ON THE HAUPTVERMUTUNG FOR A CLASS OF OPEN MANIFOLDS(1)

BY

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1. Introduction. The Hauptvermutung for PL manifolds is the conjecture that homeomorphic PL manifolds are PL homeomorphic. No counterexample to this conjecture is known. It is known to be true for manifolds of dimension three or less ([13], [14]) and for many high dimensional compact manifolds ([20], [21]), but the only high dimensional result for open (i.e. noncompact with empty boundary) manifolds is that it is true in case the manifolds are topologically $E^n$ ($n \geq 5$) [19].

The main result of this paper is that the conjecture is true whenever the manifolds are topologically $S^n$ minus a nonempty tame [8] compact 0-dimensional subset and $n > 5$. The analogous result in the differentiable category holds if $n = 6, 7$ (Theorem 5.2), but there exist many counterexamples to the complete transfer of the theorem, there is one in dimension 8. We also obtain a connectivity characterization of $S^n$ ($n > 5$) minus a nonempty tame compact 0-dimensional subset (Theorem 4.3).

2. Definitions and basic facts. An end of a manifold $M$ is a function $\varepsilon : \{\text{compact subsets of } M\} \rightarrow \{\text{open subsets of } M\}$ such that

1. $\varepsilon(C)$ is a nonempty component of $M - C$ and
2. $\varepsilon(C_1) \supseteq \varepsilon(C_2)$ whenever $C_1 \subseteq C_2$.

This definition is equivalent to that given by Siebenmann in [16].

Throughout this paper $\mathbb{E}X$, $\mathbb{F}X$, $\mathbb{F}_a(X - A)$ and $\sigma X$ will denote the set of ends of $X$, the set of components of $X$, the set of unbounded components of $X - A$ (i.e. components with noncompact closure in $X$), and the cardinality of $X$ respectively. The following elementary lemma is given without proof.

**Lemma 2.1.** Let $M^n$ ($n \geq 2$) be a compact manifold and let $K$ be a 0-dimensional closed subset of $M$. Then $\sigma \mathbb{E}(M - K) = \sigma K$.

A manifold $M$ is said to be $q$-connected at infinity if and only if given any compact subset $C$ of $M$, there is a compact subset $D$ (depending upon $C$) of $M$ such that $C \subseteq D$ and each component of $M - D$ is $q$-connected. $M$ is $(p, q)$-connected at infinity.

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Throughout this paper $E^n$, $B^n$, and $S^n$ will denote the PL manifolds, Euclidean $n$-space, a standard $n$-simplex, and the boundary of $B^{n+1}$ respectively. A cored $n$-ball $CB^p_n$ of index $k$ ($k$ a positive integer) is a PL $n$-manifold which is PL homeomorphic to $S^n$ minus the interiors of $k$ mutually disjoint PL $n$-balls. $M$ is the monotone union of the submanifolds $M_1$, $M_2$, ... if and only if $M = \bigcup M_i$ and $M_i \subseteq \text{int } M_{i+1}$ for all $i$.

**Theorem 2.2.** If $U$ is an open PL $n$-manifold ($n > 5$) which is the monotone union of cored $n$-balls, then $U$ is PL homeomorphic to $S^n$ minus a nonempty tame compact 0-dimensional subset.

**Remark.** We will not give all the details of the proof of Theorem 2.2. The theorem actually can be proven in every dimension except four (in question here because we use the Hauptvermutung for balls). Edwards [6] gives a proof of the 3-dimensional case. One proves Theorem 2.2 in a straightforward manner using the following three lemmas and a theorem of McMillan [8] which tells us that the subset is tame.

**Lemma 2.3.** Let $h: M \rightarrow S^n$ be a PL embedding where $M$ is a compact connected PL $n$-manifold ($n > 5$) with $\text{Bd } M$ being a disjoint union of $k$ PL $(n-1)$-spheres, $S_1$, $S_2$, ..., $S_k$. Then $\text{Cl } (S^n - h(M))$ is the disjoint union of $k$ PL $n$-balls, $B_1$, ..., $B_k$ with $h(S_i) = \text{Bd } B_i$; and hence $M$ is a cored $n$-ball.

