PIERCING POINTS OF CRUMPLED CUBES

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McMillan [20] proved that a free 2-sphere in $S^3$ can be pierced by a tame arc at each of its points. Since each complementary domain of a free 2-sphere is an open 3-cell [22], it seems natural to attempt to prove some theorem analogous to McMillan’s with this weaker hypothesis. We show that a crumpled cube $C$ in $S^3$ has at most one nonpiercing point if $\text{Int} \ C$ is an open 3-cell. A crumpled cube $C$ is the union of a 2-sphere and one of its complementary domains in $S^3$, and a point $p$ of $\text{Bd} \ C$ is a piercing point of $C$ if there is a homeomorphism $h$ of $C$ into $S^3$ such that $h(\text{Bd} \ C)$ can be pierced by a tame arc at $h(p)$. It follows that a 2-sphere $S$ in $S^3$ has at most two points where $S$ cannot be pierced by a tame arc if each component of $S^3 - S$ is an open 3-cell (Corollary 3). McMillan [21] obtained the same result independently using an entirely different approach. Our proof follows from Lemmas 1–3 and the main result of [8].

Other results follow from the methods used in the proofs of Lemmas 1–3. For example, if a Cantor set $W$ lies in a 2-sphere $S$ in $S^3$ such that each component of $S^3 - S$ is an open 3-cell, then $W$ is tame (Theorem 3). Thus each Cantor set on a free 2-sphere is tame. We also show that a continuum $F$ is cellular if $F$ lies in the boundary $S$ of a cellular 3-cell and $F$ does not separate $S$ (Theorem 4).

We also obtain two characterizations of piercing points of crumpled cubes. One that is useful in this paper is that a point $p$ in the boundary $S$ of a crumpled cube $C$ is a piercing point of $C$ if and only if Property $(\ast, p, \text{Int} \ C)$ is satisfied (Theorem 2). If $F$ is a closed subset of $S$, $(\ast, F, \text{Int} \ C)$ is defined to mean that Bing’s Side Approximation Theorem [6] can be applied relative to $S$ and $\text{Int} \ C$ in such a way that the intersection with $S$ of the polyhedral approximation to $S$ lies in the union of a finite set of mutually disjoint small disks in $S - F$. A precise definition can be found in [11] or [15]. The other characterization of piercing points (see Corollary 4) follows as a consequence of this one and results from [17].

The references should be consulted for needed definitions.

I. Piercing points of the closure of an open 3-cell.

Lemma 1. If $\epsilon > 0$ and $p$ and $q$ are points in the boundary $S$ of a crumpled cube $C$ in $S^3$, then there exist a crumpled cube $M$, an $\epsilon$-homeomorphism $h$ taking $S$ onto $\text{Bd} \ M$, and a Sierpinski curve $X$ such that

1. $p$ and $q$ are inaccessible points of $X$,
2. $X \subset S \cap \text{Bd} \ M$,
(3) $\text{Bd } M$ is locally tame modulo $\{p, q\}$.
(4) $M$ lies in an $\varepsilon$-neighborhood of $C$.
(5) $h$ is the identity on $X$, and
(6) each component of $S - X$ has diameter less than $\varepsilon$.

**Proof.** We shall use the technique introduced by Martin [18]. Let $D$ be a disk on $S$ such that $p \in \text{Int } D$, $q \in S - D$, and $\text{Bd } D$ is tame [4], and let $J_1, J_2, \ldots$ be a sequence of tame simple closed curves in $D$ such that $\text{Bd } J_1 = \text{Bd } D$ and if $D_1, D_2, \ldots$ are the disks on $D$ bounded by $J_1, J_2, \ldots$, respectively, then $D_{i+1} \subseteq D_i$ and $p = \bigcap D_i$. Using repeatedly the results from [6] and [11] together with the techniques of [4], we obtain a collection of tame annuli $A_1, A_2, \ldots$ such that

$$\text{Bd } A_i = \text{Bd } D_i \cup \text{Bd } D_{i+1},$$

$A_i \cap D_i$ contains a Sierpinski curve $X_i$ in $D_i - \text{Int } D_{i+1}$,

$A_i \cap \text{Int } A_i = \emptyset$ if $i \neq j$,

$\text{Cl } \left( \bigcup A_i \right)$ is a disk $E$ with boundary $J_i$,

$\text{Bd } M = (S - D) \cup E$, and each component of $A_i - X_i$ has diameter less than $\varepsilon/i$.

