SOME THEOREMS ON HOPFICITY

BY

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1. **Introduction.** Let G be a group and let Aut G be the group of automorphisms of G and let End on G be the semigroup of endomorphisms of G onto G. A group G is called hopfian if End on $G = \text{Aut } G$, that is, a group G is hopfian if every endomorphism of G onto itself is an automorphism. To put this in another way, G is hopfian if G is not isomorphic to a proper factor group of itself.

The question whether or not a group is hopfian was first studied by Hopf, who using topological methods, showed that the fundamental groups of closed two-dimensional orientable surfaces are hopfian [5].

Several problems concerning hopfian groups are still open. For instance, it is not known whether or not a group H must be hopfian if $H \subset G$, G abelian and hopfian and G/H finitely generated. Also it is not known whether or not G must be hopfian, if G is abelian, $H \subset G$, H hopfian, and G/H finitely generated [2]. On the other hand, A. L. S. Corner [3], has shown the surprising result, that the direct product $A \times A$ of an abelian hopfian group A with itself need not be hopfian.

Corner's result leads us to inquire: What conditions on the hopfian groups A and B will guarantee that $A \times B$ is hopfian? We shall prove, for example, in §3, that the direct product of a hopfian group and a finite abelian group is hopfian. Also we shall prove that the direct product of a hopfian abelian group and a group which obeys the ascending chain condition for normal subgroups (for short, an A.C.C. group) is hopfian (Theorems 3 and 5 respectively).

In §4 we examine various conditions on a hopfian group A which guarantee $A \times B$ is hopfian for groups B with a principal series. For example if the center of A, $Z(A)$, is trivial or if A satisfies the descending chain condition for normal groups, (for short, A is a D.C.C. group) then $A \times B$ is hopfian.

Theorem 3 is equivalent to: The direct product of a hopfian group and a cyclic group of prime power order is hopfian. In seeking to generalize this result we note that the normal subgroups of a cyclic group $C_{p^{n+1}}$ of prime power order p^{n+1} form a chain and $C_{p^{n+1}}$ has exactly n-proper normal subgroups. We define an n-normal group as a group G with exactly n-proper normal subgroups such that the normal subgroups of G form a chain. Hence the simplest example of an n-normal group is $C_{p^{n+1}}$. (We only consider n finite.) We then consider in §5 the direct product $G = A \times B$ of a hopfian group A with an n-normal group B. In Theorem 16, we show that if G is not hopfian, several anomalies arise with respect to A. For instance if G is not hopfian we will show that there are infinitely many

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homomorphisms of A onto B. We show that if B is 0-normal or 1-normal, $A \times B$ is hopfian.

In §6 we explore briefly the concept of super-hopficity. If all homomorphic images of A are hopfian, we say that A is super-hopfian. We show for example that if G is generated by a super-hopfian normal subgroup A and a normal subgroup B such that B has finitely many normal subgroups, then G is super-hopfian.

Unless otherwise stated, A will always designate a hopfian normal subgroup of G and T will designate an element of End on G. If $g \in G$, $O(g)$ will designate the order of g, $|G|$ will designate the cardinality of G. If $H \subset G$ and j is a positive integer, HT^{-j} will designate the complete pre-image of H under T^j.

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2. Some general theorems.

We begin with a result that shows us that in some cases it suffices to consider infinitely generated hopfian groups A.

THEOREM 1. *If G is a group containing a hopfian subgroup N of index $[G : N]=r$, r finite, such that G contains only finitely many subgroups of index r, then G is hopfian.*

Proof. Suppose $G \sim G/K$, $K \neq 1$. If under an isomorphism of G onto G/K, K corresponds to K_1/K, we see $G \sim G/K \sim G/K_1$. Repeating the procedure, we see there exists subgroups K_i, where K_i is a proper subgroup of K_{i+1} such that

$$G \sim G/K_i, \qquad i \geqq 0,\ K_0 = K.$$

Hence we may write $N \sim M_i/K_i$ so that $[G : N]=[G : M_i]=r$. Hence $M_i = M_j$ for some i and j with $i<j$. But then,

$$\frac{M_i/K_i}{K_j/K_i} \sim \frac{M_i}{K_j} = \frac{M_j}{K_j} \sim N \sim \frac{M_i}{K_i}$$

so that N is not hopfian.

The following corollaries follow quite easily:

COROLLARY 1. *Let G be a group containing a hopfian normal subgroup N of index $[G : N]=r$ (r not necessarily finite) such that G contains only finitely many normal subgroups of index r, then G is hopfian.*

COROLLARY 2. *If G is a finitely generated group containing a subgroup N of finite index, N hopfian, then G is hopfian.*

COROLLARY 3. *If A is finitely generated, and $|B| < \infty$, then $A \times B$ is hopfian.*

LEMMA 1. *If G/A is hopfian and if $AT \subset A$, then $T \in$* Aut G.

Proof. T induces an endomorphism of G/A onto itself in the obvious way. Since G/A is hopfian we conclude $AT^{-1}=A$ from which the conclusion easily follows.

THEOREM 2. *Let A and G/A be hopfian and suppose one of the following holds:*

(a) *$A\subset Z(G)$, G/A centerless,*

(b) *A a periodic group, G/A torsion free,*

(c) *A and $G/A=B$ both periodic groups such that if $a\in A$, $b\in B$, then $(O(a), O(b))=1$.*

Then G is hopfian.

Proof. Apply the previous lemma.

3. G/A **an A.C.C. group.**

THEOREM 3. *If B is a finite abelian group then $G=A\times B$ is hopfian.*

Proof. It suffices to assume that B is cyclic of prime power order, say, $|B|=p^n$ $B=\langle b\rangle$. Throughout this discussion and the next one, we will use symbols a, a_i to designate elements of A.

