PROPERTIES OF H^p $(0 < p < 1)$ AND ITS CONTAINING BANACH SPACE(1)

BY

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1. **Introduction.** A function $f(z)$ analytic in the unit disk $|z| < 1$ is said to belong to the class H^p $(0<p<\infty)$ if

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}$$

is bounded for $0 \leqq r < 1$. A recent study [3] of the dual space structure of H^p in the case $p<1$ led to the consideration of the Banach space B^p $(0<p<1)$ consisting of all analytic functions f such that

$$\int_0^1 (1-r)^{1/p-2} M_1(r,f)\, dr < \infty. \tag{1}$$

Hardy and Littlewood had proved [6] that $H^p \subset B^p$. It was found in [3] that B^p is the "containing Banach space" of H^p; that is, H^p is dense in B^p, and the two spaces have the same continuous linear functionals. This made it possible to identify B^p with the closure of H^p in its second dual space $(H^p)^{**}$. Furthermore, it turned out that various properties of H^p functions extend to B^p, and that the spaces B^p are in some respects "nicer" than the H^p spaces.

The present paper contains some further illustrations of this last principle, as it applies to conjugate harmonic functions and Taylor coefficients. In §2 we discuss the conjugate function problem. In the later sections (which are independent of §2) we characterize the multipliers from B^p to l^1, and apply the result both to generalize a theorem of Hardy and Littlewood and to obtain a theorem of Paley type for lacunary coefficients. We also prove converses of these latter results, and we obtain a complete description of the B^p functions with lacunary Taylor series.

Although the present investigation was motivated by our previous paper [3], it is not heavily dependent on the results obtained there.

2. **Conjugate harmonic functions.** Let $u(z)$ be a real-valued function harmonic in $|z|<1$. It will be said that $u \in h^p$ $(0<p<\infty)$ if $M_p(r, u)$ is bounded; and that $u \in b^p$ $(0<p<1)$ if the integral corresponding to (1) is convergent. Let v be the harmonic conjugate of u, unique up to an additive constant. It is clear that $f=u+iv$ belongs to H^p (resp., B^p) if and only if u and v are both in h^p (resp., b^p). The

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theorem of M. Riesz says that for $1<p<\infty$, $u \in h^p$ implies $v \in h^p$. This is not true for $p=1$, but at least $u \in h^1$ implies $v \in h^q$ for all $q<1$. The theorem fails altogether for $p<1$; Hardy and Littlewood [7] have constructed examples showing that u may belong to h^p for all $p<1$, yet v not be in h^q for any $q>0$. Nevertheless, the theorem is true for the space b^p; in other words, b^p is a self-conjugate class.

THEOREM 1. *If* $u \in b^p$, *then* $v \in b^p$.

Proof. Let $z=re^{i\theta}$, $\rho=(1+r)/2$, and $f(z)=u(z)+iv(z)$. Then by the Poisson formula,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho e^{it}+z}{\rho e^{it}-z}\, u(\rho e^{it})\, dt + iC.$$

Thus

$$|f'(z)| \leqq \frac{1}{\pi}\int_0^{2\pi} \frac{|u(\rho e^{i(\theta+t)})|\, dt}{\rho^2-2\rho r\cos t+r^2},$$

so that

$$M_1(r,f') \leqq \frac{2M_1(\rho,u)}{\rho^2-r^2} \leqq \frac{4M_1(\rho,u)}{1-\rho}.$$

The hypothesis $u \in b^p$ therefore implies $f' \in B^q$, $1/q=1+1/p$. It now follows that $f \in B^p$, by Theorem 5 of [3].

Since $H^p \subset B^p$, it is natural to ask whether $h^p \subset b^p$; or equivalently (in view of Theorem 1), whether $u \in h^p$ implies $v \in b^p$.

THEOREM 2. *If* $u \in h^p$ $(0<p\leqq 1)$, *then* $v \in b^q$ *for all* $q<p$. *If* p *is of the form* $(k+1)^{-1}$ $(k=1,2,\ldots)$, *then* $u \in h^p$ *does not imply* $v \in b^p$.

Proof. The first statement follows from the theorem of Hardy and Littlewood [7] that

$$M_p(r,u) = O((1-r)^{-a}) \Rightarrow M_p(r,v) = O((1-r)^{-a}) \tag{2}$$

for $a>0$ and $0<p<\infty$. Indeed, $u \in h^p$ implies

$$M_p(r,u) = O((1-r)^{-a}), \qquad a = \tfrac{1}{2}(1/q-1/p).$$

Thus by (2), the function $f=u+iv$ has the property $M_p(r,f)=O((1-r)^{-a})$. By another theorem of Hardy and Littlewood [6, p. 406], this gives

$$M_1(r,f) = O((1-r)^{1-1/p-a}).$$

Therefore

$$\int_0^1 (1-r)^{1/q-2} M_1(r,f)\, dr \leqq C\int_0^1 (1-r)^{a-1}\, dr < \infty.$$

Hence $f \in B^q$, and $v \in b^q$. Alternatively, a proof can be based on the fact that $u \in h^p$ implies $M_p(r,f')=O((1-r)^{-1})$, which Hardy and Littlewood used to prove (2).

