ON THE MARX CONJECTURE FOR STARLIKE FUNCTIONS OF ORDER α

BY

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1. **Introduction.** Let D denote the open unit disk, and for $0 \leqq \alpha < 1$, let S_α^* denote the class of functions that are starlike of order α in D; in other words, a function f belongs to S_α^* if and only if $f(z) = z + \sum_{n=2}^\infty a_n z^n$ is analytic and univalent in D and

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) \geqq \alpha \quad \text{for all } z \text{ in } D.$$

The class S_0^* is the class S^* of starlike functions. In 1932, A. Marx [4] conjectured that for every $f(z) \in S^*$, the function $f'(z)$ is subordinate to the derivative of the Koebe function $k(z) = z(1-z)^{-2}$ in the sense that for each fixed $r < 1$, the image of the disk $|z| \leqq r$ under $f'(z)$ is contained in its image under $k'(z)$. In 1965, P. L. Duren [1] showed that the Marx conjecture holds in the disk $|z| \leqq 0.736\ldots$, and he showed that the Marx conjecture holds in a disk $|z| \leqq r_0$ $(r_0 \leqq 1)$ if one can show that the solution of the extremal problem

$$\max_{f \in S^*} \operatorname{Re}\{e^{i\psi} \log f'(z_0)\}$$

is a rotation of the Koebe function, for each fixed ψ $(0 \leqq \psi < 2\pi)$ and each fixed z_0 $(|z_0| < r_0)$, provided r_0 is small enough so that the function $\log k'(z)$ is convex in $|z| \leqq r_0$.

Using Goluzin's variational method [2], we shall investigate the Marx conjecture in the classes S_α^* $(0 \leqq \alpha < 1)$. Instead of Koebe functions $e^{-i\phi}k(ze^{i\phi}) = z(1-ze^{i\phi})^{-2}$, we shall find as extremal functions

$$e^{-i\phi}k_\alpha(ze^{i\phi}) = z(1-ze^{i\phi})^{-2(1-\alpha)}.$$

We show that the Marx conjecture for the class S_α^* holds in a disk $|z| \leqq r_\alpha$, where r_α is a root of a certain polynomial. For $\alpha = 0$, we obtain the same constant $r_0 = 0.736\ldots$ that Duren found earlier. As α increases, the radii of the disks increase, up to a certain value of α, and then they decrease. The results obtained by this method are not best possible, but they seem to be the best that this method will yield.

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I am indebted to Professor P. L. Duren for suggesting this problem.

2. **Goluzin's variational formula.** Let f be a function with the representation

$$f(z) = \int_0^{2\pi} g(z, t)\, d\mu(t), \tag{1}$$

where $\mu(t)$ is nondecreasing in the interval $[0, 2\pi]$ and where $\int_0^{2\pi} d\mu(t) = 1$. Goluzin's variational method consists of varying the function $\mu(t)$ in two different ways. Since the method has been described elsewhere [2], [5], we give only an outline of it.

Goluzin's first variation changes a function f with representation (1) by changing $\mu(t)$ on a subinterval (t_1, t_2) of $[0, 2\pi]$. As variational function we obtain

$$f_*(z) = f(z) + \lambda \int_{t_1}^{t_2} \frac{\partial g(z, t)}{\partial t} |\mu(t) - c|\, dt \qquad (-1 \leqq \lambda \leqq 1), \tag{2}$$

where the constant c is independent of t and λ.

The second variation alters $\mu(t)$ only between two of its jump points t_1 and t_2 $(0 \leqq t_1 < t_2 \leqq 2\pi)$. As variational function we obtain

$$f_{**}(z) = f(z) + \lambda[g(z, t_1) - g(z, t_2)] \qquad (|\lambda| < \varepsilon), \tag{3}$$

where ε is a sufficiently small positive constant, depending on the heights of the jumps.