Suppose first for a given T, we have $bT=a$. Let $b^r a_1$ be a pre-image of b under T. Let $u=ba_1$ and let $v=ba^{-r+1}$. We may then verify,

$$G = \langle b\rangle\times A = \langle u\rangle A = \langle v\rangle\times A$$

and $uT=v$. Let $A^1=\langle v\rangle T^{-1}\cap A$ so that $\langle v\rangle T^{-1}=\langle u\rangle A^1$. Hence,

$$A \sim (G/\langle v\rangle) \sim (G/\langle v\rangle T^{-1}) = (\langle u\rangle A)/(\langle u\rangle A^1) \sim A/A^1.$$

Hence $A^1=1$. Hence T is an isomorphism on A and without too much difficulty, one sees that $T\in \operatorname{Aut} G$.

Now suppose $bT\notin A$, say $bT=b^q a$. If $(q,p)=1$, we can find an automorphism S of G such that $bTS=b$, so that by Lemma 1, $TS\in \operatorname{Aut} G$ and a fortiori, $T\in\operatorname{Aut} G$. Hence we may assume $(q,p)\neq 1$.

If $aT\in A$ and if f_p designates the greatest power of p dividing the integer f then $bT^2=b^r a_2$ where $r_p>q_p$. If $aT=b^s a_3$, and $aT\notin A$, and if $s_p\leqq q_p$, then for a suitable integer u, if $z=ba^u$, $zT\in A$ and $G=\langle z\rangle\times A$. If $s_p>q_p$ then $bT^2=b^v a_4$ where $v_p>q_p$. Hence if $(q,p)\neq 1$, we see that we may find an element w of G, such that $G=\langle w\rangle\times A$ and $wT^i\in A$ for some integer i, $1\leqq i<2^n$. Hence T^i and T are automorphisms.

THEOREM 4. *If A is abelian and B is finitely generated and abelian then $G=A\times B$ is hopfian.*

Proof. By the previous theorem, we may assume $B=\langle b\rangle\sim C_\infty$.

By Lemma 1, if $A\subset AT^{-1}$ then $T\in\operatorname{Aut} G$. Hence we may assume $A|(A\cap AT^{-1})$ is infinite cyclic, that is,

$$A = \langle a\rangle\times A\cap AT^{-1}.$$

But $A/(A \cap AT)$ is contained isomorphically in G/AT which in turn is a homomorphic image of G/A. Hence we may write, $A = \langle a_1 \rangle A \cap AT$. Hence there is an element S, $S \in \text{End}$ on A which agrees with T on $A \cap AT^{-1}$ such that $aS = a_1$. It easily follows that $T \in \text{Aut } G$.

COROLLARY. *If A is abelian and if B is finitely generated and B' the commutator group of B is hopfian then $A \times B = G$ is hopfian.*

Proof. $B'T = B'$ so that $B'T^{-1} = B'$ or else $(A \times B)/B' \sim A \times (B/B')$ is not hopfian.

COROLLARY. *If $Z(A)$ and $A/Z(A)$ are hopfian and if B is a finitely generated abelian group, then $A \times B$ is hopfian.*

Proof. $[Z(A) \times B]T^{-1} = Z(A) \times B$. Now apply the theorem.

We present here some general observations concerning T in relation to G/A, where G and T are arbitrary and G/A is an A.C.C. group. (A need not be hopfian in this discussion.)

We note T induces in a natural way, a homomorphism of G/AT^i onto G/AT^{i+1}. Since G/A is an A.C.C. group we see that ultimately all these homomorphisms are isomorphisms that is, for $s \geqq r$

$$(AT^{s+j})T^{-j} = AT^s, \qquad j \geqq 1$$

so that kernel $T^j \subset AT^s$. Hence

$$\text{kernel } T^j \subset \bigcap_{s \geqq r} AT^s, \qquad j \geqq 1.$$

It follows that a necessary and sufficient condition that $T \in \text{Aut } G$ is that T^i be an isomorphism on A for all $i \geqq 1$. Moreover in seeking to prove that $T \in \text{Aut } G$ it is not restrictive to assume that, for $i \geqq 1$ and $j \geqq 1$,

$$(1) \qquad G/AT^i \sim G/AT^{i+1}, \qquad (AT^{i+j})T^{-j} = AT^i, \qquad \text{kernel } T^j \subset AT^i.$$

For if T does not obey the above conditions some power T_1 of T does and we could work with T_1 instead of T. We will assume (1) whenever it is convenient.

We now resume our convention that A is hopfian.

THEOREM 5. *If every proper homomorphic image of A is abelian and B is an A.C.C. group then $G = A \times B$ is hopfian.*

Proof. Deny. Then we may find T, T not an isomorphism on A such that the conditions (1) hold. Let,

$$G_1 = \text{gp}\,(A, AT, AT^2, AT^3, \ldots).$$

Then $G_1T \subset G_1$, so that $G_1T^{-1} = G_1$. However $AT^i \subset Z(G)$, $i \geqq 1$ because $G = AT^i \cdot BT^i$ and AT^i is abelian. Hence $G_1 = A \times B_1$ where $B_1 \subset Z(B)$. Hence B_1 is finitely generated so that $A \times B_1$ is hopfian which implies T is an isomorphism on G_1, a contradiction of our hypothesis.

COROLLARY. *If A is abelian and B is an A.C.C. group, then $A \times B$ is hopfian.*

In view of the last theorem, it might be of some interest to give an example of a hopfian group A, which is not an A.C.C. group and which is not abelian, but yet every proper homomorphic image of A is abelian. We proceed to do this.

