BOUNDARY BEHAVIOR OF POISSON INTEGRALS ON SYMMETRIC SPACES

BY

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Introduction. Let D be a product of halfplanes,

$$D = \{(z_1, \ldots, z_n) \mid z_j = x_j + iy_j, y_j > 0 \ (1 \leqq j \leqq n)\}.$$

Its distinguished boundary is the space of real points $(x_1, \ldots, x_n)$. An unrestricted nontangential domain at $(x_1^0, \ldots, x_n^0)$ is a set $\{(z_1, \ldots, z_n) \mid |x_j - x_j^0| < \alpha y_j \ (1 \leqq j \leqq n)\}$ for some $\alpha > 0$. A restricted nontangential domain is a subset of the above, satisfying $y_j \leqq M y_k$ $(1 \leqq j, k \leqq n)$ for some $M > 1$. With the aid of these one defines the notion of restricted or unrestricted convergence of a function F on D to a boundary function f. Generalizing the classical Fatou theorem Marcinkiewicz and Zygmund have shown [18, Chapter XVII] that if F is the Poisson integral of f then F converges to f almost everywhere, unrestrictedly if $f \in L^p$ $(1 < p \leqq \infty)$ and restrictedly if $f \in L^1$.

For a product of discs one can make analogous definitions and one has analogous results. Since halfplanes and discs are equivalent under conformal transformations which extend to the boundary almost everywhere, the results about products of discs are equivalent to the results about products of halfplanes.

The results of Marcinkiewicz and Zygmund were generalized in [9] to products of unit balls in complex n-space. In this case the natural generalization of nontangential convergence is no longer nontangential in the geometric sense; in [9] it was given the name *admissible convergence*. One still has the distinction between restricted and unrestricted admissible convergence.

In trying to generalize these notions to arbitrary symmetric spaces of noncompact type several new features appear. First, a space may have more than one boundary (the Furstenberg-Satake boundaries). To each boundary there corresponds a different Poisson integral, and to each one has to define a different notion of restricted (and unrestricted) admissible convergence. Second, since a symmetric space in general has no natural imbedding as a domain in Euclidean space, one has to define restricted and unrestricted admissible convergence in an intrinsic way. Finally, the notion of restricted admissible convergence which seems to be natural depends (for each fixed boundary) on the arbitrary choice of an element H in a certain Weyl chamber. In fact, this arbitrary choice also appears in the case of a

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product of halfplanes or discs as soon as one tries to make the definition intrinsic, i.e., independent of the geometry of the ambient Euclidean space; it is a reflection of the fact that there are infinitely many geodesic lines going from an interior point to a given boundary point. We note, however, that there are cases in which there is no arbitrary choice to be made. Such are the symmetric spaces of rank one (where there is only one boundary) and the irreducible Hermitian symmetric spaces with the Bergman-Šilov boundary.

Several generalizations of Fatou's theorem involving more special notions of convergence have been found recently [5], [7], [16], [17]. The purpose of the present paper is to extend all these results to the case of restricted admissible convergence. In the case of [16] and [17] this does not involve an actual extension, we shall only show that the type of convergence used there is equivalent with our general notion of restricted admissible convergence.

In §1 we transform the Poisson integral from an integral on the boundary to an integral on a nilpotent group; this generalizes the transformation from the unit circle to the real line, and is an idea which has much been used by Harish-Chandra. The results of this section could all be extracted from the work of Karpelevič [6], but we prefer to give a concise, more or less self-contained treatment. In §2 we define restricted admissible convergence, then proceed to give an equivalent characterization of it, more suitable for computations. This again corresponds to transforming nontangential domains in the disc to nontangential domains in a halfplane. Proposition 2.5 generalizes the fact that Fatou's theorem on the disc is equivalent to Fatou's theorem on the halfplane. Theorem 2.6 is the generalization of Fatou's theorem for Poisson integrals of L^∞-functions. In §3 we discuss Hermitian symmetric spaces with the Bergman-Šilov boundary. We show that the notion of convergence used in [9], [16], [17] is equivalent with restricted admissible convergence. So in this case Fatou's theorem holds for L^p-functions $(1<p\leqq\infty)$; in fact, by [9] and by some recent, as yet unpublished results of E. M. Stein and N. J. Weiss, even for L^1-functions. In §4 we make some remarks on unrestricted admissible convergence without being able to settle the question completely. In §5 we discuss symmetric spaces of rank one; in this case there is no difference between restricted and unrestricted convergence. For these spaces A. W. Knapp [7] proved generalized radial convergence for Poisson integrals of L^1-functions; we extend his result to admissible convergence, and we also get a generalization of Privalov's local version of the Fatou theorem.

1. **The Poisson integral.** Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup, $X=G/K$ the corresponding symmetric space. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition. We denote by $\mathfrak{a}$ a maximal abelian subspace in $\mathfrak{p}$, and by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. We consider the root system of the complexification $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$; the real subspace spanned by the roots is then

$\mathfrak{a} \oplus \mathfrak{h}^+$, with $i\mathfrak{h}^+ \subset \mathfrak{h} \cap \mathfrak{k}$. We fix a lexicographic ordering of the roots by choosing a basis in $\mathfrak{a}$ and in $\mathfrak{h}^+$. The restriction of a root to $\mathfrak{a}$ we call a restricted root. We denote by F the set of distinct nonzero restrictions to $\mathfrak{a}$ of the simple roots. $\mathfrak{a}^+$, the positive Weyl chamber in $\mathfrak{a}$, is the set of those points in $\mathfrak{a}$ on which every element of F (and hence also every positive restricted root) is positive.

Following Satake [13] and Moore [12] we fix a subset E of F and denote by $\mathfrak{a}(E)$ the subspace of $\mathfrak{a}$ on which all elements of E vanish. $\mathfrak{g}$ is the direct sum of weight spaces for the adjoint representation of $\mathfrak{a}(E)$ on $\mathfrak{g}$, and the weights are the restrictions of the restricted roots to $\mathfrak{a}(E)$. The weight space for 0 is a reductive subalgebra of $\mathfrak{g}$, we denote its semisimple part by $\mathfrak{g}^E$ and its intersection with $\mathfrak{k}$ by $\mathfrak{m}(E)$. (So $\mathfrak{m}(E)$ is the centralizer of $\mathfrak{a}(E)$ in $\mathfrak{k}$.) Let $\mathfrak{k}^E = \mathfrak{g}^E \cap \mathfrak{k}$; this is a maximal compact subalgebra of $\mathfrak{g}^E$, also contained in $\mathfrak{m}(E)$. The sum of the positive, resp. negative, weight spaces we denote by $\mathfrak{n}(E)$, resp. $\bar{\mathfrak{n}}(E)$; these are nilpotent subalgebras of $\mathfrak{g}$. The sum of the nonnegative weight spaces will be denoted $\mathfrak{b}(E)$. The sum of the positive weights, with multiplicities counted, we denote by $2\rho_E$.

The analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{a}(E)$, $\mathfrak{g}^E$, $\mathfrak{k}^E$, $\mathfrak{m}(E)$, $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$, $\mathfrak{b}(E)$ will be denoted by A, $A(E)$, G^E, K^E, $M_0(E)$, $N(E)$, $\bar{N}(E)$, $B_0(E)$. $M(E)$ will be the centralizer of $a(E)$ in K, and $B(E)$ the normalizer of $n(E)$ in G; their identity components are then $M_0(E)$ and $B_0(E)$. If $E = \varnothing$ we have $A = A(E)$, $G^E = K^E = \{e\}$. In this case we write $M_0, M, N, \bar{N}, B_0, B$ instead of $M_0(E), M(E), \ldots$. Clearly $N, \bar{N}$ are then ordinary Iwasawa subgroups of G. It is known [12, Theorem 3] that $B(E)$ equals the semidirect product $M(E)\cdot A\cdot N$. By the Iwasawa decomposition it follows that $K/M(E)$ and $G/B(E)$ are isomorphic as K-spaces. These spaces, for the possible different choices of E, are the boundaries of X in the sense of Furstenberg and Satake; the case $E = \varnothing$ gives the "maximal boundary" of Furstenberg.

We shall repeatedly use the Iwasawa decomposition $G = KAN$. For $g \in G$, we shall use the notation $g = k(g)(\exp H(g))n$ where $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n \in N$. $k(g)$ and $H(g)$ are uniquely defined continuous functions of g.

1.1. Lemma. *For all* $b \in B(E)$,

$$|\det \mathrm{Ad}(b)| = e^{2\rho_E(H(b))}$$

where Ad *denotes the adjoint representation of* $B(E)$ *on* $\mathfrak{b}(E)$.

Proof. We write $b = man$ with $m \in M(E)$, $a = \exp(H(b))$, $n \in N$. Using that $M(E)$ is compact and N is nilpotent it follows that $|\det \mathrm{Ad}(b)| = |\det \mathrm{Ad}(a)|$. Denoting by $\mathfrak{g}_0$ the weight space for the weight 0 of $\mathfrak{a}(E)$ on $\mathfrak{g}$, we have the decomposition $\mathfrak{b}(E) = \mathfrak{g}_0 + \mathfrak{n}(E)$. Since $\mathfrak{g}_0$ is reductive and contains $\mathfrak{a}$, $\mathrm{Ad}(a)$ acts on it trivially. Therefore $\det \mathrm{Ad}(a)$ is equal to the determinant of the restriction of $\mathrm{Ad}(a)$ to $\mathfrak{n}(E)$ and the assertion follows.

1.2. PROPOSITION. *Let μ_E be the normalized K-invariant measure on $G/B(E)$, μ_E is quasi-invariant under G and its multiplier is given by*

$$P_E(g, \dot{u}) = \frac{d\mu_E(g^{-1}\dot{u})}{d\mu_E(\dot{u})} = e^{2\rho_E(H(u)-H(g^{-1}u))}$$

where $\dot{u}$ denotes the coset $uB(E)$.

Proof. (Trivial extension of a proof sketched in [14].) For any $\varphi \in C_0(G)$ we have

$$\int_{G/B(E)} d\mu_E(\dot{u}) \int_{B(E)} \varphi(ub)\, db = \int_{K\times B(E)} \varphi(kb)\, dk\, db \tag{1.1}$$

where dk, db are left Haar measures. Also, by a known formula [1, Chapter 7, §2, Corollary to Proposition 13] for any $\psi \in C_0(G)$ we have

$$\int_G \psi(u)\, du = \int_{K\times B(E)} \psi(kb) \frac{\Delta_G(b)}{\Delta_B(b)}\, dk\, db \tag{1.2}$$

where Δ_G, Δ_B are the modular functions of G, $B(E)$. In the present case $\Delta_G \equiv 1$ since G is semisimple, and Δ_B is given by Lemma 1.1. Applying (1.2) to the function $\psi(g) = \varphi(g)e^{-2\rho_E(H(g))}$ and using (1.1) we obtain

$$\int_{G/B(E)} d\mu(\dot{u}) \int_{B(E)} \varphi(ub)\, db = \int_G \varphi(u)e^{-2\rho_E(H(u))}\, du. \tag{1.3}$$

Applying (1.3) to the left translate $\varphi^g(u) = \varphi(gu)$ and changing the variable

$$\int_G \varphi(u)e^{-2\rho_E(H(g^{-1}u))}\, du = \int_{G/B(E)} d\mu_E(g^{-1}\dot{u}) \int_{B(E)} \varphi(ub)\, db.$$

Applying (1.3) again, the right-hand side is equal to

$$\int_G \frac{d\mu_E(g^{-1}\dot{u})}{d\mu_E(\dot{u})} \varphi(u)e^{-2\rho_E(H(u))}\, du$$

and the assertion follows.

We denote the identity coset of $G/K = X$ by o. Following Furstenberg [3] we define the Poisson integral of a function $f \in L^1(G/B(E))$ by

$$F(g\cdot o) = \int_{G/B(E)} f(g\dot{u})\, d\mu_E(\dot{u}) = \int_{G/B(E)} P_E(g, \dot{u}) f(\dot{u})\, d\mu_E(\dot{u})$$

for all $g \in G$. It is clear that the integral depends only on the coset $g \cdot o$.

By the lemma of Bruhat and Harish-Chandra the restriction of the natural map $G \to G/B(E)$ is an injective analytic map of $\bar{N}(E)$ onto a dense open subset (cf. also [12, p. 208]). Therefore every integral on $G/B(E)$ can be transformed to an integral on $\bar{N}(E)$ with respect to the Haar measure of $\bar{N}(E)$.

Given any function f on $G/B(E)$, we shall denote by f_1 the function defined on $\bar{N}(E)$ by $f_1(s)=f(\dot{s})$. This notation will be used throughout the paper. Because of typographical reasons we use the notations $s, s', s_0, \ldots$ for elements of $\bar{N}(E)$.

1.3. LEMMA. *For every integrable function f on $G/B(E)$ we have*

$$\int_{G/B(E)} f(\dot{u})\, d\mu_E(\dot{u}) = \int_{\bar{N}(E)} f_1(s)e^{-2\rho_E(H(s))}\, ds.$$

Proof. (This argument, for the case $E=\varnothing$, was communicated to me by S. Helgason.) It is clear that a formula of this type must hold with some function $\psi(s)$ in place of $e^{-2\rho_E(H(s))}$. By obvious transformations we have, for fixed $s \in \bar{N}(E)$,

$$\begin{aligned}\int_{\bar{N}(E)} f_1(s_1)\psi(s^{-1}s_1)\, ds_1 &= \int_{\bar{N}(E)} f_1^s(s_1)\psi(s_1)\, ds_1 \\ &= \int_{G/B(E)} f^s(\dot{u})\, d\mu(\dot{u}) = \int_{G/B(E)} f(\dot{u})P_E(s, \dot{u})\, d\mu(\dot{u}) \\ &= \int_{\bar{N}(E)} f_1(s_1)P_E(s, \dot{s}_1)\psi(s_1)\, ds_1.\end{aligned}$$

This is true in particular for every continuous f_1 with compact support. Therefore $\psi(s^{-1}s_1)=P_E(s, \dot{s}_1)\psi(s_1)$. Setting $s_1=e$ and using Lemma 1.2 we find that $\psi(s) = e^{-2\rho_E(H(s))}$.

