DIRECTED BANACH SPACES OF AFFINE FUNCTIONS(1)

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0. **Introduction.** Let X be a compact convex set and let F be a closed face of X. In this paper we develop a technique which yields sufficient conditions for F to be a peak-face of X (a subset of X where a continuous affine function on X attains its maximum).

The theory is based on a duality between certain types of ordered Banach spaces. This duality is an extension of the results of [6] (see also [17]) and relates the directness of an ordered Banach space E to the degree to which the triangle inequality can be reversed on the positive elements of E^*. A precise formulation of this is given in §1.

In §2 we define a compact convex set X to be *conical* at an extreme point x if there is a bounded nonnegative affine function f on X such that $f(x)=0$ and $X=\operatorname{conv}(\{x\} \cup \{y \in X : f(y) \geqq 1\})$. If X is conical at the G_δ extreme point x then the results of §1 are applied to show that x is a peak-point of X.

Every compact convex set X has a natural identification with the positive elements of norm one in $A(X)^*$, where $A(X)$ is the space of continuous affine functions on X. If N is the subspace of $A(X)^*$ spanned by the closed face F of X then by making use of the quotient map from $A(X)^*$ to $A(X)^*/N$ we can extend the definition of "conical" to the closed face F. This is then used to establish a sufficient condition for F to be a peak-face. This procedure of using the quotient map is used repeatedly throughout and as a by-product yields different (and possibly simpler) proofs of some known results. For example we use this approach (see Proposition 4.2) to reprove a result of Alfsen's [2] concerning the complementary face of a closed face of a Choquet simplex.

In §3 we define a class $\mathscr{P}$ of compact convex sets X for which it turns out that (1) every closed G_δ face F of X is a peak-face and (2) every continuous affine function on F can be extended to a continuous affine function on X. It is known that Choquet simplexes have these two properties and we show that $\mathscr{P}$ in fact contains the simplexes. In addition it is proved that $\mathscr{P}$ contains the α-polytopes. (These are defined by R. Phelps [19] and he proves that they correspond exactly to the polyhedrons defined by Alfsen [1].)

In [19] Phelps also defines the β-polytopes as the intersection of a simplex S with a closed subspace of $A(S)^*$ of finite codimension. In §4 we show that the β-polytopes are conical at each extreme point and that those β-polytopes which are

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the intersection of a simplex with a closed hyperplane are in the class $\mathscr{P}$. We also give an example of a compact convex set which is in $\mathscr{P}$ but is neither an α-polytope nor a β-polytope. As a further illustration we show that the compact convex set of positive normalized functionals over a function algebra (on a compact Hausdorff space) is conical at each extreme point.

We are indebted to Professor Phelps for valuable discussions concerning his work on polytopes.

1. **Directed Banach spaces.** Let E be a Banach space partially ordered by the closed convex cone P. Let E_α denote the closed ball of radius α about the origin and let E^* be the dual space of E. Then the dual cone, P^*, is a weak* closed convex cone in E^* consisting of those functionals which are nonnegative on P.

The space E is said to be *positively-generated* if each element of E can be represented as the difference of two positive elements, i.e., $E=P-P$. It is known (see [4]) that if E is positively generated then E is in fact *boundedly positive-generated* (there exists a constant M such that for any $x \in E$, $x=x_1-x_2$; $x_1, x_2 \in P$ and $\|x_1\|+\|x_2\| \leqq M\|x\|$). An ordered Banach space is *normal* if there is a constant M such that if $y \leqq x \leqq z$ then $\|x\| \leqq M \max\{\|y\|, \|z\|\}$. A classical theorem of Grosberg and Krein [14] establishes the duality between normality on E and positive-generation in E^*. On the other hand Andô [4] and Ellis [13] have shown the reverse duality, i.e., E is positively generated if and only if E^* is normal. In addition a close connection is known to hold between the constants involved in the definitions of "bounded positively generated" and normality ([14], [13]).

In this section we give an analogous duality theorem involving somewhat different ordering conditions on E and E^* which we now define.

DEFINITION. Let $\alpha \geqq 1$ and n be a positive integer. The ordered Banach space E is said to be (α, n)-*directed* if whenever $x_1, \ldots, x_n \in E_1$ there exists $x \in E_\alpha$ such that $x \geqq x_1, \ldots, x_n$. If E is (α', n)-directed for all $\alpha' > \alpha$ then E is said to be *approximately* (α, n)-*directed.*

The space E is called (α, n)-additive if whenever $x_1, \ldots, x_n \in P$ then $\sum_{i=1}^n \|x_i\| \leqq \alpha\|\sum_{i=1}^n x_i\|$.

If E satisfies one of these properties where $\alpha=1$ and $n=2$ the property then holds for any positive integer n. This case was dealt with in [6]. Note that if E is positively generated then E is (α, n)-directed for any n and an appropriate choice of α. Conversely if E is (α, n)-directed $(n \geqq 2)$ then E is positively generated. In this context Theorem 1.1 is related to the results of Andô and Ellis mentioned above.

THEOREM 1.1. *Let E be an ordered Banach space with closed positive cone P. Then E is approximately (α, n)-directed if and only if E^* is (α, n)-additive.*

The following proof was suggested by Namioka and Phelps represents a considerable economy over the author's original argument.

Proof. Let $\boldsymbol{E}$ be the n-fold Cartesian product of E with itself. With $\|\boldsymbol{x}\| = \max\{\|x_i\| : 1 \leqq i \leqq n\}$ where $\boldsymbol{x}=(x_1, \ldots, x_n)$ and the closed positive cone $\boldsymbol{P}=$

$\{x : x_i \geqq 0;\ 1 \leqq i \leqq n\}$ E is an ordered Banach space. Let D be the diagonal in E, i.e. $D = \{x : x_1 = x_2 = \cdots = x_n\}$. For each positive β let

$$A_\beta = D_\beta - P = \{x \in E : x = z - p;\ z \in D,\ \|z\| \leqq \beta \text{ and } p \in P\}.$$

In this context it is easily verified that E is approximately (α, n)-directed if and only if

$$E_1 \subset \bigcap \{A_\beta : \beta > \alpha\}. \tag{1}$$

The dual space E^* can be identified with the n-fold Cartesian product of E^* with itself where $f(x) = \sum_{k=1}^n f_k(x_k)$; $f = (f_1, \ldots, f_n) \in E^*$ and $x \in E$. Then the dual norm is given by $\|f\| = \sum_{k=1}^n \|f_k\|$ and $f \geqq 0$ if and only if $f(x) \geqq 0$ for all $x \in P$. Now E^* is (α, n)-additive if and only if for all $f \geqq 0$

$$\begin{aligned} \|f\| \leqq \alpha \left\| \sum_{k=1}^n f_k \right\| &= \sup\{f(z) : z \in D_\alpha\} \\ &= \sup\{f(z) : z \in D_\alpha - P = A_\alpha\} \quad (\text{since } f \geqq 0) \\ &= \sup\{f(z) : z \in (A_\alpha)^-\}. \end{aligned}$$

It now follows from the separation theorem that E^* is (α, n)-additive if and only if

$$E_1 \subset (A_\alpha)^-. \tag{2}$$

To prove the theorem it remains to show that (1) and (2) are equivalent. Suppose (1) holds and $x \in E_1$ and $1 > \varepsilon > 0$ are given. Choose β such that $\alpha < \beta < \alpha/(1-\varepsilon)$. Then $x \in A_\beta$ so that $y = (\alpha/\beta)x \in A_\alpha$ and $\|y - x\| < \varepsilon$. Thus $x \in (A_\alpha)^-$.

