ON THE COEFFICIENT PROBLEM FOR BOUNDED UNIVALENT FUNCTIONS(1)

BY

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1. **Introduction.** Let $S(b_1)$ be the class of functions

$$f(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu \tag{1}$$

which are analytic and univalent in the unit disc $|z|<1$ and bounded:

$$|f(z)| < 1. \tag{2}$$

We assume the normalization $b_1>0$ and keep b_1 fixed. We can choose b_1 freely in the interval

$$0 < b_1 \leqq 1. \tag{3}$$

In this paper we shall develop a technique for estimating the coefficients b_ν $(\nu>1)$ of the power series (1) of the class $S(b_1)$ which is particularly efficient in the case where b_1 is close to 1.

The method is based on a generalization of an inequality, due to Nehari, which represents a necessary and sufficient condition for a function (1) to belong to the class $S(b_1)$ [4], [10]. Nehari's inequality is a counterpart of Grunsky's necessary and sufficient condition for an analytic function to be univalent [2]. It is of methodological interest to derive the generalized Nehari inequality in a manner which shows its close relation to the original Grunsky inequality. We will prove it in two different ways. In §2 we will establish it by the variational method, while in §3 we shall obtain it by the Grunsky method, using Faber polynomials and their generalizations. In §4 the results proved will then be applied to the coefficient problem for the class $S(b_1)$. We shall treat, in particular, the cases of the coefficients b_ν with $\nu=3$ and 5. The extension of the considerations for higher indices can be achieved by analogous arguments.

To define the terms of our inequality, consider the power series

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} A_{mn} z^m \zeta^n, \tag{4}$$

and

$$-\log (1-f(z)\overline{f(\zeta)}) = \sum_{m,n=1}^{\infty} B_{mn} z^m \bar{\zeta}^n \tag{5}$$

in the bicylinder $|z|<1$, $|\zeta|<1$. The fact that $f(z) \in S(b_1)$ guarantees evidently that

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these power series converge in that bicylinder and, conversely, the fact that these power series converge there and that $A_{00}=\log b_1$ guarantees that $f(z)\in S(b_1)$. The numbers A_{mn} and B_{mn} are polynomials of the b_ν and the $\bar{b}_\nu$; they form a symmetric and hermitian matrix, respectively.

The inequality which shall be established is as follows: Let $x_0, x_1, \ldots, x_N$ be an arbitrary vector of complex numbers with the only restriction $x_0=$real. Then we have the inequality

$$\operatorname{Re}\left\{\sum_{m,n=0}^{N} A_{mn}x_m x_n\right\}+\sum_{m,n=1}^{N} B_{mn}x_m\bar{x}_n \leqq \sum_{m=1}^{N}\frac{|x_m|^2}{m}. \tag{6}$$

The original Nehari inequality is obtained for the case $x_0=0$. However, the fact that the additional real parameter x_0 is at our disposal in (6) will be of decisive significance in our applications.

It is clear that the inequalities (6) in their totality guarantee the convergence of the power series (4) and (5) and are thus sufficient to ensure the univalence and boundedness of the function $f(z)$ to which the matrices $((A_{mn}))$ and $((B_{mn}))$ belong.

2. **Proof by method of variation.** We choose an arbitrary but fixed N-vector $x_0, x_1, \ldots, x_N$ with x_0 real and consider the functional

$$\Phi[f] = \operatorname{Re}\left\{\sum_{m,n=0}^{N} A_{mn}x_m x_n\right\}+\sum_{m,n=1}^{N} B_{mn}x_m\bar{x}_n \tag{7}$$

for all bounded univalent functions in the unit disc. We observe that $x_0^2A_{00}=x_0^2\log b_1$ is negative; hence, in view of the normality of all bounded univalent functions, we can assert that there exists a function of this type for which $\Phi[f]$ takes its maximum value.

We shall characterize such an extremum function by the calculus of variations, determine its matrices $((A_{mn}))$ and $((B_{mn}))$ and thus calculate explicitly the maximum value of $\Phi[f]$. The result will be the estimate (6).

Let us consider the extremum function $f(z)$ for which $\Phi[f]$ attains the maximum. We denote the range of the function $f(z)$ by D. If w_0 lies outside or on the boundary of D and if $|w_0|<1$, there exist univalent functions $f^*(z)$ of the form

$$f^*(z) = f(z)+a\rho^2\frac{f(z)}{[f(z)-w_0]w_0}-\bar{a}\rho^2\frac{f(z)^2}{[1-\bar{w}_0f(z)]\bar{w}_0}+o(\rho^2) \tag{8}$$

with $|a|\leqq 1$ and $0<\rho$ arbitrarily small, which are also univalent and bounded. The estimate $o(\rho^2)$ of the remainder term is uniform in each compact subdomain of the unit disc [6], [9].