1.4. LEMMA. *Every $x \in X$ can be written in the form $x=sma\cdot o$ ($s \in \bar{N}(E)$, $m \in K^E$, $a \in A$). x determines s uniquely and a up to an element of the centralizer of $\mathfrak{a}(E)$ in the small Weyl group. If x varies in a compact set, then s, a also stay in compact sets.*

Proof. By the Iwasawa decomposition $G=\bar{N}AK$ we know that $x=s_1a_1\cdot o$ with unique $s_1 \in \bar{N}$, $a_1 \in A$, depending continuously on x. We have an induced Iwasawa decomposition, $G^E=\bar{N}^EA^EK^E$, and $\bar{N}=\bar{N}(E)\cdot\bar{N}^E$ (continuous semidirect decomposition), $A=A(E)\cdot A^E$ (continuous direct decomposition). Hence $x=ss'a_0a'\cdot o= sa_0s'a'\cdot o$ with $s \in \bar{N}(E)$, $s' \in \bar{N}^E$, $a_0 \in A(E)$, $a' \in A^E$, each depending continuously on x. We also have $G^E=K^EA^EK^E$, the part in A^E being determined up to the small Weyl group of G^E/K^E. So we have $s'a'\cdot o=ma''\cdot o$ ($m \in K^E$, $a'' \in A^E$) and $x= sa_0ma''\cdot o=sm(a_0a'')\cdot o$, proving the assertion.

REMARK. We have $G^E\cdot o \cong G^E/K^E$; this is a symmetric subspace of X which can be denoted by X^E. The lemma can be reformulated as saying that $x \in X$ has the unique continuous decomposition $x=sa_0\cdot x^E$ ($s \in \bar{N}(E)$, $a_0 \in A(E)$, $x^E \in X^E$). In this form it is much used by Karpelevič [6].

1.5. PROPOSITION. *Let f be an integrable function on $G/B(E)$ and let F be its Poisson integral. Then*

$$F(s_0ma\cdot o) = \int_{\bar{N}(E)} f_1(s_0s^{ma})e^{-2\rho_E(H(s))}\, ds$$

for all $s_0 \in \bar{N}(E)$, $m \in K^E$, $a \in A$, (also for $m \in M(E)$).

Proof. By definition of the Poisson integral and by Lemma 1.3,

$$F(s_0 ma\cdot o) = \int_{G/B(E)} f^{s_0 ma}(\dot u)\, d\mu(\dot u)$$
$$= \int_{\bar N(E)} f^{s_0 ma}(\dot s) e^{-2\rho_E(H(s))}\, ds.$$

Since $s_0 manB(E) = s_0 s^{ma}B(E)$, we have $f^{s_0 ma}(\dot s) = f_1(s_0 s^{ma})$, finishing the proof.

REMARKS. 1. Defining the function $\mathscr{P}^E_{m,a}$ on $\bar N(E)$ by

$$\mathscr{P}^E_{m,a}(s^{-1}) = e^{-2\rho_E(H(s^{(ma)^{-1}}) - \log a)}$$

the statement can be rewritten as a convolution

$$F(s_0 ma\cdot o) = (f * \mathscr{P}^E_{m,a})(s_0).$$

2. $\mathscr{P}^E_{m,a}$ has the (trivial) homogeneity property

$$\mathscr{P}^E_{m,aa_1}(s^{a_1}) = \mathscr{P}^E_{m,a}(s) e^{2\rho_E(\log a_1)}$$

for all $a_1 \in A(E)$.

2. **Restricted admissible convergence.** Let E be a subset of F, which will be fixed for this section. Let $\mathfrak{a}^+(E)$ be the subset of $\mathfrak{a}(E)$ on which all positive $\mathfrak{a}(E)$-weights have positive value. Let $H \in \mathfrak{a}^+(E)$, $\tau \in \boldsymbol{R}$, $g \in G$, and let $C \subset X$ be a compact $M(E)$-invariant set with nonempty interior. We define a *truncated restricted admissible domain* at $\dot g \in G/B(E)$ by

$$\mathscr{R}^\tau_{H,C}(\dot g) = \{k(g)(\exp tH)\cdot x \mid t \geqq \tau,\ x \in C\}.$$

It is easy to check that this set depends only on the coset of g.

We shall say that a function F on X converges at $\dot g$ *admissibly and restrictedly* with respect to H to the number r if for all $\varepsilon > 0$ and all C there exists τ such that $x \in \mathscr{R}^\tau_{H,C}(\dot g)$ implies $|F(x) - r| < \varepsilon$.

If f is a function on $G/B(E)$ and F a function X, we say that F *converges to f admissibly and restrictedly* (*with respect to H*) *a.e.* if for almost all $\dot g$ (with respect to μ_E) F converges at $\dot g$ to $f(\dot g)$ admissibly and restrictedly.

2.1. PROPOSITION. *Given $C \subset X$, compact, $M(E)$-invariant with nonempty interior and a compact set $C_G \subset G$, there exists $C' \subset X$, compact, $M(E)$-invariant with nonempty interior, such that*

$$g'\mathscr{R}^\tau_{H,C}(\dot g) \subset \mathscr{R}^\tau_{H,C'}(g'\dot g)$$

for all $g' \in C_G$ and all τ, H, $\dot g$.

Proof. Using the definition and the Iwasawa decomposition $G = KAN$, we have

$$g'\mathscr{R}^\tau_{H,C}(\dot g) = g'k(g)\mathscr{R}^\tau_{H,C}(\dot e)$$
$$= k(g'k(g))a(g'k(g))n(g'k(g))\mathscr{R}^\tau_{H,C}(\dot e).$$

It is clear that $k(g'k(g))=k(g'g)$ and that $a(g'k(g))$, $n(g'k(gg))$ stay in compact subsets of A, N as g' varies in C_G and g in G. Hence it suffices to show that there exists a compact $C'\subset X$ such that

$$an\mathscr{R}^{\tau}_{H,C}(\dot{e}) \subset \mathscr{R}^{\tau}_{H,C'}(\dot{e})$$

for all a, n varying in given compact subsets C_A, C_N of A, N. For this we note that

$$\begin{aligned} an\mathscr{R}^{\tau}_{H,C}(\dot{e}) &= \{an(\exp tH)\cdot x \mid t \geqq \tau, x \in C\} \\ &= \{(\exp tH)an^{\exp -tH}\cdot x \mid t \geqq \tau, x \in C\}. \end{aligned}$$

This shows that

$$C' = \{an^{\exp -tH}\cdot x \mid a \in C_A, n \in C_N, t \geqq \tau, H \in \mathfrak{a}^+(E), x \in C\}$$

has the required properties.