It is well known that every function $f \in S_\alpha^*$ has the representation

$$f(z) = z \exp\left[-2(1-\alpha) \int_0^{2\pi} \log(1 - ze^{-it})\, d\mu(t)\right], \tag{4}$$

where $\log z$ denotes that branch of the logarithm for which $\log 1 = 0$, and where $\mu(t)$ is a nondecreasing function with total variation 1. Applying formulas (2) and (3) to functions in the class S_α^*, we find that for sufficiently small values of λ, the functions

$$f_*(z) = f(z) + 2(1-\alpha)\lambda f(z) \int_{t_1}^{t_2} \frac{ize^{-it}}{1 - ze^{-it}} |\mu(t) - c|\, dt + O(\lambda^2) \tag{2'}$$

and

$$f_{**}(z) = f(z) + 2(1-\alpha)\lambda f(z)[\log(1 - ze^{-it_2}) - \log(1 - ze^{-it_1})] + O(\lambda^2) \tag{3'}$$

belong to the class S_α^*. (In formula (3′), t_1 and t_2 are jump points of $\mu(t)$.)

3. **Known results.** Suppose the function $\mu(t)$ corresponding to a function $f \in S_\alpha^*$ has n jump points $0 \leqq t_1 < t_2 < \cdots < t_n \leqq 2\pi$ with jumps a_k $(\sum_{k=1}^n a_k = 1)$. Then it follows from (4) that $f(z)$ has the representation

$$f(z) = z \prod_{k=1}^{n} (1 - ze^{-it_k})^{-2a_k(1-\alpha)}. \tag{5}$$

In §3 of his paper, Goluzin [2] solves several general extremal problems in the class S^*. His proofs can easily be used to prove the corresponding statements for S_α^*. We need the modified version of two of his theorems, and we state them for the sake of completeness. In all cases, log stands for the branch of the logarithm that vanishes at 1.

THEOREM 1. *For a fixed entire function $\Phi(w)$ and a fixed point z in $|z|<1$, either of the functionals*

$$\operatorname{Re}\left\{\Phi\left(\log\frac{f(z)}{z}\right)\right\} \quad \text{and} \quad \left|\Phi\left(\log\frac{f(z)}{z}\right)\right|$$

attains its maximum in S_α^* only for a function of the form

$$f(z) = z(1-ze^{it_0})^{-2(1-\alpha)} \qquad (t_0 \text{ real}).$$

THEOREM 2. *For a fixed entire function $\Phi(w)$ and a fixed point z in $|z|<1$, either of the functionals*

$$\operatorname{Re}\,[\Phi(\log f'(z))] \quad \textit{and} \quad |\Phi(\log f'(z))|$$

attains its maximum in S_α^* only for a function of the form

$$f(z) = z(1-ze^{it_0})^{-b(1-\alpha)}(1-ze^{it_1})^{-(2-b)(1-\alpha)},$$

where $0 \leqq b \leqq 2$ and t_0, t_1 are real.

(The original versions of Theorems 1 and 2 in [2] contain the restrictions that $\Phi'(\log z^{-1}f(z)) \neq 0$, respectively, that $\Phi'(\log f'(z)) \neq 0$ for an extremal function f. These restrictions are unnecessary, in view of a theorem of Kirwan [3]. Theorem 1 in [5] contains the above theorems as a special case.)

4. **An extremal problem.** The extremal problem

$$\max_{f\in S_\alpha^*} \operatorname{Re}\,\{e^{i\psi}\log f'(r)\}, \tag{6}$$

for a fixed r $(0<r<1)$ and a fixed ψ $(0 \leqq \psi < 2\pi)$, is a special case of Theorem 2 in §3. Hence an extremal function for problem (6) has the form

$$f(z) = z(1-ze^{it_0})^{-b(1-\alpha)}(1-ze^{it_1})^{-(2-b)(1-\alpha)} \qquad (0 \leqq b \leqq 2). \tag{7}$$

