CARDINAL ALGEBRAS AND MEASURES INVARIANT UNDER EQUIVALENCE RELATIONS([1])

BY
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Introduction. There have been discussions from time to time of "abstract measures" the values of which need not be numerical (e.g. [2], [3], [4], [6], [7]). One of the purposes of this paper is to present arguments in favor of the use of cardinal algebras as values for these measures. Cardinal algebras were introduced and developed by A. Tarski in [8]. They have many of the good properties of real numbers and arise naturally in situations like the following:

A (pseudo) group G of one-one functions is given with domain and range in a σ-ring of sets $\mathscr{K}$. An equivalence relation between members of $\mathscr{K}$ is defined as follows:

$A \simeq B$ iff there are A_i, $B_i \in \mathscr{K}$, $f_i \in G$ for $i < \infty$ such that $A_i \cap A_j = 0 = B_i \cap B_j$ for $i \neq j$, $A = \bigcup_{i<\infty} A_i$, $B = \bigcup_{i<\infty} B_i$, $A_i \subseteq \operatorname{Dom} f_i$ and $f_i^*(A_i) = B_i$ for all $i < \infty$.

This is equivalence by countable decomposition. Equivalence relations like $\simeq$ have been considered in [2], [3], [4], [6], [7].

When the aim is to obtain a measure that is faithful to an equivalence relation of this form, the first natural step is to consider the equivalence classes determined by the relation. It happens that these equivalence classes with suitably defined finite and infinite addition form a generalized cardinal algebra. For instance, the "measure algebras" considered in [4] and [6] are generalized cardinal algebras.

My second purpose is to determine in what conditions we can obtain a numerical measure faithful to the equivalence relation; that is, a countably additive measure that satisfies the following:

(a) The only sets with measure zero are those that necessarily have to have it. That is, sets that have infinitely many disjoint equivalent sets contained in a set of measure one. These sets I call negligible.

(b) For sets with positive measure, it should be valid that two sets have the same measure iff they are equivalent.

These characteristics are specially important with respect to probability measures where we want to be as faithful as possible to the equal likelihood relation (cf. [3]).

Theorem 2.11, below, gives sufficient (and almost necessary) conditions on the equivalence relation to obtain such a measure.

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Requirement (b), however, is not satisfied by Lebesgue measure when G is the group of translations. However, this measure satisfies (a) and

(b′) Two sets have the same measure iff they are equivalent modulo a negligible set.

Sufficient hypotheses for the existence of a measure that satisfies (a) and (b′) are given by Theorem 2.10.

The results of this paper depend heavily on the theory of cardinal algebras (see [8]). The first section of this paper contains some new results about cardinal algebras that are of interest in their own right. One of these results gives a necessary and sufficient condition for a generalized cardinal algebra to be isomorphic to a generalized cardinal subalgebra of the nonnegative real numbers with addition.

In the second section, the main theorems are proved that give the construction of measures that satisfy (a), (b) or (a), (b′). The works of Maharam [4] and [6] use equivalence relations with some properties like mine to construct strictly positive measure on σ-complete complemented distributive lattices (or σ-complete Boolean algebras). That is, measures that only vanish for the 0 of the algebra. There is a discussion of the relation between Maharam's work and mine in this section.

The last section contains some applications and examples of the main theorems. As pointed out above, it is possible in many cases to obtain a probability measure from the equal likelihood relation (cf. [3]).

By applying the main theorems to Lebesgue measure in $\boldsymbol{R}^n$ it is possible to prove some new facts about the relation between the measure and translations. Particularly interesting are two characterizations of sets of measure zero and a refinement of the Banach-Tarski result given in [1].

I. **Real multiples in cardinal algebras.** The results in this paper depend on the book *Cardinal algebras* by A. Tarski [8]. I shall identify the theorems taken from that book by their number and a T.

The possibility of defining real multiples of elements in a cardinal algebra is mentioned in Tarski's book and some properties of the real multiples, namely 1.7, 1.8, 1.9, 1.10 below, are stated without proof. I shall give an indication of the construction of real multiples and a sketch of the proofs of these properties.

In all this section I shall assume a fixed cardinal algebra $\mathfrak{A}=\langle A, +, \Sigma\rangle$. a, b, c, d, e will be elements of A with or without subscripts. n, m, N, i, j will be nonnegative integers or ∞.

First, I shall define rational multiples.

DEFINITION 1.1. Let p be a nonnegative rational number, $p=m/n$ with m, n nonnegative integers. Then $pa=b$ iff $ma=nb$.

By Theorem 2.34T and 2.36T it is easy to show that this definition is correct. From the same theorems it is easy to deduce the following two theorems:

THEOREM 1.2. *Let p, q be nonnegative rational numbers. Suppose that qa and $p(qa)$ are defined. Then $(pq)a$ is defined and $(pq)a=p(qa)$.*

THEOREM 1.3. *Let p, q be nonnegative rational numbers, $p \leqq q$, pa and qa defined. Then $pa \leqq qa$.*

THEOREM 1.4. *Let $N<\infty$. If for every $i<N$, p_i is a nonnegative rational number and $p_i a$ is defined, then $(\sum_{i<N} p_i)a$ is defined and*

$$\left(\sum_{i<N} p_i\right)a = \sum_{i<N} p_i a.$$

Proof. We can assume that $N \neq 0$. As for $N=0$ the result is trivial.

Let $p_i = m_i/n_i$ with m_i, n_i relatively prime integers and let $n_i b_i = m_i a$.

First we prove by induction, using 2.34T and 2.36T, the following lemma:

If the hypotheses of the theorem are satisfied and $N \neq 0$, then there is a c such that

$$a = kc, \qquad b_i = (p_i k)c$$

where k is the least common multiple of $n_0, n_1, \dots, n_{N-1}$.

Having proved the lemma, we have

$$\sum_{i<N} p_i a = \sum_{i<N} b_i = \sum_{i<N} (p_i k)c.$$

But $p_i k$ is an integer. So

$$\sum_{i<N} p_i a = \left(\sum_{i<N} p_i k\right)c.$$

On the other hand $a=kc$, so by 1.2

$$\left(\sum_{i<N} p_i k\right)a = \left(\sum_{i<N} p_i k\right)kc = \left(\sum_{i<N} p_i k^2\right)c.$$

Then by 2.34T

$$\left(\sum_{i<N} p_i\right)a = \left(\sum_{i<N} p_i k\right)c = \sum_{i<N} p_i a.$$

The next theorem provides the justification for the definition of real multiples. I shall adopt the convention that when $\sum_{i<\infty} p_i$ diverges for p_i real numbers, $\sum_{i<\infty} p_i = \infty$.

THEOREM 1.5. *Let p_i, q_i be nonnegative rational numbers for every $i<\infty$ such that $\sum_{i<\infty} p_i = \sum_{i<\infty} q_i$.*

Then if $p_i a$, $q_i a$ exist for all $i<\infty$, we have

$$\sum_{i<\infty} p_i a = \sum_{i<\infty} q_i a.$$

Proof. *Case* 1. $\sum_{i<\infty} p_i = \sum_{i<\infty} q_i = \infty$. Then for every $n<\infty$ there is an $m<\infty$ such that

$$\sum_{i<n} p_i \leqq \sum_{i<m} q_i.$$

Then by 1.2 and 1.3

$$\sum_{i<n} p_i a = \Big(\sum_{i<n} p_i\Big)a \leqq \Big(\sum_{i<m} q_i\Big)a = \sum_{i<m} q_i a \leqq \sum_{i<\infty} q_i a$$

for every $n<\infty$.

Thus, $\sum_{i<\infty} p_i a \leqq \sum_{i<\infty} q_i a$ by 2.21T.

Similarly we can prove the converse inequality.

Case 2. $\sum_{i<\infty} p_i = \sum_{i<\infty} q_i < \infty$.

