ABSOLUTE GAP-SHEAVES AND EXTENSIONS OF COHERENT ANALYTIC SHEAVES

BY

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Thimm introduced the concept of gap-sheaves for analytic subsheaves of finite direct sums of structure-sheaves on domains of complex number spaces (Definition 9, [13]) and proved that these gap-sheaves are coherent if the subsheaves themselves are coherent (Satz 3, [13]). This concept of gap-sheaves can be readily generalized to analytic subsheaves of arbitrary analytic sheaves on general complex spaces (Definition 1, [12]). All the gap-sheaves of coherent analytic subsheaves of arbitrary coherent analytic sheaves on general complex spaces are coherent (Theorem 3, [12]). The gap-sheaves of a given analytic subsheaf depend not only on the subsheaf itself but also on the analytic sheaf in which the given subsheaf is embedded as a subsheaf.

In this paper we introduce a new notion of gap-sheaves which we call absolute gap-sheaves (Definition 3 below). These gap-sheaves arise naturally from the problem of removing singularities of local sections of a coherent analytic sheaf. They depend only on a given analytic sheaf and neither require nor depend upon an embedding of the given sheaf as a subsheaf in another analytic sheaf. We give here a necessary and sufficient condition for the coherence of absolute gap-sheaves of coherent sheaves (Theorem 1 below). This yields some results concerning removing singularities of local sections of coherent sheaves (see Remark following Corollary 2 to Theorem 1). Then we use absolute gap-sheaves to derive a theorem (Theorem 2 below) which generalizes Serre’s Theorem on the extension of torsion-free coherent analytic sheaves (Theorem 1, [11]). Finally a result on extensions of global sections of coherent analytic sheaves is derived (Theorem 4 below).

Unless specified otherwise, complex spaces are in the sense of Grauert (§1, [5]). If $\mathcal{F}$ is an analytic subsheaf of an analytic sheaf $\mathcal{F}$ on a complex space $(X, \mathcal{H})$, then $\mathcal{I} : \mathcal{F}$ denotes the ideal-sheaf $\mathcal{I}$ defined by $\mathcal{I}_x = \{ s \in \mathcal{H}_x \mid s \mathcal{F}_x \subseteq \mathcal{I}_x \}$ for $x \in X$. $\mathcal{E}(\mathcal{F}, \mathcal{I})$ denotes $\{ x \in X \mid \mathcal{I}_x \neq \mathcal{F}_x \}$. Supp $\mathcal{F}$ denotes the support of $\mathcal{F}$. If $t \in \Gamma(X, \mathcal{F})$, then Supp $t$ denotes the support of $t$. For $x \in X$, $t_x$ denotes the germ of $t$ at $x$. By the annihilator-ideal-sheaf $\mathcal{A}$ of $\mathcal{F}$ we mean the ideal-sheaf $\mathcal{A}$ defined by $\mathcal{A}_x = \{ s \in \mathcal{H}_x \mid s \mathcal{F}_x = 0 \}$ for $x \in X$. If $\theta : (X, \mathcal{H}) \to (X', \mathcal{H}')$ is a holomorphic map (i.e. a morphism of ringed spaces) from $(X, \mathcal{H})$ to another complex space $(X', \mathcal{H}')$, then $R^0 \theta(\mathcal{F})$ denotes the zeroth direct image of $\mathcal{F}$ under $\theta$. If $f \in \Gamma(X, \mathcal{H})$ and $x \in X$, we say that $f$ vanished at $x$ if $f_x$ is not a unit in $\mathcal{H}_x$. 

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I. Absolute gap-sheaves.

Definition 1. Suppose \( \mathcal{I} \) is an analytic subsheaf of an analytic sheaf \( \mathcal{F} \) on a complex space \((X, \mathcal{H})\) and \( \rho \) is a nonnegative integer. The \( \rho \)-th gap-sheaf of \( \mathcal{I} \) in \( \mathcal{F} \), denoted by \( \mathcal{I}_{(\rho)} \), is the analytic subsheaf of \( \mathcal{F} \) defined as follows: For \( x \in X \), \( s \in (\mathcal{I}_{(\rho)})_x \) if and only if there exist an open neighborhood \( U \) of \( x \) in \( X \), a subvariety \( A \) in \( U \) of dimension \( \leq \rho \), and \( t \in \Gamma(U, \mathcal{F}) \) such that \( t_x = s \) and \( t_y \in \mathcal{I}_y \) for \( y \in U - A \).

Denote the set \( \{ x \in X \mid \mathcal{I}_x \neq (\mathcal{I}_{(\rho)})_x \} \) by \( E^\rho(\mathcal{I}, \mathcal{F}) \).

Remark. When \( \mathcal{I} \) and \( \mathcal{F} \) are both coherent, then \( x \in E^\rho(\mathcal{I}, \mathcal{F}) \) if and only if \( \mathcal{I}_x \) as an \( \mathcal{H}_x \)-submodule of \( \mathcal{F}_x \) has an associated prime ideal of dimension \( \leq \rho \) (Theorem 4, [12]). \( E^\rho(\mathcal{I}, \mathcal{F}) = \emptyset \) means that for every \( x \in X \) \( \mathcal{I}_x \) as an \( \mathcal{H}_x \)-submodule of \( \mathcal{F}_x \) has no associated prime ideal of dimension \( \leq \rho \).

Definition 2. Suppose \( \mathcal{I} \) is an analytic subsheaf of an analytic sheaf \( \mathcal{F} \) on a complex space \((X, \mathcal{H})\) and \( A \) is a subvariety of \( X \). Then the gap-sheaf of \( \mathcal{I} \) in \( \mathcal{F} \) with respect to \( A \), denoted by \( \mathcal{I}[A]_\mathcal{F} \), is defined as follows: For \( x \in X \), \( s \in (\mathcal{I}[A]_\mathcal{F})_x \) if and only if there exist an open neighborhood \( U \) of \( x \) in \( X \) and \( t \in \Gamma(U, \mathcal{F}) \) such that \( t_x = s \) and \( t_y \in \mathcal{I}_y \) for \( y \in U - A \).

Proposition 1. Suppose \( \mathcal{I} \) is a coherent analytic subsheaf of a coherent analytic sheaf \( \mathcal{F} \) on a complex space \((X, \mathcal{H})\) and \( \rho \) is a nonnegative integer. Then \( \mathcal{I}_{(\rho)} \) is coherent and \( E^\rho(\mathcal{I}, \mathcal{F}) \) is a subvariety of dimension \( \leq \rho \) in \( X \).

Proof. See Theorem 3 [12]. This can also be derived easily from Satz 3 [13]. Q.E.D.

Proposition 2. Suppose \( \mathcal{I} \) is a coherent analytic subsheaf of a coherent analytic sheaf \( \mathcal{F} \) on a complex space \((X, \mathcal{H})\) and \( A \) is a subvariety of \( X \). Then \( \mathcal{I}[A]_\mathcal{F} \) is coherent.

Proof. See Theorem 1 [12]. This can also be derived easily from [13, Satz 9]. Q.E.D.

Definition 3. Suppose \( \mathcal{F} \) is an analytic sheaf on a complex space \( X \) and \( \rho \) is a nonnegative integer. The \( \rho \)-th absolute gap-sheaf of \( \mathcal{F} \), denoted by \( \mathcal{F}^{[\rho]} \), is the analytic sheaf on \( X \) defined by the following presheaf: Suppose \( U \subseteq V \) are open subsets of \( X \). Then

\[
\mathcal{F}^{[\rho]}(U) = \text{ind lim}_{A \in \mathcal{A}(U)} \Gamma(U - A, \mathcal{F}),
\]

where \( \mathcal{A}(U) \) is the directed set of all analytic subvarieties in \( U \) of dimension \( \leq \rho \) directed under inclusion. \( \mathcal{F}^{[\rho]}(V) \to \mathcal{F}^{[\rho]}(U) \) is induced by restriction.

Remarks. (i) \( \mathcal{F}^{[\rho]} = (\mathcal{F}[0_{(\rho)}])^{[\rho]} \), where \( 0 \) is the zero-subsheaf of \( \mathcal{F} \).

(ii) There is a natural sheaf-homomorphism \( \mu: \mathcal{F} \to \mathcal{F}^{[\rho]} \). The kernel of \( \mu \) is \( 0_{(\rho)} \). When \( E^\rho(0, \mathcal{F}) = \emptyset \), \( \mu \) is injective and we can regard \( \mathcal{F} \) as a subsheaf of \( \mathcal{F}^{[\rho]} \). In this case we denote the set \( \{ x \in X \mid \mathcal{F}_x \neq (\mathcal{F}^{[\rho]})_x \} \) by \( E^\rho(\mathcal{F}) \).
Lemma 1. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a reduced complex space $(X, \mathcal{O})$ of pure dimension $n$. Suppose $0 \leq p \leq n-2$. If $E^{n-1}(0, \mathcal{F}) = \emptyset$, then $E^p(\mathcal{F})$ is coherent and $E^p(\mathcal{F})$ is a subvariety of dimension $\leq p$.

Proof. Let $\pi: (\bar{X}, \bar{\mathcal{O}}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\mathcal{F}$ be the inverse image of $\mathcal{F}$ under $\pi$ (Definition 8, [6]). Let $\mathcal{I}$ be the torsion-subsheaf of $\mathcal{F}$ and $\mathcal{G} = \mathcal{F}/\mathcal{I}$. Let $\mathcal{J} = \text{Supp} \mathcal{F}$ and $\mathcal{I}$ are both coherent and $\mathcal{I}$ is torsion-free (Proposition 6, [1]). $\dim \mathcal{J} \leq n-1$ (Proposition 7, [1]). We claim that

$$\mathcal{I}^{(\alpha)}$$

is coherent and $E^p(\mathcal{I})$ is a subvariety of dimension $\leq p$ in $\bar{X}$.

Take $x \in \bar{X}$. On some open neighborhood $U$ of $x$ in $\bar{X}$ $\mathcal{I}$ can be regarded as a coherent subsheaf of $\mathcal{O}_x$ for some $p$ (Proposition 9, [1]). It is clear that $\mathcal{I}^{(\alpha)}$ is isomorphic to $\mathcal{I}^{(\alpha)}$ on $U$ and $E^p(\mathcal{I}, \mathcal{O}_x) \cap U = E^p(\mathcal{I}) \cap U$. (1) follows from Proposition 1.