DEFINITION. Let H be a group and F a group of automorphisms. We will say G is an extension of H by F, if G consists of elements fh, $f \in F$, $h \in H$, where multiplication in G is defined by

$$(f_1h_1)(f_2h_2) = (f_1f_2)(h_1^{f_2}h_2)$$

for $f_i \in F$ and $h_i \in H$, where $h_1^{f_2}$ is the image of h_1 under f_2.

THEOREM 6. *Let H be a simple group and let L be a hopfian group of automorphism of H. Furthermore, suppose*

(2) *$L \cap$ inner-automorphism $H = 1$.*

Then if G is an extension of H by L then G is hopfian. In fact if L is super-hopfian, then G is super-hopfian.

Proof. If $N \triangle G$ and $N \neq 1$ then $H \subset N$, for if $H \cap N = 1$ the elements of H and N commute element-wise, which leads to a contradiction of (2). Hence, if $T \in$ End on G, by Lemma 1, $HT \neq 1$. Hence $H \subset HT$. But $HT \sim H$ since H is simple. Hence $H = HT$. By Lemma 1 again, $T \in$ Aut G. If L is super-hopfian, every proper homomorphic image of G is a homomorphic image of L so that G is super-hopfian.

As an application, let H be the alternating group on an infinite countable set. Let p_i, $i = 1, 2, 3, \ldots$, be a sequence of distinct primes. Then H has a group of automorphisms L which is the restricted direct product of cyclic groups of order p_i, $i = 1, 2, \ldots$, and such that (2) holds. L is super-hopfian. Hence G is not an A.C.C. group, every proper homomorphic image of G is abelian and G is super-hopfian.

Somewhat along the lines of the previous theorem, we have

THEOREM 7. *Let every proper normal subgroup of A be an A.C.C. group. Let every normal subgroup of B be an A.C.C. group. Then if $G/A \sim B$, then G is hopfian.*

Proof. Deny. Suppose T is not an isomorphism on A and kernel $T \subset AT$. Hence

$$B_1 = (A \cdot AT)/A \sim AT/A \cap AT \sim A/A_1.$$

Now B_1 is contained isomorphically in B as a normal subgroup. Hence A/A_1 is an A.C.C. group. But $A_1 \neq A$ or else $AT \subset A$ contradicting Lemma 1. Hence A_1 is an A.C.C. group. But then so is A and certainly then so is G implying that G is hopfian after all.

We now present some observations concerning the group G where G/A has finitely many normal subgroups.

Suppose G is not hopfian. Then we may choose T satisfying the conditions (1), T not an isomorphism on A. Moreover, we may choose positive integer r and k, $r<k$ such that

$$A\cdot AT^{-k} = A\cdot AT^{-r} = L,$$
$$AT^{k}\cdot A = AT^{r}\cdot A = M.$$

Hence, $MT^{r-k}=M$, so that M is not hopfian. If G/A is finite, but G is not hopfian, we might begin by choosing $[G:A]$ as small as possible so that if M is constructed as above, $M=G$. But then G/A is a homomorphic image of A. We may summarize part of the previous remarks as

THEOREM 8. *The statement,*

If A is hopfian and G/A is finite then G is hopfian is universally true if and only if the statement,

If A is hopfian and G/A is a finite homomorphic image of A, then G is hopfian, is universally true.

4. $A\times B$, **where B has a principal series.**

DEFINITION. We say that a group B may be cancelled in direct products if whenever

$$C\times B \sim C^{1}\times B^{1} \quad\text{and}\quad B\sim B^{1}$$

then $C\sim C^{1}$ (for any C).

LEMMA 2. *If B has a principal series, B may be cancelled in direct products.*

Proof. See [4].

THEOREM 9. *If B has a principal series, a necessary and sufficient condition for $A\times B$ to be hopfian is that $AT\cap BT=1$ for arbitrary T of* End *on* $(A\times B)$.

Proof. The necessity part of the theorem is clear. Now suppose that $AT\cap BT=1$ for any $T\in$ End on $(A\times B)$. By the remarks preceding (1), we can choose $r>0$ such that

$$\text{kernel } T^{j}\subset AT^{s}\quad\text{for } j\geqq 1 \text{ and } s\geqq r,$$

where r depends on T. By hypothesis, $AT^{r}\cap BT^{r}=1$ so,

$$A\times B = AT^{r}\times BT^{r}.$$

Hence if $K=\text{kernel } T^{r}\cap B$, then

$$B/K\sim BT^{r} \quad\text{and}\quad A\times(B/K)\sim(AT^{r}/K)\times BT^{r}.$$

Hence by Lemma 2 we see $A\sim AT^{r}/K$. It follows without difficulty that T^{r} and T are automorphisms.

Corollary 1. *If B has a principal series, then a sufficient condition for T to be an automorphism, for T in* End *on* $(A\times B)$, *is*

$$AT^i\cap BT^i = 1, \qquad i\geqq 1.$$

Corollary 2. *A sufficient condition for $T\in$ Aut $(A\times B)$ is kernel $T^i\subset A$, $i\geqq 1$ (where B has a principal series).*

Corollary 3. *If G has a principal series and if $T\in$ End on $(A\times B)$, and if $AT\cap BT=1$, and if kernel $T\cap B\subset AT$ then T is an automorphism.*

Theorem 10. *If B has a principal series and if there are only finitely many possible kernels for homomorphisms of A into normal subgroups of B, then $A\times B$ is hopfian.*

Proof. Choose T obeying the conditions (1). Then write,

$$A\cdot AT^k = A\times B_k, \qquad k\geqq 1,\ B_k\triangle B.$$

The above gives rise to a homomorphism of A onto B_k, whose kernel is $A\cap AT^{-k}$. Hence we have for say $0<r<s$,

$$A\cap AT^{-r} = A\cap AT^{-s}.$$

Hence, $AT^s\cap AT^{s-r}=AT^s\cap A$, so that kernel $T^i\subset A$, $i\geqq 1$ and we may apply Corollary 2, of the previous theorem.