Then there is an increasing sequence of finite positive integers $n_0, n_1, n_2, \ldots$ such that $(1/n_i)a$ is defined for every $i<\infty$ and $1/n_i \to 0$ as $i \to \infty$.

So by 1.3 we have for each $j<\infty$ and $N<\infty$ such that for every $n<\infty$

$$\Big(\sum_{i<n} p_i\Big)a \leqq \Big(\sum_{i<N} q_i + \frac{1}{n_j}\Big)a.$$

By 1.5

$$\sum_{i<n} p_i a \leqq \sum_{i<N} q_i a + \Big(\frac{1}{n_j}\Big)a \leqq \sum_{i<\infty} q_i a + \Big(\frac{1}{n_j}\Big)a$$

for every $n<\infty$. Hence by 2.12T

$$\sum_{i<\infty} p_i a \leqq \sum_{i<\infty} q_i a + \Big(\frac{1}{n_j}\Big)a \quad \text{for every } j<\infty.$$

So by 2.29T there is a b such that

$$\sum_{i<\infty} p_i a \leqq \sum_{i<\infty} q_i a + b \quad \text{and} \quad b \leqq \Big(\frac{1}{n_j}\Big)a \quad \text{for every } j<\infty.$$

As $n_i \to \infty$ as $i \to \infty$ we have $nb \leqq a$ for every $n<\infty$. So, by 2.21T $\infty b \leqq a$ and by 1.29T $a+b=a$. Using 2.36T and 1.46T we obtain $q_i a + b = q_i a$ and hence by 1.28T

$$\sum_{i<\infty} p_i a \leqq \sum_{i<\infty} q_i a.$$

By a similar argument we can prove the converse inequality, and hence the theorem.

DEFINITION 1.6. Let r be a nonnegative real number. Then $ra=b$ iff there is a sequence of nonnegative rational numbers $p_0, p_1, \ldots$ such that $p_i a$ is defined for all $i<\infty$, $r=\sum_{i<\infty} p_i$ and $b=\sum_{i<\infty} p_i a$.

The next two theorems show a way of determining which real multiples are defined.

THEOREM 1.7. *If for a given a, there is a largest integer $n<\infty$ such that $a=nb$ for some b, then if r is a nonnegative real number, ra exists in case rn is an integer, and does not exist otherwise.*

Proof. Based on 2.34T and 2.36T.

THEOREM 1.8. *If, for a given a, there is no largest integer $n<\infty$ such that $a=nb$ for some b, then ra exists for every nonnegative real number r.*

Proof. The set of rational numbers p for which pa is defined is dense in the nonnegative reals. Hence, for every nonnegative real r there are rationals $p_0, p_1, \ldots$ with $p_i a$ defined and $r=\sum_{i<\infty} p_i$.

THEOREM 1.9. *If $n\leqq\infty$ and for every $i<n$, r_i are real numbers with $r_i a$ defined, then $(\sum_{i<n} r_i)a$ is defined and*

$$\Big(\sum_{i<n} r_i\Big)a = \sum_{i<n} r_i a.$$

Proof. Let $r_i=\sum_{j<\infty} p_{ij}$ and $r_i a=\sum_{j<\infty} p_{ij}a$ for rational p_{ij}. Then from the definitions:

$$\sum_{i=n} r_i a = \sum_{i<n}\Big(\sum_{j<\infty} p_{ij}a\Big) = \Big(\sum_{i<n}\sum_{j<\infty} p_{ij}\Big)a = \Big(\sum_{i<n} r_i\Big)a.$$

THEOREM 1.10. *If $n\leqq\infty$, r is a real number and ra_i exists for every $i<n$, then $r\sum_{i<n} a_i$ is defined and*

$$r\sum_{i<n} a_i = \sum_{i<n} ra_i.$$

Proof. (a) Let $r=p/q$ with p, q integers. We have, for every $i<n$, some b_i with $pa_i=qb_i$. So, $\sum_{i<n} pa_i=\sum_{i<n} qb_i$ and $p\sum_{i<n} a_i=q\sum_{i<n} b_i$.

(b) Suppose $r=\sum_{j<\infty} p_j$, with $ra_i=\sum_{j<\infty} p_j a_i$, p_j rational. Then,

$$\begin{aligned}
\sum_{i<n} ra_i &= \sum_{i<n}\Big(\sum_{j<\infty} p_j a_i\Big) & \\
&= \sum_{j<\infty}\Big(\sum_{i<n} p_j a_i\Big) & \text{by commutative law}\\
&= \sum_{j<\infty}\Big(p_j\sum_{i<n} a_i\Big) & \text{by (a)}\\
&= r\sum_{i<n} a_i & \text{by definition.}
\end{aligned}$$

THEOREM 1.11. *Let r, r' be nonnegative real numbers. If ra and $r'a$ are defined and $r\leqq r'$, then $ra\leqq r'a$.*

Proof. The result is obtained from 1.2, 1.3, 1.4 and 2.21T.

THEOREM 1.12. *If there is a finite nonnegative real number $r\neq 0$ for which $ra+b=ra$, then for every nonnegative real number $r'\neq 0$, with $r'a$ defined, we have $r'a+b=r'a$.*

Proof. Let m be an integer with $m\geqq r$. Then by 1.11, $ma\geqq ra$, and by 1.28T, $ma+b=ma$. Hence by 2.12T, $a+b=a$.

Let now n be an integer with $(1/n)a$ defined. Then $n(1/n)a+b=n(1/n)a$. So by 2.12T $(1/n)a+b=(1/n)a$.

For any real r' such that $r'a$ is defined, there is an integer n such that $(1/n)a$ is defined and $(1/n)\leqq r'$ (by 1.7, 1.8). Then by 1.28T $r'a+b=r'a$.

DEFINITION 1.13. a is completely divisible iff there is no largest integer $n<\infty$ such that $a=nb$ for some b. Otherwise, a is incompletely divisible.

Next we have two special theorems for completely divisible elements.

THEOREM 1.14. *Let a be completely divisible. Then $a+c=a$ iff for every positive real number r, $c\leqq ra$.*

Proof. (a) Suppose $a+c=a$, $r>0$. Then by 1.12 $ra+c=ra$. So by 1.29T, $c\leqq ra$.

(b) Suppose $c\leqq(1/n)a$ for every positive integer $n<\infty$. So, $nc\leqq a$ for every $n<\infty$. Then by 2.21T, $\infty c\leqq a$, and by 1.29T, $a+c=a$.

THEOREM 1.15. *Let a be completely divisible. If for every positive real number r, $b\leqq ra$ or $ra\leqq b$, $a+b\neq a$ and $b\leqq\infty a$, then $b=pa$ for some positive real number p.*

Proof. Let $Q=\{r : ra\leqq b\}$, $p=$least upper bound of Q. Then by 1.14, $p>0$. If $p=\infty$ then $b\geqq na$ for every $n<\infty$ and $b=\infty a$. So let $p<\infty$, $p=\sum_{i<\infty} q_i$, $pa=\sum_{i<\infty} q_i a$ with q_i rational, different from 0. This is possible, as pa is defined, by 1.8.

Then $\sum_{i<n} q_i a\leqq b$ for every $n<\infty$. So by 2.21T $pa\leqq b$. On the other hand,

$$b \leqq (p + 1/n)a = pa+(1/n)a \quad \text{for every } n < \infty.$$

Then by 2.28T, there is a c such that

$$b \leqq pa+c \quad \text{and} \quad c \leqq (1/n)a \quad \text{for every } n < \infty.$$

Hence by 1.14 and 1.12, $pa+c=pa$. So $b\leqq pa$ and $b=pa$.

In general, the relation $\leqq$ is only a partial ordering in cardinal algebras. When it is a simple ordering, we have a very interesting situation, as is shown by the following theorems.