Let $\mathcal{F}^* = R^0\pi_!(\mathcal{F})$, $\mathcal{G}^* = R^0\pi_!(\mathcal{G})$, and $(\mathcal{I}^{(\alpha)})^* = R^0\pi_!(\mathcal{I}^{(\alpha)})$. Let $\alpha: \mathcal{F}^* \to \mathcal{G}^*$ and $\beta: \mathcal{F}^* \to (\mathcal{I}^{(\alpha)})^*$ be induced respectively by the quotient map $\mathcal{F} \to \mathcal{I}$ and the inclusion map $\mathcal{I} \to \mathcal{I}^{(\alpha)}$. We have a natural sheaf-homomorphism $\lambda: \mathcal{F} \to \mathcal{F}^*$ (Satz 7(b), [6]). Let $Z$ be the set of all singular points of $X$. Let $\mathcal{H}$ be the kernel of $\alpha\lambda$. Then $\text{Supp} \mathcal{H} \subset Z \cup \pi(Y)$. Since $E^{n-1}(0, \mathcal{F}) = \emptyset$ and $\dim \mathcal{H} \leq n-1$, $\mathcal{H} = 0$. $\gamma = \beta\alpha\lambda: \mathcal{F} \to (\mathcal{I}^{(\alpha)})^*$ is injective. It is easily seen that $((\mathcal{I}^{(\alpha)})^*|_\mathcal{O} = (\mathcal{I}^{(\alpha)})^*$. $\gamma$ induces a sheaf-monomorphism $\gamma_1: \mathcal{F}^{(\alpha)} \to (\mathcal{I}^{(\alpha)})^*$, $\mathcal{F}^{(\alpha)} = \gamma_1(\mathcal{F}^{(\alpha)}) = \gamma(\mathcal{F}^{(\alpha)}|_{\mathcal{O}^{(\alpha)}})$, and $E^p(\mathcal{F}) = E^p(\gamma(\mathcal{F}^{(\alpha)}), (\mathcal{I}^{(\alpha)})^*)$. Since by Proposition 1 $\gamma(\mathcal{F}^{(\alpha)})$ is coherent and $E^p(\gamma(\mathcal{F}^{(\alpha)}), (\mathcal{I}^{(\alpha)})^*)$ is a subvariety of dimension $\leq p$ in $X$, the Lemma follows. Q.E.D.

Lemma 2. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{O})$. Suppose $x \in X$ and $f \in \mathcal{O}_x$ such that for every nonnegative integer $p$ either $x \notin E^p(0, \mathcal{F})$ or $f$ does not vanish identically on any branch-germ of $E^p(0, \mathcal{F})$ at $x$. Then $f$ is not a zero-divisor for $\mathcal{F}_x$.

Proof. Suppose the contrary. Then there exist $s \in \Gamma(U, \mathcal{F})$ and $g \in \Gamma(U, \mathcal{O})$ for some open neighborhood $U$ of $x$ such that $g_s = f$, $gs = 0$, and $s_x \neq 0$. Let $Z = \text{Supp} s$ and $\dim \text{Z} = \rho$. By shrinking $U$, we can assume that $\dim Z = \rho$. Hence $Z \subset E^\rho(0, \mathcal{F})$. Since $\dim E^\rho(0, \mathcal{F}) \leq \rho$, the union $Z_0$ of all $\rho$-dimensional branches of $Z$ is equal to the union of some $\rho$-dimensional branches of $E^\rho(0, \mathcal{F}) \cap U$. By assumption $g$ does not vanish identically on $Z_0$. For some $y \in Z_0$, $g_x$ is a unit in $\mathcal{H}_y$, $s_y = 0$, contradicting that $Z = \text{Supp} s$. Q.E.D.

Lemma 3. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $X$ and $\rho$ is a nonnegative integer. If $E^\rho(0, \mathcal{F}) = \emptyset$, then for any nonnegative integer $\sigma$ either $E^\sigma(0, \mathcal{F}) = \emptyset$ or every branch of $E^\sigma(0, \mathcal{F})$ has dimension $> \rho$.

Proof. Suppose $Y$ is a nonempty $m$-dimensional branch of $E^\rho(0, \mathcal{F})$ for some nonnegative integer $\sigma$ such that $m \leq \rho$. Take a Stein open subset $U$ of $X$ such that $U \cap E^\rho(0, \mathcal{F}) = U \cap Y \neq \emptyset$. Take $x \in U \cap Y$. Since $(E^\rho(0, \mathcal{F}))_x \neq 0$, there exists
Lemma 4. Suppose $F_i$, $1 \leq i \leq 3$, are coherent analytic sheaves on a complex space $(X, \mathcal{A})$ and $p$ is a nonnegative integer such that $E^p(0, F_i) = 0$ for $1 \leq i \leq 3$. Suppose $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of sheaf-homomorphisms. If $(F_i)^{[\alpha_1]}$ is coherent and $E^p(F_i)$ is a subvariety of dimension $\leq \rho$ for $i = 1, 3$, then $(F_2)^{[\alpha_1]}$ is coherent and $E^p(F_2)$ is a subvariety of dimension $\leq \rho$.

Proof. Let $X_i = E^p(F_i)$, $i = 1, 3$. The problem is local in nature. Take $x_0 \in X$ and take an open Stein neighborhood $U$ of $x_0$ in $X$. $F_i$ is a coherent analytic subsheaf of $(F_i)^{[\alpha_1]}$, $i = 1, 3$. Let $A_i = (F_i)^{[\alpha_1]}$, $i = 1, 3$. $E(A_i, F_i) = X_i$, $i = 1, 3$. Let $J_i$ be the ideal-sheaf for $X_i$, $i = 1, 3$. By Hilbert Nullstellensatz, after shrinking $U$, we can find a natural number $m$ such that $J_i^m \subset A_i$ on $U$, $i = 1, 3$. By Lemma 3 for any nonnegative integer $\sigma$ every nonempty branch of $E^\sigma(0, F_2)$ has dimension $> \rho$. Since $\dim X_i \leq \rho$, $i = 1, 3$, we can choose $f \in \Gamma(U, F_i^\sigma \cap F_3)$ such that $f_{x_0}$ does not vanish identically on any nonempty branch-germ of $E^\sigma(0, F_2)$ at $x_0$ for any nonnegative integer $\sigma$. By Lemma 2 $f_{x_0}$ is not a zero-divisor for $(F_2)^{[\sigma]}$. Let $\mathcal{A}_i$ be the kernel of the sheaf-homomorphism $\alpha : F_2 \to F_3$ on $U$ defined by multiplication by $f$. Then $X_{x_0} = 0$. By shrinking $U$, we can assume that $X = 0$ on $U$. $\alpha$ induces a sheaf-homomorphism $\beta : (F_2)^{[\sigma]} \to (F_2)^{[\sigma]}$. Let $y = \beta \circ \beta$. We claim that $y((F_2)^{[\sigma]}) \subset F_2$ on $U$. Take $s \in ((F_2)^{[\sigma]})_x$ for some $x \in U$. $s$ is defined by some $t \in \Gamma(W - A, F_2)$, where $W$ is an open neighborhood of $x$ in $U$ and $A$ is a subvariety of dimension $\leq \rho$ in $W$. $\eta(t) \in \Gamma(W - A, F_2)$ defines an element $a$ of $((F_2)^{[\sigma]})_x$. $f_x a \in (F_3)_x$. By shrinking $W$ we can find $u \in \Gamma(W, F_3)$ such that $u$ agrees with $f(t)$ on $W - A$ and we can find $v \in \Gamma(W, F_2)$ such that $\eta(v) = u$. $\eta(v - ft) = 0$ on $W - A$. $v - ft$ defines an element $b$ of $((F_2)^{[\sigma]})_x$. By shrinking $W$ we can find $w \in \Gamma(W, F_2)$ such that $w$ agrees with $f(v - ft)$ on $W - A$. $f^w x = f^w w$ on $W - A$. $\eta(s) = b(v) - w x \in (F_3)_x$. Hence $y((F_2)^{[\sigma]}) \subset F_2$. It is easily seen that $y((F_2)^{[\sigma]}) = y(\mathcal{A}_2)^{[\sigma]} F_2$ on $U$ and $E^p(F_2) \cap U = E^p(y(F_2)_x, F_2) \cap U$. The Lemma follows from Proposition 1. Q.E.D.

Lemma 5. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{A})$ of pure dimension $n$ and $0 \leq \rho \leq n - 2$. If $E^{n-1}(0, \mathcal{F}) = \varnothing$, then $\mathcal{F}^{[\alpha]}$ is coherent and $E^\rho(\mathcal{F})$ is a subvariety of dimension $\leq \rho$.

Proof. Let $\mathcal{A}$ be the subsheaf of all nilpotent elements of $\mathcal{A}$ and $\mathcal{O} = \mathcal{A} \mid \mathcal{A}$. Since the lemma is local in nature, we can suppose that for some nonnegative integer $k \mathcal{A}^k = 0$. For $0 \leq l \leq k$ define $\mathcal{F}^{(l)}$ inductively as follows: $\mathcal{F}^{(0)} = \mathcal{F}$ and, for $1 \leq l \leq k$, $\mathcal{F}^{(l)} = (X \mathcal{F}^{(l-1)}(\mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)}))$. $Y$ is a subvariety of dimension $\leq n - 1$. On $X - Y \mathcal{F}^{(l)} = \mathcal{F}^{(l-1)}$ for $1 \leq l \leq k$. Hence $\mathcal{F}^{(k)} = 0$ on $X - Y$. Since $\mathcal{F}^{(k)} \subset \mathcal{F}$ and $E^{n-1}(0, \mathcal{F}) = \varnothing$, $\mathcal{F}^{(k)} = 0$. From the definition of $\mathcal{F}^{(l)}$ we see that $E^{n-1}(\mathcal{F}^{(l)}(\mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)})) = \varnothing$ for $1 \leq l \leq k$. Hence $E^{n-1}(0, \mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)}(\mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)})) = \varnothing$ implies that $E^{n-1}(0, \mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)}(\mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)})) = \varnothing$. $E^{n-1}(0, \mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)}(\mathcal{F}^{(l-1)} \cap \mathcal{F}^{(l-1)}))$ can be regarded as a coherent analytic sheaf on $(X, \mathcal{O})$.
1 ≤ l ≤ k. By Lemma 1 \((\mathcal{F}^{(i-1)}/\mathcal{F}^{(i)})^{(o)}\) is coherent and \(E^{o}(\mathcal{F}^{(i-1)}/\mathcal{F}^{(i)})\) is a subvariety of dimension \(≤ ρ\). Since \(\mathcal{F}^{(k)} = 0\), from Lemma 4 and the exact sequences \(0 → \mathcal{F}^{(i)} → \mathcal{F}^{(i-1)} → \mathcal{F}^{(i-1)}/\mathcal{F}^{(i)} → 0\), 1 ≤ l ≤ k, we conclude by backward induction on \(l\) that \((\mathcal{F}^{(i)})^{(o)}\) is coherent and \(E^{o}(\mathcal{F}^{(i)})\) is a subvariety of dimension \(≤ ρ\) for 0 ≤ l ≤ k. The Lemma follows from \(\mathcal{F} = \mathcal{F}^{(0)}\). Q.E.D.

