m-BOUNDED UNIFORMITIES BETWEEN TWO GIVEN UNIFORMITIES

BY
ELLEN E. REED AND W. J. THRON

Introduction. This paper is an attempt to gain some information about the lattice of the uniformities which can be defined on a set X. Our results are stated in terms of the m-boundedness of uniformities, where m is an infinite cardinal. We define a pseudo-metric ρ on X to be m-bounded if for $\varepsilon > 0$, X can be written as a union of fewer than m sets, each of ρ-diameter not exceeding ε. If P is a family of m-bounded pseudo-metrics on X we say that the associated uniformity $\mathfrak{H}(P)$ is m-bounded. We call $\mathfrak{H}(P)$ *strictly* m-bounded if m is its least (infinite) bound.

Roughly speaking, we found that if two uniformities $\mathfrak{H}_0$ and $\mathfrak{H}_1$ have different strict bounds, with $\mathfrak{H}_0 \subset \mathfrak{H}_1$, then there are at least c uniformities between them. (Here c is the cardinality of the continuum.) The lower bound on the number of uniformities in-between can be improved as the strict bound on $\mathfrak{H}_1$ increases. Moreover, these uniformities can be chosen to be strictly m-bounded, provided m exceeds the bound on $\mathfrak{H}_0$ (but not the bound on $\mathfrak{H}_1$).

These results throw some light on the structure of a proximity class. For example, if a proximity class has a non-totally-bounded member then it must contain at least c uniformities. If a proximity class has a member which is strictly m-bounded then for every cardinal n between $\aleph_0$ and m the class has a strictly n-bounded member (in fact, quite a few). Thus, no gaps are allowed as far as n-boundedness is concerned. Finally, no proximity class has a minimal m-bounded uniformity, for $m > \aleph_0$.

We would like to express our thanks to the referee for simplifying our proofs and our results, and indeed for strengthening our results. From his improvements we were able to obtain further extensions. Theorem 2.1 is in fact a combined effort.

1. **Some properties of m-bounded uniformities.** We will find it helpful to have on hand several equivalent definitions of m-boundedness. This is accomplished in the next two theorems.

THEOREM 1.1. *A pseudo-metric ρ on a set X is m-bounded iff for $\varepsilon > 0$ there is a subset A of X with less than m elements such that every element of X is within ε (under ρ) of some element of A.*

Now we wish to define an m-bounded uniformity more carefully than we did in the introduction. Recall that for every uniformity on X there is a family P of

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pseudo-metrics on X such that the sets $H_{\rho,\varepsilon}=[(x, y) : \rho(x, y)<\varepsilon]$ are a subbase for $\mathfrak{H}$. Here ρ ranges over P and ε over the positive reals. We call P an associated family of pseudo-metrics (for $\mathfrak{H}$), and in general P is not unique. However, there is a *largest* associated family, which we will call the *gage* of $\mathfrak{H}$. It consists of all pseudo-metrics ρ on X such that $H_{\rho,\varepsilon}$ is in $\mathfrak{H}$ for all $\varepsilon>0$. We now *define* $\mathfrak{H}$ to be m-bounded iff every pseudo-metric in its gage is m-bounded. It is part of the next theorem that the definition of m-boundedness is actually independent of the choice of an associated family of pseudo-metrics.

We obtain a useful equivalent definition using the concept of a uniformly discrete subspace.

Definition 1.1. A subset D of a set X is called *uniformly discrete* with respect to a uniformity $\mathfrak{H}$ on X iff the gage of $\mathfrak{H}$ contains a pseudo-metric ρ such that for some $\varepsilon>0$ the ρ-distance between any two points of D is at least ε. When this occurs we say D is ε-discrete with respect to ρ.

Theorem 1.2. *Let $\mathfrak{H}$ be a uniformity on a set X and let P be its gage. The following conditions are equivalent*:

(1) *$\mathfrak{H}$ is m-bounded*;

(2) *$\mathfrak{H}$ has an associated family of m-bounded pseudo-metrics*;

(3) *every uniformly discrete subset of X has fewer than m elements.*

Proof. It is clear that (1) $\Rightarrow$ (2). To see that (2) $\Rightarrow$ (1) let Q be an associated family of m-bounded pseudo-metrics for $\mathfrak{H}$. Let $\rho \in P$ and $\varepsilon>0$. Since $H_{\rho,\varepsilon}$ is in $\mathfrak{H}$ we can choose σ_i's in Q and $\delta>0$ so that

$$H_{\rho,\varepsilon} \supset H_{\sigma_1,\delta} \cap \cdots \cap H_{\sigma_k,\delta}.$$