For the negative part of the theorem, we borrow the elementary counterexample

$$f(z) = u(z)+iv(z) = e^{ik\pi/2}(1-z)^{-k-1}$$

from Hardy and Littlewood [7, p. 416]. They showed that $u \in h^p$ for $p=(k+1)^{-1}$ $(k=1, 2, \ldots)$. On the other hand, it is easy to show that $M_1(r,f) \geqq C(1-r)^{-k}$, from which it follows that $f \notin B^p$.

Corollary. *If $u \in h^p$ $(p \leqq 1)$, then $u \in b^q$ for all $q<p$. If $p=(k+1)^{-1}$ $(k=1, 2, \ldots)$, then h^p is not contained in b^p.*

If p is not the reciprocal of an integer, we do not know whether h^p is contained in b^p; or equivalently, whether $u \in h^p$ implies $v \in b^p$.

It would be interesting to make a study of h^p similar to the study of H^p in [3]. In particular, what is the conjugate space of h^p and what is the containing Banach space?

3. **Multipliers.** Let A and B be complex sequence spaces. A sequence $\{\lambda_n\}$ is said to be a *multiplier* of A into B (or to *multiply* A into B) if $\{\lambda_n a_n\} \in B$ for each $\{a_n\} \in A$. It is convenient to begin our study of the Taylor coefficients of B^p functions by describing the multipliers of B^p into l^1. In other words, we shall characterize the class of sequences $\{\lambda_n\}$ such that $\sum |\lambda_n a_n| < \infty$ whenever $\sum a_n z^n \in B^p$. The following lemma will be needed.

Lemma 1. *If $b_n \geqq 0$, then*

$$\sum_{n=1}^{N} n^2 b_n = O(N) \tag{3}$$

if and only if

$$\sum_{n=N}^{\infty} b_n = O\left(\frac{1}{N}\right). \tag{4}$$

Proof. Let (3) hold and let $S_n = \sum_{k=1}^{n} k^2 b_k$. Then

$$\begin{aligned}\sum_{n=N}^{M} b_n &= \sum_{n=N}^{M-1} S_n[n^{-2}-(n+1)^{-2}]+S_M M^{-2}-S_{N-1}N^{-2}\\ &\leqq C\sum_{n=N}^{M-1} n^{-2}+CM^{-1}.\end{aligned}$$

Letting $M \to \infty$, we have (4).

Conversely, assume (4) and let $R_n = \sum_{k=n}^{\infty} b_k$. Then

$$\sum_{n=1}^{N} n^2 b_n = \sum_{n=1}^{N} [n^2-(n-1)^2]R_n - N^2 R_{N+1} \leqq CN.$$

Theorem 3. *A sequence $\{\lambda_n\}$ multiplies B^p into l^1 if and only if*

$$\sum_{n=1}^{N} n^{1/p}|\lambda_n| = O(N). \tag{5}$$

Proof. If $\{\lambda_n\}$ multiplies B^p into l^1, then it certainly multiplies H^p into l^1. But the linear operator

$$\Lambda\colon \sum a_n z^n \to \{\lambda_n a_n\}$$

from H^p to l^1 is closed, since the nth coefficient a_n is a continuous linear functional on H^p. Thus Λ is bounded, by the closed graph theorem. Now let ν be the positive integer such that

$$1/(\nu+1) < p \leqq 1/\nu,$$

let $f(z)=(1-z)^{-(\nu+1)}$, and let $f_r(z)=f(rz)$. Then since Λ is bounded,

$$\|\Lambda(f_r)\| \leqq C\|f_r\| = CM_p(r,f),$$

which gives

$$\sum_{n=1}^{\infty} n^\nu|\lambda_n|r^n = O((1-r)^{-(\nu+1)+1/p}).$$

From this it follows (letting $r=1-1/N$) that

$$\sum_{n=1}^{N} n^\nu|\lambda_n| = O(N^{\nu+1-1/p}),$$

which is equivalent to (5). (See [2, p. 253]. The above argument also occurs in [1].)