**Lemma 6.** Suppose \(\mathcal{F}\) is a coherent analytic sheaf on a complex space \((X, \mathcal{H})\) and \(ρ\) is a nonnegative integer. Let \(Y\) be the union of \((ρ+1)\)-dimensional branches of \(E^{ρ+1}(0, \mathcal{F})\). Then for \(x \in Y (\mathcal{F}^{(o)})_x\) is not finitely generated over \(\mathcal{H}_x\).

**Proof.** We can assume that \(Y \neq \emptyset\). Let \(\mathcal{G} = \mathcal{F}|_{0 \times \emptyset}\). Since \(E^0(0, \mathcal{G}) = \emptyset\), by Lemma 3 and Proposition 1 every branch of \(E^{ρ+1}(0, \mathcal{G})\) is \((ρ+1)\)-dimensional. Since \(\mathcal{G}\) agrees with \(\mathcal{F}\) on \(X - E^0(0, \mathcal{F})\), \(E^{ρ+1}(0, \mathcal{G}) - E^{ρ}(0, \mathcal{F}) = E^{ρ+1}(0, \mathcal{F}) - E^{ρ}(0, \mathcal{F})\). \(\dim E^0(0, \mathcal{F}) ≤ ρ\) implies \(\dim E^{ρ}(0, \mathcal{F}) = Y\).

Fix \(x \in Y\). Suppose \((\mathcal{F}^{(o)})_x\) is finitely generated over \(\mathcal{H}_x\). Let \(\mathcal{J} = 0^{(ρ+1)}\mathcal{J}\). Since \(E^0(0, \mathcal{J}) = E^0(0, \mathcal{G}) = \emptyset\), \(\mathcal{J} \subset \mathcal{F}^{(o)} \subset \mathcal{G} = \mathcal{F}^{(o)}\). Since \(\text{Supp } \mathcal{J} = E^{ρ+1}(0, \mathcal{F}) = Y\), \((\mathcal{F}^{(o)})_x\) is a nonzero finitely generated \(\mathcal{H}_x\)-module. Let \((\mathcal{F}^{(o)})_x\) be generated by \(s_1, \ldots, s_m \in (\mathcal{F}^{(o)})_x\). For some open neighborhood \(U\) of \(x\) in \(X\) and for some subvariety \(A\) of dimension \(≤ ρ\) in \(U\) \(s_i\) is induced by \(t_i \in \Gamma(U - A, \mathcal{J})\), 1 ≤ i ≤ m. By shrinking \(U\), we can choose \(f \in \Gamma(U, \mathcal{J})\) such that \(W = Z(f) \cap Y\) is a subvariety of dimension \(ρ\) in \(U\) and \(x \in Z(f)\), where \(Z(f) = \{y \in U | f_y\) is not a unit in \(\mathcal{H}_x\}\). There exists a unique \(g \in \Gamma(U - Z(f), \mathcal{J})\) such that \(g \cdot f = 1\) on \(U - Z(f)\). For 1 ≤ i ≤ m define \(u_i \in \Gamma(U - (A \cup W), \mathcal{J})\) by \((u_i)_y = 0\) for \(y \in U - Y\) and \((u_i)_y = (g \cdot f)^{-1}|_y\) for \(y \in Y \cap (U - (A \cup W))\). \(u_i\) induces \(v_i \in (\mathcal{F}^{(o)})_x\), 1 ≤ i ≤ m. \(f \cdot v_i = s_i\), 1 ≤ i ≤ m. For some \(a_{ij} \in \mathcal{H}_x\), \(v_i = \sum_{n=1}^{m} a_{ij} s_j\), 1 ≤ i ≤ m. \(s_i = f \cdot v_i = \sum_{n=1}^{m} a_{ij} f \cdot s_j\), 1 ≤ i ≤ m. \((\mathcal{F}^{(o)})_x\) = \(f \cdot (\mathcal{F}^{(o)})_x\). Since \(f \cdot x\) is not a unit in \(\mathcal{H}_x\), by \([8, (4.1)]\) we have \((\mathcal{F}^{(o)})_x = 0\) (contradiction). Q.E.D.

**Theorem 1.** Suppose \(\mathcal{F}\) is a coherent analytic sheaf on a complex space \((X, \mathcal{H})\) and \(ρ\) is a nonnegative integer. Then \(\mathcal{F}^{(o)}\) is coherent if and only if \(\dim E^{ρ+1}(0, \mathcal{F}) < ρ+1\). In that case \(E^{o}(\mathcal{F}|_{0 \times \emptyset})\) is a subvariety of dimension \(≤ ρ\).

**Proof.** It follows from Lemma 6 that, if \(\mathcal{F}^{(o)}\) is coherent, then \(\dim E^{ρ+1}(0, \mathcal{F}) < ρ+1\).

Suppose now \(\dim E^{ρ+1}(0, \mathcal{F}) < ρ+1\). We are going to prove that \(\mathcal{F}^{(o)}\) is coherent and \(E^{o}(\mathcal{F}|_{0 \times \emptyset})\) is a subvariety of dimension \(≤ ρ\). Since \(\mathcal{F}\) agrees with \(\mathcal{F}|_{0 \times \emptyset}\) on \(X - E^0(0, \mathcal{F}), E^{ρ+1}(0, \mathcal{F}|_{0 \times \emptyset})\) is contained in the subvariety \(E^{ρ}(0, \mathcal{F}) \cup E^{ρ+1}(0, \mathcal{F}|_{0 \times \emptyset})\) of dimension \(≤ ρ\). \(E^{ρ}(0, \mathcal{F}|_{0 \times \emptyset}) = 0\) implies \(E^{ρ+1}(0, \mathcal{F}|_{0 \times \emptyset}) = 0\) by Lemma 3. Since \((\mathcal{F}^{(o)}) = (\mathcal{F}|_{0 \times \emptyset})^{(o)}\), by replacing \(\mathcal{F}\) by \(\mathcal{F}|_{0 \times \emptyset}\), we can assume that \(E^{ρ+1}(0, \mathcal{F}) = 0\). Since the problem is local in nature, we can suppose that \(X\) is of finite dimension \(\nu\). If \(n < ρ+2\), \(E^{ρ+1}(0, \mathcal{F}) = 0\) implies that \(\mathcal{F} = 0\). \(\mathcal{F}^{(o)} = 0\) is coherent and \(E^{o}(\mathcal{F}) = 0\). So we can assume that \(n ≥ ρ+2\). For \(ρ+1 ≤ m ≤ n\) let \(\mathcal{G}^{(m)} = 0_{(m)}\). \(\mathcal{G}^{(ρ+1)} = 0\), because \(E^{ρ+1}(0, \mathcal{F}) = 0\). For \(ρ+2 ≤ m ≤ n\) let \(X_m = \text{Supp } \mathcal{G}^{(m)}/\mathcal{G}^{(m-1)}\). \(X_m\) is the union of all \(m\)-dimensional branches of \(E^{m}(0, \mathcal{F})\),
\[ p + 2 \leq m \leq n. \ E^{m-1}(0, \mathcal{O}_{X}) = 0 \text{ for } p + 2 \leq m \leq n. \] For \( p + 2 \leq m \leq n \) let \( \mathcal{A}(m) \) be the annihilator-ideal-sheaf for \( \mathcal{O}_{X} \). Then \( \mathcal{A}(m) \) can be regarded as a coherent analytic sheaf on the complex space \( (X, \mathcal{F}/\mathcal{A}(m)) \mid X \) which is either empty or of pure dimension \( m, p + 2 \leq m \leq n \). By Lemma 5

\[ ((\mathcal{O}_{X}/\mathcal{O}_{m})/\mathcal{A}(m)) \mid X \] is coherent and \( E^p((\mathcal{O}_{X})/\mathcal{A}(m)) \mid X \) is a subvariety of dimension \( \leq p, p + 2 \leq m \leq n \). Since \( \mathcal{O}_{p+2} = \mathcal{O}_{p+1} \), from Lemma 4 and the exact sequences \( 0 \rightarrow \mathcal{O}_{m-1} \rightarrow \mathcal{O}_{m} \rightarrow \mathcal{O}_{m}/\mathcal{O}_{m-1} \rightarrow 0, p + 3 \leq m \leq n \), we conclude by induction on \( m \) that \( \mathcal{A}(m)/\mathcal{O}_{m} \) is coherent and \( E^p(\mathcal{O}_{m}) \) is a subvariety of dimension \( \leq p, p + 2 \leq m \leq n \). The Theorem follows from \( F = \mathcal{O}_{n} \). Q.E.D.

**Corollary 1.** Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \), \( p \) is a nonnegative integer, and \( x \in X \). \( \mathcal{F}(p) \) is coherent at \( x \) if and only if \( x \) does not belong to a \( (p+1) \)-dimensional branch of \( E^{p+1}(0, \mathcal{F}) \). Hence the set of points where \( \mathcal{F}(p) \) is not coherent is either empty or it is a subvariety of pure dimension \( p + 1 \).

**Remark.** Under the assumption of Corollary 1 to Theorem 2 \( x \) does not belong to a \( (p+1) \)-dimensional branch of \( E^{p+1}(0, \mathcal{F}) \) if and only if the zero submodule of \( \mathcal{F}_x \) has no associated prime ideal of dimension \( p + 1 \) [12, Theorem 4]. This gives us an algebraic criterion for the coherence of \( \mathcal{F}(p) \) at \( x \).

**Corollary 2.** Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \) and \( p \) is a nonnegative integer. Let \( \mu: \mathcal{F} \rightarrow \mathcal{F}(p) \) be the natural sheaf-homomorphism. Then \( Z = \{ x \in X \mid \mu_x \text{ is not surjective} \} \) is a subvariety of dimension \( \leq p + 1 \).