For each σ_i we now choose $\mathscr{A}_i$ to be a family of fewer than m subsets of X, each with σ_i-diameter less than δ and such that $\bigcup \mathscr{A}_i = X$. To obtain a similar set for ρ we define

$$\mathscr{A} = [A_1 \cap \cdots \cap A_k : A_i \in \mathscr{A}_i].$$

Now suppose $\mathfrak{H}$ is m-bounded and D is uniformly discrete. We wish to show $|D|<m$. Let ρ be in P and $\varepsilon>0$ such that no two points of D are within ε of each other (under ρ). We now choose a family $\mathscr{A}$ of subsets of X each of ρ-diameter less than ε such that $|\mathscr{A}|<m$ and $\bigcup \mathscr{A}=X$. There is a natural 1-1 correspondence from D into $\mathscr{A}$, and from this we conclude $|D|<m$.

Finally, suppose that every uniformly discrete subset has cardinality less than m. We will show $\mathfrak{H}$ is m-bounded. Let ρ be in P, and choose $\varepsilon>0$. Now consider the family of all sets which are ε-discrete with respect to ρ. By Zorn's lemma, this set has a maximal element D_0. By hypothesis, D_0 has less than m elements. Moreover, since D_0 is maximal, every element of X is within ε of D_0 under ρ. Using the alternate definition given in Theorem 1.1, we see that ρ is m-bounded.

Notice that from the equivalence (1) $\Leftrightarrow$ (2) above it follows easily that the l.u.b. of m-bounded uniformities is also m-bounded.

2. **Construction of m-bounded uniformities between given uniformities.** The idea of the construction is simple. Given two uniformities $\mathfrak{H}$ and $\mathfrak{K}$ on a set X with $\mathfrak{H} \subset \mathfrak{K}$ we shrink $\mathfrak{K}$ down around a $\mathfrak{K}$-discrete subset of the proper cardinality. This gives the new uniformity the right boundedness. It may not be large enough to include $\mathfrak{H}$, but we remedy this by taking its l.u.b. with $\mathfrak{H}$. Choice of sufficiently separated $\mathfrak{K}$-discrete sets gives rise to different uniformities. This "shrinking down" of a uniformity about a discrete subspace is accomplished in the next two lemmas.

LEMMA 2.1. *Let ρ be a pseudo-metric on a set X and let D be an infinite 1-discrete set with respect to ρ. For d in D let $f_{\rho,d}\colon X \to R$ be defined by*

$$f_{\rho,d}(x) = \rho(x, d) \wedge \tfrac{1}{2};$$

finally, let $\rho_D\colon X \times X \to R$ be given by

$$\rho_D(x, y) = \sup\, [|f_{\rho,d}(x) - f_{\rho,d}(y)| : d \in D].$$

Then

(1) *ρ_D is a pseudo-metric on X with $\rho_D \leqq \rho$;*

(2) *ρ_D is strictly m^+-bounded, where $m = |D|$ and m^+ is m's successor;*

(3) *D is $\frac{1}{2}$-discrete relative to ρ_D;*

(4) *$\rho_D(x, y) = 0$ for x and y outside $H_{\rho,1/2}(D)$, the set of points within $\frac{1}{2}$ under ρ of some element of D.*

Proof. Clearly ρ_D is a pseudo-metric on X. To see that $\rho_D \leqq \rho$ note that for x and y in X and d in D we have

$$|f_{\rho,d}(x) - f_{\rho,d}(y)| \leqq |\rho(x, d) - \rho(y, d)| \leqq \rho(x, y).$$

Now observe that if d and d' are distinct points in D we have $f_{\rho,d}(d') = \frac{1}{2}$. Therefore

$$\rho_D(d, d') \geqq |f_{\rho,d}(d') - f_{\rho,d}(d)| = \tfrac{1}{2},$$

and hence D is $\frac{1}{2}$-discrete with respect to ρ_D. From Theorem 1.2 it follows that ρ_D is not m-bounded.