**Proof.** By duality $h(S_i)$ separates $S^n$ into two components. One of these, say $Q_i$, does not intersect $h(M)$. Since $h(S_i)$ is a PL sphere, $\text{Cl } Q_i = B_i$ is a topological ball by applying a theorem of Brown [3]. Obviously $B_i$ is a component of $\text{Cl } (S^n - h(M))$ and hence is a PL manifold by [1]. Since for $n > 5$ the PL Hauptvermutung for balls is true [18], $B_i$ is a PL $n$-ball. The lemma now follows easily.

**Lemma 2.4.** Let $CB^p_i \subseteq \text{int } CB^p_n$ ($n > 5$). Let $S_1$, $S_2$, ..., $S_r$ be the boundary spheres of $CB^p_n$. Then $\text{Cl } (CB^p_n - CB^p_i)$ has $r$ components, $Q_1$, ..., $Q_r$ with

$$Q_i \cap CB^p_n = \text{Bd } Q_i \cap \text{Bd } CB^p_n = S_i$$

and each $Q_i$ is a cored $n$-ball.

**Lemma 2.5.** Let $M$ be a cored $n$-ball ($n > 5$) with boundary spheres $S_1$, $S_2$, ..., $S_k$. Let $h: S_1 \rightarrow S^n$ be a PL embedding such that $\text{Cl } Q$ is a PL n-ball where $Q$ is a component of $S^n - h(S_1)$. Let $M' = \text{Cl } Q$ be a cored n-ball with boundary spheres, $S'_1$, $S'_2$, ..., $S'_k$ and $S'_i = \text{Bd } \text{Cl } Q$. Then $h$ extends to a PL homeomorphism $h': M \rightarrow M'$ such that $h'(S_i) = S'_i$ for $i = 1, ..., k$.

The PL $n$-manifold $M_2$ is said to be obtained from the PL $n$-manifold $M_1$ by **surgery of index $k$** ($0 \leq k < n$) if and only if there are PL embeddings $f_1: S^k \times B^{n-k} \rightarrow M_1$ and $f_2: B^{k+1} \times S^{n-k-1} \rightarrow M_2$ such that

$$M_1 \cap M_2 = M_1 - f_1(S^k \times \text{int } B^{n-k}) = M_2 - f_2(\text{int } B^{k+1} \times S^{n-k-1}),$$

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and

\[ f_1|S^k \times S^{n-k-1} = f_2|S^k \times S^{n-k-1}. \]

**Proposition 2.6.** Let \( f: B^{k+1} \to M^n \) be a proper (i.e. \( f^{-1}(\text{Bd } M) = S^k \)) PL embedding. Then there is a regular neighborhood \( N \) of \( f(B^{k+1}) \) in \( M \) such that

1. \( M' = C_1(M-N) \) is a PL \( n \)-manifold.
2. If \( j < n-k-2 \) and \( M \) is \( j \)-connected, so is \( M' \).
3. \( N \cap \text{Bd } M \approx S^k \times B^{n-k-1} \).
4. \( N \cap \text{Bd } M' \approx B^{k+1} \times S^{n-k-2} \).
5. \( \text{Bd } M' \) is obtained from \( \text{Bd } M \) by surgery of index \( k \).
6. If \( 2k+2 < n \) and \( \lambda \) is the homotopy class of \( f|S^k \) in \( \pi_k(\text{Bd } M) \), then \( \pi_i(\text{Bd } M') \approx \pi_i(\text{Bd } M) \) for \( i < k \) and \( \pi_k(\text{Bd } M') \approx \pi_k(\text{Bd } M)/(\lambda) \) where \( \lambda \) is a subgroup containing \( \lambda \).