Corollary. "*F converges to f admissibly and restrictedly with respect to H*" *is a G-invariant notion.*

We proceed to define a kind of norm on $\bar{N}(E)$, depending on H (which we consider fixed in $\mathfrak{a}^+(E)$). $\bar{\mathfrak{n}}(E)$ is the sum of weight spaces $\mathfrak{g}_\lambda$ with λ a negative linear form on $\mathfrak{a}(E)$. Each $\mathfrak{g}_\lambda$ is invariant under the compact group $\mathrm{Ad}_{\mathfrak{g}}(M(E))$ since $M(E)$ centralizes $\mathfrak{a}(E)$; we put an $M(E)$-invariant norm $|\ \ |$ on each $\mathfrak{g}_\lambda$. Any $s \in \bar{N}(E)$ can uniquely be written in the form $s=\exp \sum_{\lambda<0} U_\lambda$ ($U_\lambda \in \mathfrak{g}_\lambda$), and we define

$$|s|_H = \operatorname*{Max}_{\lambda<0} \{|U_\lambda|^{-1/\lambda(H)}\}.$$

2.2. Lemma. *For any $t \in R$,*

$$|s^{\exp tH}|_H = e^{-t}|s|_H.$$

The proof is immediate from the definitions.

2.3. Lemma. *There exists a constant c, depending on the choice of H, such that*

$$|ss'|_H \leqq c(|s|_H+|s'|_H)$$

for all s, $s' \in \bar{N}(E)$.

Proof. Let $V_t=\{s \mid |s|_H \leqq e^t\}$. The sets V_t are compact and they exhaust $\bar{N}(E)$ as $t\to\infty$. Hence $V_1\cdot V_1\subset V_r$ for some $r\in \boldsymbol{R}$. Let $c=e^r$.

By Lemma 2.2 we have $V_t=V_1^{\exp -tH}$. Now write $|s|_H=e^t$, $|s'|_H=e^{t'}$, and let $\tau=\operatorname{Max}\{t,t'\}$. Then $ss' \in V_t\cdot V_{t'}\subset V_\tau\cdot V_\tau=(V_1\cdot V_1)^{\exp -\tau H}\subset V_{r+\tau}$, and so $|ss'|_H \leqq e^{r+\tau}\leqq e^{+}(|s|_H+|s'|_H)$.

Now we define the analogues of truncated nontangential domains on the upper halfplane. Let $|\ \ |$ be any norm on the vector group A. For $s_0 \in \bar{N}(E)$, $\alpha>0$, $\tau\in\boldsymbol{R}$ we define

$$\Gamma^{\tau}_{H,\alpha}(s_0) = \{s_0(\exp tH)sma\cdot o \mid t \geqq \tau; m \in K^E; |a|, |s|_H \leqq \alpha\}.$$

2.4. Proposition. *Let H be fixed. Let F be a function on X, f a function on $G/B(E)$. Then F converges to f admissibly and restrictedly a.e. if and only if for almost all $s \in \bar{N}(E)$ (with respect to the Haar measure) and for all $\alpha > 0$, $\varepsilon > 0$ there exists $\tau \in \mathbf{R}$ such that $x \in \Gamma^{\tau}_{H,\alpha}(s)$ implies $|F(x)-f_1(s)| < \varepsilon$.*

Proof. The cosets of the elements $s \in \bar{N}(E)$ form a set of full measure in $G/B(E)$, so it is enough to consider these. We wish to show that at each $\dot{s}$ restricted admissible convergence of F is equivalent with the analogous notions defined in terms of the sets Γ instead of the sets $\mathscr{R}$. By the obvious fact that $\Gamma^{\tau}_{H,\alpha}(s) = s \cdot \Gamma^{\tau}_{H,\alpha}(e)$ and by Proposition 2.1 it suffices to show this at $\dot{e}$.

For this we note that each $\Gamma^{\tau}_{H,\alpha}(e)$ is contained in (even equal to) an $\mathscr{R}^{\tau}_{H,C}(\dot{e})$, namely the one with $C = \{sma \cdot o \mid |s|_H, |a| \leqq \alpha; m \in K^E\}$. Conversely, every $\mathscr{R}^{\tau}_{H,C}(\dot{e})$ is contained in some $\Gamma^{\tau}_{H,\alpha}(e)$, since any compact set C is contained in one of the special type given above. This finishes the proof.

2.5. Proposition. *Let $1 \leqq p \leqq \infty$. The following statements are equivalent:*

(i) *For all $f \in L^p(G/B(E))$ the Poisson integral of f converges to f admissibly and restrictedly a.e.*

(ii) *For all functions f on $G/B(E)$ such that $f_1 \in L^p(\bar{N}(E))$ the Poisson integral of f converges to f admissibly and restrictedly, a.e.*

Similarly, we have equivalent statements if we replace L^p by the space of all signed measures in (i) *and by the space of all finite signed measures in* (ii).

Proof. By Lemma 1.3 the class of f occurring in (ii) is a subclass of $L^p(G/B(E))$; hence (i) implies (ii).

To show the converse, note that the image of $\bar{N}(E)$ under $s \to \dot{s}$ is open in $G/B(E)$. Let U be an open set whose closure is contained in this image. By compactness there exist elements $e = g_1, g_2, \ldots, g_l$ in G such that $g_1 \cdot U, \ldots, g_l \cdot U$ cover $G/B(E)$. Now any $f \in L^p(G/B(E))$ can be written as $f = \sum f^{(j)}$ with $f^{(j)} \in L^p(G/B(E))$ and the support of $f^{(j)}$ contained in $g_j U$ $(1 \leqq j \leqq l)$. Using G-invariance (Corollary to Proposition 2.1) it follows that it is enough to show that (ii) implies (i) for functions $f \in L^p(G/B(E))$ such that f_1 has compact support in $\bar{N}(E)$. This however is obvious, since in this case $f_1 \in L^p(\bar{N}(E))$.

The proof for measures is similar.

2.6. Theorem. *Let $f \in L^\infty(G/B(E))$ and let F be its Poisson integral. Then, with respect to any fixed $H \in \mathfrak{a}^+(E)$, F converges to f admissibly and restrictedly a.e.*

Proof. If V is any compact neighborhood of the identity in $\bar{N}(E)$, we denote $V^{\exp tH} = \{s^{\exp tH} \mid s \in V\}$; it is clear that the Haar measure of $V^{\exp tH}$ is $ke^{-2\rho_E(tH)}$ with a constant k. It is known [2], [15] that for almost all $s_0 \in \bar{N}(E)$

$$\lim_{t \to \infty} e^{2\rho_E(tH)} \int_{V^{\exp tH}} |f_1(s_0 s) - f_1(s_0)| \, ds = 0.$$

Let s_0 be such a point.