**Proof.** Let \( Y \) be the union of all \( (p+1) \)-dimensional branches of \( E^{p+1}(0, \mathcal{F}) \). By Lemma 6 \( Y \subseteq Z \). Since \( \mathcal{F}(p) \) agrees with \((\mathcal{O}_{0}/\mathcal{O}_{p+1})_{\mathcal{F}(p)} \) on \( X - Y \), \( Z - Y = E^p(\mathcal{F}/\mathcal{O}_{0}/\mathcal{O}_{p+1}) \) is a subvariety of dimension \( \leq p + 1 \). Q.E.D.

**Remark.** Corollary 2 to Theorem 1 can be stated alternatively in the following way: The set of points where we cannot always remove closed singularities contained in subvarieties of dimension \( p \) for local sections of a coherent analytic sheaf \( \mathcal{F} \) satisfying \( E^p(0, \mathcal{F}) = \mathcal{O} \) is a subvariety of dimension \( \leq p + 1 \).

The weaker statement that this set of points is contained in a subvariety of dimension \( \leq p + 1 \) is an easy consequence of Satz III, [9] and Satz 5, [10].

**II. Extension of coherent sheaves.** Suppose \( S \) is a subvariety of a complex space \( X \) and \( \mathcal{F} \) is a coherent analytic sheaf on \( X - S \). \( \mathcal{F} \) is said to satisfy \((*) \) if for every \( x \in S \) there exists some open neighborhood \( U \) of \( x \) in \( X \) such that \( \Gamma(U - S, \mathcal{F}) \) generates \( \mathcal{F} \) on \( U - S \).

**Lemma 7.** Suppose \( S \) is a subvariety of codimension \( \geq 2 \) in a reduced complex space \( (X, \mathcal{O}) \) of pure dimension \( n \). Let \( \theta: X - S \rightarrow X \) be the inclusion map. Suppose \( \mathcal{F} \)
is a coherent analytic sheaf on $X - S$ such that $E^{s+1}(0, \mathcal{F}) = \emptyset$. If $\mathcal{F}$ satisfies $(\ast)_{x,S}$, then $R^0\theta(\mathcal{F})$ is coherent.

**Proof.** Let $\pi : (\bar{X}, \bar{\mathcal{O}}) \rightarrow (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\bar{S} = \pi^{-1}(S)$ and $\pi' = \pi|(\bar{X} - \bar{S})$. Let $\theta : \bar{X} - \bar{S} \rightarrow \bar{X}$ be the inclusion map. Let $\mathcal{F}$ be the inverse image of $\mathcal{O}$ under $\pi'$. Let $\mathcal{F}'$ be the torsion-subsheaf of $\mathcal{F}$, $\mathcal{F}' = \mathcal{F}/\mathcal{F}$, and $Y = \text{Supp} \mathcal{F}$. Since $\mathcal{F}$ satisfies $(\ast)_{x,S}$, $\mathcal{F}'$ satisfies $(\ast)_{x,\bar{S}}$. This implies that $\mathcal{F}$ satisfies $(\ast)_{x,S}$. By Theorem 1, $R^0\theta(\mathcal{F}')$ is coherent on $\bar{X}$. Let $\mathcal{F}^* = R^0\pi'(\mathcal{F}')$ and $\mathcal{F}^* = R^0\pi(R^0\theta(\mathcal{F}')). \mathcal{F}^*$ is coherent on $X$. Let the sheaf-homomorphism $\alpha : \mathcal{F}^* \rightarrow \mathcal{F}$ on $X - S$ be induced by the quotient map $\mathcal{F} \rightarrow \mathcal{F}$. We have a natural sheaf-homomorphism $\lambda : \mathcal{F} \rightarrow \mathcal{F}^*$. Let $Z$ be the set of all singular points on $X$. Let $\mathcal{X}'$ be the kernel of $\alpha \lambda$. Then $\text{Supp} \mathcal{X}' \subset Z \cup \pi(Y)$. Since $E^{s-1}(0, \mathcal{F}) = \emptyset$ and $\dim \text{Supp} \mathcal{X}' \leq n - 1$, $\mathcal{X}' = 0$. $\alpha \lambda$ is injective. Since $R^0\theta(\mathcal{F}^* | X - S) = \mathcal{G}^*$, $\alpha \lambda$ induces a sheaf-homomorphism $\beta : R^0\theta(\mathcal{F}) \rightarrow \mathcal{G}^*$. Take $x \in S$. There exists an open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U - S, \mathcal{F})$ generates $\mathcal{F}$ on $U - S$. Let $s \in \Gamma(U - S, \mathcal{F})$ be an element of $\mathcal{O}(U - S)$. $s$ can be extended uniquely to an element of $\mathcal{O}(U)$. Define a sheaf-homomorphism $\phi : \mathcal{H} \rightarrow \mathcal{F}$ on $U$ which agrees with $\eta s$ on $U - S$. Let $\mathcal{X}$ be the kernel of $\psi$. $\mathcal{X}$ is coherent. There exist $u_1, \ldots, u_n$ generating $\mathcal{X}$. Let $\eta = \phi(u_i | (W - S))$, $1 \leq i \leq n$. Then $\eta \in \Gamma(W - S, \mathcal{F})$, $1 \leq i \leq n$, and $R^0\theta(\mathcal{F})$ is coherent. Hence $R^0\theta(\mathcal{F})$ is coherent. Q.E.D.

**Lemma 8.** Suppose $S$ is a subvariety in a complex space $(X, \mathcal{H})$. Let $\theta : X - S \rightarrow X$ be the inclusion map. Suppose $\mathcal{F}_i$, $1 \leq i \leq 3$, are coherent analytic sheaves on $X - S$ such that $R^0\theta(\mathcal{F}_j)$ is coherent. Suppose $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is an exact sequence of sheaf-homomorphisms on $X - S$. If $\mathcal{F}_2$ satisfies $(\ast)_{x,S}$, then $\mathcal{F}_1$ satisfies $(\ast)_{x,S}$.

**Proof.** Take $x \in S$. There is an open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U - S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $U - S$. Let $W$ be a Stein open neighborhood of $x$ in $U$. We claim that $\Gamma(W - S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $W - S$. Take $y \in W - S$. There exist $s \in \Gamma(W - S, \mathcal{F}_2)$, $1 \leq i \leq m$, generating $(\mathcal{F}_2)_y$. Define a sheaf-homomorphism $\varphi : \mathcal{H} \rightarrow \mathcal{F}_2$ on $U - S$ by $\varphi(s_1, \ldots, s_m) = \sum_{i=1}^m \alpha(s_i)$. $\alpha$ and $z \in U - S$. $\eta(s_i)$ can be extended uniquely to an element of $\Gamma(U, R^0\theta(\mathcal{F}_2))$, $1 \leq i \leq m$. There exists a unique sheaf-homomorphism $\psi : \mathcal{H} \rightarrow R^0\theta(\mathcal{F}_2)$ on $U$ which agrees with $\eta \psi$ on $U - S$. Let $\mathcal{X}'$ be the kernel of $\psi$. $\mathcal{X}'$ is coherent. There exist $u_1, \ldots, u_n$ generating $\mathcal{X}'$. Let $v_i = \varphi(u_i | (W - S))$, $1 \leq i \leq n$. Then $v_i \in \Gamma(W - S, \mathcal{F}_2)$, $1 \leq i \leq n$, and $(\mathcal{F}_2)_y$ is generated by $v_1, \ldots, v_n$. Q.E.D.

**Lemma 9.** Suppose $S$ is a subvariety of dimension $\rho$ in a complex space $X$. Let $\theta : X - S \rightarrow X$ be the inclusion map. Suppose $\mathcal{F}_i$, $1 \leq i \leq 3$, are coherent analytic sheaves on $X - S$ such that $R^0\theta(\mathcal{F}_j)$ is coherent for $j = 1, 3$. Suppose $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is an exact sequence of sheaf-homomorphisms on $X - S$. If $\mathcal{F}_2$ satisfies $(\ast)_{x,S}$ and $E^{s+1}(0, \mathcal{F}_2) = \emptyset$, then $R^0\theta(\mathcal{F}_2)$ is coherent.

**Proof.** Take $x \in S$. We need only prove that $R^0\theta(\mathcal{F}_2)$ is coherent at $x$. There is a Stein open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U - S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $U - S$.
The exact sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) induces the exact sequence \( 0 \to R^0\theta(\mathcal{F}_1) \to R^0\theta(\mathcal{F}_2) \to R^0\theta(\mathcal{F}_3) \). For \( s \in \Gamma(U-S, \mathcal{F}_2) \) let \( \tilde{s} \in \Gamma(U, R^0\theta(\mathcal{F}_2)) \) be the unique extension of \( s \) and let \( \tilde{s} = \eta(\tilde{s}) \). Let \( \mathcal{I} \) be the subsheaf of \( R^0\theta(\mathcal{F}_2) \) on \( U \) generated by \( \{ \tilde{s} \mid s \in \Gamma(U-S, \mathcal{F}_2) \} \) and \( \mathcal{F} \) be the subsheaf of \( R^0\theta(\mathcal{F}_3) \) on \( U \) generated by

\[ \{ s \mid s \in \Gamma(U-S, \mathcal{F}_3) \}. \]

\( \eta(\mathcal{I}) = \mathcal{F} \). Since \( R^0\theta(\mathcal{F}_2) \) is coherent, \( \mathcal{I} \) being generated by global sections is coherent. Since \( R^0\theta(\mathcal{F}_1) \) is coherent and \( U \) is Stein, on \( U \) \( R^0\theta(\mathcal{F}_2) \) is generated by \( \Gamma(U, R^0\theta(\mathcal{F}_1)) \approx \Gamma(U-S, \mathcal{F}_1) \subset \Gamma(U-S, \mathcal{F}_2) \). We have an exact sequence \( 0 \to R^0\theta(\mathcal{F}_2) \xrightarrow{\iota} \mathcal{I} \xrightarrow{\mathcal{I}} R^0\theta(\mathcal{F}_3) \to 0 \), where \( \eta'' \) is induced by \( \eta' \) and \( \mathcal{I} \) is the inclusion map. Since \( R^0\theta(\mathcal{F}_1) \) and \( \mathcal{I} \) are both coherent, \( \mathcal{I} \) is coherent. \( E^{p+1}(0, \mathcal{I}) \subset E^{p+1}(0, \mathcal{F}_2) = \mathcal{O} \). By Theorem 1 \( \mathcal{F}^{(p)} \) is coherent. Since \( \dim S = p \), \( R^0\theta(\mathcal{F}^{(p)}) = \mathcal{I}^{(p)} \). The inclusion map \( \mathcal{F}_2 \to \mathcal{I} \) on \( U-S \) induces on \( U \) a sheaf-monomorphism \( \beta: R^0\theta(\mathcal{F}_2) \to \mathcal{F}^{(p)}. \beta(\mathcal{R}^0\theta(\mathcal{F}_2)) = \mathcal{F}[S] \mathcal{R}^{(p)}. \) Since \( \mathcal{F}[S] \mathcal{R}^{(p)} \) is coherent by Proposition 2, \( R^0\theta(\mathcal{F}_3) \) is coherent on \( U \). Q.E.D.