**Proof.** Let \( N \) be a regular neighborhood of \( f(B^{k+1}) \) in \( M \) such that \( N \cap \text{Bd } M \) is a regular neighborhood of \( f(S^k) \) in \( \text{Bd } M \). By [15], there is a block bundle \( \xi \) over \( f(B^{k+1}) \) whose total space is \( N \) and whose restriction to \( f(S^k) \) has total space \( N \cap \text{Bd } M \). Also by [15], since \( f(B^{k+1}) \) is collapsible, \( \xi \) and \( \xi f(S^k) \) are product bundles. Hence, there is a PL homeomorphism \( h: B^{k+1} \times B^{n-k-1} \to N \) such that \( h(x,0)=f(x) \) for all \( x \in B^{k+1} \), and \( h(S^k \times B^{n-k-1}) = N \cap \text{Bd } M \). Hence (3) is satisfied, and by [1], it follows that (1) is satisfied. Now \( M' \) and \( M-f(B^{k+1}) \) have the same homotopy type and hence (2) follows easily using standard general position techniques [23]. An elementary point-set argument yields (4), and (5) follows. (6) is an immediate consequence of a theorem of Milnor [10].

The following elementary lemmas are given without proof.

**Lemma 2.7.** The complement of a compact subpolyhedron in a connected polyhedron has only finitely many components.

**Lemma 2.8.** Let \( C \) be a compact subset of a connected PL \( n \)-manifold \( M \). Then \( C'=C \cup \bigcup \{ Q : Q \text{ is a bounded component of } M-C \text{ (i.e. } \text{Cl } Q \text{ is compact) is compact and } M-C' \text{ has only finitely many components.} \)

3. Approximating open manifolds by compact submanifolds.

**Proposition 3.1.** Let \( U^n (n>5) \) be an open \((0,1)\)-connected PL manifold. Let \( C \) be a compact subset of \( U \) such that each component of \( U-C \) is \( 1 \)-connected and \( i: C \subset U \) induces onto homomorphisms \( i_*: H_r(C) \to H_r(U) \) for \( r \leq n-2 \).

Then there is a compact connected PL \( n \)-submanifold \( N \) of \( U \) such that

1. \( C \subset \text{Int } N \),
2. \( \sigma \in \text{Bd } N = \sigma \in (U-N) = \sigma \in (U-C) \),
3. each component of \( \text{Bd } N \) and of \( U-N \) is \( 1 \)-connected, and
4. \( i: N \subset U \) induces isomorphisms \( i_*: H_r(N) \to H_r(U) \) for \( r \leq n-3 \).

The proof of this proposition parallels closely that of Proposition 4 of [2].
For $1 \leq k \leq n-3$ let $\mathcal{S}_k$ be the statement that there is a compact PL $n$-submanifold $N_k$ of $U$ such that (1), (2), and (3) of 3.1 hold if $N_k$ replaces $N$, and (4) $i:N_k \hookrightarrow U$ induces isomorphisms $i_*: H_r(N_k) \rightarrow H_r(U)$ for $r \leq k$ and is onto if $k < r \leq n-2$.

Let $\mathcal{S}'_k$ be the statement that if $P$ is a compact subpolyhedron of $U$ with dim $P \leq n-2$ and $N_k$ is as above then there is a compact connected PL $n$-submanifold $N'_k$ of $U$ such that (1), (2), (3) and (4) hold if $N'_k$ replaces $N_k$, $N_k \cup P \subseteq \text{Int } N'_k$, and $(N'_k - N_k)$ intersected with any component of $U-N_k$ is 1-connected.

The proof will proceed through the following steps.

Step 1. Verify $\mathcal{S}'_1$.

Step 2. Verify $\mathcal{S}'_2$.

Step 3. Show that $\mathcal{S}'_k$ implies $\mathcal{S}'_{k'}$.

Step 4. Show that $\mathcal{S}'_k$ implies $\mathcal{S}'_{k+4}$ if $2k \leq n-5$.