Let $\varepsilon>0$ and $\alpha>0$. By Proposition 2.4 the proof will be finished if we show that there exists $\tau\in\boldsymbol{R}$ such that $|F(x)-f_1(s_0)|<\varepsilon$ for all $x\in\Gamma^{\tau}_{H,\alpha}(s_0)$. Now $x\in\Gamma^{\tau}_{H,\alpha}(s_0)$ means $x=s_0(\exp tH)s_1ma\cdot o=s_0s_1^{\exp tH}ma(\exp tH)\cdot o$, with $|s_1|_H$, $|a|\leqq\alpha$ and $t\geqq\tau$. By Proposition 1.5 we have

$$|F(x)-f_1(s_0)| = \left|\int_{\bar{N}(E)} (f_1(s_0s_1^{\exp tH}s^{ma\exp tH})-f_1(s_0))e^{-2\rho_E(H(s))}\,ds\right|$$

$$\leqq \left(\int_{\bar{N}(E)-U_1}+\int_{U_1}\right)|f_1(s_0s_1^{\exp tH}s^{ma\exp tH})-f_1(s_0)|e^{-2\rho_E(H(s))}\,ds.$$

Choosing U_1 to be a compact neighborhood of the identity large enough such that

$$\int_{\bar{N}(E)-U_1} e^{-2\rho_E(H(s))}\,ds < \frac{\varepsilon}{4\|f\|_\infty}$$

we have that the first integral on the right is $<\varepsilon/2$. Noting that $e^{-2\rho_E(H(s))}$ is bounded on U_1 by some number M_1, we see that the second integral is majorized by

$$M_1\int_{U_1} |f_1(s_0s_1^{\exp tH}s^{ma\exp tH})-f_1(s_o)|\,ds$$

$$= M_1e^{2\rho_E(tH+\log a)}\int_{(s_1U_1^{ma})^{\exp tH}} |f_1(s_0s)-f_1(s_0)|\,ds$$

(we have made a change of variable). Now let

$$M = M_1\operatorname*{Max}_{|a|<\alpha} e^{2\rho_E(\log a)}$$

and let

$$V = \{s_1s^{ma}\mid |s_1|_H,\ |a|\ \leqq\ \alpha;\ s\in U_1;\ m\in K^E\}.$$

Our last expression is majorized by

$$Me^{2\rho_E(tH)}\int_{V^{\exp tH}} |f_1(s_0s)-f(s)|\,ds.$$

We can choose $\tau\in\boldsymbol{R}$ so that this is $<\varepsilon/2$ for all $t>\tau$, finishing the proof.

3. **Hermitian symmetric spaces.** Hermitian symmetric spaces of noncompact type can be realized as generalized halfplanes in a complex vector space [11]. The distinguished boundary in Bergman's sense of the generalized halfplane can then be identified with the orbit of $\bar{N}(E)$ in $G/B(E)$ for a certain special E. Restricted admissible convergence to the distinguished boundary has been defined in geometric terms in [9], [10], [17]; we shall compare this definition with our general definition given in §2.

First we consider the irreducible case. The group corresponding to our G in [11] is $\operatorname{ad}(c)G^o$; our $\mathfrak{a}$ is $i\mathfrak{h}^-$, i.e. the linear span of $\{H_\gamma\}_{\gamma\in\Delta}$ (cf. [11, Lemma 4.3]). As was shown by C. C. Moore [12], if we order Δ as $\gamma_1>\cdots>\gamma_l$, then F consists of $\frac{1}{2}(\gamma_i-\gamma_{i+1})$ $(1\leqq i<l)$ and of γ_l or $\frac{1}{2}\gamma_l$ depending on whether the latter appears among the restricted roots. The subset E corresponding to the distinguished

boundary consists of $\frac{1}{2}(\gamma_1-\gamma_2), \ldots, \frac{1}{2}(\gamma_{l-1}-\gamma_l)$. $\mathfrak{a}(E)$ is the span of the element $\frac{1}{2}\sum_{\gamma\in\Delta} H_\gamma$. The element H is now determined up to a constant multiple. Since our notion of restricted admissible convergence is clearly unchanged if we replace H by a positive constant multiple, we can fix H once and for all to be equal to $\frac{1}{2}\sum_{\gamma\in\Delta} H_\gamma$. Our $M(E)$ is the same as L in [11], we also have $K^E=M_0(E)$ in this case. By [11, Lemma 5.3] there are one or two negative weights on $\mathfrak{a}(E)$. On H they have the value -1 and (maybe) $-\frac{1}{2}$; the corresponding weight spaces are $\mathfrak{n}_1^-$ and $\mathfrak{n}_2^-$. Their sum is our $\bar{\mathfrak{n}}(E)$; in [11] it is denoted by $\mathfrak{n}^-$, the corresponding group by N^-. The norm $|n|_H$ introduced in §2 is clearly the same as the norm used in [9], [10], [17]. The symmetric space G/K is realized [11, Theorem 6.8] as the domain

$$D^c = \{(z_1, z_2) \in \mathfrak{p}_1^- + \mathfrak{p}_2^- \mid \operatorname{Im} z_1 - \Phi(z_2, z_2) \in \mathfrak{c}\}$$

where $\mathfrak{c}$ is a cone, equal in our present notations to the orbit $K^E A\cdot o$.

The *geometric definition* of restricted admissible convergence in [9], [10], [17] is given in terms of the sets

$$\Gamma^\tau_{\alpha,\omega}(s_0\cdot 0) = \{s_0 s_1\cdot(iy, 0) \mid y \in \omega,\ |s_1|_H \leqq \alpha|y|,\ |y| \leqq e^{-\tau}\}$$

where $\tau \in \boldsymbol{R}$, $\alpha>0$, and ω is any subcone of $\mathfrak{c}$ such that $\bar{\omega}\subset \mathfrak{c} \cup \{0\}$. Writing $|y|=e^{-t}$ and using Lemma 2.2 we see at once that

$$\Gamma^\tau_{\alpha,\omega}(s_0\cdot 0) = \{s_0(\exp tH)s_1\cdot(iy_1, 0) \mid y_1 \in \omega_1,\ |s_1|_H \leqq \alpha,\ t \geqq \tau\}$$

where ω_1 is the set of all $y_1 \in \omega$ such that $|y_1|=1$. It is clear that the notion of convergence does not change if instead of the sets ω_1 we use arbitrary compact subsets of $\mathfrak{c}$, which is equivalent to using the sets $\Gamma^\tau_{H,\alpha}(s_0)$ defined in §2. So, taking into account Proposition 2.4, we have proved the following result.

3.1. Proposition. *For irreducible Hermitian symmetric spaces realized as generalized halfplanes the geometric definition of restricted admissible convergence a.e. is equivalent with the definition given in* §2.