Suppose (2) holds. Then given any $x \in E$ and $\varepsilon > 0$ there exists $z - p \in \|x\| A_\alpha$ such that $\|x - (z-p)\| < \varepsilon$. Let $y = x - (z - p)$. Then

$$x \leqq z + y \quad \text{where } z \in \|x\| D_\alpha \quad \text{and} \quad \|y\| < \varepsilon. \tag{3}$$

Given $x \in E_1$ and $\beta > \alpha$ choose ε such that $\beta = \alpha/(1-\varepsilon)$. By (3) there exist $z_1 \in D_\alpha$ and $y_1 \in E$ $(\|y_1\| < \varepsilon)$ such that $x < z_1 + y_1$. By (3) again there exist $z_2 \in \varepsilon D_\alpha$ and y_2 $(\|y_2\| < \varepsilon^2)$ such that $x \leqq z_1 + z_2 + y_2$. Continuing by induction there exist sequences $\{z_n\}_{n=1}^\infty$ in D and $\{y_n\}_{n=1}^\infty$ in E such that for each n

$$x \leqq z_1 + \cdots + z_n + y_n; \qquad z_n \in \varepsilon^{n-1} D_\alpha, \quad \|y_n\| < \varepsilon^n. \tag{4}$$

The series $\sum_{n=1}^\infty z_n$ converges to $z \in D$ with $\|z\| \leqq \sum_{n=1}^\infty \|z_n\| \leqq \alpha/(1-\varepsilon) = \beta$. Since P is closed (4) holds in the limit so that $x \leqq z$, i.e., $x \in D_\beta - P = A_\beta$. Thus (1) holds and the proof is complete.

2. **Faces of compact convex sets.** Let X be a compact convex set. We assume throughout that X is a subset of a Hausdorff locally convex space. Let $A(X)$ be the ordered Banach space of affine continuous functions on X. Under the usual evaluation map X is affinely homeomorphic to the set of positive elements of norm one in $A(X)^*$ with the (relative) weak* topology and we shall regularly identify

X with this set. Use will also be made of the fact that if $0 \in X$ and $[X]$ denotes the linear subspace spanned by X then $[X]$ can be identified with $A_0(X)^*$ where $A_0(X) \equiv \{f \in A(X) : f(0)=0\}$. In each case the unit ball of the function space dual is conv $(X \cup -X)$.

Let F be a closed convex subset of X and let $[F]$ be the linear subspace (not necessarily closed) spanned by F in $A(X)^*$. If $[F] \cap X = F$ and $X \backslash F$ is convex then F is called a *face* of X. If in addition $[F]^- \cap X = F$ (closure in the weak* topology) then F will be said to be *self-determining*, i.e., F is its own set of determinacy (see Alfsen [1]). Let $A_F(X)$ denote $\{f \in A(X) : f \equiv 0 \text{ on } F\}$. We call F a *peak-face* if there is an $f \in A_F(X)$ such that $f(x) > 0$ for all $x \in X \backslash F$. Equivalently F is exposed in the weak* topology on X.

THEOREM 2.1. *Let F be a self-determining face of the compact convex set X and let $A_F(X)$ be $(\alpha, n+1)$-directed. Let $f_1, \ldots, f_n$ be functions in the unit ball of $A_F(X)$ and let K be a compact subset of $X \backslash F$. Then there exists $f \in A_F(X)$ such that*

(i) *$f(x) > 0$ for all $x \in K$,*

(ii) *$f \geqq f_1, \ldots, f_n$*

(iii) *$\|f\| \leqq \alpha$.*

Proof. Let $U_1, \ldots, U_k$ be compact convex neighborhoods in $X \backslash F$ covering K and let $G = \text{conv}\,(U_1 \cup \cdots \cup U_k)$. Then G is a compact convex subset of X and since F is a face $G \cap F = \varnothing$. If $N = [F]^-$ then $N \cap G = \varnothing$ and so there is a separating functional g in $A(X)$ of norm one such that $g \equiv 0$ on F and $g(x) > 0$ for all $x \in G \supset K$. But since $A_F(X)$ is $(\alpha, n+1)$-directed there is an $f \in A_F(X)$ such that $f \geqq f_1, \ldots, f_n, g$ and $\|f\| \leqq \alpha$.

COROLLARY 2.2. *If F is a self-determining G_δ face of X and $A_F(X)$ is $(\alpha, n+1)$-directed then given $f_1, \ldots, f_n$ in the unit ball of $A_F(X)$ there exists $f \in A_F(X)$ such that*

(i) *F is a peak-face with respect to f,*

(ii) *$f \geqq f_1, \ldots, f_n$,*

(iii) *$\|f\| \leqq \alpha$.*

We now give conditions on a face F which guarantee that $A_F(X)$ is directed.

DEFINITION. Let 0 be an extreme point of the compact convex set X and let p be the Minkowski functional of X. Let α be a real number and n a positive integer such that $\alpha \geqq 1$ and $n > 1$. We say X is *(α, n)-additive* at 0 if $p(x_1) + \cdots + p(x_n) \leqq \alpha p(x_1 + \cdots + x_n)$ for any $x_1, \ldots, x_n \in X$. Call X *α-additive* at 0 if X is (α, n)-additive for all n. We say X is *α-conical* at 0 if there is a (not necessarily continuous) linear functional f on $[X]$ such that $0 \leqq f \leqq \alpha$ on X and $x \in X$ implies $x \in f(x)X$.

As the following propositions will show, if $\alpha = 1$ then all of the above properties are equivalent and this case X is a universal cap [6].

PROPOSITION 2.3. *If X is α-conical at 0 then X is α-additive at 0. If X is α-additive at 0 then X is α^2-conical at 0.*

Proof. Let f be a linear functional on $[X]$ such that $0 \leqq f \leqq \alpha$ on X and $x \in f(x)X$ for all $x \in X$. Let $Q = \bigcup_{n=1}^{\infty} nX = \{x \in [X] : p(x) < \infty\}$. From the definition of p we have $p(x) \leqq f(x) \leqq \alpha p(x)$ for all $x \in Q$. Thus if $x_1, \ldots, x_n \in X$ then $\sum_{i=1}^{n} p(x_i) \leqq \sum f(x_i) = f(\sum x_i) \leqq \alpha p(\sum x_i)$.

