) for
\( m' \in \mathcal{M}(A, B) \)
so \([e_A]\) is the (unique) identity morphism of \( A \).

A.3. Let each morphism in a word \( m \in \mathcal{M}(A, B) \) be either a member of \( \mathcal{M} \) or an
identity. Then \([m]\) is invertible in \( \mathcal{X}/\mathcal{M} \); and its inverse is represented by \( m \)
written in reverse order.

\(^{(14)}\) If \( \mathcal{X} \) is not a small category, \( \text{hom}(A, B) \) need not be a set. However, in all the choices
of \( \mathcal{X} \) and \( \mathcal{M} \) considered in this paper, the class \( \text{hom}(A, B) \) can be proved to be a set, so that the
construction of this category is legitimate.
Proof. Letting $m'$ be the word in reverse order, we have $m' \in \mathcal{W}(B, A)$, and

$$m \otimes m' = A \leftarrow X \rightarrow \cdots \leftarrow Y \rightarrow B \leftarrow Y \rightarrow \cdots \leftarrow X \rightarrow A.$$ 

Starting from the middle with $PB(Y, B; 1, Y, 1)$ and repeating, shows $m \otimes m' \sim A \leftarrow X \rightarrow A$; inserting $PO(A, X, A; 1, A, 1)$ we find $m \otimes m' \sim e_A$. Similarly $m' \otimes m \sim e_B$ and the proof is complete.

Observe that any word

$$A \leftarrow X_1 \rightarrow \cdots \rightarrow X_{2n-2} \leftarrow X_{2n-1} \rightarrow X \rightarrow \cdots \rightarrow Y \rightarrow B$$

is equivalent to

$$(A \leftarrow X_1 \rightarrow X_2 \leftarrow \cdots \leftarrow X_{2n-2} \leftarrow X_{2n-1} \rightarrow X) \otimes (X_1 \leftarrow X_2 \rightarrow X_3 \leftarrow \cdots \leftarrow X_{2n-2} \leftarrow X_{2n-1} \rightarrow X).$$

Thus, each morphism in $\mathcal{H}/\mathcal{W}$ can be factored as $f_n \circ \alpha_n \circ \cdots \circ f_1 \circ \alpha_1$ where, according to A.3, each $\alpha_i$ is invertible.

The transformation $\eta : \mathcal{H} \rightarrow \mathcal{H}/\mathcal{W}$ given by rules

$$\eta(A) = A \text{ on the objects},$$

$$\eta(f) = [A \leftarrow A \rightarrow B] \text{ for each } f \in \mathcal{H}(A, B)$$

is easily seen to be a covariant functor, called the canonical projection. Because of A.3, $\eta(\alpha)$ is invertible in $\mathcal{H}/\mathcal{W}$ for each $\alpha \in \mathcal{W}$; and the known [3] universality property of $(\eta, \mathcal{H}/\mathcal{W})$, in a formulation adapted for our purposes, is

A.4. Theorem. Let $T : \mathcal{H} \rightarrow \mathcal{L}$ be any covariant functor to any category $\mathcal{L}$. Then

(a) There exists a covariant functor $\Delta : \mathcal{H}/\mathcal{W} \rightarrow \mathcal{L}$ with $T = \Delta \circ \eta$ if and only if $T(\alpha)$ is invertible in $\mathcal{L}$ for each $\alpha \in \mathcal{W}$.

(b) If $T = \Delta \circ \eta$, then $\Delta$ is unique.

Proof. (a) If $T = \Delta \circ \eta$ then, because $\eta(\alpha)$ is invertible for each $\alpha \in \mathcal{W}$ and functors preserve invertible morphisms, $T$ has the required property. Conversely, assume $T(\alpha)$ invertible for each $\alpha \in \mathcal{W}$. Define $\Delta$ on the objects of $\mathcal{H}/\mathcal{W}$ by $\Delta(A) = T(A)$. For each word

$$m = A \leftarrow X_1 \rightarrow \cdots \leftarrow X_{2n-1} \rightarrow X \rightarrow \cdots \rightarrow Y \rightarrow B$$

in $\mathcal{W}(A, B)$, let

$$\Delta(m) = T(f_1) \circ T(\alpha_n)^{-1} \circ \cdots \circ T(f_1) \circ T(\alpha_1)^{-1} \in \mathcal{L}(\Delta(A), \Delta(B))$$

where $T(\alpha_i)^{-1}$ is the inverse of $T(\alpha_i)$ in $\mathcal{L}$. If $m \sim m'$, then $\Delta(m) = \Delta(m')$ because the operations in changing $m$ to $m'$ are governed by commutative diagrams, and all
morphisms "in the wrong direction" belong to $\mathfrak{M}$. Thus, $[m] \mapsto \Delta(m)$ is a well-defined\(^{(15)}\) map $\Delta: \text{hom}(A, B) \to \mathcal{L}(\Delta(A), \Delta(B))$ for each $A, B \in \text{ob}(\mathcal{X}/\mathfrak{M})$ which with the above indicated correspondence of the objects, is easily seen to determine a covariant functor $\Delta: \mathcal{X}/\mathfrak{M} \to \mathcal{L}$. Clearly, $\Delta \circ \eta(A) = A$ on objects, and $\Delta \circ \eta(f) = \Delta[A \leftarrow A \rightarrow B] = T(f)$ for each $f \in \mathcal{X}(A, B)$, so $\Delta \circ \eta = T$ and the proof is complete.