Let us denote by $((A^*_{mn}))$ and $((B^*_{mn}))$ the corresponding matrices for the varied functions (8). We easily calculate the asymptotic development

$$\begin{aligned}\log\frac{f^*(z)-f^*(\zeta)}{z-\zeta} &= \log\frac{f(z)-f(\zeta)}{z-\zeta}-\frac{a\rho^2}{[f(z)-w_0][f(\zeta)-w_0]}\\ &\quad -\frac{\bar{a}\rho^2[f(z)+f(\zeta)-\bar{w}_0f(z)f(\zeta)]}{[1-\bar{w}_0f(z)][1-\bar{w}_0f(\zeta)]\bar{w}_0}+o(\rho^2).\end{aligned} \tag{9}$$

To obtain the power series development of this expression, we introduce polynomials $P_n(t)$ of degree $n-1$ by means of the generating function

$$\frac{1}{1-tf(z)} = 1+\sum_{n=1}^{\infty} tP_n(t)z^n, \qquad \frac{f(z)}{1-tf(z)} = \sum_{n=1}^{\infty} P_n(t)z^n. \tag{10}$$

These polynomials are closely related to the Faber polynomials of the function $f(z)$ [5], and their coefficients are easily expressed in terms of the b_ν.

By comparing the coefficients of $z^m\zeta^n$ on both sides, we derive from (9) and (10)

$$A^*_{mn} = A_{mn}-\frac{a\rho^2}{w_0^4} P_m\left(\frac{1}{w_0}\right)P_n\left(\frac{1}{w_0}\right)-\bar{a}\rho^2 P_m(\bar{w}_0)P_n(\bar{w}_0)+o(\rho^2) \quad \text{for } m, n > 0, \tag{11}$$

$$A^*_{m0} = A_{m0}-\frac{a\rho^2}{w_0^3} P_m\left(\frac{1}{w_0}\right)-\frac{\bar{a}\rho^2}{\bar{w}_0} P_m(\bar{w}_0)+o(\rho^2) \quad \text{for } m > 0, \tag{11'}$$

$$A^*_{00} = A_{00}-a\rho^2/w_0^2+o(\rho^2). \tag{11''}$$

These formulas describe the variation of the matrix $((A_{mn}))$ under a change (8) of the corresponding function $f(z)$.

We consider next the asympotic identity

$$\begin{aligned} -\log(1-f^*(z)\overline{f^*(\zeta)}) &= -\log(1-f(z)\overline{f(\zeta)})+\frac{a\rho^2}{w_0}\frac{f(z)\overline{f(\zeta)}}{(f(z)-w_0)(1-w_0\overline{f(\zeta)})} \\ &\quad +\frac{\bar{a}\rho^2 f(z)\overline{f(\zeta)}}{\bar{w}_0(1-\bar{w}_0 f(z))(\overline{f(\zeta)}-\bar{w}_0)}+o(\rho^2) \end{aligned} \tag{12}$$

and derive from it the coefficient relations

$$B^*_{mn} = B_{mn}-\frac{a\rho^2}{w_0^2} P_m\left(\frac{1}{w_0}\right)\overline{P_n(\bar{w}_0)}-\frac{\bar{a}\rho^2}{\bar{w}_0^2} P_m(\bar{w}_0)\overline{P_n\left(\frac{1}{w_0}\right)}+o(\rho^2), \qquad m, n > 0. \tag{13}$$

We are now in a position to determine the variation of the functional $\Phi[f]$ defined in (7) under the variation (8) of the argument function. An easy rearrangement of terms and the use of (7), (11) and (13) leads to

$$\Phi[f^*] = \Phi[f]-\operatorname{Re}\left\{\frac{a\rho^2}{w_0^2} M(w_0)^2\right\}+o(\rho^2) \tag{14}$$

with

$$M(w) = x_0+\frac{1}{w}\sum_{m=1}^{N} x_m P_m\left(\frac{1}{w}\right)+w\sum_{m=1}^{N} \bar{x}_m\overline{P_m(\bar{w})}. \tag{15}$$

The expression $M(w)$ is a rational function of the complex variable w with a pole of order N at $w=0$ and at $w=\infty$.

Since we assume that $\Phi[f]$ is the maximum value of the functional for all bounded univalent functions, we are led to the inequality

$$\operatorname{Re}\{(a\rho^2/w_0^2)M(w_0)^2\}+o(\rho^2) \geqq 0 \tag{16}$$

for all admissible variations (8). The fundamental lemma of the calculus of variations for univalent functions leads then to the conclusion that the complement of the range of $f(z)$ inside the unit disc consists of analytic arcs with the differential equation

$$(w'(\tau)^2/w(\tau)^2)M(w(\tau))^2 \leqq 0 \tag{17}$$

if $w(\tau)$ is a parametric representation for the arc considered, and $w'(\tau)$ has therefore the direction of the tangent vector to the arc at the point $w(\tau)$ [6].