Suppose now X is α-additive and let $Y = \operatorname{conv}\{x \in X : p(x) = 1\}$. If $x \in Y$ then $p(x) \geqq 1/\alpha$. Let $W = (1/\alpha^2)(X - X)$. Then $0 \in \operatorname{core} W$ (relative to $[X]$) and if $x \in W \cap Q$ then $p(x) \leqq 1/\alpha$: let $\alpha^2 x = x_1 - x_2$; $x_1, x_2 \in X$. Then $x_1 = x_2 + \alpha^2 x$ and so

$$\alpha \geqq \alpha p(x_1) \geqq p(x_2) + \alpha^2 p(x) \geqq \alpha^2 p(x).$$

Thus $p(x) \leqq 1/\alpha$ and hence $(\operatorname{core} W) \cap Y = \varnothing$. There exists a linear functional f on $[X]$ by the separation theorem such that $\sup f(W) \leqq 1 \leqq \inf f(Y)$. If $x \in Y$ then $x \in (p(x)Y) \cap (\alpha^2 p(x) W)$ and consequently $p(x) \leqq f(x) \leqq \alpha^2 p(x)$. Therefore X is α^2-conical at 0.

PROPOSITION 2.4. *If X is (α, n)-additive at 0 then $A_0(X)$ is approximately (α^2, n)-directed. If X is α-conical at 0 then $A_0(X)$ is approximately α-directed.*

Proof. Suppose X is (α, n)-additive. Let $[X]$ be normed with unit ball $U = \operatorname{conv}(X \cup -X)$. Let $B = U \cap Q$ where $Q = \bigcup_{n=1}^{\infty} nX$. If $x \in B$ then $x = \lambda x_1 + (1-\lambda)(-x_2)$; $x_1, x_2 \in X$ and so $\lambda \alpha p(x_1) \geqq p(x)$. Thus $p(x) \leqq \alpha$ and therefore $X \subset B \subset \alpha X$. Hence for $x \in Q$, $\|x\| \leqq p(x) \leqq \alpha \|x\|$ and from this it follows that $([X], \|\cdot\|, Q)$ is an (α^2, n)-additive ordered Banach space.

If X is α-conical then for $x \in Q$

$$\|x\| \leqq p(x) \leqq f(x) \leqq \alpha \|x\|$$

and therefore $([X], \|\cdot\|, Q)$ is α-additive.

Since $B \subset \alpha X$, $B = Q \cap U = (\alpha X) \cap U$ is weak* compact. Thus Q is weak* closed. If P is the positive cone of $A_0(X)$ then it follows from the separation theorem that $Q = P^*$ and so $([X], \|\cdot\|, Q)$ is the Banach space order dual of $A_0(X)$. Thus in both cases the desired result follows from Theorem 1.1.

We now wish to extend these additivity properties to self-determining faces of X.

DEFINITION. Let F be a self-determining face of X and let N be the closure of the linear span of F in $A(X)^*$ (weak* topology). Let p be the Minkowski functional of the closed convex set $X + N$. Then F will be said to be an (α, n)-*additive* face of X if whenever $x_1, \ldots, x_n \in X + N$ then $\sum_{i=1}^{n} p(x_i) \leqq \alpha p(\sum_{i=1}^{n} x_i)$. The face F is α-*additive* if the inequality holds for all n. The face F is α-*conical* if there is a linear functional f on $A(X)^*$ such that $0 \leqq f \leqq \alpha$ on $X + N$, $N \subset f^{-1}(0)$ and $x \in f(x)X + N$ for all $x \in X + N$.

We note that if in particular there is a linear functional f on $A(X)^*$ such that $0 \leqq f \leqq \alpha$ on X, $N \subset f^{-1}(0)$ and $X = \operatorname{conv}(F \cup K)$, where $K = \{x \in X : f(x) \geqq 1\}$, then F is α-conical under f.

Let F be a self-determining face of X with $N = [F]^-$ and let $q: A(X)^* \to A(X)^*/N$ be the quotient map.

PROPOSITION 2.5. *The face F is (α, n)-additive if and only if qX is (α, n)-additive at* 0. *The face F is α-conical if and only if qX is α-conical at* 0.

Proof. Let p be the Minkowski functional of $X+N$ and let $\bar{p}$ be the Minkowski functional of qX. Then $p=\bar{p}\circ q$. Similarly in the "conical" case $\bar{f}\circ q=f$ well-defines a functional $\bar{f}$ in terms of f with the required properties and vice-versa.

THEOREM 2.6. *Let F be a self-determining face of X. If F is an (α, n)-additive face of X then $A_F(X)$ is approximately (α^2, n)-directed. If F is an α-conical face of X then $A_F(X)$ is approximately α-directed.*

Proof. Let $\bar{q}=q|X$ where $q: A(X)^* \to A(X)^*/N$ is the quotient map. Then $\Phi: A_0(qX) \to A_F(X)$ defined by $\Phi(f)=f\circ\bar{q}$ is an isometric order-preserving isomorphism of the two function spaces and so the result follows from Theorem 1.1 and Proposition 2.4.

3. **Simplexes and α-polytopes.** If X is a compact convex set then $A(X)^*$ is an ordered Banach space with closed positive cone $P=\bigcup_{n=1}^{\infty} nX$ and unit ball $B=\operatorname{conv}(X\cup -X)$. If an addition $A(X)^*$ is a Kakutani L-space then X is called a Choquet *simplex*. This is equivalent to P being a lattice in its own ordering since in this context the additivity of the norm on P is automatic and $B=\operatorname{conv}(X\cup -X)$ guarantees that $\|x\|\leqq\|y\|$ whenever $|x|\leqq|y|$ [8], [6].

Recent results have shown the facial structure of simplexes to be quite interesting. For example it is shown by Edwards [11] and Lazar [16] that any continuous affine function on a closed face of a simplex X can be extended to a continuous affine function on X. Another property of simplexes is that every closed G_δ face is a peak-face [9], [11], [15].

Let us say that a compact convex set X has the *extension property* if for any closed face F of X there is a real constant r such that every affine continuous function f on F can be extended to a continuous affine function $\hat{f}$ on X such that $\|\hat{f}\|\leqq r\|f\|_F$. We shall say X has the *peak-face property* if for any closed face F and G_δ V containing F there is a nonnegative continuous affine function which is identically zero on F and strictly positive on $X\backslash V$. Thus every simplex has the extension property (where in fact the extension is norm-preserving) and the peak-face property.

Let $\mathscr{P}$ denote the class of compact convex sets such that (a) for each closed face F of a member X of $\mathscr{P}$ the subspace $[F]$ spanned by F is weak* closed in $A(X)^*$, and (b) each closed face of X is conical.

By virtue of (a) each closed face of a member of $\mathscr{P}$ is self-determining and thus (b), together with Theorem 2.6, assures that each member of $\mathscr{P}$ has the peak-face property. In addition it is shown below that (a) is sufficient to guarantee the extension property. We will show also that every simplex is a member of $\mathscr{P}$ and that $\mathscr{P}$ also includes the α-polytopes defined by Phelps [19], or equivalently the polyhedrons of Alfsen [1] (definitions below).

Alfsen [3] introduces the notion of *strongly Archimedean* faces and gives a complete discussion of the extension property for such faces, including a characterization of the "best" value for r in terms of a geometric relationship between F and X. The essential observation is that the extension property for F is equivalent to $[F]$ being weak* closed in $A(X)^*$. We summarize the situation in the following theorem which is an adaptation of Edwards [10, Corollary 2] and Alfsen [3] to our present context.