**Lemma 10.** Suppose \( S \) is a subvariety of codimension \( \geq 2 \) in a complex space \( (X, \mathcal{H}) \) of pure dimension \( n \). Let \( \theta: X-S \to X \) be the inclusion map. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( X-S \). If \( \mathcal{F} \) satisfies \((*)_{X,S} \) and \( E^{n-1}(0, \mathcal{F}) = \mathcal{O} \), then \( R^0\theta(\mathcal{F}) \) is coherent on \( X \).

**Proof.** Let \( \mathcal{H} \) be the subsheaf of all nilpotent elements of \( \mathcal{H} \) and \( \mathcal{O} = \mathcal{H} | X \). Since the Lemma is local in nature, we can suppose that for some nonnegative integer \( k \mathcal{H}^k = 0 \). For \( 0 \leq l \leq k \) define coherent analytic sheaves \( \mathcal{F}^{(l)} \) on \( X-S \) inductively as follows: \( \mathcal{F}^{(0)} = \mathcal{F} \) and, for \( 1 \leq l \leq k \), \( \mathcal{F}^{(l)} = (\mathcal{H} \mathcal{F}^{(l-1)})_{(n-1)\mathcal{F}^{(l-1)}}. \)

Let

\[ Y = \bigcup_{i=1}^{k} E^{n-1}(X \mathcal{F}^{(l-1)}, \mathcal{F}^{(l-1)}). \]

\( Y \) is a subvariety in \( X-S \) of dimension \( \leq n-1 \). On \( X-(S \cup Y) \), \( \mathcal{F}^{(l)} = \mathcal{X} \mathcal{F}^{(l-1)} \) for \( 1 \leq l \leq k \). Hence \( \mathcal{F}^{(k)} = 0 \) on \( X-(S \cup Y) \). Since \( \mathcal{F}^{(k)} \subset \mathcal{F} \) and \( E^{n-1}(0, \mathcal{F}) = \mathcal{O} \), \( \mathcal{F}^{(k)} = 0 \) on \( X-S \). From the definition of \( \mathcal{F}^{(l)} \) we see that \( E^{n-1}(\mathcal{F}^{(l)}, \mathcal{F}^{(l-1)}) = \mathcal{O} \) for \( 1 \leq l \leq k \). Hence \( E^{n-1}(0, \mathcal{F}^{(l-1)}, \mathcal{F}^{(l)}) = 0 \) for \( 1 \leq l \leq k \). \( E^{n-1}(0, \mathcal{F}) = \mathcal{O} \) implies that \( E^{n-1}(0, \mathcal{F}) = \mathcal{O} \) for \( 1 \leq l \leq k \). Since \( \mathcal{F}^{(l-1)} \subset \mathcal{F}^{(l)} \), \( \mathcal{F}^{(l-1)} \subset \mathcal{F}^{(l)} \) can be regarded as a coherent analytic sheaf on \( (X-S, \mathcal{O} | (X-S)) \), \( 1 \leq l \leq k \).

Set \( \mathcal{F}^{(k+1)} = 0 \). We are going to prove (2), for \( 0 \leq l \leq k \) by induction on \( l \):

(2), \( \mathcal{F}^{(l)} \) satisfies \((*)_{X,S} \) and \( R^0\theta(\mathcal{F}^{(l)}/\mathcal{F}^{(l+1)}) \) is coherent.

Since \( \mathcal{F}^{(0)} = \mathcal{F} \), \( \mathcal{F}^{(0)} \) satisfies \((*)_{X,S} \), \( \mathcal{F}^{(0)}/\mathcal{F}^{(1)} \) satisfies \((*)_{X,S}. \) By Lemma 7

\[ R^0\theta(\mathcal{F}^{(0)}/\mathcal{F}^{(1)}) \]

is coherent. (2)0 is true. Suppose for some \( 0 \leq m < k \) (2)m is true. By Lemma 8 and
the exact sequence \( 0 \rightarrow \mathcal{F}(m+1) \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}(m+1)/\mathcal{F}(m+2) \rightarrow 0 \), we conclude that \( \mathcal{F}(m+1) \) satisfies \((*)_{X,S}\). Hence \( \mathcal{F}(m+1)/\mathcal{F}(m+2) \) satisfies \((*)_{X,S}\). By Lemma 7

\[ R^0\theta(\mathcal{F}(m+1)/\mathcal{F}(m+2)) \]

is coherent. \((2)_{m+1}\) is true. Hence \((2)_{l}\) holds for \(0 \leq l \leq k\).

Now we are going to prove \((3)_{l}\) for \(0 \leq l \leq k\) by backward induction on \(l\):

\[ (3)_l \]

\[ R^0\theta(\mathcal{F}(i)) \text{ is coherent.} \]

Since \( \mathcal{F}(k) = 0 \), \((3)_k\) is true. Suppose \((3)_m\) is true for some \(0 < m \leq k\). From \((2)_{m-1}\), \((3)_m\), Lemma 10 and the exact sequence \( 0 \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}(m-1) \rightarrow \mathcal{F}(m-1)/\mathcal{F}(m) \rightarrow 0 \), we conclude that \((3)_{m-1}\) is true. Hence \((3)_l\) holds for \(0 \leq l \leq k\). The Lemma follows from \((3)_0\). Q.E.D.

**Lemma 11.** Suppose \( S \) is a subvariety of dimension \( p \) in a complex space \((X, \mathcal{H})\). Suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( X-S \) such that \( \text{Supp} \mathcal{F} \) is a subvariety of pure dimension \( n > p \) and \( E^{n-1}(0, \mathcal{F}) = \emptyset \). Then there exists a complex subspace \((Y, \mathcal{H})\) of pure dimension \( n \) in \((X, \mathcal{H})\) such that \( Y-S=\text{Supp} \mathcal{F} \) and \( \mathcal{F}|(Y-S) \) can be regarded as a coherent analytic sheaf on \((Y-S, \mathcal{H}|(Y-S))\).

**Proof.** By [7, V.D.5] the topological closure \( Y \) of \( \text{Supp} \mathcal{F} \) in \( X \) is a subvariety of pure dimension \( n \). Let \( Y = \bigcup_{a \in A} Y_a \) be the decomposition into irreducible branches. Let \( \mathcal{I}_a \) be the ideal-sheaf for \( Y_a \), \( a \in A \). Choose \( x_a \in Y_a-(S \cup (\bigcup_{b \in B, b \neq a} Y_b)) \). Let \( \mathcal{I} \) be the annihilator-ideal-sheaf for \( \mathcal{F} \). Then \( E(\mathcal{I}, \mathcal{H}|(X-S)) = Y-S \). By Hilbert Nullstellensatz, there exists a natural number \( m_a \) such that \( (\mathcal{I}^{m_a})_{x_a} \subseteq \mathcal{I}_{x_a} \), \( a \in A \). Let \( \mathcal{I} = \prod_{a \in A} \mathcal{I}^{m_a} \). Then \( \mathcal{I} \) is coherent and \( (\mathcal{I} \mathcal{F})_{x_a} = 0 \) for \( a \in A \). \( \text{Supp} \mathcal{I} \mathcal{F} \) is a subvariety of dimension \( < n \) in \( X-S \). \( E^{n-1}(0, \mathcal{F}) = \emptyset \) implies that \( \mathcal{I} \mathcal{F} = 0 \). Set \( \mathcal{H} = (\mathcal{H} | \mathcal{I}) | Y \). Then \((Y, \mathcal{H})\) satisfies the requirements. Q.E.D.

**Theorem 2.** Suppose \( S \) is a subvariety of dimension \( p \) in a complex space \((X, \mathcal{H})\). Let \( \theta : X-S \rightarrow X \) be the inclusion map. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( X-S \) such that \( E^{n-1}(0, \mathcal{F}) = \emptyset \) or equivalently for every \( x \in X-S \) the zero \( \mathcal{H}_x \)-submodule of \( \mathcal{F}_x \) has no associated prime ideal of dimension \( \leq p+1 \). Then the following conditions are equivalent:

(i) \( R^0\theta(\mathcal{F}) \) is coherent.

(ii) There exists a coherent analytic sheaf on \( X \) which extends \( \mathcal{F} \).

(iii) \( \mathcal{F} \) satisfies \((*)_{X,S}\).

**Proof.** It is clear that (i) implies (ii) and (ii) implies (iii). We need only prove that (iii) implies (i). Suppose \( \mathcal{F} \) satisfies \((*)_{X,S}\). We are going to prove that \( R^0\theta(\mathcal{F}) \) is coherent. Since the problem is local in nature, we can suppose that \( X \) is of finite dimension \( n \). If \( n < p+2 \), then \( E^{n+1}(0, \mathcal{F}) = \emptyset \) implies that \( \mathcal{F} = 0 \). \( R^0\theta(\mathcal{F}) = 0 \) is coherent. So we can assume that \( n \geq p+2 \). For \( p+1 \leq m \leq n \) let \( \mathcal{F}(m) = 0 \), \( \mathcal{F}(p+1) = 0 \), because \( E^{n+1}(0, \mathcal{F}) = \emptyset \). For \( p+2 \leq m \leq n \) let \( X_m=\text{Supp} \mathcal{F}(m)/\mathcal{F}(m-1) \). Then \( X_m \) is the union of all \( m \)-dimensional branches of \( E^n(0, \mathcal{F}) \), \( p+2 \leq m \leq n \).
$E^{m-1}(0, \mathcal{G}(m)/\mathcal{G}(m-1)) = \emptyset$ for $\rho + 2 \leq m \leq n$. By Lemma 11 there exists a complex subspace $(Y_m, \mathcal{H}_m)$ of pure dimension $m$ in $(X, \mathcal{H})$ such that $Y_m - S = X_m$ and $(\mathcal{G}(m)/\mathcal{G}(m-1))(Y_m - S)$ can be regarded as a coherent analytic sheaf on

$$(Y_m - S, \mathcal{H}_m|(Y_m - S)), \rho + 2 \leq m \leq n.$$ 

Let $\theta_m: Y_m - S \to Y_m$ be the inclusion map $\rho + 2 \leq m \leq n$. $E^{\rho + 1}(0, \mathcal{F}) = \emptyset$ implies that $E^{\rho + 1}(0, \mathcal{G}(m)) = 0$ for $\rho + 2 \leq m \leq n$.