Step 5. Show that $\mathcal{S}'_{n-4}$ implies $\mathcal{S}'_{n-3}$.

It is clear that $\mathcal{S}'_{n-3}$ implies the proposition.

Step 1. The validity of $\mathcal{S}'_1$.

Proof. Let $C' = C \cup \bigcup \{Q : Q$ is a bounded component of $U-C\}$. By Lemma 2.8, $C'$ is compact and $U-C'$ is composed of finitely many components, $Q_1, \ldots, Q_r$, which are precisely the unbounded components of $U-C$. Since $C'$ is compact, let $N_0$ be a compact PL $n$-submanifold of $U$ such that $C' \subseteq \text{Int } N_0$. Without loss of generality we assume that $N_0$ is connected; for if not, we can join components of $N_0$ by arcs, take regular neighborhoods, and let the new $N_0$ be the old $N_0$ union these neighborhoods.

Now let $a$ and $a'$ be the number of components of $\text{Cl}(U-N_0)$ and $U-C'$ respectively. By Lemma 2.7, $a$ is finite; and since $C' \subseteq \text{Int } N_0$, $a \geq a'$. We must alter $N_0$ so that $a = a'$. But if $a > a'$, then two components $V_1$ and $V_2$ of $\text{Cl}(U-N_0)$ lie in the same component $Q_i$ of $U-C$. Let $\alpha$ be a polygonal arc in $Q_i$ such that $\alpha \cap V_i = \text{Bd } \alpha \cap \text{Bd } V_i = \{x_i\}$ ($i = 1, 2$) and $\text{Int } \alpha \subseteq \text{Int } N_0$. Let $P$ be a regular neighborhood of $\alpha$ in $N_0$ such that $P \cap C' = \emptyset$. Let $N'_0 = \text{Cl}(N_0-P)$. Since an arc cannot separate $N_0$, it now follows that $N'_0$ is a compact connected PL $n$-submanifold of $U$ such that $C' \subseteq \text{Int } N'_0$ and $\text{Cl}(U-N'_0)$ has one less component than $\text{Cl}(U-N_0)$. By finite induction we now assume that $\alpha \in \text{Cl}(U-N'_0) = \alpha \in \text{Cl}(U-C)$. But $\alpha \in \text{Cl}(U-N'_0) = \alpha \in \text{Cl}(U-N_0)$, and hence $N'_0$ almost satisfies (2).

Now let $b$ be the number of components of $\text{Bd } N'_0$ and $a'$ as above. It follows that $b \geq a'$. If $b > a'$, then since $\text{Bd } N'_0 = \text{Bd } \text{Cl}(U-N'_0)$, two components $B_1$ and $B_2$ of $\text{Bd } N'_0$ are components of $\text{Bd } V$ for some component $V$ of $\text{Cl}(U-N'_0)$. Let $\beta$ be a polygonal arc in $V$ such that $\beta \cap \text{Bd } V = \text{Bd } \beta \cap (B_1 \cup B_2) = \{y_1, y_2\}$ with $y_1 \in B_1$. Let $W$ be a regular neighborhood of $\beta$ in $V$. Let $N_0 = N'_0 \cup P$. Since $\beta$ cannot separate $V$, it is clear, by finite induction, that we may now assume that $N_0$ satisfies (1) and (2). Now let $M$ be a component of $\text{Bd } N_0$. Let $Q$ be the component of $U-C$ which contains $M$. Let $\lambda$ be a generator of $\pi_1(M)$. By general position let $f: S^1 \rightarrow M$ be a PL embedding such that $[f] = \lambda([f]$ is the homotopy class of $f$). Since $Q$ is 1-connected extend $f$ to $\tilde{f}: B^2 \rightarrow Q$. By Irwin's embedding theorem we
assume that \( f \) is a PL embedding, and putting \( f \) into general position with respect to \( M \) keeping \( f|S^1 \) fixed, we may assume that \( f(B^2) \cap M \) is a finite number of PL 1-spheres \( S_1, S_2, \ldots, S_r \) with \( S_r = f(S^1) \). Now let \( S_i \) be an innermost embedded 1-sphere. It follows that \( f(f^{-1}(S_i) \cup \text{Int} f^{-1}(S_i)) \) is a PL properly embedded 2-ball \( B^2 \) in \( N_0 \) or in a component of \( \text{Cl}(U - N_0) \). By taking a regular neighborhood of \( B^2 \), it follows from Proposition 2.6 that we can replace \( N_0 \) by a compact connected PL \( n \)-submanifold \( N'_0 \) such that the component \( M' \) of \( \text{Bd} N'_0 \) which replaces \( M \) by surgery satisfies the condition