THEOREM 3.1. *Let X be compact and convex and let F be a compact convex subset of X such that if M is the linear subspace spanned by F in $A(X)^*$ then $M \cap X = F$. Let B denote the unit ball* ($=\operatorname{conv}(X \cup -X)$) *in $A(X)^*$ and let $T = \operatorname{conv}(F \cup -F)$. The following are equivalent*:

(1) *M is weak* closed in $A(X)^*$,*

(2) *M is norm closed in $A(X)^*$,*

(3) *there exists s, $0 < s < \infty$, such that $M \cap B \subset sT$,*

(4) *there exists r, $1 \leqq r < \infty$ such that every $f \in A(F)$ has an extension $\hat{f} \in A(X)$ such that $\|\hat{f}\|_X \leqq r\|f\|_F$.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): Since T is weak* compact and hence norm closed and $M = \bigcup_{n=1}^{\infty} nT$, 0 must be a norm-relative interior point of T and (3) follows.

(3) $\Rightarrow$ (1): We have $M \cap B = sT \cap B$ is weak* compact and hence M is weak* closed by the Krein-Šmulyan theorem.

We show (4) is equivalent to (2). Let $\|\cdot\|_1$ be the norm on M with unit ball $M \cap B$ and let $\|\cdot\|_2$ be the norm on M with unit ball T. Then $(M, \|\cdot\|_2) \cong A(F)^*$. Let j: $A(X) \to A(F)$ be defined by $j(f) = f|_F$. Then j is a bounded linear map and (4) is equivalent to j being onto. It is well known (see [18, Proposition 4.5]) that j has dense range. Hence (4) holds if and only if j has closed range. The adjoint j^* is easily identified as the identity map i: $(M, \|\cdot\|_2) \to (M, \|\cdot\|_1)$ whose range is closed if and only if (2). But the range of j is closed if and only if the range of j^* $(=i)$ is closed.

COROLLARY 3.2. *If $X \in \mathscr{P}$ then X has the extension property and the peak-face property.*

We show next that the simplexes are contained in $\mathscr{P}$.

LEMMA 3.3. *Let X be a compact convex set with* 0 *as an extreme point and let p be the Minkowski functional of X. If $x \in X$ then $p(x) = 1$ if and only if every probability measure on X representing x has mass zero at* $\{0\}$.

Proof. Suppose μ_x represents $x \neq 0$ and $\mu(\{0\}) = a > 0$. Then $(\mu_x - a\varepsilon_0)/(1-a)$ (ε_0 the point mass at 0) is a probability measure on X representing $x/(1-a)$. Thus $x/(1-a) \in X$ and $p(x) \leqq 1 - a < 1$. If $p(x) < 1$ then $x = ay$ $(a < 1)$. Let μ be a probability measure representing y. Then $(1-a)\varepsilon_0 + a\mu$ represents x and has positive mass at $\{0\}$.

THEOREM 3.4. *If S is a simplex then S is 1-conical at each extreme point.*

Proof. We assume $0 \in \operatorname{ext} S$ and show S is 1-conical at 0. By Proposition 2.3 it suffices to prove S is 1-additive at 0, or equivalently, $\{x : p(x)=1\}$ is convex. Suppose $z=\lambda x+(1-\lambda)y$ where $p(x)=1=p(y)$ and $0<\lambda<1$. Suppose $p(z)=a<1$ and let $w=z/a \in S$. Let μ_x, μ_y and μ_w be maximal (as in [16]) probability measures on S representing x, y and w respectively. If $\mu_1=\lambda\mu_x+(1-\lambda)\mu_y$ and $\mu_2=(1-a)\varepsilon_0+a\mu_w$ then μ_1 and μ_2 are both maximal measures representing z. By the lemma $\mu_1(\{0\})=0$. But $\mu_2(\{0\})=1-a>0$ so that μ_1 and μ_2 are different maximal representing measures for z contradicting the fact that S is a simplex.

COROLLARY 3.5. *If S is a simplex then $S \in \mathscr{P}$ and in fact S is 1-conical at each closed face.*

Proof. Let $N=[F]$ in $A(S)^*$. From a result of Edwards [10] N is weak* closed and from a result of Effros [12] $A(S)^*/N$ is again an L-space. Let $q: A(S)^* \to A(S)^*/N$ be the quotient map and let $B=\operatorname{conv}(S \cup -S)$ be the ball in $A(S)^*$. We show qS is the positive part of the ball B' in $A(S)^*/N$. Since B is weak* compact $B'=qB$. If $\bar{x} \in B'$ and $\bar{x} \geqq 0$ then there exist $x \in B$ and $n \in N$ such that $qx=\bar{x}$ and $x+n \geqq 0$. Then since $n^+ \in N$ (N is an order ideal) $x+n^+ \geqq x, 0$ and so $x+n^+ \geqq x^+$. Thus $x-x^+ \in N$ and $\|x^+\| \leqq \|x\| \leqq 1$. Let $y \in F$ and set $z=x^++(1-(\|x^+\|)y$. Then $z \in S$ and $q(z)=q(x^+)=q(x)=\bar{x}$. Since qS is clearly contained in the positive part of B' the two are equal. Thus qS is a simplex and hence 1-conical at 0. Therefore F is a 1-conical face of S.

We consider now pairs (X, Y) of compact convex sets for which there exists an affine continuous map ϕ from X onto Y. The map ϕ extends to a map $\bar{\phi}: A(X)^* \to A(Y)^*$ which is linear, weak* continuous and onto. If $M=\{f \in A(X) : f$ annihilates $\ker \bar{\phi}\}$ then M is isometrically isomorphic to $A(Y)$. It follows that the quotient topology on $A(Y)^*$ induced by $\bar{\phi}$ is the same as the weak* topology.

In [19] Phelps defines an *α-polytope* to be a compact convex set X for which there exists a compact Choquet simplex S and an affine continuous map ϕ of S onto X such that $\phi^{-1}(x)$ is finite dimensional for all $x \in X$. It is shown in [19] that for α-polytopes this is equivalent to the extended map $\bar{\phi}$ having finite dimensional kernel. Phelps also shows in [19] that the α-polytopes are exactly the polyhedrons defined by Alfsen in [1] as the compact convex sets X such that for any point $x \in X$ the set of maximal probability measures representing x is finite dimensional.

The next theorem shows the equivalence between the finite dimensionality of $\phi^{-1}(x)$ and $\ker \bar{\phi}$ holds in general.