We are going to prove $(4)_m$ for $\rho + 2 \leq m \leq n$ by backward induction on $m$:

$$(4)_m \quad \text{is coherent}.$$ 

Since $\mathcal{G}(\rho) = \mathcal{F}$, $(\mathcal{G}(\rho)/\mathcal{G}(\rho-1))(Y_m - S)$ satisfies $(*)_{Y_m, Y_m - S}$. By Lemma 10 $R_0\theta(\mathcal{G}(\rho)/\mathcal{G}(\rho-1))(Y_m - S)$ is coherent. $(4)_n$ is true. Suppose for some $\rho + 2 < q \leq n$, $(4)_q$ is true. From Lemma 8, $(4)_q$ and the exact sequence $0 \to \mathcal{G}(q+1) \to \mathcal{G}(q) \to \mathcal{G}(q)/\mathcal{G}(q-1) \to 0$ we conclude that $\mathcal{G}(q-1)$ satisfies $(*)_{Y_m, Y_m - S}$. $(\mathcal{G}(q-1)/\mathcal{G}(q-2))(Y_m - S)$ satisfies $(*)_{Y_m - Y_m - S}$. By Lemma 10 $R_0\theta(\mathcal{G}(q-1)/\mathcal{G}(q-2))(Y_m - S)$ is coherent. $(4)_{q-1}$ is true. Hence $(4)_m$ holds for $\rho + 2 \leq m \leq n$.

Now we are going to prove $(5)_m$ for $\rho + 1 \leq m \leq n$ by induction on $m$:

$$(5)_m \quad \text{is coherent.}$$ 

Since $\mathcal{G}(\rho + 1) = \mathcal{F}$, $(\mathcal{G}(\rho + 1))_{Y_m, Y_m - S}$ is true. Suppose $(5)_q$ is true for some $\rho + 1 < q < n$. From $(4)_{q+1}$, $(5)_q$, Lemma 9, and the exact sequence $0 \to \mathcal{G}(q) \to \mathcal{G}(q+1) \to \mathcal{G}(q+1)/\mathcal{G}(q) \to 0$ we conclude that $R_0\theta(\mathcal{G}(q+1))$ is coherent. $(5)_{q+1}$ is true. Hence $(5)_m$ holds for $\rho + 1 \leq m \leq n$. Since $\mathcal{G}(m) = \mathcal{F}$, $(5)_n$ implies that $R_0\theta(\mathcal{F})$ is coherent. Q.E.D.

**COROLLARY.** Suppose $S$ is a subvariety of dimension $\rho$ in a complex space $(X, \mathcal{H})$ and $\theta: X - S \to X$ is the inclusion map. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X - S$ such that the homological codimension $(p. 358, [9])$ of the $\mathcal{H}_x$-module $\mathcal{F}|_x \geq \rho + 2$ for $x \in X$. Then the following conditions are equivalent:

(i) $R^p\theta(\mathcal{F})$ is coherent.
(ii) There exists a coherent analytic sheaf on $X$ which extends $\mathcal{F}$.
(iii) $\mathcal{F}$ satisfies $(*)_{X, S}$.

**Proof.** Follows from Theorem 2 and Satz I [9]. Q.E.D.

**Remark.** [14, (4.1)] is a special case of the Corollary to Theorem 2.

**III. Extensions of global sections of coherent sheaves.**

**Definition 4.** Suppose $\rho$ is a natural number. A real-valued function $v$ on a complex space $X$ is said to be *-strongly $\rho$-convex at $x \in X$ if there exist a nowhere degenerate holomorphic map $\varphi$ from some open neighborhood $U$ of $x$ in $X$ to an open subset $D$ of $\mathbb{C}^n$ and a real-valued $C^2$ function $\delta$ on $D$ such that $v = \delta\varphi$ on $U$ and at every point in $D$ the Hermitian matrix $(\partial^2\delta/\partial z_i \partial z_j)_{1 \leq i, j \leq n}$ has at least $n - \rho + 1$ positive eigenvalues.

**Definition 5.** Suppose $\rho$ is a natural number. An open subset $D$ of a complex space $X$ is said to be *-strongly $\rho$-concave at $x \in X$ if there is a *-strongly $\rho$-convex
LEMMA 12. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a reduced complex space \( (X, \mathcal{O}) \) of pure dimension \( n \) such that \( E^{*^{-1}}(0, \mathcal{F}) = \emptyset \). Suppose \( 1 \leq p < n, x \in X, \) and \( D \) is an open subset of \( X \) which is \(*\)-strongly \( p \)-concave at \( x \). Then there exist an open neighborhood \( U \) of \( x \) in \( X \), a subvariety \( V \) of dimension \( < p \) in \( U \), and a natural number \( m \) satisfying the following: If for some open neighborhood \( W \) of \( x \) in \( U \) \( f \in \Gamma(W, \mathcal{O}) \) vanishes identically on \( V \cap W \) and \( s \in \Gamma(W \cap D, \mathcal{F}) \), then \( f^m s | W' \cap D \) can be extended to an element of \( \Gamma(W', \mathcal{F}) \) for some open neighborhood \( W' \) of \( x \) in \( W \).

Proof. Let \( \pi : (\tilde{X}, \tilde{\mathcal{O}}) \to (X, \mathcal{O}) \) be the normalization of \( (X, \mathcal{O}) \). Let \( \tilde{\mathcal{F}} \) be the inverse image of \( \mathcal{F} \) under \( \pi \), \( \mathcal{F} = \mathcal{I} \mathcal{O} \mathcal{F} \mathcal{I} \) be the torsion subsheaf of \( \tilde{\mathcal{F}} \), and \( \mathcal{G} = \mathcal{F} / \mathcal{F} \) be the quotient map \( \tilde{\mathcal{F}} \to \mathcal{G} \). Let \( \pi^{-1}(x) = \{ y_1, \ldots, y_k \} \). For every \( 1 \leq i \leq k \), there exists a sheaf-monomorphism \( \alpha_i : \mathcal{G} \to \tilde{\mathcal{O}} y_i \) on some open neighborhood \( U_i \) of \( y_i \) in \( X \). By shrinking \( U_i \), \( 1 \leq i \leq k \), we can suppose that \( U_i \cap U_j = \emptyset \) for \( i \neq j \). There is an open neighborhood \( U^* \) of \( x \) in \( X \) such that \( \pi^{-1}(U^*) \subset \bigcup_{i=1}^{k} U_i \). Define a coherent analytic sheaf \( \mathcal{F} \) on \( \pi^{-1}(U^*) \) by setting \( \mathcal{F} = \tilde{\mathcal{O}} y_i \) on \( U_i \cap \pi^{-1}(U^*) \). Define \( \alpha : \mathcal{G} \to \mathcal{F} \) on \( \pi^{-1}(U^*) \) by setting \( \alpha = \alpha_i \) on \( \pi^{-1}(U^*) \cap U_i \) for \( 1 \leq i \leq k \). Let \( \beta : R^0 \pi(\mathcal{F}) \to R^0 \pi(\mathcal{G}) \) and \( \gamma : R^0 \pi(\mathcal{G}) \to R^0 \pi(\mathcal{F}) \) on \( U^* \) be induced respectively by the quotient map \( \mathcal{F} \to \mathcal{G} \) and \( \alpha \). Let \( \lambda : \mathcal{F} \to R^0 \pi(\mathcal{F}) \) be the natural map. \( E^{*^{-1}}(0, \mathcal{F}) = \emptyset \) implies that \( \xi = \gamma \beta \lambda \) is injective. Let \( V^* = E^{*^{-1}}(\xi(\mathcal{F}), R^0 \pi(\mathcal{F})) \) and let \( \mathcal{G} \) be the ideal-sheaf on \( U^* \) for \( V^* \). By Proposition 1 dim \( V^* < \rho \). Let \( \mathcal{A} = \xi(\mathcal{F}) : \xi(\mathcal{F}) \to R^0 \pi(\mathcal{G}) \). Then \( E(\mathcal{A}, \emptyset | U^*) = V^* \). Let \( U \) be a relatively compact open neighborhood of \( x \) in \( U^* \). By Hilbert Nullstellensatz there is a natural number \( m \) such that \( \mathcal{F}^m \subset \mathcal{A} \) on \( U \). Let \( V = V^* \cap U \). We claim that \( U, V \) and \( m \) satisfy the requirements.

Suppose for some open neighborhood \( W \) of \( x \) in \( U \) we have \( f \in \Gamma(W, \mathcal{O}) \) vanishing identically on \( V \cap W \) and \( s \in \Gamma(W \cap D, \mathcal{F}) \). By Proposition 6.1, [3], for some open neighborhood \( W' \) of \( x \) in \( W \) \( \xi(t) \cap W' \cap D \) can be extended to \( t \in \Gamma(W', R^0 \pi(\mathcal{F})) \). Let \( Z = \{ y \in W' | t_y \notin \xi(\mathcal{G}) \} \). \( Z = E(\xi(\mathcal{F}) : \emptyset t, \emptyset | W') \) is a subvariety in \( W' \). Since \( D \) is \(*\)-strongly \( p \)-concave at \( x \), every subvariety-germ of dimension \( \geq \rho \) at \( x \) intersects \( D \) (4.9 of Definition 2.8 and Proposition 2.9, [3]). Hence \( Z \cap D = \emptyset \) implies that \( \dim Z < \rho \). By shrinking \( W' \), we can assume that \( \dim Z < \rho \). By shrinking \( t \in \Gamma(W', \xi(\mathcal{F})) \), \( f^m t \in \Gamma(W', \xi(\mathcal{F})) \). \( f^{-1}(f^m t) \in \Gamma(W', \mathcal{F}) \) extends \( f^m s \mid W' \cap D \). Q.E.D.