We now take a closer look at the proof of Theorem 2, with $\Phi(w)=we^{i\psi}$. Goluzin's first variation reveals that if f is an extremal function for (6), then the corresponding $\mu(t)$ is a step function with at most 4 discontinuities in $[0, 2\pi)$, and the jumps of μ occur at the roots of the equation

$$F(t) = \operatorname{Re}\left\{\frac{e^{i\psi}}{f'(r)}\frac{d}{dz}\left[f(z)\frac{ize^{-it}}{1-ze^{-it}}\right]\Bigg|_{z=r}\right\} = 0. \tag{8}$$

Goluzin's second variation shows that the function

$$G(t) = \operatorname{Re}\left\{e^{i\psi}\left(\log(1-re^{-it}) - \frac{f(r)}{rf'(r)}\frac{re^{-it}}{1-re^{-it}}\right)\right\} \tag{9}$$

must assume the same value at each discontinuity of μ. Since $G'(t)=F(t)$, the function μ has at most 2 discontinuities in $[0, 2\pi)$.

We wish to find conditions under which μ has exactly one discontinuity in $[0, 2\pi)$. To this end, set $C=f(r)/rf'(r)$. Since the constant C satisfies the condition $\operatorname{Re} 1/C \geqq \alpha$, C lies in the disk $|z-1/2\alpha| \leqq 1/2\alpha$. But computing C from equation (7) yields a sharper estimate. Equation (7) implies that

$$\frac{1}{C} = 1+2(1-\alpha)\left[\frac{b}{2}\frac{re^{it_0}}{1-re^{it_0}}+\frac{2-b}{2}\frac{re^{it_1}}{1-re^{it_1}}\right]. \tag{10}$$

Let Q denote the quantity in brackets. Then Q lies on the line segment between $z_0=re^{it_0}/(1-re^{it_0})$ and $z_1=re^{it_1}/(1-re^{it_1})$; and z_0 and z_1 lie on the image of the circle $|z|=r$ under the map $z/(1-z)$. Hence Q lies inside a certain disk. It is now easy to find the disk in which $1+2(1-\alpha)Q$ must lie. Finally, it follows that C lies in a disk with center $(1+r^2(1-2\alpha))/(1-r^2(1-2\alpha)^2)$ and radius $2r(1-\alpha)/(1-r^2(1-2\alpha)^2)$.

Suppose now that the function

$$h(z) = \frac{z}{1-z}+C\frac{z}{(1-z)^2} \tag{11}$$

is starlike with respect to the origin in a disk $|z|<r_0$. Then the equation

$$F(t) = \operatorname{Re}\{ie^{i\psi}h(re^{-it})\} = 0$$

has at most two roots in the interval $[0, 2\pi)$, for each fixed r $(0<r<r_0)$. Now the condition $G'(t)=F(t)$ implies that μ has only one discontinuity. Hence an extremal function for problem (6) has the form

$$f(z) = z(1-ze^{it_0})^{-2(1-\alpha)} = e^{-it_0}k_\alpha(ze^{it_0}),$$

provided the fixed point r lies in the interval $(0, r_0)$.

It remains to investigate where the function $h(z)$, given by (11), is starlike. A simple calculation shows that

$$\frac{zh'(z)}{h(z)} = \frac{2}{1-z}-\frac{1+C}{1+C-z}.$$

The number $(1+C)/(1+C-z)=1+z/(1+C-z)$ lies in a disk with center

$$1+\frac{2z(1+r^2\alpha(1-2\alpha))-r^2(1-r^2(1-2\alpha)^2)}{4-4r^2\alpha^2-4\operatorname{Re} z[1+r^2\alpha(1-2\alpha)]+r^2(1-r^2(1-2\alpha)^2)}$$

and radius

$$\frac{2r^2(1-\alpha)}{4-4r^2\alpha^2-4\operatorname{Re} z[1+r^2\alpha(1-2\alpha)]+r^2(1-r^2(1-2\alpha)^2)}$$