(b) Assume $T = \Delta \circ \eta = \Gamma \circ \eta$; then $\Delta(A) = T(A) = \Gamma(A)$ for each $A \in \text{ob}(\mathcal{X}/\mathfrak{M})$. For each morphism of type $f = [A \leftarrow A \rightarrow B]$ we have $\Delta(f) = \Delta \circ \eta(f) = T(f) = \Gamma \circ \eta(f) = \Gamma(f)$. Each morphism of type $\tilde{a} = [A \leftarrow A \rightarrow B]$ is invertible in $\mathcal{X}/\mathfrak{M}$, with inverse $\tilde{a}^{-1} = [B \leftarrow B \rightarrow A]$. Thus $\Delta(\tilde{a})$, $\Gamma(\tilde{a})$ are invertible in $\mathcal{L}$ and, from what we have already shown, $\Delta(\tilde{a}) = \Delta(\tilde{a}^{-1})^{-1} = \Gamma(\tilde{a}^{-1})^{-1} = \Gamma(\tilde{a})$. Since each morphism in $\mathcal{X}/\mathfrak{M}$ is representable as a composition of morphisms of these two types, we find $\Delta[m] = \Gamma[m]$ for each morphism $[m]$ and the proof is complete.

**Remark.** If $\mathfrak{M}$ is the family of all morphisms in $\mathcal{X}$, then all the morphisms of $\mathcal{X}/\mathfrak{M}$ are invertible therefore each hom $(X, X)$ is a group, $\pi(X)$. Furthermore, for each $\phi \in \text{hom}(X, Y)$ there is a homomorphism $\phi^*: \pi(X) \to \pi(Y)$ given by $\phi^*(f) = \phi \circ f \circ \phi^{-1}$, and clearly $\phi^*$ is an isomorphism. Thus the groups $\pi(X)$ belonging to a "component" of $\mathcal{X}$ are all isomorphic; if $\mathcal{X}$ is "connected" then, as in topology, we can define the fundamental group $\pi(\mathcal{X})$ of a category. From C2 it follows that a covariant functor $\Phi: \mathcal{X} \to \mathcal{L}$ on "connected" categories induces a homomorphism $\pi(\mathcal{X}) \to \pi(\mathcal{L})$. It is easy to see that for $\text{Top}_0$, the category of based topological spaces with base-point preserving continuous maps, each word from $(X, x_0)$ to $(X, x_0)$ becomes equivalent to $(X, x_0) \leftarrow (x_0, x_0) \to (X, x_0)$ and therefore that $\pi(\text{Top}_0) = 0$. Similarly, it can be shown that $\pi(\text{Top}) = 0$.

We now show

**A.5. Proposition.** There exists a category $\mathcal{X}$ and a class $\mathfrak{M}$ such that $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$ and $E(\mathfrak{M}) \neq E(\mathfrak{M})$, where $\mathfrak{M}$ is the maximal stable subclass in $\mathfrak{M}$.

**Proof.** Consider the following diagram

$$
\begin{array}{cccc}
*_{A} & \xrightarrow{h'} S_{0}^2 & \xrightarrow{\alpha} S_{1}^2 & \xrightarrow{h} *_{B} \\
\beta & & & \\
\end{array}
$$

where $S_{0}^2$, $S_{1}^2$ are two distinct 2-spheres, $*_{A}$, $*_{B}$ are distinct points, $h'(*)_{A} =$ north pole of $S_{0}^2$, $\alpha$ is the rotation about the $z$-axis with angle $\pi$, and $\beta$ is the rotation about the same axis with angle $\pi/u$, $u$ an irrational. Let $\mathcal{X}$ be the category with these four distinct objects, and with morphisms generated by $h'$, $\alpha$, $\beta$, $h$.

Let $\mathfrak{M} = \{h, h'\}$; then it is immediate that $\mathfrak{M}$ = all morphisms in $\mathcal{X}$ and $E(\mathfrak{M}) = \mathfrak{M} - \mathfrak{M}$. Moreover

\(^{(15)}\) The set of all morphisms $A \to B$ in a category $\mathcal{L}$ will be denoted by $\mathcal{L}(A, B)$; if $\mathcal{L} = \mathcal{X}/\mathfrak{M}$, this set is denoted by hom $(A, B)$. \[\text{License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use}\]
(a) \( E(\mathcal{H}) \neq E(\mathcal{E}) \). Let \( \mathcal{L} \) be the smallest subcategory of \( \text{Top} \) containing \( \mathcal{H} \) and the inverses \( \alpha, \beta \) of \( \alpha, \beta \). Let \( \lambda : \mathcal{H} \to \mathcal{L} \) be the inclusion functor. Since \( \lambda \) sends every member of \( E(\mathcal{E}) \) to an isomorphism in \( \mathcal{L} \), there exists (A.4) a functor \( \Delta : \mathcal{H}/E(\mathcal{E}) \to \mathcal{L} \) such that \( \Delta \circ \eta = \lambda \). Now, \( \alpha \notin E(\mathcal{E}) \): for if \( \eta(\alpha)\eta(g) = 1 \) for some \( g \) in \( \mathcal{H} \), then \( \lambda(\alpha \circ g) = 1 \) in \( \mathcal{L} \), i.e., \( \alpha \circ g = 1 \) in \( \mathcal{L} \); however \( \alpha \) does not have a right inverse in \( \mathcal{H} \), so such a \( g \) cannot exist in \( \mathcal{H} \). Thus, \( E(\mathcal{H}) \neq E(\mathcal{E}) \). Moreover, since \( \alpha \in E(\mathcal{E}) \subset [E(\mathcal{E})]^\sim \), this shows also that \( E(\mathcal{E}) \) is not stable; furthermore, \( \mathcal{H} = \emptyset \), since \( h \) and \( h' \) do not have inverses in \( \mathcal{H} \), so we also have \( E(\mathcal{E}) \neq E(\mathcal{H}) \).

Bibliography


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