The remarkable fact about the functional $\Phi[f]$ is that its Fréchet derivative is a perfect square and that the differential equation (17) can be simplified to the form

$$(w'(\tau)/w(\tau))M(w(\tau)) = \text{imaginary}. \tag{17'}$$

We observe next that for $|w|=1$ we have the equation $M(w)=\text{real}$ and that the unit circumference satisfies the differential equation

$$w'(\tau)/w(\tau) = \text{imaginary}. \tag{18}$$

Hence the equation (17′) is not only satisfied by the boundary arcs of the range of $f(z)$, which lie inside the unit disc, but also by the boundary arcs on the unit circumference.

We can use as the parametric representation of the various boundary arcs the same formula

$$w(\tau) = f(e^{i\tau}), \qquad w'(\tau) = ie^{i\tau}f'(e^{i\tau}) \tag{19}$$

and find that the function

$$\begin{aligned} H(z) &= \frac{zf'(z)}{f(z)} M[f(z)] \\ &= \frac{zf'(z)}{f(z)} \left[x_0 + \frac{1}{f(z)} \sum_{m=1}^{N} x_m P_m\left(\frac{1}{f(z)}\right) + f(z) \sum_{m=1}^{N} \bar{x}_m \overline{P_m[\overline{f(z)}]}\right] \end{aligned} \tag{20}$$

takes real values for $|z|=1$. It is regular analytic for $|z|<1$, except for a pole of order N at the origin. Hence $H(z)$ can be extended into the entire complex plane, having a pole of order N also at infinity.

To determine $H(z)$ explicitly, we derive from (4) by differentiation the identity

$$\frac{zf'(z)}{f(z)-f(\zeta)} = \frac{z}{z-\zeta} + \sum_{m,n=0}^{\infty} mA_{mn}z^m\zeta^n \tag{21}$$

and comparing equal powers of ζ on both sides of (21), we obtain by use of definition (10)

$$\frac{zf'(z)}{f(z)^2} P_n\left[\frac{1}{f(z)}\right] = z^{-n} + \sum_{m=1}^{\infty} mA_{mn}z^m, \quad \text{for } n \geqq 1 \tag{22}$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \sum_{m=1}^{\infty} mA_{m0}z^m, \quad \text{for } n = 0. \tag{22'}$$

Similarly, we derive from (5) the identity

$$\frac{zf'(z)\overline{f(\zeta)}}{1-f(z)\overline{f(\zeta)}} = \sum_{m,n=1}^{\infty} mB_{mn}z^m\bar{\zeta}^n \tag{23}$$

and in the same fashion the equations

$$zf'(z)\overline{P_n[\overline{f(z)}]} = \sum_{m=1}^{\infty} mB_{mn}z^m, \quad \text{for } n \geqq 1. \tag{24}$$

Combining all these results, we arrive at the following series development for $H(z)$ near the origin:

$$H(z) = \sum_{m=1}^{N} \frac{x_m}{z^m} + x_0 + \sum_{k=1}^{\infty} kC_k z^k \tag{25}$$

with

$$C_k = \sum_{m=1}^{N} (x_m A_{km} + \bar{x}_m B_{km}) + x_0 A_{k0}, \qquad k \geqq 1. \tag{26}$$

In view of the reality of $H(z)$ for $|z|=1$ and by the Schwarz reflection principle we find

$$H(z) = \sum_{m=1}^{N} \frac{x_m}{z^m} + x_0 + \sum_{m=1}^{N} \bar{x}_m z^m. \tag{27}$$

Observe here the decisive role of our assumption $x_0 = \text{real}$. We compare (25) with (27) and arrive at the equalities

$$\begin{aligned} \frac{1}{k}\bar{x}_k &= C_k = x_0 A_{k0} + \sum_{m=1}^{N} (x_m A_{km} + \bar{x}_m B_{km}), \qquad 1 \leqq k \leqq N, \\ C_k &= 0, \qquad k \geqq N+1. \end{aligned} \tag{28}$$

Finally we introduce the Faber polynomials $F_k(t)$ by means of the generating function [5]

$$-\log(1 - tf(z)) = \sum_{k=1}^{\infty} \frac{1}{k} F_k(t) z^k. \tag{29}$$

Evidently, $F_k(t)$ is a polynomial in t of degree k. Differentiating the identity (29) with respect to t yields

$$\frac{f(z)}{1 - tf(z)} = \sum_{k=1}^{\infty} \frac{1}{k} F_k'(t) z^k, \tag{30}$$

and comparison of (30) with (10) leads to the relation

$$\frac{1}{k} F_k'(t) = P_k(t). \tag{31}$$

Since (29) clearly implies

$$F_k(0) = 0, \tag{32}$$

we can obtain $F_k(t)$ by integration of $P_k(t)$.