(this can be seen by looking at the successive quantities $w_0=1+C$, $w_1=w_0-z$, $w_2=1/w_1$, $w_3=zw_2$, $w_4=1+w_3$). Now

$$(12)\qquad \begin{aligned} \operatorname{Re}\frac{zh'(z)}{h(z)} &= \operatorname{Re}\frac{2}{1-z}-\operatorname{Re}\frac{1+C}{1+C-z} \\ &\geqq \operatorname{Re}\frac{2}{1-z}-[\operatorname{Re}(\text{center})+\text{radius}] = \frac{N}{D}, \end{aligned}$$

where

$$D = \{1-2\operatorname{Re} z+r^2\}\{4-4r^2\alpha^2-4\operatorname{Re} z[1+r^2(1-2\alpha)]+r^2(1-r^2(1-2\alpha)^2)\}$$

and where

$$\begin{aligned} N/2 = 2(\operatorname{Re} z)^2[1+r^2\alpha(1-2\alpha)]+\operatorname{Re} z[r^4(1-2\alpha)(1-\alpha)+r^2(2-5\alpha+6\alpha^2)-3] \\ +2+r^2[-2\alpha^2+\alpha-2-r^2(2-5\alpha+2\alpha^2)]. \end{aligned}$$

It is easy to check that D is positive. Therefore, $\operatorname{Re} zh'(z)/h(z)\geqq 0$ if and only if $N/2\geqq 0$. Set $z=re^{i\theta}$, and set $N/2=H(\cos\theta, r)$. We regard $H(\cos\theta, r)$ as a second degree polynomial in $\cos\theta$ and find the roots of the equation $H(\cos\theta, r)=0$.

Set

$$\begin{aligned} B = B(r,\alpha) = r^8(1-2\alpha)^2(1-\alpha)^2+2r^6(1-2\alpha)(2+\alpha-9\alpha^2+2\alpha^3) \\ +r^4(14-26\alpha+13\alpha^2-28\alpha^3+4\alpha^4)+2r^2(2+3\alpha+6\alpha^2)-7. \end{aligned}$$

Then the roots of $H(\cos\theta, r)=0$ are formally given by the expression

$$(13)\qquad {}^*\!\cos\theta = \frac{-[r^4(1-2\alpha)(1-\alpha)+r^2(2-5\alpha+6\alpha^2)-3]\pm\sqrt{B}}{4r(1+r^2\alpha(1-2\alpha))}.$$

Note that $H(\cos\theta, 0)=2>0$ and that for $r=0$, we have $B=B(0,\alpha)=-7$.

Expression (13) only defines actual roots of the equation $H(\cos\theta, r)=0$ if $-1\leqq {}^*\!\cos\theta\leqq 1$. This is equivalent to the conditions ${}^*\!\cos^2\theta\leqq 1$ and $B\geqq 0$. Since B is negative for small values of r, the conditions ${}^*\!\cos^2\theta\leqq 1$ and $B\geqq 0$ can only hold for $r\geqq r_\alpha$, where r_α is the smallest positive root of the equation $B(r,\alpha)=0$ (for a fixed α). For $r=r_\alpha$, the condition ${}^*\!\cos^2\theta\leqq 1$ holds if and only if

$$(14)\qquad 2+r_\alpha^2(-4+\alpha-2\alpha^2)+r_\alpha^4(\alpha+2)(2\alpha-1) \leqq 0.$$

A tedious, but straightforward computation shows that relation (14) is always satisfied. Hence r_α is the smallest possible value of r for which the relation $H(\cos\theta, r)=0$ can hold.

We computed the polynomials $B(r,\alpha)$ for various values of α and used a computer to find the roots r_α. The following table lists the results. (Note that

$$B(r,0) = (r^2+1)(r^6+3r^4+11r^2-7),$$

so that r_0 is the same constant that Duren [1] obtained.)