LEMMA 1.16. *Suppose that for all $a, b\leqq c$ we have $a\leqq b$ or $b\leqq a$. Then for all nonnegative integers $n\leqq\infty$ and for all $a, b\leqq c$, we have $na\leqq b$ or $b\leqq na$.*

Proof. We prove it first for $n<\infty$ by induction on n.

By applying 2.21T, we obtain the case $n=\infty$.

THEOREM 1.17. *Suppose that for all $a, b\leqq a_0$, $a\leqq b$ or $b\leqq a$. If there is a sequence $a_0, a_1, \dots$ and two sequences of finite positive integers $k_0, k_1, \dots, l_0, l_1, \dots$ such that $a_n=k_n a_{n+1}+l_n a_{n+2}$ for every $n<\infty$, then a_0 is completely divisible.*

Proof. If $a_0=2a_0$, there is nothing to prove, because $a_0=ma_0$ for every positive integer m.

So, assume that

(0) $a_0\neq 2a_0$.

We have

$$a_n = a_{n+1}+(k_n-1)a_{n+1}+l_n a_{n+2} \quad \text{for every } n < \infty.$$

So, by the remainder postulate, there is a c such that

$$a_n = c+(k_n-1)a_{n+1}+\sum_{i<\infty} p_{i+n+2}a_{i+n+2} \quad \text{for all } n<\infty,$$

where $p_{i+n+2}=l_i+k_i-1\neq 0$. Hence, $c\leqq a_n$ for all $n<\infty$.
So, by 2.21T, $\infty c\leqq\sum_{i<\infty} a_{i+n+2}\leqq\sum_{i<\infty} p_{i+n+2}a_{i+n+2}$. So

(1) $a_n=\sum_{i<\infty} p_{i+n+2}a_{i+n+2}+k_{n-1}a_{n+1}$.

By induction we can prove

(2) $a_n=\sum_{i<\infty} q_{i+n+m,n}a_{i+n+m}$ for every finite positive integer $m>1$. The $q_{i,m}$'s are finite positive integers.

Suppose now that for some n, $k<\infty$ we have

(3) $a_n+a_k=a_n$.

So, $a_n+a_l=a_n$ for all $l\geqq k$ by 1.30T. Hence $a_0+pa_l=a_0$ for all $p\leqq\infty$ and all $l\geqq k$, by 1.46T and 1.28T.

Hence by 1.47T

$$a_0+\sum_{i<\infty} q_{i+k,0}a_{i+k} = a_0.$$

So by (2), $a_0+a_0=a_0$, contradicting (0).

Then we have that (3) is false, i.e.

(4) $a_n+a_k\neq a_n$ for all k, $n<\infty$.

Let now the finite positive integer m be given. Let $b_0=a_0$ and define by recursion elements b_n, c_n of A and nonnegative integers j_n for $n>0$ such that

(i) $c_n=a_k$ for some k or $c_n=0$,

(ii) $b_n\leqq mj_nc_n$,

(iii) $mj_nc_n+b_n=b_{n-1}$,

(iv) $b_n\neq 0$ and $mj_nc_n+b_n\neq mj_nc_n$, or $b_n=0$.

(a) $n=1$. There is a k such that $ma_k\leqq a_0$. Because if not, from the simple ordering hypotheses and 1.16, we would have $a_0\leqq ma_k$ for every $k<\infty$. Then $\infty a_0\leqq m\sum_{i<\infty} q_{i+1,0}a_{i+1}$. So by (2), $ma_0+a_0=m_0$, and by 2.12T $a_0+a_0=a_0$, contradicting (0).

Let j be the smallest k with $ma_k\leqq a_0$. Now, if $na_j\leqq a_0$ for every $n<\infty$ we would have by 2.12T $\infty a_j\leqq a_0$, contradicting (4), by 1.29T. So let $c_1=a_j$ and $j_1=$ the largest l for which $mla_j\leqq a_0$. Thus there is a d such that

$$mj_1c_1+d = a_0 = b_0 \quad\text{and}\quad d\leqq mj_1c_1.$$

If $mj_1c+d=mj_1c_1$, then define $b_1=0$.

If $mj_1c+d\neq mj_1c_1$, then define $b_1=d$.

(b) Suppose b_n, c_n, j_n are defined with the required properties.

Case 1. $b_n=0$. Then define $b_{n+1}=0=c_{n+1}, j_{n+1}=0$.

Case 2. $b_n\neq 0$. Then $mj_nc_n\neq mj_nc_n+b_n$. So, $c_n+b_n\neq c_n$ and $c_n=a_j$ for some j. In this case we proceed exactly as in (a), taking b_n for a_0, to define b_{n+1}, c_{n+1}, j_{n+1}. Thus, the definition is complete.

So, we have $b_n = mj_{n+1}c_{n+1} + b_{n+1}$ for every $n < \infty$. Hence, by the remainder postulate, there is an e such that

$$b_n = e + \sum_{i<\infty} mj_{n+i+1}c_{n+i+1}.$$

Then $e \leqq b_n \leqq mj_n c_n$ for every $n < \infty$. So, $\infty e \leqq \sum_{i<\infty} mj_{n+i+1}c_{n+i+1}$. So by 1.29T

$$a_0 = b_0 = m \sum_{i<\infty} j_{i+1}c_{i+1}$$

and the proof is completed.

THEOREM 1.18. *Suppose that for all $c, d \leqq a$, we have $c \leqq d$ or $d \leqq c$. Then if $b \leqq na$ for some integer $n \leqq \infty$ and $a + b \neq a$, there is a positive real number r such that $b = ra$.*

Proof. Suppose first $b \leqq a$. Let $a_0 = a$, $a_1 = b$. We define by recursion for $n > 0$, a_n, and nonnegative finite integers k_n, such that either

(1) $k_n a_n + a_{n+1} = a_{n-1}$, $a_{n+1} \leqq a_n$, $a_n + a_{n+1} \neq a_n$, $k_n \neq 0$, or

(2) $a_{n+1} = 0$.

(a) $n = 1$. Definition of k_1, a_2. If for every $k < \infty$, $kb \leqq a_0$, we would have $\infty b \leqq a_0$ contradicing $a + b \neq a$. So by 1.16, there is a largest l such that $la_1 \leqq a_0$. Let k_1 be this l. So, $k_1 a_1 + c = a_0$ for some c with $c \leqq a_1$.

If $c + a_1 = a_1$, define $a_2 = 0$.

If $c + a_1 \neq a_1$, define $a_2 = c$.

(b) Suppose k_{n-1} is defined and a_i is defined for $i \leqq n$.

Case 1. Suppose $a_n = 0$. Then define $a_{n+1} = 0$, $k_n = 0$.

Case 2. Suppose $a_n \neq 0$. Then $k_{n-1}a_{n-1} + a_n = a_{n-1}$, $a_n \leqq a_{n-1}$, $k_{n-1} \neq 0$, $a_{n-1} + a_n \neq a_{n-1}$.

We proceed as in (a) to get a_{n+1}, k_n, and the definition is completed. There are two cases:

Case 1. There is an n such that $a_n = 0$. Let j be the smallest such n; then $a_n = 0$ for $n \geqq j \geqq 1$. In this case we have

$$a = a_0 = pa_{j-1} \quad \text{for some integer } p < \infty, p > 0,$$

$$b = a_1 = qa_{j-1} \quad \text{for some integer } q < \infty, q > 0.$$

So $b = (q/p)a$.

Case 2. There is no n such that $a_n = 0$. Then we have

$$a_n = k_{n+1}a_{n+1} + a_{n+2} \quad \text{for all } n < \infty, k_{n+1} \neq 0.$$

So by the previous theorem a is completely divisible. Hence by 1.15, as b is comparable with all other elements $\leqq a$, $b = ra$ for some positive real number r.