We utilize this relation by deriving from (20) the identity

$$(33)\qquad \int H(z)\frac{dz}{z} = x_0 \log f(z) - \sum_{m=1}^{N}\frac{x_m}{m}F_m\left(\frac{1}{f(z)}\right) + \sum_{m=1}^{N}\frac{\bar{x}_m}{m}\overline{F_m(\overline{f(z)})} + \text{const.}$$

We insert on the left the known value (27) of $H(z)$ and find

$$(34)\qquad x_0 \log\frac{f(z)}{z} + \sum_{m=1}^{N}\frac{x_m}{m}\left(\frac{1}{z^m} - F_m\left(\frac{1}{f(z)}\right)\right) + \sum_{m=1}^{N}\frac{\bar{x}_m}{m}\left(\overline{F_m(\overline{f(z)})} - z^m\right) = \text{const.}$$

Since the range of $f(z)$ is a slit domain in the unit disc, there exist surely points z_0 on the unit circumference which are mapped into the unit circumference. We have then

$$(35)\qquad \overline{f(z_0)} = 1/f(z_0), \qquad \bar{z}_0 = 1/z_0.$$

Hence (34) is pure imaginary for such points z_0, and it follows that the constant on the right hand of (34) is an imaginary number.

We will utilize this fact for $z=0$. For this purpose we will have to determine

$$(36)\qquad \lim_{z\to 0}\left[\frac{1}{z^m} - F_m\left(\frac{1}{f(z)}\right)\right] = l_m.$$

We start with the identity (4) in the form

$$(37)\qquad \log\frac{f(z)}{z} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{\zeta^m}{z^m} - \sum_{m=1}^{\infty}\frac{1}{m}F_m\left(\frac{1}{f(z)}\right)\zeta^m = \sum_{m,n=0}^{\infty}A_{mn}z^m\zeta^n,$$

which follows immediately from the definition (29). Letting $z\to 0$, we obtain by (36)

$$(38)\qquad \log b_1 + \sum_{m=1}^{\infty}\frac{1}{m}l_m\zeta^m = \sum_{n=0}^{\infty}A_{0n}\zeta^n,$$

whence

$$(39)\qquad l_m = mA_{0m}.$$

Thus we find from (34) the equation

$$(40)\qquad C_0 = x_0\log b_1 + \sum_{m=1}^{N}x_mA_{0m} = \text{imaginary}.$$

Now we are ready to derive the estimate (6) by explicitly calculating the maximum value of $\Phi[f]$. Indeed, we multiply the kth equation (28) by x_k, multiply (40) by x_0 and add all results. In view of $A_{00}=\log b_1$ we find

$$(41)\qquad \sum_{k=1}^{N}\frac{1}{k}|x_k|^2 = \sum_{m,n=0}^{\infty}x_mx_nA_{mn} + \sum_{m,n=1}^{\infty}x_m\bar{x}_nB_{mn} + \text{imaginary term.}$$

Hence

$$(42)\qquad \Phi[f] = \operatorname{Re}\left\{\sum_{m,n=0}^{\infty}x_mx_nA_{mn}\right\} + \sum_{m,n=1}^{\infty}x_m\bar{x}_nB_{mn} = \sum_{k=1}^{N}\frac{1}{k}|x_k|^2.$$

Since this is the value of the functional in the maximum case, the inequality (6) has been established.

According to (28) and (40) we have in the maximum case

$$(42')\qquad \begin{aligned} C_k &= \frac{1}{k}\bar{x}_k, \qquad k = 1,\dots,N,\\ \operatorname{Re}\{C_0\} &= 0,\\ C_k &= 0, \qquad k = N+1,\dots. \end{aligned}$$

The preceding proof is, of course, analogous to the proof given for the Grunsky inequalities in [5].

3. **An elementary proof for the generalized Nehari condition.** It is instructive to consider an alternative proof for the inequality (6) which is patterned after the original Grunsky method and based on the ideas of the Bieberbach area theorem. This method can also be applied to various other interesting subclasses of univalent functions [3]. Let D_r be the subdomain of the unit disc in the w-plane which corresponds to the disc $|z|<r<1$ under the mapping $w=f(z)$ where $f(z)\in S(b_1)$. We denote by Δ_r the complement of D_r in $|w|<1$ and by γ_r the analytic curve which separates D_r from Δ_r.

We consider an arbitrary analytic function $q(w)$ in Δ_r which may even be multi-valued in that ring domain, but we suppose that it possesses a single-valued real part in Δ_r. We have the obvious inequality

$$(43)\qquad 0 \leqq \iint_{\Delta_r} |q'(w)|^2\, d\tau,$$

where $d\tau$ is the area element in the w-plane. By the complex version of Green's identity we then have

$$(44)\qquad 0 \leqq \frac{1}{i}\int_{\partial\Delta_r} \operatorname{Re}\{q(w)\}\, dq(w),$$

and since the boundary of Δ_r consists of the unit circumference K_1 and the curve γ_r, we have the inequality

$$(45)\qquad \frac{1}{i}\int_{\gamma_r} \operatorname{Re}\{q(w)\}\, dq(w) \leqq \frac{1}{i}\int_{K_1} \operatorname{Re}\{q(w)\}\, dq(w).$$

We shall now choose the arbitrary function $q(w)$ in such a way that it becomes pure imaginary on K_1; this is easily achieved by setting up

$$(46)\qquad q(w) = x_0 \log w - \sum_{n=1}^{N}\left[\frac{x_n}{n} F_n\left(\frac{1}{w}\right) - \frac{\bar{x}_n}{n}\overline{F_n(\bar{w})}\right]$$

with x_0 real and arbitrary polynomials $F_n(t)$. We thus arrive at the estimate

$$(47)\qquad \frac{1}{i}\int_{\gamma_r} \operatorname{Re}\{q(w)\}\, dq(w) \leqq 0.$$

We select, in particular, the polynomials $F_n(t)$ to be the Faber polynomials of the function $f(z)$ considered and defined by the identity (29).