Now we wish to establish that ρ_D is m^+-bounded. Let E be δ-discrete relative to ρ_D. We will show E has at most m members. Notice that for x not in $H_{\rho,1/2}(d)$ we have $f_{\rho,d}(x) = \frac{1}{2}$. Thus we have $\rho_D(x, y) = 0$ for x and y outside $H_{\rho,1/2}(D)$. Besides establishing (4), this means we need only concern ourselves with points of E in $H_{\rho,1/2}(D)$.

The idea is to divide each $H_{\rho,1/2}(d)$ into concentric "annuli" of width $\delta/2$. Altogether there are m of these, and each member of $E \cap H_{\rho,1/2}(D)$ lies in exactly one. More precisely, for e in $E \cap H_{\rho,1/2}(D)$ define d_e to be the unique member of D within $\frac{1}{2}$ of e under ρ. Then take i_e to be the largest integer i satisfying $i\delta/2 \leqq f_{\rho,d_e}(e)$. Note $0 \leqq i_e < 1/\delta$. This defines a 1-1 map from $E \cap H_{\rho,1/2}(D)$ into $D \times [0, 1, \ldots, 1/\delta]$. The latter set has cardinality m.

LEMMA 2.2. *Let $\mathfrak{H}$ be an m-bounded uniformity on a set X, and let ρ be a pseudo-metric on X. Assume D is a* 1*-discrete set relative to ρ and $n \geqq m$. Define*

$$P_D(\rho, n) = [\rho_A : A \subset D \text{ and } |A| < n].$$

Let $\mathfrak{H}_D = \mathfrak{H} \vee \mathfrak{H}(P_D(\rho, n))$. Then

(1) $\mathfrak{H} \subset \mathfrak{H}_D \subset \mathfrak{H} \vee \mathfrak{H}(\rho)$;

(2) $\mathfrak{H}_D$ *is n-bounded*;

(3) *every subset of D of cardinality less than n is uniformly discrete relative to* $\mathfrak{H}_D$;

(4) *every set uniformly discrete relative to $\mathfrak{H}_D$ has fewer than m points outside* $H_{\rho,1/2}(D)$.

Proof. The first three parts follow easily from the corresponding parts of the preceding lemma. Regarding (4), notice that any set which lies outside $H_{\rho,1/2}(D)$ and is uniformly discrete relative to $\mathfrak{H}_D$ is also uniformly discrete relative to $\mathfrak{H}$.

THEOREM 2.1. *Let $\mathfrak{H}_0$ and $\mathfrak{H}_1$ be uniformities on a set X with $\mathfrak{H}_0 \subset \mathfrak{H}_1$. Suppose $\mathfrak{H}_0$ is m_0-bounded and $\mathfrak{H}_1$ is strictly m_1-bounded, with $m_0 < m_1$. Then for m and n between m_0 and m_1, with $n < m_1$ and $m > m_0$, there are at least 2^n strictly m-bounded uniformities between $\mathfrak{H}_0$ and $\mathfrak{H}_1$.*

Proof. Since $\mathfrak{H}_1$ is not n-bounded, it has a uniformly discrete subspace D of cardinality n. The idea is to break D up into a family $\mathscr{A}$ of 2^n sets of cardinality n and to build up a uniformity about each of these sets. For $n^+ \geqq m$ these uniformities will be strictly m-bounded. For $n^+ < m$ we need to build on sets of higher cardinality than D has. The trick in this case is to pick these sets well away from D. Then the uniformities we construct can still be distinguished via our specially chosen family of subsets of D.

Now let $\mathscr{A}$ be a family of 2^n subsets of D, each of cardinality n, and such that the symmetric difference of distinct members of $\mathscr{A}$ has cardinality n. Such a family can be obtained as follows: using the fact that $D \times D$ is in 1-1 correspondence with D we can obtain a partition $\mathscr{D}$ of D into n subsets each of cardinality n. For D in $\mathscr{D}$ let $\mathscr{E}_D$ be a partition of D into n subsets of cardinality n. To obtain a set in $\mathscr{A}$, we choose a member of $\mathscr{E}_D$, for each D, and then take the union of these sets. Then $\mathscr{A}$ is in 1-1 correspondence with $\prod[\mathscr{E}_D : D \in \mathscr{D}]$, which has cardinality 2^n. Moreover, the difference of any two sets in $\mathscr{A}$ has cardinality n, and so does each set in $\mathscr{A}$.

Now let ρ be a pseudo-metric in the gage of $\mathfrak{H}_1$ such that D is 1-discrete relative to ρ. (Notice that for some $\varepsilon > 0$, D is ε-discrete relative to some σ in the gage of $\mathfrak{H}_1$. We can let $\rho = (1/\varepsilon)\sigma$.)