Let now $b \leqq na$ for some integer $n \leqq \infty$. Then there are b_i for $i < n$ such that $b_i \leqq a$ and $b = \sum_{i<n} b_i$, by 2.2T. So we have from what was previously proved, $b_i + a = a$ or $b_i = r_i a$ for some positive real number r_i. We cannot have $b_i + a = a$ for

all $i<n$, because we would then have $b+a=a$. If $b_i+a=a$, we have $b_i+r_ja=r_ja$ by 1.12. So if in this case we put $r_i=0$, we have

$$b = \sum_{i<n} r_i a = \Big(\sum_{i<n} r_i\Big)a \quad \text{by 1.9, 1.47T.}$$

The next theorem gives the uniqueness of the real number that determines a multiple.

THEOREM 1.19. *Let $a\neq 2a$. Then for all nonnegative real numbers, r and r', if ra and $r'a$ are defined and $ra=r'a$, then $r=r'$.*

Proof. Suppose $r>r'$, $r-r'=s$. Then sa is defined, by 1.9 and 1.8. Suppose $ra=r'a$; then $ra=r'a+sa=ra+sa$.

Then by 1.12, as $r>r'\geqq 0$, $a+sa=a$. Again by 1.12, as $s>0$, $a=a+a$.

As a summary of the theorems of this section, we can state the following theorem:

THEOREM 1.20. *The following condition is necessary and sufficient for a generalized cardinal algebra $\mathfrak{A}=\langle A, +\rangle$ to be isomorphic to a generalized cardinal subalgebra of $\mathscr{N}$, the cardinal algebra of nonnegative real numbers with addition: There is a finite element $a\in A$ such that*

(i) *For every $b\in A$, there is an integer $n\leqq\infty$ such that $b\leqq na$.*

(ii) *For every $b, c\in A$, with $b\leqq a$, $c\leqq a$, we have $b\leqq c$ or $c\leqq b$.*

Where a is finite means, if $a+b=a$, then $b=0$.

II. Construction of the measure.

DEFINITION 2.1. Let $\mathscr{K}$ be a σ-ring of sets, $\cong$ an equivalence relation between elements of $\mathscr{K}$. Then

(i) $\cong$ is (finitely) refining if whenever $A\cong B$, $A, B\in\mathscr{K}$ $A=A_1\cup A_2$, $A_1\cap A_2=0$, $A_1, A_2\in\mathscr{K}$, there are $B_1, B_2\in\mathscr{K}$ such that $B=B_1\cup B_2$, $B_1\cap B_2=0$, $A_1\cong B_1$, $A_2\cong B_2$,

(ii) $\cong$ is countably additive if whenever $A_i, B_i\in\mathscr{K}$ for every $i<\infty$, $A_i\cap A_j=0$ $=B_i\cap B_j$ for $i\neq j$, $A_i\cong B_i$ for all $i<\infty$, we have $\bigcup_{i<\infty} A_i\cong\bigcup_{i<\infty} B_i$.

DEFINITION 2.2. Let $\mathscr{K}$ be a σ-ring of sets, $\cong$ an equivalence relation between elements of $\mathscr{K}$. Then

(i) For every $A\in\mathscr{K}$, put $\tau(A)=\{B\in\mathscr{K} : B\cong A\}$;

(ii) $\Gamma'=\{\tau(A) : A\in\mathscr{K}\}$;

(iii) Let $\alpha_0, \alpha_1, \ldots, \gamma\in\Gamma'$. Then

(a) $\alpha_0+\alpha_1=\gamma$ iff there are $A, B, C\in\mathscr{K}$ such that $\alpha_0=\tau(A)$, $\alpha_1=\tau(B)$, $\gamma=\tau(C)$, $A\cap B=0$ and $A\cup B\cong C$;

(b) $\sum_{i<\infty}\alpha_i=\gamma$ iff there are $A_i\in\mathscr{K}$ for every $i<\infty$ and $C\in\mathscr{K}$ such that $A_i\cap A_j$ $=0$ for $i\neq j$, $\alpha_i=\tau(A_i)$, $\gamma=\tau(C)$ and $\bigcup_{i<\infty} A_i\cong C$.

It turns out that if $\cong$ is refining and countably additive, $\langle\Gamma', +, \Sigma\rangle$ is a generalized cardinal algebra (16.4T). The equivalence relations determined as follows are refining and countably additive.

DEFINITION 2.3. Let $\mathscr{K}$ be a σ-ring, G, a set of biunique functions. Then G is a pseudogroup of functions over $\mathscr{K}$ if:

(i) $\mathscr{D}f \in \mathscr{K}$, $\mathscr{D}f^{-1} \in \mathscr{K}$ for all $f \in G$.

(ii) If $f, g \in G$, then $f \circ g, f^{-1} \in G$.

(iii) For every $A \in \mathscr{K}$, the identity restricted to A belongs to G.

DEFINITION 2.4. Let $\mathscr{K}$ be a σ-ring, G a pseudogroup of function over $\mathscr{K}$. Define for all A, $B \in \mathscr{K}$:

$A \cong B$ iff there are A_i, $B_i \in \mathscr{K}$, $f_i \in G$ for all $i < \infty$ such that $A_i \cap A_j = 0 = B_i \cap B_j$ for $i \neq j$, $A = \bigcup_{i<\infty} A_i$, $B = \bigcup_{i<\infty} B_i$, $A_i \subseteq \mathscr{D}f_i$ and $B_i = f_i^*(A_i)$ for all $i < \infty$.

It is easy to see (it is also contained in Tarski's book) that $\cong$ defined as above is a refining and countably additive equivalence relation. In the last section of this paper I will consider the equivalence relation determined by a quite important pseudogroup of functions. In the rest of this section, $\mathscr{K}$ will be a fixed σ-ring and $\cong$ a refining and countably additive equivalence relation between elements of $\mathscr{K}$. S is a fixed element of $\mathscr{K}$.

As was mentioned in the introduction the algebra defined in 2.2 is the natural way to assign "abstract measure" values. This algebra is a generalized cardinal algebra (16.4T). It is more convenient to work with the closed algebra, i.e. a cardinal algebra that is the closure of the algebra of 2.2 as defined in 7.1T. This closure is very "conservative" and retains most of the properties of the original system (cf 7.2T, 7.3T, 7.4T, 7.5T, 7.6T, 7.8T). So I shall assume $\mathfrak{A} = \langle \Gamma, +, \Sigma \rangle$ to be this closure. This closure coincides with the original algebra for elements of this last one and does not add any new elements that are smaller than old elements.

I shall also consider another cardinal algebra obtained from $\mathfrak{A}$ by the following process. Let $\Phi \subseteq \Gamma$ be the ideal (see 9.1T for definition of ideal) of all elements of Γ absorbed by $\tau(S)$ $(A(\tau(S))$ with the terminology of 9.15T). Then the algebra $\mathfrak{B}$ will be the quotient algebra given by the following equivalence relation:

$$\alpha \equiv \beta \quad \text{iff there is a } \gamma \in \Phi \text{ (i.e. } \tau(s) + \gamma = \tau(s))$$

such that $\alpha + \gamma = \beta + \gamma$.

Then $\mathfrak{B} = \mathfrak{A}/\equiv$.

$\mathfrak{B}$ is also a cardinal algebra (cf. 9.1T, 9.15T, 9.28T, 9.29T).

I will put $\rho(A) = \tau(A)/\equiv$ for every $A \in \mathscr{K}$. Lower case Greek letters will be used for elements of $\mathfrak{A}$ or $\mathfrak{B}$

DEFINITION 2.5. (i) Let $A \in \mathscr{K}$. Then A is negligible if there is a sequence $A_0, A_1, A_2, \ldots$ of pairwise disjoint elements of $\mathscr{K}$ such that $A_i \subseteq S$ for every $i < \infty$ and $A \cong A_i \cong A_j$ for every $i, j < \infty$.

(ii) Let A, $B \in \mathscr{K}$. Then $A \leqq B$ if there is a $C \in \mathscr{K}$ such that $A \cong C \subseteq B$.