LEMMA 13. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( (X, \mathcal{O}) \) of pure dimension \( n \) such that \( E^{*^{-1}}(0, \mathcal{F}) = \emptyset \). Suppose \( 1 \leq p < n, x \in X, \) and \( D \) is an open subset of \( X \) which is \(*\)-strongly \( p \)-concave at \( x \). Then there exist an open neighborhood \( U \) of \( x \) in \( X \), a subvariety \( V \) of dimension \( < p \) in \( U \), and a natural number \( m \) satisfying the following: If for some open neighborhood \( W \) of \( x \) in \( U \) \( f \in \Gamma(W, \mathcal{O}) \) function \( v \) on some open neighborhood \( U \) of \( x \) in \( X \) such that \( D \cap U = \{ y \in U | v(y) > v(x) \} \).
vanishes identically on $V \cap W$ and $s \in \Gamma(W \cap D, \mathcal{F})$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F})$ for some open neighborhood $W'$ of $x$ in $W$.

Proof. Let $\mathcal{N}$ be the subsheaf of all nilpotent elements of $\mathcal{N}$ and $\mathcal{O} = \mathcal{N}/\mathcal{N}$. Since the Lemma is local in nature, we can suppose that $\mathcal{N} = 0$ for some natural number $k$. For $0 \leq l \leq k$ define $\mathcal{F}(l)$ inductively as follows:

$$\mathcal{F}(0) = \mathcal{N}, \quad \text{and, for } 1 \leq l \leq k, \quad \mathcal{F}(l) = (\mathcal{N}/\mathcal{F}(l-1))_{\mathcal{F}(l-1)}.$$

As in the Proof of Lemma 5, we have the following:

$$\mathcal{F}(k) = 0; \quad E^{n-1}(0, \mathcal{F}(l-1)/\mathcal{F}(l)) = \emptyset \quad \text{for } 1 \leq l \leq k;$$

and $\mathcal{F}(l) = \mathcal{F}(l)/\mathcal{F}(l+1)$, $0 \leq l \leq k-1$, can be regarded as a coherent analytic sheaf on the reduced complex space $(X, \mathcal{O})$. By Lemma 12 for $0 \leq l \leq k-1$ we have a subvariety $V_l$ of dimension $< \rho$ in some open neighborhood $U_l$ of $x$ in $X$ and a natural number $p_l$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U_l$ $f \in \Gamma(W, \mathcal{N})$ vanishes identically on $V_l \cap W$ and $s \in \Gamma(W \cap D, \mathcal{F}(l))$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F}(l))$ for some open neighborhood $W'$ of $x$ in $W$.

Let $U = \bigcap_{l=0}^{k-1} U_l$ and $V = \bigcup_{l=0}^{k-1} (V_l \cap U)$. Let $m_l = \sum_{i=1}^{k_n} p_i$, $0 \leq l \leq k-1$. By considering the exact sequences $0 \rightarrow \mathcal{F}(l+1) \rightarrow \mathcal{F}(l) \rightarrow \mathcal{F}(l) \rightarrow 0$, $0 \leq l \leq k-1$, and by backward induction on $l$, we conclude the following for $0 \leq l \leq k-1$: If $f \in \Gamma(W, \mathcal{N})$ vanishes identically on $W \cap V$ and $s \in \Gamma(W \cap D, \mathcal{F}(l))$ for some open neighborhood $W$ of $x$ in $U$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F}(l))$ for some open neighborhood $W'$ of $x$ in $W$. Hence $U$, $V$, and $m = m_0$ satisfy the requirements. Q.E.D.

Lemma 14. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{O})$ and $\rho$ is a natural number such that $E^\rho(0, \mathcal{F}) = \emptyset$. Suppose $x \in X$ and $D$ is an open subset of $X$ which is *-strongly $\rho$-concave at $x$. Then there exist an open neighborhood $U$ of $x$ in $X$, a subvariety $V$ of dimension $< \rho$ in $U$, and a natural number $m$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U$ $f \in \Gamma(W, \mathcal{N})$ vanishes identically on $W \cap V$ and $s \in \Gamma(W \cap D, \mathcal{F})$ for some open neighborhood $W$ of $x$ in $U$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F})$ for some neighborhood $W'$ of $x$ in $W$.

Proof. Since the problem is local in nature, we can suppose that $X$ is of finite dimension $n$. If $n \leq \rho$, $E^\rho(0, \mathcal{F}) = \emptyset$ implies that $\mathcal{F} = 0$ and what is to be proved is trivial. So we can suppose that $n > \rho$. Define $\mathcal{G}(k) = \mathcal{F}^{k-1}$ for $k \leq n$, $\mathcal{G}(0) = 0$. For $\rho < k \leq n$ let $X_k = \text{Supp} \mathcal{G}(k)/\mathcal{G}(k-1)$ and let $\mathcal{A}(k)$ be the annihilator-ideal-sheaf for $\mathcal{G}(k)/\mathcal{G}(k-1)$. For $\rho < k \leq n$ $X_k$ is empty or of pure dimension $k$, $E^{k-1}(0, \mathcal{G}(k)/\mathcal{G}(k-1)) = \emptyset$, and $(\mathcal{G}(k)/\mathcal{G}(k-1))|X_k$ can be regarded as a coherent analytic sheaf on the complex space $(X_k, (\mathcal{O}/\mathcal{A}(k))|X_k)$. By Lemma 13, for $\rho < k \leq n$, if $x \in X_k$, there exist a subvariety $V_k$ of dimension $< \rho$ in some open neighborhood $U_k$ of $x$ in $X_k$ and a
natural number $p_k$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U_k f \in \Gamma(W, (\mathcal{M}/\mathcal{O}(k))^X_k)$ vanishes identically on $W \cap V_k$ and
\[ s \in \Gamma(W \cap D, \mathcal{O}(k)/\mathcal{O}(k-1)), \]
then $f^{p_k} | W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{O}(k)/\mathcal{O}(k-1))$ for some open neighborhood $W'$ of $x$ in $W$. For $\rho < k \leq n$, if $x \in X_k$, choose an open neighborhood $\bar{U}_k$ of $x$ in $X$ such that $\bar{U}_k \cap X_k = U_k$; and, if $x \notin X_k$, let $\bar{U}_k = X$, $V_k = \emptyset$, and $p_k = 1$.

Let $U = \bigcap_{k=\rho}^{n+1} \bar{U}_k$ and $V = \bigcup_{k=\rho}^{n+1} (U \cap V_k)$. Set $m_k = \sum_{k=\rho+1}^{n+1} p_i$. By considering the exact sequences $0 \to \mathcal{O}(k) \to \mathcal{O}(k+1) \to \mathcal{O}(k+1)/\mathcal{O}(k) \to 0$, $\rho \leq k \leq n-1$, and by induction on $k$, we conclude the following for $\rho < k \leq n$: If for some open neighborhood $W$ of $x$ in $U$ if $f \in \Gamma(W, \mathcal{M})$ vanishes on $V \cap W$ and $s \in \Gamma(W \cap D, \mathcal{O}(k))$, then $f^{p_k} | W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{O}(k))$ for some open neighborhood $W'$ of $x$ in $W$. The Lemma follows from $F = \mathcal{O}(n)$ and $m = m_k$. Q.E.D.

**Theorem 3 (Local Extension).** Suppose $F$ is a coherent analytic sheaf on a complex space $(X, \mathcal{M})$ and $\rho$ is a natural number such that $F = F^{(\rho-1)}$. Suppose $x \in X$ and $D$ is an open subset of $X$ which is $*-$strongly $\rho$-concave at $x$. Then the following is satisfied: If $s \in \Gamma(W \cap D, F)$ for some open neighborhood $W$ of $x$ in $X$, then $s | W' \cap D$ can be extended to an element $t$ of $\Gamma(W', F)$ for some open neighborhood $W'$ of $x$ in $W$ and $t_x$ is uniquely determined.

**Proof.** Since $F = F^{(\rho-1)}$, by Theorem 1, and the definition of $F^{(\rho-1)}$, $E^0(0, F) = \emptyset$. There exist an open neighborhood $U$ of $x$ in $X$, a subvariety $V$ of dimension $< \rho$ in $U$, and a natural number $m$ satisfying the requirements of Lemma 14. By Lemma 3 every branch of $E^0(0, F)$ has dimension $> \rho$ for every nonnegative integer $\sigma$. By shrinking $U$ we can assume that there is $f \in \Gamma(U, \mathcal{M})$ such that $f$ vanishes identically on $V$ and $f$ does not vanish identically on any branch of $E^0(0, F) \cap U$ for any nonnegative integer $\sigma$. By Lemma 2 the sheaf-homomorphism $\alpha: F \to F$ on $U$ defined by multiplication by $m$ is injective.

Suppose $s \in \Gamma(W \cap D, F)$. For some open neighborhood $W'$ of $x$ in $W$ $\alpha(s) | W'$ \cap D can be extended to an element $t \in \Gamma(W', F)$. $Z = \{ y \in W' : \text{dim } Z_y < \rho \}$ is a subvariety in $W'$. Since $D$ is $*-$strongly $\rho$-concave at $x$ and $Z \cap D = \emptyset$, either $x \notin Z$ or $\text{dim } Z_x < \rho$. By shrinking $W'$, we can assume that either $Z \cap W' = \emptyset$ or $\text{dim } Z < \rho$. $t \in \Gamma(W', \alpha(F))$, $\alpha(F) = F^{(\rho-1)}$ implies that $\alpha(F) \alpha^{-1}(F) = \alpha(F)$. Hence $t \in \Gamma(W', \alpha(F))$, $t = \alpha^{-1}(t) \in \Gamma(W', F)$ extends $s | W' \cap D$.

Suppose for some other open neighborhood $W''$ of $x$ in $W$ there is $t' \in \Gamma(W'', F)$ extending $s | W'' \cap D$. We are going to prove that $t'_x = t_x$. By shrinking both $W'$ and $W''$, we can assume that $W'' = W''$. $Y = \{ y \in W' : t'_y \neq t_y \}$ is a subvariety in $W'$. Since $D$ is $*-$strongly $\rho$-concave at $x$ and $Y \cap D = \emptyset$, either $x \notin Y$ or $t'_x - t_x \in (0_{(\rho-1), F}) = 0$. Q.E.D.
Theorem 4 (Global Extension). Suppose $p$ is a natural number and $v$ is a $\ast$-strongly $p$-convex function on a complex space $X$ such that $\{x \in X \mid \lambda < v(x) < \mu\}$ is relatively compact in $X$ for any two real numbers $\lambda < \mu$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X$ satisfying $\mathcal{F} |_{\{x \in X \mid v(x) > \lambda\}}$ is uniquely extendible to a section of $\mathcal{F}$ on $X$.