CASE I. $n^+ \geqq m$. For each A in $\mathscr{A}$ let

$$\mathfrak{H}_A = \mathfrak{H}_0 \vee \mathfrak{H}(P_A(\rho, m)),$$

where $P_A(\rho, m)$ is defined as in the preceding lemma.

Clearly each $\mathfrak{H}_A$ lies between $\mathfrak{H}_0$ and $\mathfrak{H}_1$ and is m-bounded. To see that each $\mathfrak{H}_A$ is strictly m-bounded, note that for $m' < m$ we have $m' \leqq n$, and hence A has a subset of cardinality m'. By the preceding lemma, this set is uniformly discrete relative to $\mathfrak{H}_A$.

All that remains to be shown is that the $\mathfrak{H}_A$'s are all distinct. This will be accomplished by distinguishing among their discrete subspaces. Assume A and B are distinct elements of $\mathscr{A}$, with $|A-B| = n$. Let C be a subset of $A-B$ of power m_0. Since $m_0 < m$, C is uniformly discrete relative to $\mathfrak{H}_A$. However, C misses $H_{\rho,1/2}(B)$ and hence by (4) of Lemma 2.2, C cannot be uniformly discrete relative to $\mathfrak{H}_B$.

CASE II. $n^+ < m$. The only case of interest in which $n^+ < m$ is that for which $m = m_1$ and m_1 is a limit cardinal. In all other cases we can replace n by a larger cardinal—one for which the first case applies. So we assume now that m is a limit cardinal.

For $n < p < m$ choose D_p and ρ_p so that D_p has p elements, ρ_p is in the gage of $\mathfrak{H}_1$, and D_p is 1-discrete relative to ρ_p. We can assume without loss of generality that D is also 1-discrete relative to ρ_p. (Replace ρ_p by $\rho_p \vee \rho$ if necessary.) Now let $E_p = D_p - H_{\rho_p,1/2}(D)$.

We will show first that E_p also has p elements. Since $n < p$, it is sufficient to establish that we are removing at most n points from D_p. For x in $D_p \cap H_{\rho_p,1/2}(D)$ let d_x be an element of D within $\frac{1}{2}$ of x under ρ_p. The map $x \to d_x$ is 1-1, since D_p is 1-discrete relative to ρ_p. The result now follows from the cardinality of D.

Now let $F_p(A) = E_p \cup A$, for A in $\mathscr{A}$. These are the sets on which we will construct our uniformities. For simplicity, let $P(p, A)$ denote the family of pseudo-metrics with parameters $F_p(A)$, $2\rho_p$, and p. We then define

$$\mathfrak{K}_p(A) = \mathfrak{H}_0 \vee \mathfrak{H}(P(p, A)).$$

From Lemma 2.2 it follows that $\mathfrak{K}_p(A)$ lies between $\mathfrak{H}_0$ and $\mathfrak{H}_1$, and is p-bounded. To see that this uniformity is strictly p-bounded, note that $F_p(A)$ has a subset of power p', for $p' < p$; again by Lemma 2.2, such subsets are discrete relative to $\mathfrak{K}_p(A)$.

Now for each A let $\mathfrak{K}(A)$ be the l.u.b. of the $\mathfrak{K}_p(A)$'s (over all cardinals p strictly between n and m). Clearly $\mathfrak{K}(A)$ also lies between $\mathfrak{H}_0$ and $\mathfrak{H}_1$ and is m-bounded. Since m is a limit cardinal, $\mathfrak{K}(A)$ is strictly m-bounded.

We will show now that the $\mathfrak{K}(A)$'s are all distinct. First we will establish that for A in $\mathscr{A}$, every subset of $D-A$ uniformly discrete relative to $\mathfrak{K}(A)$ has fewer than m_0 elements. Notice that for each p, $D-A$ misses $H_{\rho_p,1/2}(F_p(A))$. Thus the pseudo-metrics in $P(p, A)$ are zero on $(D-A)\times(D-A)$—Lemma 2.1. It follows that the same conclusion holds for an arbitrary pseudo-metric σ in the gage of $\mathfrak{H}(P(p, A))$. (Show $\sigma(d, d') < \varepsilon$ for $\varepsilon > 0$ and d and d' in $D-A$.) Using this, we see that every subset of $D-A$ which is uniformly discrete relative to $\mathfrak{K}(A)$ is also uniformly discrete relative to $\mathfrak{H}_0$. Since $\mathfrak{H}_0$ is m_0-bounded, the conclusion follows.