THEOREM 3.6. *Let X and Y be compact convex sets and $\phi: X \to Y$ a continuous affine surjection. The extended map $\bar{\phi}: A(X)^* \to A(Y)^*$ has finite dimensional kernel if and only if $\phi^{-1}(y)$ is finite dimensional at each $y \in Y$.*

Proof. The only if part is obvious. Suppose then $\phi^{-1}(y)$ is finite dimensional for all $y \in Y$. For each $x \in X$ let $Z_x = \{z \in X : \phi(z) = \phi(x)\}$ and let $A_x = R^+(Z_x - Z_x)$. Then $\mathscr{A} = \{A_x\}_{x \in X}$ is a family of finite dimensional subspaces of $A(X)^*$. If $x_1, x_2 \in X$ and $x = \lambda x_1 + (1-\lambda)x_2$ $(0 < \lambda < 1)$ then $A_{x_1} \cup A_{x_2} \subset A_x$. For suppose $w \in A_{x_1}$. Then $w = r(x_1' - x_1'')$; $x_1', x_1'' \in A_{x_1}$. Let $w' = \lambda x_1' + (1-\lambda)x_2$ and $w'' = \lambda x_1'' + (1-\lambda)x_2$. Then $\phi(w') = \phi(x) = \phi(w'')$ so that $w', w'' \in A_x$. But $w = (r/\lambda)(w' - w'')$ and hence $w \in A_x$. We show next there is a positive integer N such that $\dim A_x \leqq N$ for all $x \in X$. Suppose not. Then for each n there exists $x_n \in X$ such that $\dim A_{x_n} \geqq n$. The series $\sum_{n=1}^{\infty} 2^{-n} x_n$ converges (in norm) to a point $x \in X$ and by the above calculation $A_{x_n} \subset A_x$ for each n which contradicts the finite dimensionality of A_x. Thus we have $\mathscr{A}$ is a directed family of subspaces whose dimensions are bounded above by N. If $A = \bigcup_{x \in X} A_x$ then A is a finite dimensional subspace which in fact equals $\ker \bar{\phi}$. For clearly each $A_x \subset \ker \bar{\phi}$. Suppose $0 \neq z \in \ker \bar{\phi}$. Then $z = ax - bx'$; $x, x' \in X$ and $a\phi(x) = b\phi(x')$. By applying $1 \in A(Y)$ we get $a = b$ so that $z = a(x - x')$. Since $a \neq 0$, $\phi(x) = \phi(x')$ and hence $z \in A_x \subset A$.

We show next that $\mathscr{P}$ is closed under continuous affine surjections with finite dimensional kernel.

THEOREM 3.7. *Let X and Y be compact convex sets such that $X \in \mathscr{P}$ and let ϕ be an affine continuous surjection from X to Y such that $\phi^{-1}(y)$ is finite dimensional for all $y \in Y$. Then $Y \in \mathscr{P}$.*

Proof. Let $\bar{\phi}: A(X)^* \to A(Y)^*$ be the extension of ϕ and let F' be a closed face of Y. If $F = \phi^{-1}(F')$ then F is a closed face of X and $N = [F]$ is weak* closed. Let $N' = [F']$. Then $\bar{\phi}^{-1}(N') = N + \ker \bar{\phi}$ and since $\dim(\ker \bar{\phi}) < \infty$, $\bar{\phi}^{-1}(N')$ is closed. It follows that N' is closed in the quotient topology induced by $\bar{\phi}$ and hence is weak* closed.

To show that ϕ preserves property (b) of $\mathscr{P}$ requires the following lemma.

LEMMA 3.8. *Let X and Y be compact convex sets both containing* 0. *Let ϕ be an affine continuous mapping of X onto Y such that $\phi(0) = 0$. Let $\bar{\phi}: [X] \to [Y]$ be the natural extension of ϕ. Suppose* $\dim(\ker \bar{\phi}) < \infty$ *and* $X \cap \ker \bar{\phi} = \{0\}$. *If X is conical at* 0 *then Y is also conical at* 0.

Proof. Assume $\dim(\ker \bar{\phi}) = 1$. The general result will follow by finite induction. Let f be a linear functional on $[X]$ such that $0 \leqq f \leqq \alpha$ on X and $x \in X$ implies $x \in f(x)X$. Let $\ker \bar{\phi} = Rx_0$ where either $f(x_0) = 0$ or $f(x_0) = 1$. If $f(x_0) = 0$ then $\bar{f} = f \circ \bar{\phi}^{-1}$ well-defines a linear function $\bar{f}$ on $[Y]$ with the necessary properties. If $f(x_0) = 1$ then since $Rx_0 \cap X = \{0\}$ there exists a convex neighborhood U of 0 such that $(x_0 + U) \cap X = \varnothing$. Let β be a positive number such that $(x_0 + \beta U) \supset X$ and let $K = \{x \in X : f(x) \geqq 1\}$. Define the projection $p: [X] \to f^{-1}(0)$ by $px = x - f(x)x_0$. If $U_1 = U \cap f^{-1}(0)$ then $0 \in \operatorname{core}(U_1)$ with respect to $f^{-1}(0)$. Also $U_1 = pU_1$ is disjoint from pK since if $x \in K$ and $px \in U_1$ then

$$x/f(x) \in x_0 + (1/f(x))U_1 \subset x_0 + U_1 \subset x_0 + U.$$

But $x/f(x) \in X$, contradicting the choice of U. Thus there is a linear functional g on $f^{-1}(0)$ such that $\sup g(U_1) \leqq 1 \leqq \inf g(pK)$. For $x \in X$ we have $x \in f(x)X \subset f(x)\cdot(x_0+\beta U)$ so that $x-f(x)x_0 = px \in f(x)\beta U_1 \subset \alpha\beta U_1$. Hence $pX \subset \alpha\beta U_1$. Also $pX = \operatorname{conv}(pK \cup \{0\})$. Thus we have $0 \leqq g \leqq \alpha\beta$ on pX and if $y \in pX$ then $y \in g(y)pX$. Thus pX is conical at 0 under g and $\bar{g} = g \circ p \circ \bar{\phi}^{-1}$ is the required functional on $[Y]$.

Continuing with the proof of the theorem let $q_1: A(X)^* \to A(X)^*/N$ and $q_2: A(Y)^* \to A(Y)^*/N'$ be quotient maps. Since F is a conical face of X, q_1X is conical at 0. Let $\bar{\phi}' = q_2 \circ \bar{\phi} \circ q_1^{-1}$. Then $\bar{\phi}'$ is a well-defined continuous linear map from $A(X)^*/N$ onto $A(Y)^*/N'$. From the equation $\bar{\phi}' \circ q_1 = q_2 \circ \bar{\phi}$ it is seen that $\bar{\phi}'(q_1X) = q_2Y$ and $\ker \bar{\phi}' = q_1(\ker \bar{\phi})$. Since $N' \cap Y = F'$ we have $(N + \ker \bar{\phi}) \cap X = F$ and consequently $q_1(\ker \bar{\phi}) \cap q_1X = \{0\}$. It follows from the lemma that q_2Y is conical at 0 and hence F' is a conical face of Y.

COROLLARY 3.9 (see also [19]). *If X is an α-polytope then $X \in \mathscr{P}$ and hence has the extension property and the peak-face property.*

4. **Further examples.** In [19] Phelps defines a β-polytope X to be the intersection of a simplex S (in $A(S)^*$) with a weak* closed subspace of finite codimension. Equivalently $X = \{x \in S : f_1(x) = \cdots = f_n(x) = 0;\ f_1, \ldots, f_n \in A(S)\}$. It is shown in [19] that the β-polytopes have the extension property and the peak-face property. In view of this it is natural to ask to what degree the results of the previous sections on conical faces can be applied to β-polytopes. Although we do not have a complete answer it is shown below that a β-polytope is conical at each extreme point and that the intersection of a simplex S with a closed hyperplane (in the terminology of [19] *slices* of S of codimension one) belongs to the class $\mathscr{P}$ of §3.