Corollary 1. *If B is finite and A has only finitely many normal subgroups A_* such that $[A:A_*]$ is a divisor of $[B:1]$, then $A\times B$ is hopfian.*

Corollary 2. *If B has a principal series and if there are only finitely many homomorphisms of A into B, then $A\times B$ is hopfian.*

Lemma 3. *Let B be a group with a principal series. Let P be a property of groups such that:*

(a) *A has a nontrivial normal group A_* such that A/A_* has property P.*

(b) *If A has property P, then $A\times B$ is hopfian.*

(c) *If $A^*\triangle A$ and A/A^* has P and if $T\in$ End on $(A\times B)$, then $A/(A^*\cap A^*T^j)$ has property P for all integers j.*

(d) *A satisfies the descending chain condition for normal subgroups A^* such that A/A^* has property P.*

Then $A\times B$ is hopfian.

Proof. Choose a minimal normal group A^* such that $A^*\neq 1$ and A/A^* has property P. Then we may assume $A^*T^j\cap A^*=A^*$ for any j so that $A^*T^{-j}=A^*$ for $j\geqq 0$. Now apply Corollary 2 of Theorem 9.

Theorem 11. *Suppose either*

(a) *B is finite, and A satisfies the descending chain condition for normal subgroups of finite index, or*

(b) *B has a composition series and A satisfies the descending chain condition for normal subgroups A^* such that A/A^* has a composition series, or*

(c) *B has a principal series and A satisfies the descending chain condition for normal groups A^* such that A/A^* has a principal series.*

Then $G = A \times B$ is hopfian.

Proof. For instance, for (c) take P the property of having a principal series. Let A/A^* have property P. Let

$$H = A/A_1, \qquad E = A^*/A_1, \qquad F = A/A^*$$

where $A_1 = A^*T^j \cap A^*$. One can show E obeys the ascending and descending chain conditions for normal subgroups of H, that is any ascending or descending chain of subgroups of E which are normal in H terminates. It follows that H has a principal series.

Corollary. *If A is a D.C.C. group, and if B has a principal series, then $A \times B$ is hopfian.*

Theorem 12. *If A satisfies the ascending chain condition for normal nonhopfian subgroups, and if B has a principal series, then $G = A \times B$ is hopfian.*

Proof. Deny. Choose T satisfying the conditions (1), but T not an isomorphism on A. Let,

$$A_i = \bigcap AT^{q\cdot 2^i}, \qquad i = 0, 1, 2, \ldots$$

where q ranges over all integers. Then $A_iT^{2^i} = A_i$ so that the A_i are nonhopfian. Hence we may find j so that $A_j = A_{j+1}$. Hence $A_{j+1}T^{2^j} = A_j$. It follows that kernel $T^i \subset A$, $i \geqq 1$. Now apply Corollary 2 of Theorem 9 to obtain a contradiction.

In view of the former result, it might be interesting to give an example of a hopfian group G such that G contains a normal nonhopfian subgroup and such that G obeys the ascending chain condition for normal nonhopfian subgroups. (The example we give will be of special interest in Theorem 18.)

Let p be a prime and K be the field with p elements. Let m be an integer, $m \geqq 3$. Let SL (m, K) be the group of nonsingular, unimodular, linear transformations of a vector space V of dimension m over K. Let

$$\mathrm{PSL}\,(m, K) = \mathrm{SL}\,(m, K)/Z$$

where Z = center of SL (m, K).

Lemma 4. *Z is the subgroup of diagonal linear transformations of* SL (m, K) *i.e., Z consists of those transformations T which have the form*

$$xT = \lambda x, \ \lambda^m = 1 \quad \textit{for all } x \in V.$$

Also, PSL (m, K) *is simple.*

Proof. This is a special case of a more general result. See [6].

Now let $\langle a_i \rangle$ be a cyclic group of order p, $i=1, 2, 3, \ldots$. Let G be the restricted direct sum of the $\langle a_i \rangle$. Let G_r be the direct sum of the groups $\langle a_i \rangle$ for $1 \leqq i \leqq r$ and let G^r be the restricted direct sum of the groups $\langle a_i \rangle$ for $i > r$. Hence G is the direct sum of G_r and G^r. Now let F_* be the set of automorphisms T of G such that there exists an r such that T fixes the group G_r, that is $G_rT = G_r$, and such that T is the identity map on G^r, i.e., if $x \in G^r$, $xT = x$. One can see that F_* is a group of automorphisms of G. Now if $T \in F_*$ we may choose r such that $G_rT = G_r$ and T is the identity on G^r. Now on G_r, T acts as a linear transformation and we define $|T|$ as the determinant of the matrix representing T on G_r. It may be verified that $|T|$ is well defined, and independent of r. Now let F be the subgroup of F_* of those transformations T, with $|T| = 1$. We claim that F is simple. To see this let F_n be those elements T of F such that $G_{p^n}T = G_{p^n}$, and T the identity on G^{p^n}. We see F is the union of the F_n. Since the union of an ascending sequence of simple groups is simple, we need only show that the groups F_n are simple. However one can see that $F_n \sim \mathrm{SL}(p^n, K)$ and since $\lambda^{p^n} = \lambda$ in K, $\mathrm{SL}(p^n, K)$ has no center and so is simple by the previous lemma.