(iii) Let A, $B \in \mathscr{K}$. Then $A \approx B$ if there is a $C \in \mathscr{K}$ such that C is negligible and $A \cup C \cong B \cup C$.

(iv) Let A, $B \in \mathscr{K}$. Then $A \preceq B$ if there is a $C \in \mathscr{K}$ such that C is negligible and $A \cup C \leqq B \cup C$.

As an immediate consequence of the definitions and of the theorems quoted above, we have the following:

THEOREM 2.6. *Let $A, B, C \in \mathscr{K}$. Then*
(i) $\tau(S)+\tau(C)=\tau(S)$ *iff C is negligible,*
(ii) $\tau(A)\leqq\tau(B)$ *iff* $A\leqq B$,
(iii) $\rho(A)=\rho(B)$ *iff* $A\approx B$,
(iv) $\rho(A)\leqq\rho(B)$ *iff* $A\preceq B$,
(v) $\tau(S)\neq 2\tau(S)$ *iff S is not negligible,*
(vi) $\rho(C)=0$ *iff C is negligible,*
(vii) $\rho(S)$ *is finite.*

Now some theorems that show the relations between real multiples of $\tau(S)$ and $\rho(S)$ and elements of $\mathscr{K}$. They will be of use to prove the unicity of the measure.

THEOREM 2.7. *Let $A, B \in \mathscr{K}$, r_i be positive real numbers for all $i<n\leqq\infty$ such that $\sum_{i<n} r_i<\infty$. Then*
(i) *If $r_i\tau(B)$ is defined for every $i<n$, then $\tau(A)=\sum_{i<n} r_i\tau(B)$ iff there are $A_i\in\mathscr{K}$ for $i<n$ such that $A_i\cap A_j=0$ for $i\neq j$, $A=\bigcup_{i<n} A_i$ and $\tau(A_i)=r_i\tau(B)$ for all $i<n$.*
(ii) *If $r_i\rho(S)$ is defined for every $i<n$, then $\rho(A)=\sum_{i<n} r_i\rho(S)$ iff there are $A_i\in\mathscr{K}$ for $i<n$, such that $A_i\cap A_j=0$ for $i\neq j$, $A=\bigcup_{i<n} A_i$ and $\rho(A_i)=r_i\rho(S)$.*

Proof. (i) Suppose $\tau(A)=\sum_{i<n} r_i\tau(B)$. From 7.4T and 2.2, there are $B_i\in\mathscr{K}$ for $i<n$ such that $B_i\cap B_j=0$ for $i\neq j$, $\tau(B_i)=r_i\tau(B)$ and $A\cong\bigcup_{i<n} B_i$. Define pairwise disjoint $C_i\in\mathscr{K}$ for $i<n$ by recursion such that:

$$C_i\subseteq A,\ C_i\cong B_i \quad\text{and}\quad \Big(A\sim\bigcup_{j<i+1} C_j\Big)\cong\bigcup_{j<n} B_{i+j+1} \quad \text{for all } i<n.$$

This definition is performed by successive applications of the refinement property.

Now let $C=A\sim\bigcup_{i<n} C_i$. So we have $A=\bigcup_{i<n} C_i\cup C$, $C_i\cap C_j=0=C\cap C_i$ for $i\neq j$. Hence, $\tau(A)=\sum_{i<n}\tau(C_i)+\tau(C)$. But, by countable additivity,

$$\bigcup_{i<n} C_i\cong\bigcup_{i<n} B_i.$$

So, $\sum_{i<n}\tau(C_i)=\sum_{i<n}\tau(B_i)=\tau(A)$. Hence, we have $\tau(A)+\tau(C)=\tau(A)$. But $\tau(A)=\sum_{i<n} r_i\tau(B)=(\sum_{i<n} r_i)\tau(B)$. So, by 1.12, $\tau(C_0)+\tau(C)=r_0\tau(B)+\tau(C)=r_0\tau(B)=\tau(C_0)$ and $C_0\cup C\cong C_0\cong B_0$. So, let $A_0=C_0\cup C$, $A_i=C_i$ for all i, $0<i<n$. Then we have $A=\bigcup_{i<n} A_i$, $A_i\cap A_j=0$ for $i\neq j$, $\tau(A_i)=r_i\tau(B)$. The converse assertion of the theorem is immediate from the definitions.

(ii) Suppose

(0) $\rho(A)=\sum_{i<n} r_i\rho(S)$. We have

(1) $(r\alpha)/\equiv\ =r(\alpha/\equiv)$ for all $\alpha\in\Gamma$ and real numbers r for which $r\alpha$ is defined, as $\mathfrak{A}/\equiv$ is a homomorphic image of $\mathfrak{A}$ (cf. 6.6T).

So from (0) we get

$$\tau(A)/\equiv\ =\Big(\sum_{i<n} r_i\tau(S)\Big)/\equiv.$$

Hence, $\tau(A \cup C)=\sum_{i<n} r_i\tau(S)+\gamma=\sum_{i<n} r_i\tau(S)$ for some C negligible, $\gamma \in \Phi$. Hence by (i), there are D_i for $i<n$ such that

(2) $A \cup C=\bigcup_{i<n} D_i$, $D_i \cap D_j=0$ for $i\neq j$, $\tau(D_i)=r_i\tau(S)$ for all $i<n$.

Let $A_i=D_i \cap A$, $C_i=D_i \cap C$. We have $A_i \cap A_j=0$ for $i\neq j$, $A=\bigcup_{i<n} A_i$ and C_i is negligible for all $i<n$. So,

(3) $D_i \approx A_i$.

From (1) and (2) we get $\rho(D_i)=r_i\rho(S)$. So by (3), $\rho(A_i)=\rho(D_i)=r_i\rho(S)$. The converse is immediate.

COROLLARY 2.8. *Let $A \in \mathscr{K}$, n be a finite positive integer. Then*

(i) *$\tau(A)=(1/n)\tau(S)$ iff there are $S_0, S_1, \ldots, S_{n-1} \in \mathscr{K}$ such that $S_i \cap S_j=0$ for $i\neq j$, $S=\bigcup_{i<n} S_i$ and $S_i \cong S_j \cong A$ for all $i, j<n$.*

(ii) *$\rho(A)=(1/n)\rho(S)$ iff there are $S_0, S_1, \ldots, S_{n-1} \in \mathscr{K}$ such that $S_i \cap S_j=0$ for $i\neq j$, $S=\bigcup_{i<n} S_i$ and $S_i \approx S_j \approx A$ for all $i, j<n$.*

Proof. (i) is an immediate consequence of 2.7 as $\tau(S)=n\tau(A)$; (ii) we have $\rho(S)=n\rho(A)$.

So $\tau(S)=n\tau(A \cup C)$, C negligible. Hence, there are $S_0, S_1, \ldots, S_{n-1}$ pairwise disjoint such that $\bigcup_{i<n} S_i=S$ and $S_i \cong S_j \cong A \cup C$. So, $S_i \approx S_j \approx A$.

DEFINITION 2.9. (i) We say that S covers $\mathscr{K}$ if for every $A \in \mathscr{K}$ there are $S_0, S_1, S_2, \ldots, \in \mathscr{K}$ such that $S \cong S_i$ for all $i<\infty$ and $A \subseteq \bigcup_{i<\infty} S_i$.

(ii) We say that S covers * $\mathscr{K}$ if for every $A \in \mathscr{K}$ there are $S_0, S_1, \ldots \in \mathscr{K}$ such that $S \approx S_j$ for all $i<\infty$ and $A \subseteq \bigcup_{i<\infty} S_i$.

Now I am ready to state and prove the main theorems.