\[
\pi_1(M') \cong \pi_1(M)/([f|S_1], \ldots, [f|S_r]).
\]

Continuing in this fashion we may assume that

\[
\pi_1(M') \cong \pi_1(M)/([f|S_1], \ldots, [f|S_r]).
\]

Hence we can kill the generator \( \lambda \). Since \( \pi_1(M) \) is finitely generated, we may assume that \( M' \) is 1-connected. By finite induction, we can replace \( N'_0 \) by a compact connected PL \( n \)-submanifold \( N_1 \) such that each component of \( \text{Bd} N_1 \) is 1-connected. Since all alterations occurred in \( U - C \), \( N_1 \) satisfies (1). The dimension of the surgery is low so that \( N_1 \) satisfies (2). The fact that \( U - N_1 \) is 1-connected follows easily by general position or the van Kampen theorem depending upon whether in the inductive stage the new compact submanifold was formed by removing or adding on a regular neighborhood to the old compact submanifold. To verify (4), consider the exact Mayer-Vietoris sequence for the triad \((U; N_1, \text{Cl}(U - N_1))\).

\[
H_1(\text{Bd} N_1) \rightarrow H_1(N_1) \oplus H_1(\text{Cl}(U - N_1)) \rightarrow H_1(U).
\]

Since each component of \( \text{Bd} N_1 \) is 1-connected, it follows that \( i_*: H_1(N_1) \rightarrow H_1(U) \) is an injection. Since \( C \subset \text{Int} N_1 \), using the hypothesis, it follows easily that \( i_* \) is a surjection.

**Step 2. The validity of \( \mathcal{S}_2 \).**

**Proof.** Let \( N_1 \) satisfy \( \mathcal{S}_1 \). We will obtain \( N_2 \) from \( N_1 \). Let \( Q_1, Q_2, \ldots, Q_s \) be the components of \( \text{Cl}(U - N_1) \). Consider the excision map \( e: (\bigcup Q_i, \bigcup \text{Bd} Q_i) \rightarrow (U, N_1) \) and the following commutative diagram with exact rows.

\[
\begin{array}{c}
H_{k+1}(U, N_1) \xrightarrow{\partial_k} H_k(N_1) \xrightarrow{i_k} H_k(U) \xrightarrow{j_k} H_k(U, N_1) \\
\approx \xrightarrow{e} \xrightarrow{i} \xrightarrow{i} \xrightarrow{e} \approx \\
H_{k+1}(\bigcup Q_i, \bigcup \text{Bd} Q_i) \xrightarrow{\partial_k} H_k(\bigcup \text{Bd} Q_i) \xrightarrow{i'_k} H_k(\bigcup Q_i) \xrightarrow{j'_k} H_k(\bigcup Q_i, \bigcup \text{Bd} Q_i)
\end{array}
\]