We compare the coefficients of ζ^m on both sides of (37) and find the power series development near the origin

$$F_n\left(\frac{1}{f(z)}\right) = \frac{1}{z^n} - n \sum_{k=0}^{\infty} A_{kn} z^k \quad \text{for } n \geqq 1 \tag{48}$$

and

$$\log \frac{f(z)}{z} = \sum_{k=0}^{\infty} A_{k0} z^k \quad \text{for } n = 0. \tag{49}$$

In a similar way, we can also derive the Taylor series for $F_n(\overline{f(z)})$. Indeed, using the definitions (5) and (29), we find

$$\sum_{n=1}^{\infty} \frac{1}{n} F_n(\overline{f(z)})\zeta^n = \sum_{m,n=1}^{\infty} \bar{B}_{mn} \bar{z}^m \zeta^n \tag{50}$$

and comparing equal powers of ζ on both sides,

$$F_n(\overline{f(z)}) = n \sum_{k=1}^{\infty} \bar{B}_{kn} \bar{z}^k. \tag{51}$$

We combine the series developments (48), (49) and (51) to calculate the series development for $q(f(z))$ near $z=0$:

$$q(f(z)) = x_0 \log z + \sum_{n=1}^{N} \frac{x_n}{nz^n} + \sum_{k=0}^{\infty} C_k z^k \tag{52}$$

with

$$C_k = x_0 A_{k0} + \sum_{n=1}^{N} (x_n A_{kn} + \bar{x}_n B_{kn}), \qquad k \geqq 0. \tag{53}$$

We split $q(f(z))$ in the form

$$q(f(z)) = x_0 \log z + Q(z) \tag{52'}$$

$$Q(z) = \sum_{n=1}^{N} \frac{x_n}{nz^n} + \sum_{k=0}^{\infty} C_k z^k \tag{52''}$$

such that $Q(z)$ is single valued in $|z| < 1$. We can then express the integral estimate (47) in the form

$$\frac{1}{i} \oint_{|z|=r} \operatorname{Re}\{x_0 \log z + Q(z)\}\left(\frac{x_0}{z} + Q'(z)\right) dz < 0 \tag{54}$$

and putting $z = re^{i\varphi}$, we arrive at

$$2\pi x_0^2 \log r + x_0 \int_0^{2\pi} \operatorname{Re}\{Q(re^{i\varphi})\}\, d\varphi + \frac{1}{2} \int_0^{2\pi} \overline{Q(re^{i\varphi})} Q'(re^{i\varphi}) re^{i\varphi}\, d\varphi < 0. \tag{55}$$

We made here use of the fact that $Q(z)$ is single valued.

In view of (52″) and (53) we have

$$\int_0^{2\pi} \operatorname{Re}\{Q(re^{i\varphi})\}\,d\varphi = 2\pi \operatorname{Re}\{C_0\} = 2\pi \operatorname{Re}\left\{x_0A_{00}+\sum_{n=1}^{N} x_nA_{n0}\right\}. \tag{56}$$

We use the usual Fourier relations and the series development (52″) of $Q(z)$ to calculate

$$\frac{1}{2}\int_0^{2\pi} \overline{Q(re^{i\varphi})}Q'(re^{i\varphi})re^{i\varphi}\,d\varphi = \pi\left[-\sum_{n=1}^{N}\frac{1}{n}|x_n|^2r^{-2n}+\sum_{k=1}^{\infty}k|C_k|^2r^{2k}\right]. \tag{57}$$

Collecting terms, we transform the inequality (55) into

$$2x_0^2\log r+2x_0\operatorname{Re}\{C_0\}+\sum_{k=1}^{\infty}k|C_k|^2r^{2k} < \sum_{n=1}^{N}\frac{1}{n}|x_n|^2r^{-2n}. \tag{58}$$

Since this estimate is valid for all values $r<1$, we can pass to the limit $r=1$ and arrive finally at the result

$$2x_0\operatorname{Re}\{C_0\}+\sum_{k=1}^{\infty}k|C_k|^2 \leqq \sum_{n=1}^{N}\frac{1}{n}|x_n|^2. \tag{59}$$