THEOREM 2.10. *Suppose $\mathscr{K}$ is a σ-ring of sets, $\cong$ a countably additive and refining equivalence relation between elements of $\mathscr{K}$, S an element of $\mathscr{K}$ such that*

(i) *S is not negligible,*

(ii) *for every $A, B \in \mathscr{K}$, $A \subseteq S$, $B \subseteq S$ we have $A \precsim B$ or $B \precsim A$,*

(iii) *S covers * $\mathscr{K}$.*

Then there is a unique countably additive measure μ defined on all sets of $\mathscr{K}$ such that

(a) *$\mu(S)=1$, and (b) $\mu(A)=\mu(B)$ iff $A \approx B$.*

Proof. Let the cardinal algebra $\mathfrak{B}$ be defined as before. Then we have:

(1) $\rho(S)\neq 2\rho(S)$ by (i).

(2) For all $\alpha, \beta \leqq \rho(S)$ we have $\alpha \leqq \beta$ or $\beta \leqq \alpha$ by (ii).

(3) For every $A \in \mathscr{K}$, $\rho(A) \leqq n\rho(S)$ for some $n \leqq \infty$ by (iii). Hence by 1.18, (2), (3) and 2.6:

For every $A \in \mathscr{K}$ there is a nonnegative real number r such that $\rho(A)=r\rho(S)$. By 1.19 and (1): if $r\rho(S)=r'\rho(S)$, then $r=r'$. So we can define for all $A \in \mathscr{K}$ $\mu(A)=r$ iff $\rho(A)=r\rho(S)$. It is clear that $\mu(S)=1$ and $\mu(A)=\mu(B)$ iff $\rho(A)=\rho(B)$ iff $A \approx B$.

Let, now, $A_i \in \mathscr{K}$ for all $i<\infty$, $A_i \cap A_j=0$ for $i \neq j$, $\mu(A_i)=r_i$. Then we have

$$\rho\Big(\bigcup_{i<\infty} A_i\Big) = \sum_{i<\infty} \rho(A_i) = \sum_{i<\infty} r_i\rho(S) = \Big(\sum_{i<\infty} r_i\Big)\rho(S).$$

So, μ is countably additive. The proof of the uniqueness of the measure is as follows:

Let ν be a countably additive measure on $\mathscr{K}$ such that (a) and (b) are satisfied. Let $A \in \mathscr{K}$. We know that $\rho(A)=r\rho(S)$ for some nonnegative real r.

Case 1. $\rho(A)=(1/n)\rho(S)$ for n a positive integer. Then by 2.8 there are $S_0, S_1, \ldots, S_{n-1} \in \mathscr{K}$ such that $S_i \cap S_j=0$ for $i \neq j$, $S=\bigcup_{i<n} S_i$ and $A \approx S_i \approx S_j$ for $i,j<n$. Hence, we have by (b) $\nu(A)=\nu(S_i)=\nu(S_j)$. By additivity and (a) we have

$$1 = \nu(S) = \sum_{i<n} \nu(S_i) = n\nu(S_0).$$

So $\nu(A)=\nu(S_0)=1/n=\mu(A)$.

Case 2. $\rho(A)=r\rho(S)$ for a finite positive real number r. Then $r=\sum_{i<m}(1/n_i)$ for some $m \leqq \infty$, where n_i is a positive integer for $i<m$ and $r\rho(S)=\sum_{i<m}(1/n_i)\rho(S)$. So by 2.7, $A=\bigcup_{i<m} A_i$ with $A_i \cap A_j=0$ for $i \neq j$ and $\rho(A_i)=(1/n_i)\rho(S)$ for all $i<m$. By countable additivity and case 1, we have

$$\nu(A) = \sum_{i<m} \nu(A_i) = \sum_{i<m} (1/n_i) = r = \mu(A).$$

Case 3. $\rho(A)=0$. Then $A \approx 0$ and $\nu(A)=0=\mu(A)$.

Case 4. $\rho(A)=\infty\rho(S)$. Then there are disjoint $A_i \subseteq A$ for $i<\infty$ such that $A_i \approx S$. Hence $\nu(A) \geqq \sum_{i<\infty} \nu(A_i)=\infty\nu(S)=\infty$. So $\nu(A)=\infty=\mu(A)$.

A similar theorem with a parallel proof that uses $\mathfrak{A}$ instead of $\mathfrak{B}$, is the following:

THEOREM 2.11. *Suppose $\mathscr{K}$ is a σ-ring of sets, $\cong$ a countably additive and refining equivalence relation between elements of $\mathscr{K}$,* S *an element of $\mathscr{K}$ such that*:

(i) *S is not negligible,*

(ii) *for all $A, B \in \mathscr{K}$, $A \subseteq S$, $B \subseteq S$, we have $A \leqq B$ or $B \leqq A$,*

(iii) *S covers $\mathscr{K}$.*

Then there is a unique countably additive measure μ defined on all members of $\mathscr{K}$, such that:

(a) $\mu(S)=1$,

(b) $\mu(A)=0$ *iff A is negligible,*

(c) *if $\mu(A) \neq 0$, then $\mu(A)=\mu(B)$ iff $A \cong B$.*

Hypothesis (i) (i.e. S is not negligible) in Theorems 2.10 and 2.11 is not only partially sufficient but also necessary. It is even necessary when we weaken requirement (b) of μ of Theorem 2.10 (or (b) and (c) of Theorem 2.11) to the following (always assuming that we have a refining and countably additive equivalence relation):

(*) If $A \cong B$ then $\mu(A)=\mu(B)$.

It is easy to see that "S is not negligible" is equivalent to the following condition:

(**) There are no sets $S_1, S_2 \in \mathscr{K}$ such that $S = S_1 \cup S_2$, $S_1 \cap S_2 = 0$ and $S \simeq S_1 \simeq S_2$. This can be paraphrased as "S has no paradoxical decomposition".

It was proved in 16.12T [8] that this is a necessary and sufficient condition for the existence of a finitely additive measure that satisfies (a) and (*) when $\simeq$ is equivalence under finite decomposition with respect to a group of functions G. My conjecture is that this is also necessary and sufficient when you require a countably additive measure and $\simeq$ is equivalence as defined in 2.4. Necessary and sufficient conditions for the existence of a strictly positive countably additive measure μ on a nonatomic Boolean algebra (i.e. μ vanishes only for 0), have been obtained in [5]. But our situation is different as we do not require the measure to be strictly positive.

Hypothesis (ii) and (iii) in Theorems 2.10 and 2.11 are not necessary for the existence of the measure, as can be shown by easy examples. However, if we add to the requirements of the measure μ, the following:

$$\mu(A) \leqq \mu(B) \quad \text{iff there is a } C \in \mathscr{K},\ C \subseteq B \text{ such that } \mu(A) = \mu(C),$$

then these hypotheses become necessary.

It is very simple to show that we could replace the ring of sets $\mathscr{K}$ by a σ-complete distributive complemented lattice with a zero element. This is the set-up chosen by Maharam in [4]. However, there are important differences between my work and Maharam's [4]. The main difference is that she does not obtain the characterization of elements of measure zero as my negligible elements. As a matter of fact, her measure is strictly positive. To prove her theorems, she makes extensive use of the fact that the elements for which a measure is obtained are bounded (finite in Tarski's terminology). That is, they are not equivalent to any proper subset. For instance, a pivotal theorem is the following (in my terminology):

(†) If $a \vee x \simeq b \vee x$ where $a \vee x$, $b \vee x$ are bounded and $a \wedge x = b \wedge x = 0$, then $a \simeq b$.

This is certainly false if $a \vee x$ and $b \vee x$ are not bounded. If they are bounded (finite) the proof of this theorem is very simple using the theory of cardinal algebras (cf. 4.19T).

In none of the proofs of §I, do I use any fact about finite elements. On the other hand I use extensively and essentially the remainder postulate (1.1 VII T) and the characterization of elements that are absorbed (1.29T). These theorems were not apparently known by Maharam. Maharam's system $\mathscr{M}^*$ of the totality of measure values is certainly a generalized cardinal algebra. Most of her theorems could be easily proved using this theory.