The procedure for obtaining the annuli $A_i$ is given roughly by Martin in [18], so we do not pursue the details.

The same procedure as outlined above relative to $q$ and the disk $S - \text{Int } D$ yields a sequence of Sierpinski curves $Y_i$ such that $\{p, q\} \cup \left( \bigcup X_i \right) \cup \left( \bigcup Y_i \right)$ is the Sierpinski curve $X$ required in Lemma 1.

**Lemma 2.** If the closure of $S^3 - C$ is a 3-cell, then, in addition to the conditions in Lemma 1, $M$ and $X$ can be selected such that

(7) $C \subseteq M$,

(8) $\text{Bd } M \cap S = X$, and

(9) the closure of each component of $M - C$ is a 3-cell.

**Proof.** We use Lemma 1 to obtain a Sierpinski curve $X$ such that $X$ contains $p$ and $q$ inaccessibly, $X$ is locally tame modulo $\{p, q\}$, and each component of $S - X$ is small. Since $S$ is tame from $S^3 - C$ we can obtain a 2-sphere $\text{Bd } M$ by pushing each component of $S - X$ slightly into $S^3 - C$. Since $\text{Bd } M$ is locally tame modulo $X$, we see that $\text{Bd } M$ is locally tame modulo $\{p, q\}$ [5]. If we identify $M$ as the crumpled cube containing $C$ and bounded by $\text{Bd } M$, conditions (7) and (8) follow. If $Z$ is the closure of a component of $M - C$, then $\text{Bd } Z$ is a 2-sphere that is locally tame from $\text{Int } Z$ modulo a tame simple closed curve in $X$. It follows that $\text{Bd } Z$ is tame from $\text{Int } Z$ (see [11, Theorem 2] and [15, Theorem 14]), so $Z$ is a 3-cell.

**Lemma 3.** Suppose $K$ is a crumpled cube in $S^3$ such that $S^3 - K$ is an open 3-cell. If $K_1, K_2, \ldots$ is a sequence of mutually disjoint 3-cells in $K$ such that, for each $i$, $K_i \cap \text{Bd } K$ is a disk $D_i$, then $S^3 - \text{Cl } \left( K - \bigcup_{i=1}^n K_i \right)$ is an open 3-cell.

**Proof.** For each integer $n$ we let $M_n = \text{Cl } \left( K - \bigcup_{i=1}^n K_i \right)$ and $V_n = S^3 - M_n$. The theorem will follow from [7] once we show that each $V_n$ is an open 3-cell or equivalently that each $M_n$ is cellular. In the remainder of the proof we show that
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$M = M_1$ is cellular. Since the $K_i$ are disjoint the same procedure can be used inductively to show the cellularity of each $M_n$.

Let $\alpha > 0$ and let $A$ be a polyhedral arc in $S^3 - K_1$ from a point $k$ in $\text{Int} \ K - K_i$ to a point $m$ in $S^3 - K$. Since $K_1$ is a 3-cell, there exist disks $D$ and $E$ such that $D \subset \text{Int} \ D_1$, $\text{Bd} \ D$ is tame, $\text{Bd} \ D = \text{Bd} \ E$, $E - \text{Bd} \ E$ lies in $\text{Int} \ K_1$, and $E$ is homeomorphically within $\alpha/2$ of $\text{Bd} \ K_1 - \text{Int} \ D_1$. Using Bing's Side Approximation Theorem [6] relative to the open set $\text{Bd} \ K - D$ in $\text{Bd} \ K$, we obtain an annulus $F$ such that $\text{Bd} \ D \subset \text{Bd} \ F$, $\text{Bd} \ F - \text{Bd} \ D \subset S^3 - K$, $F \cap (A \cup M \cup \text{Int} \ E \cup \text{Int} \ D) = \varnothing$, $F$ lies within $\alpha/2$ of $M$, and $F$ is locally polyhedral modulo $\text{Bd} \ D$. Without loss in generality we may assume that $G = F \cup E$ is a polyhedral disk near $M$ [10].