Now let M be the extension of the group G by F. One sees that if g_1 and g_2 are elements in G, $g_1 \neq 1$, $g_2 \neq 1$, there exists $T \in F$ such that $g_1T = g_2$. One can now see that G is the only normal subgroup of M so that certainly M is hopfian and has a nonhopfian normal subgroup, namely G, and M obeys the ascending chain condition for normal nonhopfian groups.

Lemma 5. *Let $C \triangle G$ and suppose that C has finitely many normal subgroups. If $T \in$ End on G, then either $C \cap CT^i = 1$ for all positive i sufficiently large, or we can find C^*, $C^* \subset C$, $C^* \triangle G$, $C^* \neq 1$, and a positive integer j such that $C^*T^j = C^*$.*

Proof. If $C \cap CT^i \neq 1$, for all i sufficiently large, we may find positive integers r and s, $r < s$, and normal groups C_* and C^* of C such that if u is either r or s,

$$CT^u \cap C = C_*T^u = C^* \neq 1.$$

Hence if $j = s - r$, $C^*T^j = C^*$.

We note at this point that if A is hyper-hopfian, that is if every normal subgroup of A is hopfian, then certainly Theorem 12 guarantees $A \times B$ is hopfian if B has a principal series. For instance if the groups M_i are torsion-hyper-hopfian groups such that elements m_i, m_j of M_i and M_j respectively, $i \neq j$, have relatively prime orders, then the restricted direct product of the M_i is hyper-hopfian. In particular one may choose the M_i to be finite groups.

Theorem 13. *If $A \times B$ is not hopfian and B has a principal series, then there exists a homomorphic image C of B such that $A \times C$ is not hopfian, and $Z(C) \neq 1$, and if T is an arbitrary element of* End *on $(A \times C)$, then T is an isomorphism on C. Also if $C_1 \triangle C$, $C_1 \neq 1$, then $A \times (C/C_1)$ is hopfian. Furthermore, if B has finitely many normal subgroups, and $A \times B$ is not hopfian, we can find C with the former* *properties, and in addition with the property that if* $T \in \text{End}$ *on* $(A \times C)$, $T \notin \text{Aut}(A \times C)$, *then* $CT^i \cap C = 1$, *for all positive i sufficiently large.*

Proof. Choose a group C, C a homomorphic image of B, with the number of terms in a principal series for C minimal with respect to $A \times C$ being nonhopfian. This guarantees that for all $T \in \text{End}$ on $(A \times C)$, T is an isomorphism on C and $A \times C/C_1$ is hopfian if $C_1 \neq 1$. Furthermore, since $A \times C$ is not hopfian, we may choose $T \in \text{End}$ on $(A \times C)$ so that $AT \cap CT \neq 1$. Hence $Z(CT) \sim Z(C) \neq 1$. Furthermore if B has finitely many normal subgroups, so does C so that if T is any element of End on $(A \times C)$, $T \notin \text{Aut}(A \times C)$, then $CT^i \cap C = 1$ for all i sufficiently large or else we could choose C^* as in the previous lemma and $A \times (C/C^*)$ would not be hopfian.

Corollary. *Suppose A cannot be written in the form*

$$\begin{gathered} A = A_1 \cdot A_2,\ A_i \triangle A,\ A_i \neq A,\ i = 1, 2, \\ A_1 \text{ and } A_2 \text{ commute elementwise, } A_1 \text{ a homomorphic image of } A,\ Z(A_2) \neq 1. \end{gathered} \tag{3}$$

Then if B has finitely many normal subgroups, $A \times B$ *is hopfian. Moreover, if the homomorphic images of A are indecomposable as a direct product, then* $A \times B$ *is hopfian. Finally if B is fixed, and A cannot be written in the form* (3) *with the additional stipulation that* A_2 *be a homomorphic image of B, then* $A \times B$ *is hopfian.*

Proof. If $A \times B$ is not hopfian, choose C as in the previous theorem and $T \in \text{End}$ on $(A \times C)$, $C \cap CT = 1$, and T not an isomorphism on A. Let $N = CT^{-1}$ so that $(A \times C)/N \sim A$ so that we may take $A_1 = (AN)/N$, and $A_2 = (CN)/N \sim C$. If A is written in the form (3), then $A/A_1 \cap A_2 = A_1/A_1 \cap A_2 \times A_2/A_1 \cap A_2$.

Theorem 14. *If* $Z_0 = 1$ *and* $Z_{n+1}/Z_n = Z(A/Z_n)$, $n \geqq 0$, *and* A/Z_n *and its center are hopfian for all* $n \geqq 0$, *then if B has a principal series,* $A \times B$ *is hopfian.*

Proof. Deny. Choose a group E with a principal series and an integer $r \geqq 0$, such that $H = A/Z_r \times E$ is not hopfian and $\triangle(E)$ = length of a principal series for E is minimal. That is, if $A/Z_q \times D$ is not hopfian, and if D has a principal series, then $\triangle(E) \leqq \triangle(D)$. Consequently, the group C we may associate with E, by the previous theorem, is E itself, so $Z(E) \neq 1$. If $T \in \text{End}$ on H, but $T \notin \text{Aut}\, H$, we see from the minimality of $\Delta(E)$ that

$$Z(H)T^{-1} = Z(H) = Z_{r+1}|Z_r \times Z(E).$$

However, $Z(E)$ is finite and this contradicts Theorem 3.