We begin with some results concerning the facial structure of simplexes.

PROPOSITION 4.1. *Let F be a closed face of the simplex S and let F be any other face of S disjoint from F. Then every $x \in \operatorname{conv}(F \cup F')$ has a unique representation of the form $\lambda x_1 + (1-\lambda)x_2$; $x_1 \in F$, $x_2 \in F'$, $0 \leqq \lambda \leqq 1$. Also $[F] \cap [F'] = \{0\}$ in $A(S)^*$.*

Proof. If $z \in F'$ and μ_z is a probability measure on S representing z then $\mu_z(F) = 0$. For suppose $\mu_z(F) = a > 0$ and let $\mu = (\mu_z|F)/a$. Then μ is a probability measure on F and hence represents a point $x \in F$. If $\mu' = (\mu_z - a\mu)/(1-a)$ then μ' is a probability measure on S representing a point $y \in S$. But then $\mu_z = (1-a)\mu' + a\mu$ so that $z = (1-a)y + ax$. Hence $x, y \in F'$ which is a contradiction of $F \cap F' = \varnothing$. Let $\lambda x_1 + (1-\lambda)x_2$ and $\lambda' x_1' + (1-\lambda')x_2'$ be two representations of x. This yields two representations of the maximal measure μ_x which represents x. By considering the restriction of μ_x to F we get $\lambda = \lambda'$ and $x_1 = x_1'$ so the representation is unique. Suppose $z \in [F] \cap [F']$ and $z \neq 0$. Then $z = ax - by = cu - dv$; $x, y \in F$, $u, v \in F'$, and $u, b, c, d \geqq 0$. By applying $1 \in A(S)$ we get $a + d = b + c$. Thus $(ax+dv)/(a+d) = (by+cu)/(b+c)$ are two representations of an element of $\operatorname{conv}(F \cup F')$ and hence $ax = by$ which is a contradiction.

The next proposition is due to Alfsen [2]. We give a proof from a somewhat different point of view.

PROPOSITION 4.2 (ALFSEN). *Let F be a closed face of the simplex S. There exists a complementary face F' such that $F \cap F' = \varnothing$ and every $x \in S$ has a unique representation as an element of* conv $(F \cup F')$. *Moreover there is a lower-semicontinuous affine function f on S such that $F = f^{-1}(0)$ and $F' = f^{-1}(1)$.*

Proof. Let $N = [F]$. By Corollary 3.6 there is an f' linear on $A(S)^*$ such that $0 \leqq f' \leqq 1$ on $S+N$, $N \subset f'^{-1}(0)$ and $x \in f'(x)S + N$ for every $x \in S$. Let $f = f'|S$. Then $F = f^{-1}(0)$. Let $F' = f^{-1}(1)$. Then F' is a face of X such that $F \cap F' = \varnothing$. Given any $x \in S$, $x = f(x)y + n$; $y \in S$ and $n \in N$. Thus $f'(y) = f(y) = 1$ and so $y \in F'$. Now it is known (see [2]) that the convex hull of two faces of a simplex is again a face. Hence $x \in [F \cup F'] \cap S = [\text{conv}\,(F \cup F')] \cap S = \text{conv}\,(F \cup F')$. Thus $S = \text{conv}\,(F \cup F')$ and by Proposition 4.1 the representation of points of S in this way is unique. To show f is lsc consider qS where $q: A(S)^* \to A(S)^*/N$ is the quotient map. Then 0 is an extreme point of qS and since qS is again a simplex it is 1-conical at 0. Thus the Minkowski functional p of qS is affine and lsc on qS. Furthermore $f = p \circ q$ and hence is lsc.

A proof of the next proposition can be found in [5, Lemma 2.3].

PROPOSITION 4.3. *Let X be a compact convex set and let $Y = \{x \in X : f(x) = 0\}$ where f is an affine function on X. If x is an extreme point of Y then x is contained in the line segment $[y, z]$ where $[y, z]$ is a face of X.*

THEOREM 4.4. *If X is a β-polytope then X is conical at each extreme point.*

Proof. Let S be a simplex and assume $X = \{x \in S : g_1(x) = \cdots = g_n(x) = 0\}$ where $g_1, \ldots, g_n \in A(S)$. Let $X_k = \{x \in S : g_1(x) = \cdots = g_k(x) = 0\}$ where $1 \leqq k \leqq n$. Let x be an extreme point of $X = X_n$. By Proposition 4.3 $x \in [x_1, x_2]$ where $[x_1, x_2]$ is a face of X_{n-1}. Then $x_1, x_2 \in \text{ext}\, X_{n-1}$ and successive applications of Proposition 4.3 yields $x = \sum_{k=1}^m \lambda_k x_k$ where $x_k \in \text{ext}\, S$. If $x = x_k$ for some k then X would automatically be 1-conical at x to begin with so we can assume $m > 1$ and $0 < \lambda_k < 1$ for each k. Let us say f is the function "associated" with the face F if f is as in Proposition 4.2. Let f_k be the function associated with the face $F_k = \text{conv}\,\{x_j : 1 \leqq j \leqq m \text{ and } j \neq k\}$. Since x is in the interior of the simplex spanned by $\{x_1, \ldots, x_m\}$ and x is an extreme point of X it must be the case that $X \cap F_k = \varnothing$ for each k. Since f_k is lsc on S there exists $\alpha_k > 0$ such that $f_k \geqq \alpha_k$ on X. Let $\alpha = \min\,\{\alpha_1, \ldots, \alpha_m\}$ and define $f = (\sum_{k=1}^m f_k - 1)/\alpha$. Then $f(x_k) = 0$ for each k and hence $f(x) = 0$. Also $f \leqq (m-1)/\alpha$ on S and hence on X. Let $K = \{x \in X : f(x) \geqq 1\}$. We will show $X = \text{conv}\,(\{x\}) \cup K)$ and from this it will follow that X is $(m-1)/\alpha$-conical at x. Note that $(\sum_{j \neq k} f_j) \geqq 1$ on the complementary face $\{x_k\}'$ of $\{x_k\}$. Thus if $z \in X \cap \{x_k\}'$ then $f(z) \geqq 1$. Suppose now $z \in X$ and the line segment $[x, z]$ cannot be extended in X beyond z. Then z must be in some $\{x_k\}'$ for otherwise there is a $\lambda > 1$ such that

$y_k = x_k + \lambda(z - x_k) \in S$ for each k. But then $\sum \lambda_k y_k = x + \lambda(z - x)$ is an extension of $[x, z]$ in S (and hence in X) beyond z. Thus if z' is any point of X, $z' \in [x, z]$ where $z \in X \cap \{x_k\}'$ for some k and hence $f(z) \geqq 1$. Therefore $X = \operatorname{conv}(\{x\} \cup K)$ and the proof is complete.