It is also easy to see what happens in the reducible case. Then D^c is a product of irreducible domains, and the dimension of $\mathfrak{a}(E)$ is equal to the number of irreducible factors. H is no longer unique up to constant multiples, and depending on the choice of H we get different notions of restricted admissible convergence. There is, however, a natural choice of H, namely such that its projection onto each irreducible factor should have the normalization given above. (This corresponds to the projection of the geodesic line $(\exp tH)\cdot o$ onto each irreducible factor having a tangent of unit length at o.) Restricted admissible convergence with respect to this H is again equivalent to the geometric definition.

In [9] still one more definition of convergence was considered, this time in terms of the structure of D^c as a complex domain. For a product of discs it was shown to be equivalent with *unrestricted* nontangential convergence. This definition is meaningful for any D^c: For $\alpha>0$ and a point $s_0\cdot 0$ on the distinguished boundary one defines

$$\mathscr{A}_\alpha(s_0\cdot 0) = \left\{z\in D^c \,\middle|\, \frac{S(z,z)}{|S(s_0\cdot 0,z)|} \leqq \alpha\right\},$$

and one considers convergence $z\to s_0\cdot 0$, in the topology of the ambient vector space, under the restriction $z\in\mathscr{A}_\alpha(s_0\cdot 0)$. (One can also make an analogous definition using the canonical bounded realization D of the space and its Szegö kernel $\mathscr{S}$. By the relation connecting $\mathscr{S}$ with S [8, p. 342] it is clear that this gives the same notion of convergence a.e.) Writing $z=s_0mas_1\cdot o$ ($m\in K^E, a\in A, s_1\in\bar N(E)$) we see by [8, formula (3.4)] that $S(z,z)\cdot|S(s_0\cdot 0,z)|^{-1}=S(o,o)\cdot|S(s_1^{-1}\cdot 0,o)|^{-1}$. This shows that the condition $z\in\mathscr{A}_\alpha(s_0\cdot 0)$ is equivalent with s_1 being restricted to a compact subset of $\bar N(E)$. It is now easy to see that convergence under the restriction $z\in\mathscr{A}_\alpha(s_0\cdot 0)$ can be intrinsically redefined by using the sets

$$\Gamma_U^T(s_0) = \{s_0mas_1\cdot o \mid m\in K^E, a\in A, \log a \geqq T, s_1\in U\}$$

where U is a compact subset of $\bar N(E)$ with nonempty interior, $T\in\mathfrak{a}(E)$, and $\log a\geqq T$ means that $\beta(\log a-T)\geqq 0$ for all positive restricted roots β.

It would seem natural to define unrestricted admissible convergence in terms of the sets $\Gamma_U^T(s_0)$; this would then make sense for any symmetric space and any boundary. However, an example constructed by E. M. Stein and N. J. Weiss (as yet unpublished) shows that in the case of the tube over a 3-dimensional circular cone, i.e. the space $Sp(2,R)/U(2)$, there exist L^∞-functions on $G/B(E)$ whose Poisson integral does not converge a.e. in this sense. Therefore we do not pursue this further, at any rate not in the apparently too general case of an arbitrary subset $E\subset F$.

4. **Remarks on unrestricted convergence.** Because of the example mentioned at the close of the preceding section, we define and study unrestricted admissible convergence only in relation to the maximal boundary G/B. Even in this case our discussion is not complete; in order to prove that the condition of Proposition 4.3 is satisfied at almost every point we would need an extension of the strong differentiation theorem, known for vector spaces, to nilpotent groups.

For $g\in G$, $T\in\mathfrak{a}$ and for any compact M-invariant set $C\subset X$ with nonempty interior we define

$$\mathscr{A}_C^T(\dot g) = \{k(g)a\cdot x \mid a\in A, \log a \geqq T, x\in C\}$$

where $\log a\geqq T$ means that $\beta(\log a-T)\geqq 0$ for all positive restricted roots β.

We say that a function F on X *converges at $\dot g$ admissibly and unrestrictedly* to the number r if for all $\varepsilon>0$ and all C there exists T such that $x\in\mathscr{A}_C^T(\dot g)$ implies $|F(x)-r|<\varepsilon$.

If f is a function on G/B and F a function on X we say that F *converges to f admissibly and unrestrictedly a.e.* if for almost all $\dot g$ (with respect to μ) F converges at $\dot g$ admissibly and unrestrictedly to $f(\dot g)$.

It is clear that this implies restricted admissible convergence with respect to any choice of an element $H \in \mathfrak{a}^+$.

4.1. PROPOSITION. *Given a compact set $C \subset X$ and an element $g' \in G$, there exists a compact M-invariant set $C' \subset X$ with nonempty interior such that*

$$g' \cdot \mathscr{A}_C^T(\dot{g}) \subset \mathscr{A}_{C'}^T(g'\dot{g})$$

for all $g \in G$ and $T \in \mathfrak{a}^+$. Therefore, unrestricted admissible convergence a.e. is a G-invariant notion.

Proof. It is clear from the definitions that

$$\mathscr{A}_C^T(\dot{g}) = \bigcup_{H \geqq T} \mathscr{R}_{H,C}^1(\dot{g}).$$

Thus the assertion follows from Proposition 2.1.

For any $s_0 \in \bar{N}$, $T \in \mathfrak{a}$, and any compact set $U \subset \bar{N}$ with nonempty interior we define

$$\Gamma_U^T(s_0) = \{s_0 a s_1 \cdot o \mid \log a \geqq T, s_1 \in U\}.$$

4.2. PROPOSITION. *Let F be a function on X, f a function on G/B. Then F converges at $\dot{s}_0$ to $f(\dot{s}_0)$ admissibly and restrictedly if and only if for all $\varepsilon > 0$ and all compact $U \subset \bar{N}$ there exists $T \in \mathfrak{a}$ such that $x \in \Gamma_U^T(s_0)$ implies $|F(x) - f_1(s_0)| < \varepsilon$.*

Proof. Similarly to the proof of Proposition 2.4, we will show that at each point $\dot{s}_0$ unrestricted admissible convergence is equivalent with the analogous notion defined in terms of the sets Γ instead of $\mathscr{A}$. By the fact that $\Gamma_U^T(s_0) = s_0 \cdot \Gamma_U^T(e)$ and by Proposition 4.1 it suffices to show this at $\dot{e}$.

In fact, each $\Gamma_U^T(e)$ is contained in (even equal to) some $\mathscr{A}_C^T(\dot{e})$; we can e.g. take $C = \{a_1 s_1 \cdot o \mid 0 \leqq \log a_1 \leqq T_0, s_1 \in U\}$ with some $T_0 \in \mathfrak{a}^+$. Conversely, given $\mathscr{A}_C^T(\dot{e})$, there exist compact sets $C_A \subset A$, $U \subset \bar{N}$ such that $C \subset C' = \{a_1 s_1 \cdot o \mid a_1 \in C_A, s_1 \in U\}$, and there exists $T_0 \in \mathfrak{a}$ such that $\log a_1 \leqq T_0$ for all $a_1 \in C_A$. We have then $\mathscr{A}_C^T(\dot{e}) \subset \mathscr{A}_{C'}^T(\dot{e}) \subset \Gamma_U^{T-T_0}(e)$, finishing the proof.