Corollary. *If* $Z(A) = 1$, *and if A is hopfian and B has a principal series, then* $A \times B$ *is hopfian.*

5. $A \times B$, B ***n*-normal.** We begin by giving some examples of n-normal groups. As we have mentioned, we have the groups C_t, $t = p^{n+1}$, p a prime. Or if F is a

simple group and B is an n-normal group of automorphisms of F, such that B does not contain any inner-automorphism (different from 1), then the extension of F by B is $n+1$ normal. In particular, if $B\sim C_{p^n}$, p a prime, we can find a prime q, $q=1 \bmod p^n$ so that C_q has a group of automorphisms, B, and extending C_q by B gives us a nonabelian n-normal group. Similarly if H is the alternating group of arbitrary infinite cardinality, and if $R\in \operatorname{Aut} H$, $O(R)=p^n$, and if $\langle R\rangle$ contains no inner-automorphism except 1, if we extend H by $\langle R\rangle$ we obtain an infinite n-normal group so that nonabelian n-normal groups of arbitrary infinite cardinality exist.

Until further notice, B shall represent an n-normal group, with normal subgroups,

$$1 = B_0, B_1, \ldots, B_n, B_{n+1} = B, B_i \subset B_{i+1}.$$

LEMMA 6. *If $T\in$ End on G, $G=A\times B$, $T\notin$ Aut G, then $G=A\cdot AT$ and $B\cdot AT$ is a proper subgroup of G.*

Proof. Either $A\cdot AT\subset B\cdot AT$ or $B\cdot AT\subset A\cdot AT$. Hence all we need show is that $A\cdot AT$ is not a subgroup of $B\cdot AT$. But if $A\cdot AT\subset B\cdot AT$, then $G=B\cdot AT$ and hence $A\sim AT/B\cap AT$, from which we could easily deduce that $T\in \operatorname{Aut} G$.

LEMMA 7. *Suppose $T\in$ Aut G, and $AT\cap BT=B_iT$ where kernel $T\cap B=B_k\subset B_i$. Then if $B\cap AT=B_j$, then $j>i$.*

Proof. Use the previous lemma to see that B/B_j is contained isomorphically as a proper normal subgroup of $G/AT\sim B/B_i$.

LEMMA 8. *If $BT\nsim B$, then $AT\cap BT\neq 1$.*

Proof. Deny. Then $G=A\times B=AT\times BT$. By the previous lemma, $B_k\subset AT$. But then Corollary 3 of Theorem 9 implies that $T\in \operatorname{Aut} G$.

THEOREM 15. *A necessary and sufficient condition that $T\in$ Aut G is $AT\cap BT=1$.*

Proof. The previous lemma and Lemma 2.

In our next theorem, we show that if $A\times B$ is not hopfian A must enjoy several anomalous properties.

THEOREM 16. *Suppose $G=A\times B$ is not hopfian. Then,*

(1) *There exists infinitely many homomorphisms of A onto B.*

Also there exist normal subgroups of $A\times B$, R^*, R, R_i, R^i, $i\geqq 0$ such that

(2) *$R^*\subset R^{i+1}\subset R^i$, $R^0=R_0$, $R_i\subset R_{i+1}\subset R$ for all i.*

(3) *$R^*=\bigcap R^i$, $R=\bigcup R_i$, where the intersection and union are taken over all $i\geqq 0$. Also the containments in* (2) *are proper.*

(4) *The R^i are subgroups of A.*

(5) *R^* and R are not hopfian.*

(6) *$R^i/R^{i+1}\sim R^j/R^{j+1}\sim R_1/R_0\sim$ a normal subgroup of B for all i and j, and $R^i/R^*\sim R^j/R^*$ for all i and j.*

(7) *$R_{i+1}/R_i\sim R_{j+1}/R_j\sim$ a normal subgroup of a proper homomorphic image of B, $i\geqq 1$, $j\geqq 1$ and*

(8) *There exist normal subgroups* $A_i \subset A$, $i=1, 2, \ldots$, $A_i \subset A_{i+1}$ *properly, such that* $A_{i+1}/A_i \sim R_2/R_1$ *for all* i.

(9) *There exist normal subgroups* K_i, $K_i \subset K_{i+1}$, $i \geqq 0$, *such that if* $L = \bigcup K_i$, *then* L *is nonhopfian, and*

$$R_i/K_j \sim R_{i+j}, \qquad R_i/L \sim R_j/L, \qquad K_{i+j}/K_j \sim K_i \quad \text{and} \quad L/K_j \sim L.$$

Proof. Let $T \in \text{End}$ on G, $T \notin \text{Aut}\, G$. Then by Lemma 6, $A \cdot AT^j \neq G$ for all $j > 0$, which implies $A \cdot AT^{-j} = G$ for $j > 0$. Hence $A/A \cap AT^{-j} \sim B$ and one may show (as in Theorem 10) that if the groups $A \cap AT^{-j}$, $j = 1, 2, 3, \ldots$ are not distinct, then T is an automorphism.

Now let us assume, without loss of generality, that T satisfies the condition (1), and that $AT^r \cap BT^r = B_i T^r$ for all $r \geqq 1$ (for some fixed i, $i \geqq 1$) and that i is maximal in the sense that if

$$B_u T^q \subset AT^q \cap BT^q \quad \text{for some } q \geqq 1, \text{ then } u \leqq i.$$

(For if T does not obey these conditions, some power of T does, and we could then work with this power of T.)

Now we define,

$$R_j = \bigcap_{i \geqq j} AT^i \qquad j \geqq 0,\ R = \bigcup_{j \geqq 0} R_j.$$

With the aid of (1), we see $R_i T = R_{i+1}$ so that $RT = R$, and $R \neq 1$, since kernel $T \subset R$. Moreover, the groups R_j are all distinct, for if say $R_m = R_{m+1}$, then $R_j = R_m$ for $j > m$ and hence $R = R_m$. But then with the aid of Lemma 7, we see $B_{i+1} \subset R_1 \subset R$. Hence,

$$A \cdot R = A \times B_s, \qquad s > i.$$

Hence, $(A \cdot R)T^m = AT^m \cdot RT^m = AT^m R = AT^m = AT^m \cdot B_s T^m$ so that $B_s T^m \subset BT^m \cap AT^m$, a contradiction of the maximality of i.