Instead of using boundedness for the construction of the measure, I use a unit set S which does not have a paradoxical decomposition. If a set is bounded then it does not have a paradoxical decomposition, but not vice-versa. Another difference is in Maharam's condition (δ) [4, p. 422]. She assumes comparability for any pair of elements of $\mathscr{M}^*$. I assume only comparability for elements that are $\subseteq S$ (condition (ii)) and that every set can be covered by copies of S (condition (iii)). This does imply (δ), but the proof that it does is not trivial. (δ), on the other hand, implies (ii) and (iii) for some S.

In [6] Maharam extends her results of [4]. In this paper she works with a σ-complete Boolean algebra that satisfies the countable chain condition, and an equivalence relation that is countably additive and refining. So her "measure algebra" is again a generalized cardinal algebra. But she adds the following postulate:

(III) If $x \cong y$ then there are bounded elements x_n, y_n such that $x = \bigvee x_n$, $y = \bigvee y_n$, $x_n \wedge x_m = 0 = y_n \wedge y_m$ for $n \neq m$ and $x_n \cong y_n$ for all $n < \infty$.

This postulate does not have an easy translation into the theory of cardinal algebras. As in [4], Maharam uses the cancellation law for bounded elements (†) and does not use the remainder postulate nor the characterization of absorbed elements. Thus her proofs, in [6], as in [4], are very different from mine. She cannot obtain in [6] the characterization of null sets as she could not do it in [4].

III. **Applications.**

A. *Probability theory.* It is well known that the classical definition of probability is based upon the concept of equal likelihood, an equivalence relation between events. This definition was only possible when the sure event was decomposable into a finite number of equally likely elementary events. The theorems I have just shown give the possibility of defining a probability measure even in cases when the number of elementary events is infinite, provided the equal likelihood relation satisfies certain additional properties. For a detailed discussion of this problem, see [3].

As a simple example of the use of cardinal algebras as values for abstract measures, consider the following:

Suppose we throw a ball along the floor, following a line l perpendicular to a wall from which it will rebound: We want to determine the probability of the ball's stopping in a certain area of the floor. It is clear that two areas are equally likely iff they are symmetric with respect to the line l. It is also clear that not every set is comparable and that we cannot assign a numerical probability to all sets (or even to all Lebesgue measurable sets), that is faithful to the equal likelihood relation. It is even true that the sets to which we can assign a numerical measure do not form a σ-ring (or even a ring). In this case, it would be best to consider the cardinal algebra of equivalence classes as defined in 2.2, with respect to the group of symmetries around l, as values for the measure (cf. [3]).

B. *Lebesgue measure in $\boldsymbol{R}^n$.* Let us take $\cong$ as defined in 2.4, taking as $\mathscr{K}$ the σ-field of Borel sets, and as G, the group of translations of $\boldsymbol{R}^n$. Then, as there is a measure in $\mathscr{K}$ that is invariant under translations, namely Lebesgue measure, 2.10(i) is true, taking for S any set of positive finite measure, in particular $[0, 1]^n$.

Conditions (ii), (iii) of 2.10 are not necessary. The next few theorems give a useful way of showing that (ii) is satisfied.

We assume as before that $\mathscr{K}$ is a σ-ring of sets, $\cong$ a refining and countably additive equivalence relation between elements in $\mathscr{K}$, $S \in \mathscr{K}$. We also assume all the definitions of the last section.

LEMMA 3.1. *Suppose that* $B, A_n \in \mathscr{K}$, $B \subseteq S$, $A_{n+1} \subseteq A_n \subseteq S$ *for all* $n < \infty$. *Then if* $B \preceq A_n$ *for every* $n < \infty$, $B \preceq \bigcap_{n<\infty} A_n$.

Proof. Let $\alpha_n = \rho(A_n)$, $\beta_n = \rho(S \sim A_n)$, $\beta = \rho(B)$, $\alpha = \rho(S)$. So, α, β, α_n, $\beta_n \in \mathfrak{B}$ for all $n < \infty$. We have $\alpha_{n+1} \leqq \alpha_n$ and $\alpha_n + \beta_n = \alpha$ for all $n < \infty$. It is clear that $\alpha = \rho(S)$ is finite (cf. 4.10T). Then by 4.24T, $\bigcap_{n<\infty} \alpha_n$, $\bigcup_{n<\infty} \beta_n$ exist and

$$\bigcap_{n<\infty} \alpha_n + \bigcup_{n<\infty} \beta_n = \alpha.$$

But $\bigcup_{n<\infty} \rho(S \sim A_n) = \rho(\bigcup_{n<\infty} (S \sim A_n))$, because $\bigcup_{n<\infty} \tau(S \sim A_n) = \tau(\bigcup_{n<\infty} (S \sim A_n))$ and $\mathfrak{B}$ is homomorphic to $\mathfrak{A}$. By the same reasons:

$$\rho\Big(\bigcap_{n<\infty} A_n\Big) + \rho\Big(\bigcup_{n<\infty} (S \sim A_n)\Big) = \rho(S) = \alpha$$

and

$$\bigcap_{n<\infty} \alpha_n + \rho\Big(\bigcup_{n<\infty} (S \sim A_n)\Big) = \alpha.$$

As α is finite, $\rho(\bigcap_{n<\infty} A_n) = \bigcap_{n<\infty} \alpha_n$. As we have $\beta \leqq \alpha_n$ for all $n < \infty$, $\beta \leqq \bigcap_{n<\infty} \alpha_n = \rho(\bigcap_{n<\infty} A_n)$, i.e. $B \preceq \bigcap_{n<\infty} A_n$.

THEOREM 3.2. *Let* $\mathscr{E} \subseteq \mathscr{K}$ *and let* $\mathscr{K}$ *be the monotone class generated by* $\mathscr{E}$. *Then if for all* $A, B \in \mathscr{E}$ *with* $A \subseteq S$, $B \subseteq S$ *we have* $A \preceq B$ *or* $B \preceq A$, *the same is true for all pairs of elements of* $\mathscr{K}$ *subsets of* S.

Proof. Define $\mathscr{K}_\lambda$ for all ordinals $\lambda \leqq \Omega$ (the first uncountable ordinal) by recursion as follows:

(i) $\mathscr{K}_0 = \mathscr{E}$;

$$\mathscr{K}_\lambda = \Big\{\bigcap_{i<\infty} X_i : \exists\kappa < \lambda \ X_i \in \mathscr{K}_\kappa \text{ and } X_{i+1} \subseteq X_i \text{ for all } i < \infty\Big\} \text{ if } \lambda \text{ is odd},$$

$$\mathscr{K}_\lambda = \Big\{\bigcup_{i<\infty} X_i : \exists\kappa < \lambda \ X_i \in \mathscr{K}_\kappa \text{ and } X_{i+1} \supseteq X_i \text{ for all } i < \infty\Big\} \text{ if } \lambda \text{ is even}.$$

Then we have $\mathscr{K}_\lambda \subseteq \mathscr{K}_\kappa$ if $\lambda \leqq \kappa$ and $\mathscr{K} = \mathscr{K}_\Omega$. I shall prove by transfinite induction on λ: If $A, B \in \mathscr{K}_\lambda$ with $A \subseteq S$, $B \subseteq S$, then $A \preceq B$ or $B \preceq A$.

(i) The condition is true for $\mathscr{K}_0$ by hypothesis.

(ii) Suppose λ odd and claim proved for $\mathscr{K}_\kappa$ with $\kappa < \lambda$.

Case 1. Suppose Y, $X_i \in \bigcup_{\kappa<\lambda} \mathscr{K}_\kappa$, $X_{i+1} \subseteq X_i$ for all $i < \infty$. If $Y \preceq X_i$ for all $i < \infty$, then $Y \preceq \bigcap_{i<\infty} X_i$ by Lemma 3.1. If there is an $i < \infty$ such that $X_i \preceq Y$, then $\bigcap_{i<\infty} X_i \preceq Y$.