Given a fixed function $f(z)\in S(b_1)$, let us consider the function

$$M(x) = \operatorname{Re}\left\{\sum_{m=0}^{N}x_mC_m\right\} = \operatorname{Re}\left\{\sum_{m,n=0}^{N}A_{mn}x_mx_n\right\}+\sum_{m,n=1}^{N}B_{mn}x_m\bar{x}_n \tag{60}$$

of the vector $(x_0, x_1, \ldots, x_n\}$ and ask for the maximum of this function under the restrictions

$$x_0 = \text{real},\qquad \sum_{n=1}^{N}\frac{1}{n}|x_n|^2 = S. \tag{61}$$

Such a maximum value must exist since we know that $\operatorname{Re}\{A_{00}\}=\log b_1<0$. Using the method of Lagrange multipliers, we find the following conditions for the maximum vector

$$\operatorname{Re}\{C_0\} = 0 \qquad C_k = \lambda(1/k)\bar{x}_k, \qquad k>0 \tag{62}$$

with a fixed factor λ.

Thus, in the maximum case we can assert instead of (59) the estimate

$$\sum_{k=1}^{\infty}k|C_k|^2 \leqq \sum_{n=1}^{N}\frac{1}{n}|x_n|^2 = S. \tag{63}$$

Hence by (60) and by use of the Schwarz inequality

$$M(x)^2 \leqq \left|\sum_{m=1}^{N}x_mC_m\right|^2 \leqq \sum_{k=1}^{N}k|C_k|^2\sum_{n=1}^{N}\frac{1}{n}|x_n|^2 \leqq S^2. \tag{64}$$

Therefore we proved

$$M(x) = \operatorname{Re}\left\{\sum_{m,n=0}^{N}A_{mn}x_mx_n\right\}+\sum_{m,n=1}^{N}B_{mn}x_m\bar{x}_n \leqq \sum_{n=1}^{N}\frac{1}{n}|x_n|^2. \tag{65}$$

Since this is true in the maximum case, it is a fortiori valid for all other x-vectors and the inequality (6) is proved in general.

We may next raise the question, for which functions $f(z) \in S(b_1)$ can equality be achieved in (6). The extremum of $M(x)$ can only be achieved for such functions for which all signs in (64) are equality signs. We then have

$$\operatorname{Re}\left\{\sum_{m=0}^{N} x_m C_m\right\} = \left|\sum_{m=1}^{N} x_m C_m\right| \tag{66}$$

and in order that equality not be lost under application of the Schwarz inequality, the condition

$$C_k = x_0 A_{k0} + \sum_{n=1}^{N} (x_n A_{kn} + \bar{x}_n B_{kn}) = 0 \quad \text{for } k > N \tag{67}$$

is necessary. From (62) we already know the conditions for the C_k with $k \leqq N$. In view of (62), condition (66) becomes

$$\operatorname{Re}\{\lambda\} \sum_{k=1}^{N} \frac{1}{k} |x_k|^2 = |\lambda| \sum_{k=1}^{N} \frac{1}{k} |x_k|^2, \tag{68}$$

which shows that λ must be positive. In order that equality hold in the estimate (63), we have finally to demand

$$\lambda = 1. \tag{69}$$

Hence, for a given vector $(x_0, x_1, \ldots, x_N)$, the extremum function $f(z)$ must satisfy the conditions

$$C_k = x_0 A_{k0} + \sum_{n=1}^{N} (x_n A_{kn} + \bar{x}_n B_{kn}) = \begin{cases} (1/k)\bar{x}_k & \text{for } 0 < k \leqq N \\ 0 & \text{for } k > N \end{cases} \tag{70}$$

and

$$\operatorname{Re}\{C_0\} = \operatorname{Re}\left\{x_0 A_{00} + \sum_{n=1}^{N} x_n A_{0n}\right\} = 0. \tag{70'}$$

Thus we have again the extremum conditions (42').

In §2 we proved the formal identity

$$\begin{aligned} H(z) &= \frac{zf'(z)}{f(z)}\left[x_0 + \frac{1}{f(z)} \sum_{m=1}^{N} x_m P_m\left(\frac{1}{f(z)}\right) + f(z) \sum_{m=1}^{N} \bar{x}_m \overline{P_m[\overline{f(z)}]}\right] \\ &= \sum_{m=1}^{N} \frac{x_m}{z^m} + x_0 + \sum_{k=1}^{\infty} k C_k z^k. \end{aligned} \tag{71}$$

In view of the conditions (70) we can then assert that $f(z)$ satisfies the differential equation

$$\frac{zf'(z)}{f(z)}\left[x_0 + \frac{1}{f(z)} \sum_{m=1}^{N} x_m P_m\left(\frac{1}{f(z)}\right) + f(z) \sum_{m=1}^{N} \bar{x}_m \overline{P_m(\overline{f(z)})}\right] = \sum_{m=1}^{N}\left(\frac{x_m}{z^m} + \bar{x}_m z^m\right) + x_0. \tag{72}$$

This is the precise condition for the extremum function to which we were led by the variational method. The complete equivalence of the two derivations is thus established.