4.3. PROPOSITION. *Let $f \in L^\infty(G/B)$ and let F be its Poisson integral. Let $s_0 \in \bar{N}$, and assume that for any compact neighborhood V of the identity in $\bar{N}$ and for any > 0 there exists $T \in \mathfrak{a}$ such that $\log a \geqq T$ implies*

$$\frac{1}{\operatorname{mes}(V^a)} \int_{V^a} |f_1(\dot{s}_0 s) - f_1(\dot{s}_0)| \, ds < \varepsilon.$$

Then F converges at $\dot{s}_0$ to $f(\dot{s}_0)$ admissibly and unrestrictedly.

Proof. By Proposition 4.2 it suffices to show that for any compact $U \subset \bar{N}$ and $\varepsilon > 0$ there exists $T \in \mathfrak{a}$ such that $x \in \Gamma_U^T(s_o)$ implies $|F(x) - f_1(s_o)| < \varepsilon$. $x \in \Gamma_U^T(s_o)$ means that $x = s_o a s_1 \cdot o = s_o s_1^a a \cdot o$ with $s_1 \in U$ and $\log a \geqq T$.

Using Proposition 1.5 we have

$$\begin{aligned} |F(x) - f_1(s_0)| &= \left| \int_{\bar{N}} (f_1(s_0 s_1^a s^a) - f_1(s_0)) e^{-2\rho(H(s))} \, ds \right| \\ &\leqq \left(\int_{\bar{N} - U_1} + \int_{U_1} \right) |f_1(s_0 s_1^a s^a) - f_1(s_0)| e^{-2\rho(H(s))} \, ds. \end{aligned}$$

Choosing U_1 to be a compact neighborhood of the identity such that

$$\int_{\bar{N}-U_1} e^{-2\rho(H(s))}\, ds < \frac{\varepsilon}{4\|f\|_\infty}$$

we have that the first integral on the right is $<\varepsilon/2$. $e^{-2\rho(H(s))}$ is bounded on U_1 by some number M_1, therefore the second integral is majorized by

$$M_1 \int_{U_1} |f_1(s_0 s_1^a s^a) - f_1(s_0)|\, ds = M_1 e^{2\rho(\log a)} \int_{(s_1 U_1)^a} |f_1(s_0 s) - f_1(s_0)|\, ds$$

(after having made a change of variable). We have $s_1 \in U$; therefore, denoting $U \cdot U_1 = V$, this is further majorized by

$$M_1 e^{2\rho(\log a)} \int_{V^a} |f_1(s_0 s) - f_1(s_0)|\, ds. \tag{4.1}$$

Noticing that mes $(V^a) =$ mes $(V) e^{-2\rho(\log a)}$, it follows from our hypothesis that (4.1) is $<\varepsilon/2$ if $\log a \geqq T$ for an appropriate T. This finishes the proof.

5. **Symmetric spaces of rank one.** The first lemma is valid for symmetric spaces of arbitrary rank.

5.1. LEMMA. *For any symmetric space of noncompact type,* $e^{2\rho(H(s))}$ *is a polynomial in the canonical coordinates of* $\bar{N}$.

Proof. Let $\{X_j\}$ be a basis of $\mathfrak{n}$ such that each X_j belongs to some positive restricted root space. If we denote the Cartan involution of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ by θ and write $\bar{X}_j = \theta X_j$, $\{\bar{X}_j\}$ is a basis of $\bar{\mathfrak{n}}$ and $\{X_j + \bar{X}_j\}$ spans a complement $\mathfrak{l}$ of $\mathfrak{m}$ in $\mathfrak{k}$. [4, Lemma 3.6, p. 223].

By [7, Theorem 2.1] we have

$$e^{2\rho(H(s))} = \det (P \operatorname{Ad}(s^{-1})|_{\mathfrak{l}}) \tag{5.1}$$

where P is the projection of $\mathfrak{g}$ onto the subspace $\mathfrak{l}$ along the complementary subspace $\mathfrak{a} + \mathfrak{n}$.

Let $s = \exp \sum x_j \bar{X}_j$. Then

$$\operatorname{Ad}(s^{-1}) = \sum_\nu \frac{1}{\nu!} \operatorname{Ad}\left(-\sum x_j \bar{X}_j\right)^\nu$$

where the sum over ν is finite since each $\bar{X}_j$ is in a negative root space. We pick a basis $\{Y_k\}$ of $\mathfrak{m}$. It is then clear that each fixed $X_j + \bar{X}_j$ and Y_k is carried by $\operatorname{Ad}(s^{-1})$ into a sum of the form $\sum (\bar{p}_r(x)\bar{X}_r + p_r(x) X_r + q_s(x) Y_s)$ where the $\bar{p}_r$, p_r, q_s are polynomials in the variables $\{x_j\}$. Applying P to this, we get $\sum (\bar{p}_r(x)(X_r + \bar{X}_r) + q_s(x) Y_s)$. Thus $P \operatorname{Ad}(s^{-1})|_{\mathfrak{l}}$ has a matrix whose entries are polynomials in the $\{x_j\}$, and our assertion follows by (5.1).

From now on it will be assumed that the rank of G/K equals 1. Then $\mathfrak{a}$ is 1-dimensional, and there are at most two positive restricted roots; we shall denote them by λ and (possibly) $\frac{1}{2}\lambda$. The element $H \in \mathfrak{a}^+$ is unique up to positive constant multiples. Therefore we can fix it once and for all by the normalization $\lambda(H)=2$; the notion of restricted admissible convergence is unaffected by this. Since in this case there is no difference between restricted and unrestricted, we shall simply talk about admissible convergence. Instead of $|s|_H$ we write $|s|$. We use the definitions of §2 making only one slight simplification: Instead of the sets $\Gamma^{\tau}_{H,\alpha}(s_0)$ we use

$$\Gamma^{\tau}_{\alpha}(s_0) = \{s_0(\exp tH)s\cdot o \mid |s| \leqq \alpha,\, t \geqq \tau\}.$$

It is clear that Proposition 2.4 remains true for these.

5.2. Lemma. *Let $m=2\rho(H)$. There exists a constant M such that*

$$\begin{aligned} e^{2\rho(H(s^{\exp -tH})-tH)} &\leqq Me^{mt} && \text{for all } s \in \bar{N}, \\ &\leqq Me^{-t}/|s|^{m+1} && \text{for } |s| \geqq e^{-t}. \end{aligned}$$

Proof. The first inequality is trivially true since $e^{-2\rho(H(s))}$ is a bounded function on $\bar{N}$.