Case 2. $X_i, Y_i \in \bigcup_{\kappa<\lambda} \mathscr{K}_\kappa$, $X_{i+1} \subseteq X_i$, $Y_{i+1} \subseteq Y_i$ for all $i<\infty$. Then by Case 1, $\bigcap_{i<\infty} X_i \preceq Y_i$ for every $i<\infty$ or there is an $i<\infty$ such that $Y_i \preceq \bigcap_{i<\infty} X_i$. Again by Case 1, $\bigcap_{i<\infty} X_i$ and $\bigcap_{i<\infty} Y_i$ are comparable.

(iii) Suppose λ even and claim proved for $\mathscr{K}_\kappa$, $\kappa<\lambda$. There are two cases similar to (ii) as, clearly, if $X_i \subseteq X_{i+1} \preceq Y$ for all $i<\infty$, $\bigcup_{i<\infty} X_i \preceq Y$.

As a particular case of this theorem, we have that if $\mathscr{K}$ is the σ-ring generated by the ring $\mathscr{R}$ and all elements of $\mathscr{R}$ subsets of S are comparable, then all elements of $\mathscr{K}$, subsets of S are comparable.

Let us consider now the case of $\boldsymbol{R}^n$. Let $\mathscr{K}$ be the class of Borel sets, G the group of translation, $S=[0, 1]^n$. Define $\cong$ as in 2.4.

THEOREM 3.3. *For all $A, B \in \mathscr{K}$ $A \subseteq S$, $B \subseteq S$ we have $A \preceq B$ or $B \preceq A$.*

Proof. Consider the field $\mathscr{F}$ that contains all finite unions of cubes I^n where I is an interval of $\boldsymbol{R}$ with rational endpoints. Then $\mathscr{K}$ is the σ-field generated by $\mathscr{F}$.

It is easy to see that all hyperplanes parallel to any axis are negligible. So, cubes with sides of the same length are equivalent because faces are negligible. As all intervals in $\mathscr{F}$ have rational endpoints, any finite union of cubes can be considered as the finite union of equivalent cubes. This shows that all elements of $\mathscr{F}$ subsets of S are comparable. So the theorem follows from 3.2.

THEOREM 3.4. *There is a unique countably additive measure μ defined in $\mathscr{K}$ such that $\mu(S)=1$, $\mu(A)=\mu(B)$ iff $A \approx B$.*

Proof. 2.10(i) has to be true, as remarked before.

2.10(ii) is true by 3.3.

2.10(iii) is obviously true.

$\cong$ is a countably additive, refining equivalence relation between elements of $\mathscr{K}$ as remarked after 2.4. So, Theorem 2.10 applies and the result is obtained.

From this theorem we obtain some new results about Lebesgue measure in Borel sets and translations, namely:

COROLLARY 3.5. *Let $\mathscr{K}$ be the family of Borel sets of $\boldsymbol{R}^n$, λ, the Lebesgue measure on $\mathscr{K}$, $\cong$ the equivalence by countable decomposition over $\mathscr{K}$ under the group of translations and $S=[0, 1]^n$. Then*

(1) *$\lambda(A)=0$ iff A is negligible, i.e. there are $A_0, A_1, \ldots \in \mathscr{K}$ such that $A_i \cap A_j=0$ for $i \neq j$, $A_i \subseteq S$ and $A_i \cong A$ for all $i<\infty$.*

(2) *$\lambda(A)=\lambda(B)$ iff there is a negligible set $C \in \mathscr{K}$ such that $A \cup C \cong B \cup C$.*

It is clear that it is possible to replace $\mathscr{K}$ by the σ-algebra of measurable sets in 3.5.

3.5.(2) is similar to a theorem of Banach and Tarski [1]. They proved directly (2) with "negligible" replaced by "of measure 0". The Banach-Tarski result was also obtained as a corollary in Maharam's papers [4] and [6].

Theorem 2.11 is not applicable to Lebesgue measure. If it were we would have for sets of positive measure A, B

$$\lambda(A) = \lambda(B) \quad \text{iff } A \cong B.$$

But it is evident that if $A \cong B$ then A and B are of the same (Baire) category. On the other hand, we know that there are sets of different categories that have the same positive measure.

However we have the following:

COROLLARY 3.6. *Let A, B be Lebesgue measurable sets with nonempty interiors. Then $\lambda(A)=\lambda(B)$ iff $A \cong B$.*

Proof. I shall give the proof for Borel sets A, B with nonempty interiors. It is easy to generalize for measurable sets. It is clear, from 3.5, that if $A \cong B$ then $\lambda(A)=\lambda(B)$.

So suppose $\lambda(A)=\lambda(B)$. By 3.4 we get $A \approx B$, i.e. $\rho(A)=\rho(B)$. Hence there is a $\gamma \in \mathfrak{A}$ such that

(1) $\tau(A)+\gamma=\tau(B)+\gamma$ and $\tau(S)+\gamma=\tau(S)$.

As A, B have nonempty interior, there are cubes A', B' of volume some $r>0$, included in A and B respectively. Now, $\tau(A')=r\tau(S)$ and $\tau(B')=r\tau(S)$. From (1) we get, $r\tau(S)+\gamma=r\tau(S)$ by 1.12. As $\tau(A)\geqq r\tau(S)$ and $\tau(B)\geqq r\tau(S)$, we get $\tau(A)+\gamma =\tau(A)$, $\tau(B)+\gamma=\tau(B)$. So $\tau(A)=\tau(B)$ and $A \cong B$.

The next corollary gives a new characterization of null sets besides the one given in 3.5(1).

COROLLARY 3.7. *Let A be a Lebesgue measurable set. Then $\lambda(A)=0$ iff for every open set B there is a $B' \subseteq B$ such that $A \cong B'$, B' measurable.*

Proof. As usual, I shall prove the statement for Borel sets.

We know from 3.5 that $\lambda(A)=0$ iff A is negligible, i.e. $\tau(A)+\tau(S)=\tau(S)$. It is clear that $\tau(S)$ is completely divisible. So from 1.12 we get $\tau(A)+\tau(S)=\tau(S)$ iff $\tau(A)\leqq r\tau(S)$ for every positive real number r.

But if B is an open set, then as in 3.6 we get an $r>0$ such that $r\tau(S)\leqq\tau(B)$. So $\lambda(A)=0$ iff $\tau(A)\leqq\tau(B)$ for every open B.

It is well known that there is a set of second category and of measure 0. As it was mentioned above, the relation $\cong$ preserves category. Hence from 3.5 we can infer immediately:

COROLLARY 3.8. *In any open set there are countably many disjoint sets of second category that have measure* 0 *and are equivalent to each other.*

As Choquet remarks [2], Lebesgue measure is rather crude as it does not respect category and does not distinguish the different sorts of null sets. Sets so different as finite, countable, of the power of the continuum, of second category are all lumped together as null sets. He suggests replacing the real numbers by another structure for values of the measure. I think that a natural structure would be $\mathfrak{A}$, the closure of the algebra given by $\{\tau(A) : A \in \mathscr{K}\}$ (Definition 2.2). It is a cardinal algebra and, hence, it has many properties in common with the real numbers (see [8]). On the other hand, if $\tau(A)=\tau(B)$ then A and B are of the same cardinality and of the same category. I think it would be worthwhile to investigate this algebra. In it the equivalence classes (types) given by sets with nonempty interiors plus the empty set form a subalgebra isomorphic to the additive algebra of nonnegative real numbers. But, as was pointed out above, there are other types beside these.

The main problems left open are the following:

(a) A simplified version of the condition: "S is not negligible". As it stands now, it is very difficult to prove directly. If we had a direct proof that did not involve the construction of the measure then we would have a new construction of Lebesgue measure.

(b) The generalization of Theorem 3.4 to arbitrary locally compact topological groups.

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