THEOREM. *For all the $S(b_1)$-functions, with free complex $x_1, \ldots, x_N$ and a free real x_0,*

$$\operatorname{Re}\left\{\sum_{m,n=0}^{N}(A_{mn}x_m x_n + B_{mn}x_m\bar{x}_n)\right\} \leqq \sum_{n=1}^{N}\frac{|x_n|^2}{n}$$

holds. Equality is true if and only if

$$\operatorname{Re}\left\{\sum_{m=0}^{N}A_{m0}x_m\right\} = 0,$$

$$x_0A_{k0}+\sum_{n=1}^{N}(x_nA_{kn}+\bar{x}_nB_{kn}) = \frac{\bar{x}_k}{k} \quad \textit{for } 0 < k \leqq N,$$

$$x_0A_{k0}+\sum_{n=1}^{N}(x_nA_{kn}+\bar{x}_nB_{kn}) = 0 \quad \textit{for } k > N.$$

4. **Application to the third coefficient of** $f(z)$. We illustrate the significance of the estimate (6) by simply taking $N=1$ and writing

$$\operatorname{Re}\{A_{00}x_0^2+2A_{01}x_0x_1+A_{11}x_1^2\} \leqq (1-B_{11})|x_1|^2. \tag{73}$$

A simple calculation shows that

$$\begin{aligned} A_{00} &= \log b_1, & A_{01} &= a_2,\\ A_{11} &= a_3-a_2^2, & B_{11} &= b_1^2, \end{aligned} \tag{74}$$

where we introduced the useful notation

$$a_\nu = b_\nu/b_1, \qquad \nu = 2, 3, \ldots. \tag{75}$$

We assume that $a_3>0$; this is no loss of generality since $e^{-\alpha i}f(e^{\alpha i}z)$ lies in the same class $S(b_1)$ and has the corresponding coefficient $e^{2\alpha i}a_3$. Choosing further $x_1=1$, one obtains

$$x_0^2\log b_1+\operatorname{Re}\{a_3-a_2^2+2x_0a_2\} \leqq 1-b_1^2. \tag{76}$$

We choose x_0 so that the left side becomes as large as possible, that is,

$$x_0\log b_1+\operatorname{Re}\{a_2\} = 0. \tag{77}$$

This is precisely the condition $\operatorname{Re}\{C_0\}=0$ discussed in the general case. Now (76) becomes

$$a_3-\operatorname{Re}\{a_2^2\}+\frac{\operatorname{Re}\{a_2\}^2}{\log(1/b_1)} \leqq 1-b_1^2, \tag{78}$$

which can also be expressed in the form

$$a_3 \leqq (1-1/\log b_1^{-1})\operatorname{Re}\{a_2\}^2+(1-b_1^2)-\operatorname{Im}\{a_2\}^2. \tag{79}$$

We can then assert that for

$$e^{-1} \leqq b_1 < 1 \tag{80}$$

we have the estimate

$$a_3 \leqq 1-b_1^2. \tag{81}$$

This is a well-known sharp inequality which has been obtained earlier [11] by detailed study of the Loewner representation for the coefficients of a function $f(z) \in S(b_1)$ or by integrating the differential equation for the extremal function, which was found by variational methods. The new derivation is of considerable interest since it is based on a Grunsky-type inequality which can be derived by elementary methods. Moreover, it can be applied in an analogous way for the case of coefficients with higher odd indices and yields corresponding results.

5. **An estimate for a_5.** We let now $N=2$ and apply the inequality (6) to a function $f(z) \in S(b_1)$ for which we assume $a_5>0$. We also choose $x_2=1$; hence we have for x_0 real and x_1 arbitrary

$$\begin{aligned} \operatorname{Re}\{A_{00}x_0^2+A_{11}x_1^2+A_{22}+2A_{01}x_0x_1+2A_{02}x_0+2A_{12}x_1\} \\ \leqq (1-B_{11})|x_1|^2+(\tfrac{1}{2}-B_{22})-2\operatorname{Re}\{x_1B_{12}\}. \end{aligned} \tag{82}$$

We have

$$\begin{gathered} A_{00} = \log b_1, \qquad A_{01} = a_2, \qquad A_{02} = a_3-a_2^2/2, \\ A_{11} = a_3-a_2^2, \qquad A_{12} = a_4-2a_2a_3+a_2^3, \\ A_{22} = a_5-2a_2a_4-\tfrac{3}{2}a_3^2+4a_2^2a_3-\tfrac{3}{2}a_2^4, \end{gathered} \tag{83}$$

and

$$\begin{gathered} B_{11} = b_1^2, \qquad B_{12} = b_1^2\bar{a}_2, \\ B_{22} = b_1^2|a_2|^2+\tfrac{1}{2}b_1^4. \end{gathered} \tag{84}$$