TABLE 1

Numerical Values of r_α

α	r_α	α	r_α
0.0	0.736....	0.6	0.82459..
0.1	0.75254..	2/3	0.83448..
0.2	0.76794..	0.7	0.83964..
0.3	0.78254..	0.8	0.85713..
1/3	0.78768..	0.9	0.880657.
0.4	0.796598.	0.95	0.898583.
0.5	0.81046..	1.0	0.935415.

For each α, the number r_α represents the radius of a disk about the origin in which the function $h(z)$, given by (11), is starlike.

5. **Radius of convexity of** $\log k'_\alpha(z)$. If we can show that the function $g_\alpha(z) = \log k'_\alpha(z)$ is convex in a disk $|z| \leqq R_\alpha$, where $R_\alpha \geqq r_\alpha$, then it follows that the Marx conjecture for the class S^*_α holds in the disk $|z| \leqq r_\alpha$.

A calculation shows that

$$1+z\frac{g''_\alpha(z)}{g'_\alpha(z)} = \frac{2\{1+z(1-2\alpha)+z^2(1-2\alpha)(1-\alpha)\}}{[1+(1-2\alpha)z][2+(1-2\alpha)z](1-z)}.$$

It follows that

$$\begin{aligned}\tfrac{1}{2}|1+(1-2\alpha)z|^2|2+(1-2\alpha)z|^2|1-z|^2 \operatorname{Re}\left(1+z\frac{g''_\alpha(z)}{g'_\alpha(z)}\right)\\ = 2+|z|^2(1-2\alpha)(1-6\alpha)-2|z|^4(1-2\alpha)^2(1-\alpha^2)\\ +\operatorname{Re} z\cdot\{3-10\alpha+|z|^2(1-2\alpha)(-1-5\alpha+10\alpha^2)-|z|^4(1-2\alpha)^3(1-\alpha)\}\\ +\operatorname{Re} z^2\cdot\{-4\alpha(1-2\alpha)-|z|^2(1-2\alpha)^3\}\\ +\operatorname{Re} z^3\cdot\{-(1-2\alpha)^3\}.\end{aligned}$$

Now set $z=re^{i\theta}$ and $x=\cos\theta$. We abbreviate the left-hand side of the above equation by $P_{\alpha,r}(x)$. Thus

$$P_{\alpha,r}(x) = a_0+a_1x+a_2x^2+a_3x^3,$$

where

$$\begin{aligned} a_0 &= 2+r^2(1-2\alpha)^2+r^4(1-2\alpha)^2(-1-2\alpha+2\alpha^2),\\ a_1 &= r\{3-10\alpha+r^2(1-2\alpha)(2-17\alpha+22\alpha^2)-r^4(1-2\alpha)^3(1-\alpha)\},\\ a_2 &= -2r^2(1-2\alpha)\{4\alpha+r^2(1-2\alpha)^2\}, \quad \text{and}\\ a_3 &= -4r^3(1-2\alpha)^3.\end{aligned}$$

For a fixed α, the function $g_\alpha(z)$ is convex in a disk $|z|<R_\alpha$ if and only if the polynomials $P_{\alpha,r}(x)$ are nonnegative for all x $(-1\leqq x\leqq 1)$ and for all $r\leqq R_\alpha$.

Hence we wish to find the largest value of r (as a function of α) for which

$$P_{\alpha,r}(x) \geqq 0 \quad \text{for all } x \qquad (-1 \leqq x \leqq 1).$$

One verifies easily that

$$P_{\alpha,r}(0) > 0 \quad \text{and} \quad P_{\alpha,r}(-1) > 0$$

for all α $(0 \leqq \alpha \leqq 1)$ and all r $(0 \leqq r \leqq 1)$. A lengthy calculation shows that the inequality

$$P_{\alpha,r}(1) > 0$$

also holds for all values of r and α $(0 \leqq r, \alpha \leqq 1)$.