We now define

$$R^0 = R_0 \quad \text{and} \quad R^{n+1} = R^n T^{-1} R_0, \qquad n \geqq 0.$$

By induction and the previous lemma, we see that R^{n+1} is a proper subgroup of R^n and $R^n = \bigcap AT^j$ where j ranges over all integers $\geqq -n$ for each $n \geqq 0$. Moreover if we consider the homomorphism of R^n onto R^{n-1}, induced by T for $n \geqq 1$, we see that the preimage of R^n is exactly R^{n+1} so that

$$R^n/R^{n+1} \sim R^{n-1}/R^n, \qquad n \geqq 1.$$

Furthermore, if we consider the homomorphism of $R^0 = R_0$ onto R_1 induced by T, we see that the preimage of R_0 is exactly R^1 so that $R^0/R^1 \sim R_1/R_0$.

Now one may see that R_1/R_0 is isomorphic to a normal subgroup of $AT/A \cap AT \sim B$. Also with the aid of (1) we see,

$$R_{j+1}/R_j \sim R_{j+2}/R_{j+1}, \qquad j > 1.$$

Furthermore, R_2/R_1 is isomorphic to a normal subgroup of $AT^2/AT \cap AT^2 \sim B/B_i$.

If $A_k = R_k \cap A$, ultimately the A_k are distinct and by a suitable reindexing, the A_k may be seen to have the properties asserted in the Theorem. One may verify the remaining assertions by taking $K_j = \text{kernel } T^j$, $j \geqq 1$, and $L = \bigcup_{j \geqq 1} K_j$, and by noting that $R^* T^{-1} = R^*$.

COROLLARY. *If* $|B| > |A|$, *then* $A \times B$ *is hopfian.*

Proof. B cannot be a homorphic image of A.

We now find some particular values of n for which $A \times B$ is hopfian.

LEMMA 9. *If* $|B| = p^{n+1}$, *p a prime, then* $B \sim C_p^{n+1}$.

Proof. Use induction on n, and the fact that $Z(B) \neq 1$.

LEMMA 10. *If* $T \in \text{End}$ *on* $(A \times B)$ *and* $BT \subset A$ *and* $B \subset AT$, *then T is an isomorphism on A.*

Proof. $AT = B \times A \cap AT$ and $A = BT(A \cap AT)$. These two decompositions give rise to a homomorphism S of AT onto A such that S agrees with T on B and S is the identity on $A \cap AT$.

LEMMA 11. *Let k be the least integer,* $k \geqq 0$ *(if one exists), such that* $A \times B$ *is not hopfian for some A and for some k normal group B. Then if* $T \in \text{End}$ *on* $(A \times B)$, $T \notin \text{Aut}(A \times B)$, *then* $B \cap BT = 1$ *and T is an isomorphism on B.*

Proof. Deny. Then $B_1 T \subset B_1$ and $A \times B / B_1$ is not hopfian, which contradicts the minimality of k if $B_1 \neq B$, or the hopficity of A if $B_1 = B$.

THEOREM 17. *If B is n-normal,* $0 \leqq n \leqq 1$, *then* $A \times B$ *is hopfian.*

Proof. Let k be as in the last lemma, $A \times B$ not hopfian, B k-normal. We will show $k \geqq 2$. Let $T \in \text{End}$ on $(A \times B)$, T not an isomorphism on A. Let $A \cdot BT = A \times B_r$, $B \cdot AT = (B_q T)(AT)$, $AT \cap BT = B_i T$, $B \cap AT = B_j$ where $1 \leqq i < j$. Using Lemma 11 we see B_r, B_q and B_i are central groups of B and hence are cyclic p groups for some prime p. Furthermore, we see $A \cap BT = (B_{k-r+1})T$ and $B/B_j \sim B_q/B_i$, $B/B_{k-r+1} \sim B_r$, $q = k + i - j + 1$. Hence we must have,

$$j > r, \qquad j > q, \qquad k - r + 1 > q, \qquad k - r + 1 > r$$

or otherwise B would be a finite p group and hence B would be cyclic, a contradiction of Theorem 3. In summary we have,

$$0 \leqq r < \frac{k+1}{2} \leqq \frac{k+i}{2} < \frac{k+i+1}{2} < j \leqq k+1.$$

And with the aid of Lemma 10, we see $1 \leqq i < j - r \leqq k$. Hence we see $k = 0$ or 1 is impossible.

COROLLARY 1. *If* $C = D \cdot E$, $D \triangle C$, $D \cap E = 1$ *where D and E are simple, then* $A \times C$ *is hopfian.*

Proof. Either C is 1-normal or $C \sim D \times E$.

COROLLARY 2. *If B is 2-normal and if $T \in$ End on $(A \times B)$, T not an automorphism, then $B \subset AT$, $A \cap BT = B_2T$, $B/B_2 \sim B_1 \sim C_p$ for some prime p and $B_1 = Z(B)$.*

COROLLARY 3. *If r is a positive integer, then $A \times$ symmetric (r) is hopfian.*

Proof. Symmetric 4 is 2-normal and centerless. If $r \neq 4$, symmetric r is 1-normal.

COROLLARY 4. *If B is a group such that B has exactly one normal group in a principal series, i.e., B has a principal series of the form 1, B_*, B, then $A \times B$ is hopfian.*

Proof. Either B is 1-normal or B is the direct product of simple groups.