Since $S^3 - K$ is an open 3-cell, there is a 2-sphere $R$ within $\alpha$ of $\text{Bd} \ K$ such that $R$ separates the two boundary components of $F$, $R$ separates $k$ from $m$, and $R$ lies in $S^3 - K$. We assume that $R$ is polyhedral [1] and that $F$ and $R$ are in general position. Now we assume that $A$ and $R$ are in general position (i.e., $R \cap A$ consists of a finite number of points where $A$ pierces $R$), and we choose a component $T$ of $R - F$ such that $T \cap A$ consists of an odd number of points. Then $T$ is a disk with holes lying within $\alpha$ of $M$. We fill these holes with disks near $G$ to obtain a 2-sphere $W$ that lies within $\alpha$ of $M$. Since $A \cap W = A \cap T$, it follows that $W$ separates $k$ from $m$. This means that $W$ bounds a 3-cell $X$ such that $M = \text{Int} \ X$ and $X$ lies within $\alpha$ of $M$.

**THEOREM 1.** If $C$ is a crumpled cube in $S^3$ such that $\text{Int} \ C$ is an open 3-cell, then $C$ has at most one nonpiercing point.

**Proof.** Since there is a homeomorphism $h$ of $C$ into $S^3$ such that the closure of $S^3 - h(C)$ is a 3-cell [12], [13] and $p$ is a piercing point of $C$ if and only if $p$ is a piercing point of $h(C)$, we assume without loss in generality that the closure of $S^3 - C$ is a 3-cell $K$. Let $p$ and $q$ be two points of $\text{Bd} \ C$. We shall show that one of these two points must be a piercing point of $C$. It follows from Lemma 2 that there exists a 2-sphere $S'$, a Sierpinski curve $X$, and a crumpled cube $M$ such that $M \subset K$, $\text{Bd} \ M = S'$, $p$ and $q$ are inaccessible points of $X$, $S' \cap \text{Bd} \ C = X$, $S'$ is locally tame modulo $\{p, q\}$, and the closure of each component of $K - M$ is a 3-cell. From Lemma 3 we see that $S^3 - M$ is an open 3-cell. Since $K$ is a 3-cell it follows that $\text{Bd} \ M$ is locally tame from $\text{Int} \ M$ at both $p$ and $q$ (in fact, $M$ is a 3-cell). For more detail, see the proof of Lemma 5 to follow.

It follows from [8, Theorem 1] that $\text{Bd} \ M$ is also locally tame from $S^3 - M$ at one of the points $p$ and $q$, say $p$. Then $p$ lies in a tame arc in $X$, so $p$ is a point at which $\text{Bd} \ C$ can be pierced by a tame arc [11]. This means that $p$ is a piercing point of $C$.

Property $(\ast, p, \text{Int} \ C)$, which is used in the following theorem, was defined roughly in the introduction and can be found in either [11] or [15].

**THEOREM 2.** A point $p$ in the boundary of a crumpled cube $C$ is a piercing point of $C$ if and only if $(\ast, p, \text{Int} \ C)$ is satisfied.
Proof. If \( p \) is a piercing point of \( C \), there exists a homeomorphism \( h \) of \( C \) into \( S^3 \) such that \( h(S) \) can be pierced by a tame arc at \( h(p) \). According to Gillman [11] this means that \( h(p) \) lies in a tame arc \( A \) in \( h(S) \). Furthermore Gillman [11] proved that \((*, A, h(\text{Int } C))\) is satisfied since \( A \) is tame. Lister [14] showed that \((*, h^{-1}(A), \text{Int } C)\) follows since \( h \) is a homeomorphism. Of course this implies \((*, p, \text{Int } C)\).