In order to prove that slices of codimension one of a simplex are in $\mathscr{P}$ requires the application of a lemma of A. Lazar which is proved in [19]. We give a different proof here establishing a quotient map set-up also used in Theorem 4.8. We first prove a preliminary proposition.

PROPOSITION 4.5. *Let C be a convex subset of a tvs E and let M be the affine variety spanned by C. Let f be a linear function on E and suppose there exist $x, y \in C$ such that $f(x) < 0 < f(y)$ and $f(x) \leqq f(z) \leqq f(y)$ for all $z \in C$. If $F = \{z \in C : f(z) = 0\}$ is compact then there is a compact convex subset A of M such that $C \subset A$.*

Proof. Let $F^+ = \{z \in C : f(z) \geqq 0\}$ and $F^- = \{z \in C : f(z) \leqq 0\}$. Consider $C - x$ and the linear functional $g = f/(-f(x))$. Then $z \in C - x$ implies $0 = g(0) \leqq g(z) \leqq g(y - x)$. Also $F - x = \{z \in C - x : g(z) = 1\}$ and $F^+ - x = \{z \in C - x : g(z) \geqq 1\}$. It is easily checked that $F^+ - x \subset g(y - x) \operatorname{conv} [\{0\} \cup (F - x)] \equiv A_1'$, a compact convex subset. Thus $F^+ \subset x + A_1' \equiv A_1 \subset M$. Similarly F^- is contained in a compact convex subset A_2 of M so that $C = F^+ \cup F^+ \subset \operatorname{conv}(A_1 \cup A_2)$, a compact convex subset of M.

LEMMA 4.6 (LAZAR). *Let X be a compact convex set and let $Y = \{x \in X : f(x) = 0\}$ where $f \in A(X)$. For each closed face F of Y there is a closed face F' of X such that $F' \cap Y = F$.*

Proof. Let $N = [F]$ (N is not necessarily closed) and let $q = \bar{q}|X$ where $\bar{q}$: $A(X)^* \to A(X)^*/N$ is the quotient map. Since $N \subset f^{-1}(0)$ there exists g linear on $A(X)^*/N$ such that $g \circ \bar{q} = f$. Thus $qY = g^{-1}(0) \cap qX$ and $\{0\} = qF$ is an extreme point of qY. If $0 \in \operatorname{ext} qX$ then $F = q^{-1}(0)$ is already a closed face of X. Otherwise by Proposition 4.3, $0 \in (qx, qy)$ where $x, y \in X$ and $[qx, qy]$ is a face of qX. Then $F' = q^{-1}[qx, qy]$ is a face of X such that $F' \cap Y = F$. By Proposition 4.5 there is a compact convex set $A \subset [F']$ such that $F' \subset A$. Since F' is a face of X, $F' = [F'] \cap X = A \cap X$ which is closed.

COROLLARY 4.7. *Let X be a compact convex set and let Y be a finite codimensional slice of X. If F is a closed face of Y then there is a closed face F' of X such that $F' \cap Y = F$.*

It follows from this that if F is a closed face of a β-polytope X (derived from the simplex S) then $[F]$ is weak* closed in $A(S)^*$ and consequently X has the extension property, a fact which is also proved in [19].

THEOREM 4.8. *If X is a slice of codimension one of the simplex S then X belongs to the class $\mathscr{P}$.*

Proof. Let $X = S \cap g^{-1}(0)$ with $g \in A(S)$ and let F be a closed face of X with $N = [F]$. Repeating the set-up of Proposition 4.6 let q be the restriction of $\bar{q}$: $A(S)^* \to A(S)^*/N$ to S. Then $qX = qS \cap g'^{-1}(0)$ where $g = g' \circ q$. We have $qF = \{0\}$ is an

extreme point of qX contained in the facial line segment $[qx, qy]$ of qS and $G = q^{-1}([qx, qy])$ is a closed face of S such that $X \cap G = F$. Since $[G]$ is weak* closed and $N = [G] \cap g^{-1}(0)$ we have N is weak* closed. To show X is conical at F we must show qX is conical at 0. We will prove qS is again a simplex and then the fact that qX is conical at 0 will follow from Theorem 4.4. Let G' be the complementary face of G. Then $[G] \cap [G'] = \{0\}$. Thus q must be one-to-one on G'. Since G' is a face of S, G' is a (not necessarily compact) simplex and hence so is qG'. On the other hand qG is a line segment and thus is also a simplex. If $x \in S$ then $x = \lambda x_1 + (1-\lambda)x_2$; $x_1 \in G$, $x_2 \in G'$ and $0 \leqq \lambda \leqq 1$. Then $qx = \lambda qx_1 + (1-\lambda)qx_2$ and so $qS = \operatorname{conv}(qG \cup qG')$. We claim further that this representation of qx is unique. For suppose $qx = \mu qy_1 + (1-\mu)qy_2$. Then $(1-\mu)y_2 - (1-\lambda)x_2 = \lambda x_1 - \mu y_1 + z$ where $z \in N \subset [G]$. Thus the right hand side of this equation is in $[G]$ and the left is in $[G']$. Hence both sides are zero. By applying the function $1 \in A(S)$ we get $\lambda = \mu$ and $y_2 = x_2$. This proves the uniqueness of the representation of qx. If P_1 is a lattice ordered cone with base qG and P_2 a lattice ordered cone with base qG' then $P_1 \times P_2$ is again a lattice in the product ordering and $\operatorname{conv}[(qG \times \{0\}) \cup (\{0\} \times qG')]$ is a base for $P_1 \times P_2$ and hence is a simplex. But the uniqueness of convex combinations in $qS = \operatorname{conv}(qG \cup qG')$ yields a one-to-one affine correspondence between qS and the base of $P_1 \times P_2$. Thus qS is a simplex and this completes the proof.

We conclude our discussion of polytopes with an example of a compact convex set which is in $\mathscr{P}$ but is neither an α- nor a β-polytope.

THEOREM 4.9. *Let B be the unit ball of l^1 with the weak* topology as the dual of c_0. Then B is a member of $\mathscr{P}$.*

Proof. Let $S = \{x \in B : x \geqq 0 \text{ and } \|x\| = 1\}$. Then both S and $-S$ are (nonclosed) simplicial faces of B and $B = \operatorname{conv}(S \cup -S)$. Let F be a face of B which is contained in S. Then F is closed in B if and only if F is of the form $\operatorname{conv}\{\delta_{n_1}, \ldots, \delta_{n_k}\}$ where δ_n is the sequence with 1 in the nth place and zeros elsewhere. The if part of this assertion is clear. On the other hand suppose F is closed and let $A = \{n \in N : \delta_n \in F\}$. Then F consists exactly of those elements of S whose coordinates are zero on $N \backslash A$. But if A is infinite then 0 would be a limit point of the set $\{\delta_n\}_{n \in A}$. Now, if F is any proper closed face of B then $F = \operatorname{conv}(F_1 \cup F_2)$ where $F_1 = F \cap S$ and $F_2 = -(F \cap -S)$. Since $0 \notin F$, F_1 and F_2 are closed faces contained in S and $F_1 \cap F_2 = \varnothing$. Thus F is finite dimensional and hence spans a closed subspace in $A(B)^*$. Let f_i $(i = 1, 2)$ be the function associated with F_i in S and extend f_i to B by setting $f_i(0) = 1$. If $F_i = \varnothing$ then set $f_i \equiv 1$ on B. Let $f = 1 + f_1 - f_2$. Using the fact that $f_i(-x) + f_i(x) = 2$ it can easily be checked that $f \equiv 0$ on F_1 and $-F_2$ and hence on F. Also $f \leqq 2$ on B. Moreover $f \geqq 1$ on F_1' (the complementary face of F_1 in S) and $f \geqq 1$ on $-F_2'$. Any $x \in B$ can be written as a convex combination of y and z, where $y \in F$ and $z \in \operatorname{conv}(F_1' \cup -F_2')$. But then $f(z) \geqq 1$ so that

$$B = \operatorname{conv}(F \cup \{x \in B : f(x) \geqq 1\}).$$

Hence B is conical at F.