COROLLARY 5. *If $G = A \times B$, B n-normal and if BT is i-normal, $i = 0$ or 1, then $T \in$ Aut G.*

THEOREM 18. *Let E be a class of hopfian groups such that any hopfian group is isomorphic to a unique group of E. Then there exists a class E_* of hopfian groups such that:*

(a) *E and E_* have the same cardinality.*

(b) *No two distinct groups of E_* are isomorphic.*

(c) *Any hopfian group is contained isomorphically as a normal subgroup of some group in E_*.*

(d) *Every group in E_* has a nonhopfian normal subgroup.*

Proof. Let E_* be the set of groups which is formed by taking the direct product of groups in E with the group M of the example following Theorem 12, i.e. $E_* = \{(A \times M)/A \in E\}$.

Our assertions follow from the previous theorem, the definition of M and Lemma 2.

6. **Super-hopficity.** We terminate this paper with an investigation of the concept of super-hopficity. For an illustration of super-hopficity, we note that the restricted direct product of periodic super-hopfian groups M_i, such that $(O(m_i), O(m_j)) = 1$ for $m_i \in M_i$, $m_j \in M_j$, $i \neq j$, is super-hopfian. In particular, the M_i might be chosen as finite groups.

We no longer assume that B designates an n-normal group.

LEMMA 12. *Let A be super-hopfian and let $H = A \cdot B$, $A \triangle H$, $B \triangle H$. Suppose $T \in$ End on H and $B \subset R$, $R \triangle H$ and $RT \subset R$. Then $RT^{-1} = R$.*

Proof. If $RT^{-1} \neq R$, H/RT^{-1} is a homomorphic image of A, but H/RT^{-1} is not hopfian.

COROLLARY. *If H and T are as in the lemma, and if $r > 0$ and if L_r is the subgroup of H generated by the groups BT^{ir}, $i \geqq 0$, then $L_rT^{-r} = L_r$.*

LEMMA 13. *If H and L_r and T are as in the preceding corollary and if $B \cap BT^{ir} = 1$ for fixed r and for all $i \geqq 1$, then B abelian.*

Proof. Since $L_r T^r = L_r$, L_r is generated by the groups BT^{ir}, $i \geqq 1$, and B commutes element-wise with each BT^{ir}, $i \geqq 1$. Hence $B \subset Z(L_r)$.

THEOREM 19. *Let $H = A \cdot B$ where $A \triangle H$ and $B \triangle H$ and where A is super-hopfian. Suppose B satisfies any one of the following conditions*:

(a) *B is a finitely generated A.C.C. group.*

(b) *B has finitely many normal subgroups or,*

(c) *B is an A.C.C. group and if B_* is any homomorphic image of B and if $B_1 \triangle B_*$ and $B_2 \triangle B_*$ and if $B_1 \sim B_2$ then $B_1 = B_2$.*

Then H is super-hopfian.

Proof. It suffices to prove H is hopfian since any homomorphic image of H satisfies the same hypothesis as H in any of the three situations. Let us assume that (a) holds. Let $T \in$ End on H. In the notation of the corollary to Lemma 12, we have $B \subset L_1 = L_1 T$ and $L_1 T$ is generated by the groups BT^i, $i \geqq 1$. Hence, since B is finitely generated, we can find r such that

$$B \subset BT \cdot BT^2 \cdots BT^{r-1} BT^r = E.$$

Hence,

$$BT \subset BT^2 \cdot BT^3 \cdots BT^r \cdot BT^{r+1} = ET.$$

Consequently, $E \subset ET$ and hence,

$$ET^i \subset ET^{i+1}, \qquad i \geqq 0. \tag{4}$$

Now since B is an A.C.C. group, so is BT^i, $i \geqq 0$, and hence so is E. Consequently, T is an isomorphism on ET^i for all i sufficiently large and positive. However, L_1 is the union of the groups ET^i, $i \geqq 1$. Hence in view of (4), we see T is an isomorphism on L_1. But from the corollary to Lemma 12, $L_1 = L_1 T^{-1}$ and so T is an automorphism.

Now suppose the assertion of (b) is false and choose a counterexample $A \cdot B = H$ so that B has the fewest number of normal subgroups among all possible counterexamples. Let $T \in$ End on H, T not an isomorphism on A. Then we can find $r > 0$ such that $B \cap BT^i = 1$ for all $i \geqq r$ or else by Lemma 5, we can find $j > 0$ such that $B_* T^j = B_*$ for some normal subgroup, B_* of B, $B_* \neq 1$. Furthermore, T^j is an isomorphism on B because of the "minimality" of B. Hence,

$$H/B_* = [(AB_*)/(B_*)](B/B_*)$$

is not hopfian, which contradicts the "minimality" of B. Hence r exists as asserted, and so we see from Lemma 13 that B is abelian. Hence B is finite. This contradicts part (a) of our theorem.

Finally for (c) we may proceed by denying that G is hopfian. Hence we may choose B^* and A^* such that $H^* = A^* \cdot B^*$, $A^* \triangle H^*$, $B^* \triangle H^*$, A^* super-hopfian, B^* a homomorphic image of B, H^* not hopfian, and such that if $H_1 = A_1 \cdot B_1$, A_1 super-hopfian, B_1 a proper homomorphic image of B^*, then H_1 is hopfian.

Choose $T \in \text{End}$ on H^*, T not an isomorphism on A^*. Note T^i must be an isomorphism on B^* for $i \geqq 1$. Now if $B^* \cap B^*T^j \neq 1$ for some j, $j \geqq 1$, we may write $B_* = B_2T^j$, $B_* \subset B^*$, $B_* \neq 1$, $B_* \neq B_2$ (or else G/B_* is not hopfian, etc.) but $B_* \sim B_2$, a contradiction of our hypothesis. Hence, $B^* \cap B^*T^j = 1$ for $j \geqq 1$, so that B^* is abelian and finitely generated, a contradiction of part (a) of our theorem.

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