To prove the second, we choose a basis $\{X_j\}$ of the root space $\mathfrak{g}_{-\lambda}$ and a basis $\{Z_k\}$ of $\mathfrak{g}_{-\lambda/2}$. Then every $s \in \bar{N}$ can be written in the form $s=\exp \sum (x_jX_j+z_jZ_j)$ and we have

$$s^{\exp tH} = \exp \sum (e^{2t}x_jX_j+e^tz_kZ_k).$$

Defining the function p by

$$p(s, e^t) = e^{2\rho(H(s^{\exp -tH}))}$$

it follows now from Lemma 5.1 that p is a polynomial in s (i.e. the canonical coordinates of s) and in e^t.

A simple application of Proposition 1.2 gives

$$\frac{e^{mt}}{p(s, e^t)} = e^{-2\rho(H(s^{\exp -tH})-tH)} = P(\exp tH, \dot{s})e^{-2\rho(H(s))}.$$

By [7, Theorem 3.1] this tends to 0, for every fixed $s \neq e$, as $t \to \infty$. Hence, for fixed $s \neq e$, $p(s, \xi)$ is a polynomial in ξ of degree at least $m+1$.

It follows now easily that there exists a number M such that

$$\xi^{m+1}/p(s_1, \xi) \leqq M \tag{5.2}$$

for all $|s_1|=1$ and all $\xi \geqq 1$. Now let $s \neq e$. Then, by Lemma 2.2, $s=s_1^{\exp -t_0H}$ with $|s_1|=1$, $|s|=e^{t_0}$. In (5.2) we set $\xi=e^{t+t_0}=e^t|s|$, and we notice that $p(s_1, e^{t+t_0}) =p(s, e^t)$. This finishes the proof.

For every $r>0$ we define the sets $B(r)=\{s \in \bar{N} \mid |s| \leqq r\}$. It is clear that $B(r)^{\exp tH} =B(re^{-t})$, and hence mes $(B(r))=r^m$ mes $(B(1))$. For any integrable function φ on $\bar{N}$ one defines the *maximal function* φ^* by

$$\varphi^*(s_0) = \sup_{r>0} \frac{1}{\text{mes}\,(B(r))} \int_{B(r)} |\varphi(s_0s)|\, ds.$$

It is known [2], [15] that the Hardy-Littlewood maximal theorem is valid for φ^*.

5.3. Lemma. *Given $\alpha>0$, there exists a constant C_α such that whenever f is a function on G/B such that $f_1\in L^1(\bar{N})$ and F is the Poisson integral of f we have*

$$|F(x)| \leqq C_\alpha f_1^*(s_0)$$

for all $s_0\in\bar{N}$, $x\in\Gamma_\alpha^\tau(s_0)$, $\tau\in\mathbf{R}$.

Proof. Since everything is invariant under left translation by s_o, we may assume that $s_o=e$. Let $x\in\Gamma_\alpha^\tau(e)$. Then $x=(\exp tH)s\cdot o=s^{\exp tH}(\exp tH)\cdot o$ with $|s|\leqq\alpha$, which amounts to the same as saying that $x=s_1(\exp tH)\cdot o$ with $|s_1|\leqq\alpha e^{-t}$. Let $\delta=2c\alpha e^{-t}$ where c is the constant of Lemma 2.3. By Proposition 1.5 and a change of variable we have

$$\begin{aligned}
|F(x)| &= |F(s_1(\exp tH)\cdot o)| \\
&= \left|\int_{\bar{N}} f_1(s_1 s^{\exp tH})e^{-2\rho(H(s))}\,ds\right| \\
&= \left|\int_{\bar{N}} f_1(s)e^{-2\rho(H((s_1^{-1}s)^{\exp -tH})-tH)}\,ds\right| \\
&\leqq \left(\int_{B(\delta)}+\sum_{j=0}^{\infty}\int_{B(2^{j+1}\delta)-B(2^j\delta)}\right)|f_1(s)|e^{-2\rho(H((s_1^{-1}s)^{\exp -tH})-tH)}\,ds.
\end{aligned}$$

Now we notice that, by Lemma 2.3, $s\notin B(2^j\delta)$ i.e. $|s|>2^{j+1}c\alpha e^{-t}$ implies that $|s_1^{-1}s|>2^j\alpha e^{-t}$ $(j=0,1,\ldots)$, and we use Lemma 5.2. We obtain

$$|F(x)| \leqq Me^{mt}\int_{B(\delta)}|f_1|+\sum_{j=0}^{\infty}M\frac{e^{-t}}{(2^j\alpha e^{-t})^{m+1}}\int_{B(2^{j+1}\delta)-B(2^j\delta)}|f_1|.$$

The right-hand side is further increased by taking each integral over $B(2^{j+1}\delta)$ instead of $B(2^{j+1}\delta)-B(2^j\delta)$. By the definition of f_1^* it follows that

$$|F(x)| \leqq M\operatorname{mes}(B(1))\left((2\alpha)^m+\frac{1}{(c\alpha)^m}\sum_{j=0}^{\infty}2^{2m-j}\right)f_1^*(e)$$

and the proof is finished.

5.4. Theorem. *Let G/K have rank one, and let $f\in L^1(G/B)$. Then the Poisson integral of f converges to f admissibly a.e.*

Proof. By Proposition 2.5 it is enough to consider the case where $f_1\in L^1(\bar{N})$. In this case the theorem follows from Proposition 2.4 and Lemma 5.3 by standard methods (cf. [18, Chapter XVII] or [15]).

Still by standard methods (cf. also [7, Theorem 6.1]) one can generalize this theorem to the case of Poisson integrals of signed measures on G/B.

There is also a generalization of Privalov's local version of the Fatou theorem.

5.5. THEOREM. *Let G/K have rank one, and let F be a harmonic function on G/K. Let $S \subset \bar{N}$ be a measurable set, and assume that for each $s \in S$ there exists a set $U \subset \bar{N}$ with nonempty interior and a $\tau \in \boldsymbol{R}$ such that F is bounded in*

$$\Gamma_U^\tau(s) = \{s(\exp tH)s_1 \cdot o \mid t \geqq \tau, s_1 \in U\}.$$

Then, for almost every $s \in S$, F converges at $\dot{s}$ admissibly to a finite value.

(Of course, by the argument of Proposition 2.4, we could equivalently state this theorem starting with a measurable set $S \subset G/B$ and assuming boundedness of F in sets of the form $\mathscr{R}_{H,C}^\tau(\dot{g})$.) The proof follows the lines of a proof of Calderón [18, Chapter XVII] which was adapted to the case of Hermitian symmetric spaces of rank one in [9]. We shall not give it in detail, since the proof in [9] can very easily be translated to the present situation. There is only one point requiring special attention: In the course of the proof one needs the existence of a positive harmonic function (h in [9]) which has admissible boundary value 0 at each $\dot{s}$ ($s \in \bar{N}$) and is constant on orbits of $\bar{N}$. In the present case such a function is given by $h(sa \cdot o) = e^{-\rho(\log a)}$ which is harmonic e.g. by [6, Theorem 15.3.2].

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