If we insert these expressions in (82), we obtain a rather involved coefficient inequality. It simplifies considerably for the particular choice $x_1=a_2$ and reduces to

$$\begin{aligned} a_5-\tfrac{1}{2}(1-b_1^4) \leqq \operatorname{Re}\{\tfrac{3}{2}a_3^2-a_2^2a_3+\tfrac{1}{2}a_2^4\}+(1-b_1^2)|a_2|^2 \\ +\log b_1^{-1}x_0^2-2x_0\operatorname{Re}\{a_3+a_2^2/2\}. \end{aligned} \tag{85}$$

We minimize the right-hand side by choosing

$$x_0 = \frac{\operatorname{Re}\{a_3+a_2^2/2\}}{\log b_1^{-1}} \tag{86}$$

and introducing the notations

$$t = a_3+a_2^2/2 = u+iv, \qquad a_2^2 = U+iV, \tag{87}$$

we arrive after easy rearrangements at the estimate

$$a_5-\frac{1}{2}(1-b_1^4) \leqq \operatorname{Re}\left\{\frac{3}{2}t^2-\frac{5}{2}a_2^2t+\frac{11}{8}a_2^4\right\}+(1-4b_1^2)|a_2|^2-\frac{u^2}{\log b_1^{-1}}, \tag{88}$$

that is, with the notation (87),

$$
\begin{aligned}
(88')\qquad a_5-\frac{1}{2}(1-b_1^4) \leqq \frac{3}{2}(u^2-v^2)-\frac{5}{2}(uU-vV)+\frac{11}{8}(U^2-V^2)\\
+(1-4b_1^2)|a_2|^2-\frac{u^2}{\log b_1^{-1}}.
\end{aligned}
$$

We observe that the contribution of the imaginary parts v and V leads to the quadratic form

$$
(89)\qquad \Delta = \frac{3}{2}v^2-\frac{5}{2}vV+\frac{11}{8}V^2 \geqq 0
$$

which is positive-definite. Hence we may disregard these terms and replace (88′) by the simpler inequality

$$
(90)\qquad a_5-\frac{1}{2}(1-b_1^4) \leqq \frac{3}{2}u^2-\frac{5}{2}uU+\frac{11}{8}U^2+(1-4b_1^2)|a_2|^2-\frac{u^2}{\log b_1^{-1}}.
$$

We restrict ourselves now to the interval

$$
(91)\qquad e^{-2/3} < b_1 < 1
$$

and define the number

$$
(92)\qquad M = [\log b_1^{-1}]^{-1}-\tfrac{3}{2} > 0.
$$

Now (90) assumes the form

$$
(93)\qquad a_5-\frac{1}{2}(1-b_1^4) \leqq -M\left(u+\frac{5}{4M}U\right)^2+\left(\frac{25}{16M}+\frac{11}{8}\right)U^2+(1-4b_1^2)|a_2|^2.
$$

Observe that $U^2 \leqq |a_2|^4$ and hence

$$
(94)\qquad a_5-\frac{1}{2}(1-b_1^4) \leqq |a_2|^2\left[\left(\frac{25}{16M}+\frac{11}{8}\right)|a_2|^2+1-4b_1^2\right].
$$

Finally, we make use of the well-known estimate

$$
(95)\qquad |a_2| \leqq 2(1-b_1)
$$

valid for the class $S(b_1)$ and arrive at

$$
(96)\qquad a_5-\frac{1}{2}(1-b_1^4) \leqq |a_2|^2\left\{\left(\frac{25}{16(\log b_1^{-1})^{-1}-24}+\frac{11}{8}\right)4(1-b_1)^2+1-4b_1^2\right\}.
$$

To guarantee that the right-hand side of (96) is negative, we add the restriction

$$
(97)\qquad e^{-2/3} < e^{-1/2} \leqq b_1 < 1
$$

and find for the factor $N(b_1)$ of $|a_2|^2$ the estimate

$$
(98)\qquad N \leqq \left(\frac{25}{8}+\frac{11}{8}\right)4(1-b_1)^2+1-4b_1^2 \leqq 19-36b_1+14b_1^2
$$

and this expression is negative if

$$
(99)\qquad \frac{3-\sqrt{29/2}}{7} < b_1 < 1.
$$

We have thus proved the

THEOREM. *The fifth coefficient of a function* $f(z) \in S(b_1)$ *satisfies the inequality*

$$|a_5| \leqq \tfrac{1}{2}(1-b_1^4) \tag{100}$$

at least in the interval

$$.75 \leqq b_1 < 1 \tag{101}$$

of the characteristic parameter b_1.

It is easily seen that the extremum functions map the unit disc onto the unit disc in the image plane with four symmetrically located radial slits.

This result is in agreement with the conjecture [1], [12] that for b_1 sufficiently close to 1 the nth coefficient a_n is maximized in the case of a mapping onto the full unit disc with $(n-1)$ equal and symmetric radial slits. This is the case for $n=3$, 4 and 5 [7], [11]. Moreover, Siewiersky has proved a corresponding asymptotic result for the case of odd indexed coefficients using variational methods [8].

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