Conversely, we wish to show that (5) implies $\{\lambda_n\}$ is a multiplier of B^p into l^1. It will be sufficient to carry out the proof in the case $p=1/2$, since $\sum a_n z^n \in B^p$ if and only if $\sum n^{2-1/p} a_n z^n \in B^{1/2}$, by Theorem 5 in [3]. Thus the assumption (5) becomes

$$\sum_{n=1}^{N} n^2|\lambda_n| = O(N). \tag{6}$$

Without loss of generality we may now suppose $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. Let $s_1=0$ and

$$s_n = \sum_{k=1}^{n-1} \lambda_k, \qquad n = 2, 3, \ldots.$$

If $f(z)=\sum a_n z^n \in B^{1/2}$, then

$$\infty > \int_0^1 M_1(r,f)\,dr = \sum_{n=1}^{\infty} \int_{s_n}^{s_{n+1}} M_1(r,f)\,dr$$
$$\geqq \sum_{n=1}^{\infty} |a_n| \int_{s_n}^{s_{n+1}} r^n\,dr \geqq \sum_{n=1}^{\infty} \lambda_n|a_n|s_n^n.$$

But the hypothesis (6) gives $1-s_n \leqq C/n$, by Lemma 1. Hence

$$s_n^n \geqq (1-C/n)^n \to e^{-C} > 0,$$

so we have shown that $f \in B^{1/2}$ implies $\sum_{n=1}^{\infty} \lambda_n |a_n| < \infty$. This completes the proof of the theorem.

We mention that the second half of the proof is based on a method recently used by T. M. Flett [4] to simplify the proof of a theorem of Hardy and Littlewood (stated below as (7)).

COROLLARY 1. *If* $\sum a_n z^n \in B^p$, *then*

$$\sum_{n=1}^{\infty} n^{-1/p} |a_n| < \infty.$$

COROLLARY 2. *If* $\{n_k\}$ *is a lacunary sequence of positive integers* (*in the sense that* $n_{k+1}/n_k \geqq Q > 1$), *then* $\sum a_n z^n \in B^p$ *implies*

$$\sum_{k=1}^{\infty} n_k^{1-1/p} |a_{n_k}| < \infty.$$

COROLLARY 3. $\{\lambda_n\}$ *multiplies* H^p $(0 < p < 1)$ *into* l^1 *if and only if* (5) *holds.*

The last statement is really a corollary of the proof. The multipliers of H^1 into l^1 are not known and appear to be much more difficult to describe. In this case the condition (5) with $p=1$ is necessary, but is definitely *not* sufficient; for if it were, we could conclude

$$f \in H^1 \Rightarrow \sum_{k=1}^{\infty} |a_{n_k}| < \infty$$

for each lacunary sequence $\{n_k\}$. To see that this is false, we have only to choose $\{c_k\}$ with $\sum |c_k|^2 < \infty$ but $\sum |c_k| = \infty$; then $f(z) = \sum c_k z^{n_k}$ is a counterexample.

4. **Remarks on a Hardy-Littlewood theorem.** Let $f(z) = \sum a_n z^n$. A well known theorem of Hardy and Littlewood [5] asserts that for $0 < p \leqq 2$,

$$f \in H^p \Rightarrow \sum_{n=1}^{\infty} n^{p-2} |a_n|^p < \infty. \tag{7}$$

Since $a_n = o(n^{1/p-1})$ whenever $f \in H^p$ $(0 < p \leqq 1)$, this gives a family of weaker results:

$$f \in H^p \Rightarrow \sum_{n=1}^{\infty} n^{-\alpha} |a_n|^q < \infty, \tag{8}$$

where $p \leqq 1$, $q > p$, and $\alpha = 1 + q(1/p - 1)$. On the other hand, it is known [3, Theorem 4] that $a_n = o(n^{1/p-1})$ if only $f \in B^p$. Furthermore, Corollary 1 of Theorem 3 shows that (8) can be extended to B^p in the case $q=1$. Thus it is natural to ask whether (8) can be correspondingly improved for any $q < 1$. That this is not possible is shown by the example

$$f(z) = (1-z)^{-1/p} \left\{ \frac{1}{z} \log \frac{1}{1-z} \right\}^{-1/q}. \tag{9}$$

For this function, Littlewood [9, pp. 93–96] has shown that

$$a_n \sim Bn^{1/p-1}(\log n)^{-1/q}$$

and

$$M_1(r,f) \sim C(1-r)^{1-1/p}\left\{\log \frac{1}{1-r}\right\}^{-1/q}$$

Thus $f \in B^p$ if $q<1$, while

$$\sum_{n=2}^{\infty} n^{-\alpha}|a_n|^q \geqq A \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$

Incidentally, the same argument also shows that the Hardy–Littlewood theorem (7) cannot be improved to any statement of the form (8) with $q<p$. For if $q<p$, the function (9) belongs to H^p.

The preceding remarks show that the spaces H^p and B^p actually differ in the *moduli* of allowable coefficients. In fact, there exists a function $g(z)=\sum b_n z^n$ in B^p such that no function $f(z)=\sum a_n z^n$ with $|a_n|=|b_n|$ can belong to H^p. More generally $f \notin H^p$ if $|a_n| \geqq \delta |b_n|$ for some $\delta>0$.