It is shown in [19] that no polytope (α or β) can be centrally symmetric (and infinite dimensional). Consequently B is not a polytope.

We show next that if A is a (complex-valued) function algebra on a compact Hausdorff space Y and X is the set of probability functionals over A then X is 1-conical at each extreme point. This means in particular that if $x \in \operatorname{ext} X$ then $X-x$ is a universal cap of the cone $P=\bigcup_{n=1}^{\infty} n(X-x)$.

THEOREM 4.10. *Let A be a function algebra (Banach subalgebra of $C(Y)$ containing the constants and separating points) and let $X=\{x \in A^* : \|x\|=1=x(1)\}$ in the weak* topology. Then X is 1-conical at each extreme point.*

Proof. Let us assume for convenience that X has been translated so that 0 is an extreme point and let p be the Minkowski functional of X. To show X is 1-conical at 0 we must show that $Z=\{z \in X : p(z)=1\}$ is convex. Suppose $z_1, \ldots, z_n \in Z$. We wish to show first that given $\varepsilon>0$ there exists $h \in A(X)$ such that $\|h\| \leqq 1$, $h(0)>1-\varepsilon$ and $|h(z_i)|<\varepsilon$; $i=1, \ldots, n$. Let μ_i be probability measures supported by $(\operatorname{ext} X)^-$ which represent z_i. From Lemma 3.4 we have $\mu_i(\{0\})=0$. Thus let N be a compact neighborhood of 0 such that $\mu_i(N)<\varepsilon/2$ for $i=1, \ldots, n$. From a theorem of Bishop and de Leeuw [7] (see also [18, Chapter 8]) given any extreme point $x \in X$, a neighborhood U of x and $\varepsilon>0$ there is a function $f \in A$ such that $\|f\| \leqq 1$, $|f(x)|>1-\varepsilon$ and $|f(y)| \leqq \varepsilon$ for all $y \in (\operatorname{ext} X) \backslash U$. By multiplying by an appropriate constant and taking the real part we can assume the existence of a function $h \in A(X)$ such that $h \leqq 1$, $h(0)>1-\varepsilon$ and $|h(y)| \leqq \varepsilon/2$ for all $y \in (\operatorname{ext} X) \backslash N$. If $y \in (\operatorname{ext} X)^- \backslash N$ then $y \in (\operatorname{ext} X \backslash N)^-$ so that $|h(y)| \leqq \varepsilon/2$. Then

$$\begin{aligned} |h(z_i)| = |\mu_i(h)| \leqq \mu_i(|h|) &\leqq \int_N |h| \, d\mu_i + \int_{X \backslash N} |h| \, d\mu_i \\ &< \varepsilon/2+\varepsilon/2 = \varepsilon. \end{aligned}$$

Suppose now $x, y \in Z$ and $z=\lambda x+(1-\lambda) y$ $(0<\lambda<1)$ and $p(z)=a$, where $0<a<1$. Then $z/a \in Z$. Let $\varepsilon=(1-a)/2$ and by the above let $h \in A(X)$, $\|h\| \leqq 1$, $h(0)>1-\varepsilon$ and $|h(x)|$, $|h(y)|$, $|h(z/a)|<\varepsilon$. Thus $|h(z)| \leqq \lambda|h(x)|+(1-\lambda)|h(y)|<\varepsilon$. On the other hand $h(z)=(1-a)h(0)+ah(z/a)>(1-a)(1-\varepsilon)-a\varepsilon=1-a-\varepsilon=\varepsilon$ which is a contradiction and the proof is complete.

We conclude with an example of a somewhat different kind. Let X be a compact convex set and let $0 \in \operatorname{ext} X$. The condition that X be conical at 0 means that there is an affine function f peaking at 0 which satisfies certain algebraic conditions but has no topological restrictions. One is led to inquire whether these algebraic conditions can be weakened in favor of some topological requirements and still have 0 a peak-point of X. We give an example of a case where there is a lower-semicontinuous function on X which "peaks" at 0 but 0 is not a peak-point of any continuous affine function on X.

THEOREM 4.11. *There exists a compact, convex, metrizable set X with 0 an extreme point of X and a bounded lsc affine function f on X such that $f(0)=0$ and $f>0$ on $X \backslash \{0\}$ but 0 is not a peak-point of X.*

Proof. Let P be the positive cone of l^1 and let $C=\{x \in P : \|x\| \leqq 1\}$. Let $P'=P-(1/2, 1/4, \ldots, 1/2^n, \ldots)$ and let $C'=\{x \in P' : \sum x_n \leqq 1\}$. Let

$$X_n = \left\{x \in C' : \sum_{k=1}^{n} (x_k-1)^2 \leqq n\right\}$$

and let $X=\bigcap_{n=1}^{\infty} X_n$. In the weak* topology C' is compact and X_n is a closed convex subset of C'. Thus X is compact, convex and $C \subset X \subset C'$. Let $f(x)=\sum_{n=1}^{\infty} x_n$ for $x \in X$. Then f is lsc on C' and hence on X. Also $f \leqq 1$ on X and $f(0)=0$. If $x \in X \backslash \{0\}$ let $x_N \neq 0$ and let $n \geqq N$. Then

$$\sum_{k=1}^{n} (x_k-1)^2 = \sum_{k=1}^{n} x_k^2 - 2\sum_{k=1}^{n} x_k + n \leqq n$$

so that

$$\sum_{k=1}^{n} x_k \geqq \frac{1}{2}\sum_{k=1}^{n} x_k^2 \geqq x_N^2/2 > 0.$$

Thus $f(x) \geqq x_N^2/2 > 0$. Suppose now g is continuous on X and $g(0)=0$. Then $g|_C$ is continuous on C and hence must be in c_0. Suppose $g \geqq 0$ on X and choose positive integers N and M such that $g_N > g_M \geqq 0$. Let $Y=\{x \in X : x_n=0 \text{ for all } n \neq N, M\}$. Then looking at the two dimensional (M, N)-plane we see that Y contains an interval of the circle $(x_N-1)^2+(x_M-1)^2=2$ about 0 and the only way 0 could be a peak-point of Y would be if $g_N=g_M$. Thus $g(x)<0$ for some $x \in Y \subset X$ and so 0 is not a peak-point of X.

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