\( i_2 \) is surjective, so we will alter \( N_1 \) to \( N_2 \) so that \( N_2 \) satisfies \( \mathcal{S}_2 \) by killing \( \ker i_2 \). Let \( 0 \neq x \in \ker i_2 \). Choose \( y \in H_2(U, N_1) \) such that \( \partial_2(y) = x \). Hence \( z = \partial_2^{-1}(y) \) is a nonzero element of \( H_2(\bigcup \text{Bd} Q_i) \) such that \( i(z) = x \). But \( H_2(\bigcup \text{Bd} Q_i) \approx \)
\[ H_2(S^2 \times B^{n-3}) \xrightarrow{\alpha} H_2(N_1) \oplus H_2(P_1) \rightarrow H_2(N_1 \cup P_1) \rightarrow 0. \]

It follows that \( H_2(N_1)(\text{im } \alpha) \cong H_2(N_1 \cup P_1) \), i.e.

\[ H_2(N_1)(\{[f_1]\}) \cong H_2(N_1 \cup P_1). \]

Hence \( H_2(N_2) \cong H_2(N_1)(\{[f_1], \ldots, [f_s]\}) \cong H_2(N_1)(x) \). Since \( \text{ker } i_2 \) is finitely generated, by finite induction, there is a compact PL \( n \)-submanifold \( N_2 \) of \( U \) such that \( i : N_2 \subseteq U \) induces an isomorphism \( i_* : H_2(N_2) \rightarrow H_2(U) \). It can easily be checked that \( N_2 \) satisfies (1), (2), (3), and (4).2

Step 3. \( S_{ik} \) implies \( S_{ik+1} \).

**Proof.** Let \( N_k \) satisfy \( S_{ik} \). Since \( U \) is 1-connected at infinity, let \( C \) be a compact subset of \( U \) such that \( C \supset N_k \cup P \) and each component of \( U - C \) is 1-connected. Let \( N' \) be a compact connected PL \( n \)-submanifold of \( U \) which satisfies \( S_{ik} \) for the compact subset \( C \). Every component of \( \text{Cl}(U - N' \cup P \cup C) \) is a subset of a component of \( U - N_k \) and hence by an argument like that given in Step 1, we may assume that, in addition, \( \partial_{k+2}(y) = x \). (Note: It is here that we use that fact that \( \dim P \leq n-2 \), for recall in Step 1 we join components by arcs; and here we must be sure that the arcs do not intersect \( P \), for otherwise it may be that \( P \notin N' \).) Now let \( N'_k \) be a PL \( n \)-submanifold of \( U \) which satisfies \( S_{ik} \) for the compact set \( N' \). It is easy to check that \( N'_k \) satisfies \( S_{ik+1} \).

Step 4. \( S_{ik} \) implies \( S_{ik+1} \), \( 2 \leq k \leq n-5 \).

**Proof.** Let \( N_k \) satisfy \( S_{ik} \). Roughly, we wish to alter \( N_k \) so as to kill \( i_{k+1} \), where \( i : N_k \subseteq U \) and \( i_{k+1} : H_{k+1}(N_k) \rightarrow H_{k+1}(U) \) without disturbing the other nice properties of \( N_k \). Let \( 0 \neq x \in \ker i_{k+1} \). Choose \( y \in H_{k+2}(U, N_k) \) such that \( \partial_{k+2}(y) = x \). Since \( \dim \text{carrier } y < n-2 \), by Step 3, choose \( N'_k \) which satisfies \( S_{ik} \) for \( N_k \cup \text{carrier } y \).

Let \( Q_1, Q_2, \ldots, Q_s \) and \( Q'_1, Q'_2, \ldots, Q'_s \) be the components of \( \text{Cl}(U - N_k) \) and \( \text{Cl}(U - N'_k) \) respectively. Hence, each \( Q_i \) is 1-connected, \( \text{Bd } X_i = \text{Bd } Q_i \cup \text{Bd } Q'_i \), and \( \text{Bd } Q_i, \text{Bd } Q'_i \) are each 1-connected. Consider the excisions

\[ \varepsilon : (\bigcup Q_i \cup \text{Bd } Q_i) \subseteq (U, N_k), \quad \varepsilon' : (\bigcup Q'_i \cup X_i) \subseteq (U, N'_k) \]
and the following commutative diagram