Corollary 1 of Theorem 3 has a partial converse, which should be compared with the fact [3, Theorem 4] that $f \in B^p$ if $a_n = O(n^\alpha)$ for some $\alpha < 1/p - 3/2$.

THEOREM 4. *If* $\sum_{n=1}^{\infty} n^{1-1/p}|a_n| < \infty$, *then* $f \in B^p$. *The exponent is best possible: there exists a function* $f \notin B^p$ *such that* $\sum_{n=1}^{\infty} n^{\beta}|a_n| < \infty$ *for every* $\beta < 1-1/p$.

Proof.

$$\int_0^1 (1-r)^{1/p-2} M_1(r,f)\, dr \leqq \int_0^1 (1-r)^{1/p-2}\left\{\sum_{n=0}^{\infty} |a_n| r^n\right\} dr \leqq C \sum_{n=1}^{\infty} n^{1-1/p}|a_n|,$$

where elementary properties of the gamma function have been used. The lacunary series

$$f(z) = \sum_{k=1}^{\infty} n_k^{1/p-1} z^{n_k}$$

shows the exponent is best possible. By Corollary 2 of Theorem 3, $f \notin B^p$.

No nontrivial condition like that in Theorem 4 can imply $f \in H^p$, since whenever $\sum |a_n|^2 = \infty$, then for almost every choice of signs $\varepsilon_n = \pm 1$, the function $\sum \varepsilon_n a_n z^n$ has a radial limit almost nowhere.

5. **Lacunary series.** It is well known that a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}$$

belongs to H^p $(0<p<2)$ if and only if $f \in H^2$. (See, for example, [11, Vol. I, Chapter V, Theorem 6.4]. Recall that $f \in H^p$ has a radial limit almost everywhere, which is equivalent to saying that the Taylor series on the boundary is Abel summable.)

A theorem of Paley [10] asserts that $\sum a_n z^n \in H^1$ implies $\sum |a_{n_k}|^2 < \infty$ for every lacunary sequence $\{n_k\}$. From this it follows by fractional integration that $f \in H^p$ $(p < 1)$ implies $\sum n_k^{2-2/p} |a_{n_k}|^2 < \infty$. We can now improve this result to conclude that $\sum n_k^{1-1/p} |a_{n_k}| < \infty$, $0 < p < 1$. We also have a complete description of the lacunary series which belong to B^p.

THEOREM 5. *Let $\{n_k\}$ be a fixed lacunary sequence of positive integers, and let p be fixed* $(0 < p < 1)$. *Then for a complex sequence $\{c_k\}$ the following three statements are equivalent*:

(i) $\sum_{k=1}^{\infty} n_k^{1-1/p} |c_k| < \infty$.

(ii) $\sum_{k=1}^{\infty} c_k z^{n_k} \in B^p$.

(iii) *There exists a function* $\sum_{n=0}^{\infty} a_n z^n \in B^p$ *such that* $a_{n_k} = c_k$, $k = 1, 2, \ldots$.

Proof. Theorem 4 says that (i) implies (ii). Trivially, (ii) implies (iii). By Corollary 2 of Theorem 3, (iii) implies (i). This completes the proof.

This theorem leads to a direct sum decomposition of B^p, with one summand isomorphic to the sequence space l^1. To see this, let E be a set of nonnegative integers, let $\tilde{E}$ be its complement, and let

$$B^p(E) = \{f \in B^p : a_n = 0 \text{ for } n \notin E\},$$

where a_n is the nth Taylor coefficient of f. Then $B^p(E)$ is a closed subspace of B^p, and we can make the following statement.

COROLLARY. *If $E = \{n_k\}$ is a lacunary set, then $B^p = B^p(E) \oplus B^p(\tilde{E})$, and $B^p(E)$ is isomorphic to l^1.*

Proof. By the theorem, the operator

$$\pi : \sum a_n z^n \to \sum a_{n_k} z^{n_k}$$

maps B^p onto $B^p(E)$, and $B^p(E)$ is isomorphic to l^1.

In H^1 the analogous direct sum decomposition is also valid, the boundedness of the projection operator π following from Paley's theorem. The subspace $H^1(E)$ of gap series is isomorphic to l^2 (rather than to l^1). Thus Hilbert space is isomorphic to a direct summand of H^1. Joel Shapiro has pointed out to us that B^p, on the other hand, cannot contain any subspace isomorphic to Hilbert space. Indeed, it was noted in [3, Proposition at the end of §3] that B^p has the Schur property: each weakly convergent sequence is also norm convergent. Every subspace of B^p therefore has this property, but Hilbert space does not.

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