The other half of the proof of Theorem 2 is merely a rearrangement of the same ideas.

The following result is a consequence of Theorem 2 and a result by Martin [19].

**Corollary 1.** If \( p \) is a point in a 2-sphere \( S \) in \( S^3 \) and \( U \) and \( V \) are the components of \( S^3 - S \), then either \((*, p, U)\) or \((*, p, V)\) is satisfied.

**Corollary 2.** If \( C \) and \( L \) are the crumpled cubes bounded by a 2-sphere \( S \) in \( S^3 \) and a point \( p \in S \) is a piercing point of both \( C \) and \( L \), then \( S \) can be pierced by a tame arc at \( p \).

**Proof.** It follows from Theorem 2 that both \((*, p, \text{Int } C)\) and \((*, p, \text{Int } L)\) are satisfied. This means that \( S \) can be pierced by a tame arc at \( p \) [11].

**Corollary 3.** If each complementary domain of a 2-sphere \( S \) in \( S^3 \) is an open 3-cell, then \( S \) contains two points \( p \) and \( q \) such that \( S \) can be pierced by a tame arc at each point of \( S - \{p, q\} \).

**Corollary 4.** A point \( p \) in the boundary of a crumpled cube \( C \) is a piercing point of \( C \) if and only if \( p \) lies in an arc \( A \) in \( \text{Bd } C \) such that for each \( \varepsilon > 0 \) there is a positive number \( \delta \) such that each unknotted simple closed curve that lies in \( \text{Int } C \) and has diameter less than \( \delta \) can be shrunk to a point in an \( \varepsilon \)-subset of \( S^3 - A \).

**Proof.** From Theorem 2 we see that \((*, p, \text{Int } C)\) is satisfied if \( p \) is a piercing point of \( C \). Then \( p \) lies in an arc \( A \) satisfying the conditions of Corollary 4 (see Theorem 1 in [17] and the remark on p. 511 in [15]).

If there exists an arc \( A \) as in the statement of Theorem 2, then \((*, A, \text{Int } C)\) follows (see the remark prior to the statement of Theorem 2 in [17]). Thus it follows from Theorem 2 that \( p \) is a piercing point of \( C \).

II. Certain Cantor sets are tame. Using the methods of §I, we show that Cantor set \( W \) is tame if \( W \) lies in a 2-sphere \( S \) such that each component of \( S^3 - S \) is an open 3-cell.

**Lemma 4.** If \( \varepsilon > 0 \) and \( W \) is a closed 0-dimensional subset of a 2-sphere \( S \) in \( S^3 \), then there exists a Sierpinski curve \( X \) in \( S \) such that

1. \( W \) lies inaccessibly in \( X \),
2. \( X \) is locally tame modulo \( W \), and
3. each component of \( S - X \) has diameter less than \( \varepsilon \).

**Proof.** The proof is much the same as that given for Lemma 1. We let \( B_1 \) be a finite collection of mutually disjoint disks \( D_{11}, D_{12}, \ldots, D_{1n_1} \) in \( S \) such that each
Bd $D_{1j}$ is tame, $W \subseteq \bigcap \text{Int } D_{1j}$, and diam $D_{1j} < 1$. We inductively define, for each positive integer $n$, a similar finite collection $B_n$ of disjoint disks each of diameter less than $1/n$. If we denote the union of the disks in $B_n$ by $B_n^*$, then we insist that $B_n^*$ is in the interior of $B_{n-1}^*$ and that each point of $W$ is a component of $\bigcap_{n=1}^\infty B_n^*$.

Now we apply the procedure outlined in the proof of Lemma 1. First we obtain a tame Sierpinski curve $X_1$ in $S - \bigcup \text{Int } D_{1i}$, such that each component of $S - X_1$ has diameter less than $\varepsilon$. Then $X_2$ is a finite collection of tame Sierpinski curves each in the closure of a component of $B_n^* - B_{n+1}^*$ and having small holes. This process is continued so that, for each $n$, we obtain a finite collection $X_n$ of tame Sierpinski curves in $X_n$ by $X_n^*$, then $(\bigcup X_n^*) \cup W$ is the desired Sierpinski curve $X$.