Proof. We can assume that $X$ as a topological space is connected. Since $E^\omega(0, \mathcal{F}) = \emptyset$, we can assume that every branch of $X$ has dimension $> p$. Fix $\lambda_0 \in \mathbb{R}$ and $s \in \Gamma(X_{\lambda_0}, \mathcal{F})$. We can assume that $X_{\lambda_0} \neq \emptyset$. Let $\Lambda = \{\lambda \in \mathbb{R} \mid \lambda < \lambda_0\}$ and $s$ can be extended to $s_\lambda \in \Gamma(X_\lambda, \mathcal{F})$. Clearly, if $\lambda \in \Lambda$ and $\lambda < \mu$, then $\mu \in \Lambda$. We are going to prove:

(6) If $\lambda \in \Lambda$ and $s_\alpha, s'_\alpha \in \Gamma(X_\lambda, \mathcal{F})$ both extend $s$, then $s_\alpha = s'_\alpha$.

Suppose the contrary. Then $Z = \{x \in X_\lambda \mid (s_\alpha)_x \neq (s'_\alpha)_x\}$ is a nonempty subvariety in $X_\lambda$. Let $Z_0$ be a branch of $Z$. Take $x^* \in Z_0$ and let $\lambda^* = v(x^*)$. Let $\xi = \sup \{v(x) \mid x \in Z_0\}$. Since $Z \cap X_{\lambda_0} = \emptyset$, $\xi$ is the supremum of $v$ on the compact set $Z_0 \cap \{x \in X \mid \lambda^* \leq v(x) \leq \lambda_0\}$. $\xi = v(y)$ for some $y \in Z_0$. Since $X_\xi$ is $\ast$-strongly $p$-concave at $y$ and $Z_0 \cap X_{\xi} = \emptyset$, we have $\dim (Z_0)_y < p$. Since $Z_0$ is irreducible, $\dim Z_0 < p$. Hence $\dim Z < p$. $s_\lambda - s'_\lambda \in \Gamma(X_\xi, 0_{\mathfrak{T}^p - 1})$. (6) follows from $0_{\mathfrak{T}^p - 1} = 0$.

For $\lambda \in \Lambda$ denote the unique element of $\Gamma(X_\lambda, \mathcal{F})$ which extends $s$ by $s_\lambda$. To finish the proof, we need only prove that $\Lambda$ has no lower bound, because in that case $\Lambda = \{\lambda \in \mathbb{R} \mid \lambda \leq \lambda_0\}$ and by (6) $s^* \in \Gamma(X, \mathcal{F})$ defined by $s^*|_{X_\lambda} = s_\lambda$ for $\lambda \in \Lambda$ extends $s$. Suppose the contrary. Then $\eta = \inf \Lambda$ exists and is finite. Since $X$ is connected, this implies that $X_{\eta}$ is not closed in $X$. By Theorem 3 for every $x$ in the boundary $\partial X_n$ of $X_n$ there exists an open neighborhood $U_x$ of $x$ in $X$ such that $s_n$ can be extended to $t(x) \in \Gamma(U_x \cup X_n, \mathcal{F})$. For $x, x' \in \partial X_n$ let $Y_{(x,x')} = \{z \in U_x \cap U_{x'} \mid \{t(x)z \neq (t(x')z)_{x}\}\}$. Since $0_{\mathfrak{T}^p - 1} = \emptyset$, $Y_{(x,x')} = \emptyset$. (7) implies that $\omega_{x,x'}(x) \cap \partial X_n = \emptyset$ for $x, x' \in \partial X_n$. Since $\partial X_n$ is compact we can choose $x_1, \ldots, x_k \in \partial X_n$ such that $\partial X_n \subset \bigcup_{i=1}^k U_{x_i}$. For $1 \leq i \leq k$ choose a relatively compact open neighborhood $W_i$ of $x_i$ in $U_{x_i}$ such that $\partial X_n \subset \bigcup_{i=1}^k W_i$. Let $W_i^\ast$ be the closure of $W_i$ in $X$, $1 \leq i \leq k$. (7) implies that we can choose an open neighborhood $W$ of $\partial X_n$ in $\bigcup_{i=1}^k W_i$ such that $W$ does not intersect the closed set $\bigcup_{1 \leq i, j \leq k, i \neq j} Y_{(x_i, x_j)} \cap W_i^\ast \cap W_j^\ast$. For some $\lambda < \eta$, $X_\lambda \subset W \cup X_n$ because of Proposition 2.7 of [3]. Define $t \in \Gamma(X_\lambda, \mathcal{F})$ by setting $t = s(x_i)$ on $(U_{x_i} \cup X_n) \cap X_\lambda$. $t$ extends $s$, contradicting $\lambda \notin \Lambda$.

Uniqueness follows from (6). Q.E.D.

here have the advantage that, if \( \mathcal{F} \) does not satisfy \( \mathcal{F} = \mathcal{F}^{(0)} \), we can always construct the coherent analytic sheaf \( \mathcal{G} = \mathcal{F}/0_{\mathcal{F}}^{(0)} \) which satisfies \( \mathcal{G} = \mathcal{G}^{(0)} \).

(ii) Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \((X, \mathcal{A})\) and \( x \in X \). The condition \( \mathcal{F}_x = (\mathcal{F}^{(0)})_x \) is equivalent to the condition \( \text{codh } \mathcal{F}_x \geq 2 \). It can be proved in the following way: If \( \mathcal{F}_x = (\mathcal{F}^{(0)})_x \), then \( E^n(0, \mathcal{F}) = 0 \) and by Lemmas 2 and 3 we can find \( f \in \Gamma(U, \mathcal{F}) \) for some open neighborhood \( U \) of \( x \) in \( X \) such that \( f_x \) is not a unit of \( \mathcal{A}_x \) and \( f_x \) is not a zero-divisor for \( \mathcal{F}_x \). By shrinking \( U \), we can assume that \( f_y \) is not a zero-divisor for \( \mathcal{F}_y \) for \( y \in U \). Suppose \( x \in E^n(f, \mathcal{F}, U) \). By shrinking \( U \), we can find \( g \in \Gamma(U, \mathcal{F}) \) such that \( g_y = f_y h_y \) for \( y \in U - \{x\} \) and \( g_x \notin (f\mathcal{F})_x \). Then \( h \in \Gamma(U, \mathcal{F}^{(0)}) \) defined by \( g_y = f_y h_y \) for \( y \in U - \{x\} \) does not satisfy \( h_x \in \mathcal{F}_x \). This is a contradiction. Hence \( x \notin E^n(f, \mathcal{F}, U) \). By Lemmas 2 and 3 we can find \( s \in \mathcal{A}_x \) which vanishes at \( x \) and is not a zero-divisor for \( (\mathcal{F}^{(0)})_x \). codh \( \mathcal{F}_x \geq 2 \). On the other hand codh \( \mathcal{F}_x \geq 2 \) implies \( \mathcal{F}_x = (\mathcal{F}^{(0)})_x \) by Korollar zu Satz III, [9].

The equivalence of \( \mathcal{F}_x = (\mathcal{F}^{(0)})_x \) and \( \text{codh } \mathcal{F}_x \geq 2 \) is also a consequence of [14, (1.1)]. However, the proof presented here is more conceptual than the proof in [14].

(iii) In the case of Stein spaces we have the following stronger version of Theorem 4 which generalizes Theorem 5.4 of [4]:

Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a Stein space \( X \) such that \( \mathcal{F} = \mathcal{F}^{(0)} \). Suppose \( K \) is a compact subset of \( X \) such that, if \( A \) is a branch of \( E^n(0, \mathcal{F}) \) for any \( \sigma \geq 2 \), then \( A - K \) is irreducible.

Then for every open neighborhood \( U \) of \( K \) in \( X \) every element of \( \Gamma(U - K, \mathcal{F}) \) can be extended uniquely to an element of \( \Gamma(U, \mathcal{F}) \).

It can be proved in the following way: Suppose \( s \in \Gamma(U - K, \mathcal{F}) \). Since \( H^1(X, \mathcal{F}) = 0 \), from the Mayer-Vietoris sequence of \( \mathcal{F} \) on \( X = (X - K) \cup U \) (p. 236, [2]) we conclude that for some \( f \in \Gamma(X - K, \mathcal{F}) \) and \( g \in \Gamma(U, \mathcal{F}) \), \( f - g = s \) on \( U - K \). From Theorem 4 we can find \( f \in (X, \mathcal{F}) \) which agrees with \( f \) outside some compact subset of \( X \). Since \( E^n(0, \mathcal{F}) = 0 \) for \( \sigma \leq 1 \) and \( A - K \) is irreducible for any branch \( A \) of \( E^n(0, \mathcal{F}) \) with \( \sigma \geq 2 \), \( f \) agrees with \( f \) on \( X - K \). \( \mathcal{F} \mid U - g \) extends \( s \). The extension is clearly unique, because \( E^n(0, \mathcal{F}) = 0 \).

In view of the equivalence of \( \mathcal{F}_x = (\mathcal{F}^{(0)})_x \) and \( \text{codh } \mathcal{F}_x \geq 2 \), in the above proof we can use Theorem 15 of [2] instead of Theorem 4. So (8) can be proved also by the finiteness theorems of pseudoconvex spaces in [2].

(8) generalizes Theorem 5.4 of [4] because of the following:

Suppose \( K \) is a closed subset of an irreducible complex space \( X \) and \( U \) is an open neighborhood of \( K \) in \( X \) such that for every branch \( A \) of \( U \), \( A - K \) is irreducible. Then \( X - K \) is irreducible.

(9)
Let $R$ be the set of all regular points of $X$. To prove (9), we need only show that $R-K$ is connected. Suppose $R \cap U = \bigcup_{i \in I} R_i$ is the decomposition into topological components. Then $R_i-K$ is connected for $i \in I$. The restriction map $\Gamma(R \cap U, C) \to \Gamma(R \cap (U-K), C)$ is an isomorphism. From the following portion of the Mayer-Vietoris sequence of the constant sheaf $C$ on $R=(R \cap U) \cup (R-K)$:

$$0 \to \Gamma(R, C) \to \Gamma(R-K, C) \oplus \Gamma(R \cap U, C) \to \Gamma(R \cap (U-K), C),$$

we conclude that the restriction map $\Gamma(R, C) \to \Gamma(R-K, C)$ is an isomorphism. $R-K$ is connected.

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