Now suppose A and B are distinct elements of $\mathscr{A}$. Assume $B-A$ has n elements. Since $n \geqq m_0$, $B-A$ cannot be uniformly discrete relative to $\mathfrak{K}(A)$. However, for each p between n and m, $B-A$ is a subset of $F_p(B)$ of lower cardinality than p, and hence is uniformly discrete relative to $\mathfrak{K}_p(B)$.

COROLLARY 2.1.1. *If $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are uniformities on a set X with different strict bounds, and $\mathfrak{H}_0 \subset \mathfrak{H}_1$, then there is a strictly decreasing sequence of uniformities between $\mathfrak{H}_0$ and $\mathfrak{H}_1$. These uniformities can all be chosen to have the same strict bound as $\mathfrak{H}_1$.*

Proof. This follows by repeated application of the preceding theorem with $n=m_0$, the strict bound of $\mathfrak{H}_0$.

From the preceding theorem we can obtain a number of interesting results concerning m-bounded uniformities in a proximity class. In fact, this paper grew out of the following question: given a uniformity which is strictly m-bounded, and given $\aleph_0 < n < m$, is there a uniformity in the same proximity class which is strictly n-bounded? This in turn was inspired by a result of Samuel [3]. From the proof of Theorem XV of [3] it follows that if there are no strictly $\aleph_1$-bounded uniformities compatible with a given topology then every compatible uniformity is totally bounded.

Recall that a uniformity $\mathfrak{H}$ determines a proximity relation as follows: $A \Subset_{\mathfrak{H}} B$ iff $H(A) \subset B$ for some H in $\mathfrak{H}$. Two uniformities are in the same proximity class iff they induce the same proximity relation.

COROLLARY 2.1.2. *If $\mathfrak{H}$ is a uniformity on a set X which is not n-bounded and $\aleph_0 < m \leqq n^+$ then there are at least 2^n strictly m-bounded uniformities below $\mathfrak{H}$ in its proximity class.*

Proof. Recall that every proximity class has a unique totally-bounded member, which is its least member under containment. (For a proof, see Thron [4, p. 192].) Let $\mathfrak{H}^*$ denote the totally-bounded member of the proximity class of $\mathfrak{H}$. Note that every uniformity between $\mathfrak{H}^*$ and $\mathfrak{H}$ is in the proximity class of $\mathfrak{H}$. The result now follows from the preceding theorem by setting $m_0 = \aleph_0$.

COROLLARY 2.1.3. *If a proximity class has more than one member, then it has at least c members.*

Actually it can be proved that this estimate can be raised to 2^c. However, this is as high as it can go—in general—since in a countable space the total number of uniformities on the set is 2^c, just from set-theoretic considerations.

COROLLARY 2.1.4. *If $\mathfrak{H}$ is a strictly m-bounded uniformity on a set X and $\aleph_0 \leqq n < m$ then there is a largest strictly n-bounded uniformity below $\mathfrak{H}$ in the proximity class.*

Proof. Let $\mathfrak{H}_n$ denote the l.u.b. of the n-bounded uniformities below $\mathfrak{H}$ in the proximity class. Then $\mathfrak{H}_n$ is strictly n-bounded, by Corollary 2.1.2, at least for $n > \aleph_0$. For $n = \aleph_0$ the result is trivial.

COROLLARY 2.1.5. *No proximity class has a minimal strictly m-bounded uniformity, for $m > \aleph_0$.*

Proof. Let $m_0=n=\aleph_0$ and $m_1=m$ and apply the preceding theorem.

These last two corollaries provide an interesting way of viewing a proximity class. Suppose a given proximity class has a largest element which is strictly m-bounded. For $\aleph_0 \leqq n < m$ let $\mathfrak{H}_n$ be the largest strictly n-bounded uniformity in the class. Then for $n<p$ we have $\mathfrak{H}_n \subset \mathfrak{H}_p$; thus we can picture the class as a sort of stem with nodes. Of course, trailing down from each node—each $\mathfrak{H}_n$—are a great many filaments, the strictly n-bounded uniformities.

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University of Massachusetts,
Amherst, Massachusetts
University of Colorado,
Boulder, Colorado