**Lemma 5.** If $W$ is a closed 0-dimensional subset of the boundary $S$ of a cellular 3-cell $C$ in $S^3$ and $X$ is a Sierpinski curve in $S$ containing $W$ such that $X$ is locally tame modulo $W$, then there exists a cellular 3-cell $M$ such that

1. Bd $M$ is locally tame modulo a point of $W$,
2. $X = \text{Bd } M \cap S$, and
3. $M \subseteq C$.

**Proof.** The proof of Lemma 2 shows how to construct a crumpled cube $M$ satisfying conditions (1) through (3), and it follows from Lemma 3 that $M$ is cellular. Since $M \subseteq C$ and Int $C$ is 1-ULC it is not difficult to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each simple closed curve lying in Int $M$ and having diameter less than $\delta$ can be shrunk to a point in an $\varepsilon$-subset of Int $C$. This implies Property $(A, X, \text{Int } M)$, defined in [15], which is equivalent to $(\ast, X, \text{Int } M)$ [15, Theorems 8–10]. Hence $M$ is a 3-cell [15, Theorem 14]. It follows from [8] that Bd $M$ is locally tame modulo a point of $W$.

**Theorem 3.** If $W$ is a closed 0-dimensional subset of a 2-sphere $S$ in $S^3$ and each component of $S^3 - S$ is an open 3-cell, then $W$ is tame.

**Proof.** Let $U$ and $V$ be the components of $S^3 - S$, and let $X$ be a Sierpinski curve satisfying the conditions of Lemma 4 relative to some $\varepsilon > 0$. There is a homeomorphism $h$ of $S \cup V$ into $S^3$ such that $S^3 - h(V)$ is a cellular 3-cell [12], [13]. Since $h(X)$ is locally tame modulo $h(W)$ it follows from Lemma 5 that $h(X)$ is locally tame modulo a point $h(p) \in h(W)$. Each Sierpinski curve $h(Y)$ in $h(X) - h(p)$ satisfies $(\ast, h(Y), h(V))$ since $h(Y)$ is tame [11], so it follows from a result by Lister [14] that $(\ast, Y, V)$ is satisfied.

Applying the same argument, where $f$ is a homeomorphism of $S \cup U$ into $S^3$, we obtain a point $q \in W$ such that each Sierpinski curve $Y$ in $X - \{p, q\}$ satisfies both $(\ast, Y, V)$ and $(\ast, Y, U)$. Then $Y$ is tame [15]. This means that $X$ is locally tame modulo two points, so $W$ is locally tame modulo two points. A theorem proven by Bing [3] shows $W$ to be tame.
Corollary 5. If a closed 0-dimensional set \( W \) lies in the interior of a cellular disk \( D \) in \( S^3 \), then \( W \) is tame.

Proof. For each point \( p \in W \) there exists a disk \( D_p \) in \( \text{Int} \ D \) and a 2-sphere \( S_p \) such that \( p \in \text{Int} \ D_p \subset D_p \subset S_p \) and \( S_p \) is locally tame modulo \( D_p \) [2, Theorem 5]. It follows from the fact that \( D_p \) is cellular [21] that each component of \( S^3 - S_p \) is an open 3-cell [23]. This means that \( W \) is locally tame (Theorem 3), so \( W \) is tame [3].

Corollary 6. If \( W \) is a closed 0-dimensional subset of a free 2-sphere \( S \) in \( S^3 \), then \( W \) lies in a tame Sierpinski curve on \( S \).

Proof. The proof of Theorem 3 shows that \( W \) lies in a Sierpinski curve \( X \) in \( S \) such that \( X \) is locally tame modulo two points (each component of a free 2-sphere must be an open 3-cell [22]). Since these two points lie in tame arcs in \( S \) (see [20, Theorem 5] and [11, Theorem 6]), they each lie in tame arcs in \( X \) [11, Lemma 6.1]. Thus \( X \) is tame [10].