It is now clear that in order to determine R_α, we have to look at the relative minimum of the polynomial $P_{\alpha,r}(x)$. We distinguish three cases.

I. If $\alpha = 1/2$, then $P_{1/2,r}(x) = 2(1-rx) > 0$ $(-1 \leqq x \leqq 1, 0 \leqq r \leqq 1)$. Hence $R_{1/2} = 1$.

II. If $0 \leqq \alpha < 1/2$, then the leading coefficient a_3 of $P_{\alpha,r}(x)$ is negative, and the relative minimum of $P_{\alpha,r}(x)$ occurs at

$$x_1 = \frac{1}{3|a_3|}\,[a_2 - (a_2^2 - 3a_1a_3)^{1/2}].$$

Note that $a_2^2 - 3a_1a_3 > 0$, so that x_1 is a (negative) real number. For most points (α, r) in the rectangle $0 \leqq \alpha < 1/2$, $0 \leqq r \leqq 1$, the corresponding value of x_1 lies outside the interval $-1 \leqq x \leqq 1$. The region of points, for which $-1 \leqq x_1 \leqq 1$, is contained in the rectangle $0 \leqq \alpha < 0.26$, $1/2 < r \leqq 1$. For these points, a careful investigation of

$$P_{\alpha,r}(x_1) = \frac{1}{27a_3^2}\,[27a_3^2a_0 + 2a_2^3 - 9a_1a_2a_3 - 2(a_2^2 - 3a_1a_3)^{3/2}] \tag{15}$$

shows that this relative minimum is positive, at least for $r \leqq r_\alpha$.

III. If $1/2 < \alpha \leqq 1$, then the leading coefficient a_3 of $P_{\alpha,r}(x)$ is positive, and the relative minimum of $P_{\alpha,r}(x)$ occurs at

$$x_2 = \frac{1}{3a_3}\,[-a_2 + (a_2^2 - 3a_1a_3)^{1/2}].$$

Again, x_2 is indeed a real number, and x_2 lies in the interval $[-1, 1]$ only for certain values of r and α (α and r must be "large"). The relative minimum $P_{\alpha,r}(x_2)$ is formally given by the same expression as $P_{\alpha,r}(x_1)$ (see (15)), but in the present case, $P_{\alpha,r}(x_2)$ need not be positive. Using a computer, we find that for

$$(\alpha, r) = (0.8, 0.9), (0.9, 0.9), (0.9, 0.8), (1.0, 0.9), (1.0, 0.8), (1.0, 0.7),$$

the number x_2 lies in the interval $[-1, 1]$ and $P_{\alpha,r}(x_2)$ is negative, whereas for

$$(\alpha, r) = (0.7, 0.9), (0.8, 0.8), (0.9, 0.7), (1.0, 0.6),$$

the relative minimum $P_{\alpha,r}(x_2)$ is positive.

A careful investigation shows that the curve, enclosing the region of points (α, r) for which $-1 \leqq x_2 \leqq 1$ and $P_{\alpha,r}(x_2) < 0$, lies between the two sets of points cited above.

Summary. For $0 \leqq \alpha \leqq 0.7$, we found that $R_\alpha > r_\alpha$, where r_α is the smallest positive root of the equation $B(r, \alpha) = 0$. Hence for each function $f(z) \in S_\alpha^*$, the image of each disk $|z| \leqq r$ $(r \leqq r_\alpha)$ under $f'(z)$ is contained in the image of the disk under $k'_\alpha(z)$. For $\alpha > 0.7$, we can only assert that the Marx conjecture holds in a disk whose radius is the minimum of R_α and r_α, and R_α will furnish the minimum for large values of α.

Concluding, we remark that for $\alpha = 1/2$, other methods yield that the Marx conjecture holds in the entire unit disk; in other words, for each $f \in S_{1/2}^*$, the function $f'(z)$ is subordinate to $k'_{1/2}(z)$ (J. A. Pfaltzgraff has proved this result in an unpublished paper).

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