III. Other related results.

Lemma 6. If \( \varepsilon > 0 \) and \( F \) is a continuum in the boundary \( S \) of a crumpled cube \( C \) in \( S^3 \) such that \( F \) does not separate \( S \), then there exists a null sequence of mutually disjoint \( \varepsilon \)-disks \( \{E_i\} \) in \( S - F \) and a crumpled cube \( M \) such that

1. \( F \cup (S - \bigcup \text{Int} \ E_i) \subset \text{Bd} \ M \),
2. \( \text{Bd} \ M \) is locally tame modulo \( F \), and
3. \( M \) lies in an \( \varepsilon \)-neighborhood of \( C \).

Proof. Since \( F \) does not separate \( S \) there is a sequence \( \{D_i\} \) of disks on \( S \) such that \( F = \bigcap D_i \), \( \text{Bd} \ D_i \) is tame, and \( D_{i+1} \subset \text{Int} \ D_i \). Now we follow the procedure outlined in the proof of Lemma 1.

Lemma 7. If \( \varepsilon > 0 \) and \( F \) is a continuum in the boundary \( S \) of a cellular 3-cell \( C \) in \( S^3 \) such that \( F \) does not separate \( S \), then there exists a cellular 3-cell \( M \) satisfying all the conditions of Lemma 6 and such that \( M \subset C \).

Proof. We use Lemma 6 to obtain the disks \( E_i \) in \( S - F \). Then the construction of \( M \) is indicated in the proofs of Lemmas 2 and 3 where each \( E_i \) is replaced with a tame disk in \( C \). Since \( M \subset C \) and \( C \) is a 3-cell it is easy to see that \( M \) is also a 3-cell (see the proof of Lemma 5).

Theorem 4. If \( F \) is a continuum on the boundary \( S \) of a cellular 3-cell in \( S^3 \) such that \( F \) does not separate \( S \), then \( F \) is cellular.

Proof. It follows from Lemma 7 that there exists a cellular 3-cell \( M \) such that \( \text{Bd} \ M \) is locally tame modulo \( F \). Then Theorem 5.2 of [23] insures that \( F \) is cellular.

Remark. The hypothesis that \( S \) is the boundary of a 3-cell in the previous theorem cannot be removed. Furthermore, Theorem 4 becomes false if we require
only that each component of \( S^3 - S \) be an open 3-cell. For example, one can grow two of the "feelers" described in [9] into opposite complementary domains of a sphere and let \( F \) be any arc containing the two wild points of the resulting sphere. It then follows from [16] that \( F \) is not cellular. However, \( F \) ought to be cellular if each component of \( S^3 - S \) is an open 3-cell and \( F \) contains at most one (of the two possible) points where \( S \) cannot be pierced by a tame arc. Theorem 5 is a special case of this conjecture.

**Theorem 5.** If \( S \) is a 2-sphere in \( S^3 \) that is locally tame modulo a 0-dimensional set, \( F \) is a subcontinuum of \( S \) that does not separate \( S \), each component of \( S^3 - S \) is an open 3-cell, and \( F \) contains at most one of the two possible points where \( S \) cannot be pierced by a tame arc, then \( F \) is cellular.

**Proof.** Under the conditions of the hypothesis \( S \) can have at most two wild points [8] and \( F \) can contain at most one of these points. Then there exists a disk \( D \) on \( S \) and a point \( p \in F \) such that \( F \subseteq \text{Int} \ D \) and \( S \) is locally tame at each point of \( D - p \). It follows from Corollary 1 and Theorem 14 of [15] that \( S \) is locally tame from one component \( V \) of \( S^3 - S \) at \( p \). If \( C = S \cup V \), then \( C \) is a cellular crumpled cube. Now we are able to use the technique in the proofs of Lemmas 6 and 7 to obtain a cellular 3-cell \( M \) such that \( M \subseteq C, \ D \subseteq \text{Bd} \ M, \) and \( \text{Bd} \ M \) is locally tame modulo \( D \). Then \( \text{Bd} \ M \) is locally tame modulo \( F \) [10], and Theorem 5 follows from Theorem 4.

**References**

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