\[
\begin{array}{rcl}
H_{k+2}(\bigcup X_i, \bigcup Bd Q_i) & \xrightarrow{i_*} & H_{k+2}(U, N_k) \\
\partial & \downarrow & \downarrow \delta \kappa i + 2 \\
H_{k+1}(U Bd Q_i) & \xleftarrow{h_*} & H_{k+2}(U, N_k) \\
& \downarrow k_* & \downarrow \ell_* \\
& \bigcup Q_i, \bigcup X_i & \xrightarrow{e_*} \bigcup B_d ft & \xrightarrow{j_{k+1}} & H_{k+1}(N_{k+1}) \\
\end{array}
\]

where \(i_*, h_*, k_*, l_*, j_{k+1} \) are all induced by inclusions. Since \(N_{k+1} \supset \text{carrier } y\), \(l_*(y) = 0\). Let \(z = i_*(z) = x\) and \(h_*(z) = 0\). Now choose \(w \in H_{k+2}(\bigcup X_i, \bigcup Bd Q_i)\) such that \(\partial(w) = z\). But \(H_{k+1}(\bigcup Bd Q_i) \approx \Sigma H_{k+1}(Bd Q_i)\) and \(H_{k+2}(\bigcup X_i, \bigcup Bd Q_i) \approx \Sigma H_{k+2}(X_i, Bd Q_i)\). Hence we may think of \(z\) and \(w\) as \(z = (z_1, z_2, \ldots, z_s)\) where \(z_i \in H_{k+1}(\bigcup Bd Q_i)\), and \(w = (w_1, w_2, \ldots, w_s)\) where \(w_i \in H_{k+2}(X_i, Bd Q_i)\).

Since each \(X_i\) and \(Bd Q_i\) is 1-connected, \(\pi_1(X_i, Bd Q_i) = 0\). Since \(i_*: H_*(N_k) \rightarrow H_*(U)\) and \(i_*: H_*(N_{k+1}) \rightarrow H_*(U)\) are isomorphisms for \(r \leq k\), so is \(j_*: H_*(N_k) \rightarrow H_*(N_{k+1})\). Hence \(H_*(N_k, N_{k+1}) = 0\) for \(r \leq k\). Using excision it follows that \(H_*(X_i, Bd Q_i) = 0\) for \(1 \leq i \leq s\) and \(r \leq k\). By the relative Hurewicz theorem, \(\pi_1(X_i, Bd Q_i) = 0\) for \(r \leq k\); and hence by Lemma 8 of [2], each \(w_i\) can be represented by an embedded handle \(B_i^{k+2} \times B^{n-k-2}\) meeting \(X_i\) in \(S_i^{k+1} \times B^{n-k-2} \subseteq Bd Q_i\).

Let \(N_{k+1} = N_k \cup \bigcup_{i=1}^s (B_i^{k+2} \times B_i^{n-k-2})\). It follows that \(x = i_*(z) = i_*(\partial w)\) includes trivially in \(H_{k+1}(N_{k+1})\), where we recall that \(x\) is an arbitrary generator of \(\ker i_{k+1}\). The desired result will follow by finite induction if we do not change any of the nice properties of \(N_k\). By finite induction, we may assume that \(N_{k+1} = N_k \cup B^n\) with \(B^n \cap N_k = S^{k+1} \times B^{n-k-2}\). Considering the Mayer-Vietoris sequence for the triad \((N_{k+1}, N_k, B^n)\), one concludes that \(N_k \subseteq N_{k+1}\) induces homology isomorphisms through dimension \(k\), and since \(N_k \subseteq U\) induces homology isomorphisms through dimension \(k\); it follows that \(N_{k+1} \subset U\) induces homology isomorphisms through dimension \(k\). Furthermore, it is easy to see that \(H_{k+1}(N_{k+1})\) is being reduced so that eventually we will kill \(\ker i_{k+1}\). It now follows easily that \(N_{k+1}\) satisfies (1), (2) and (4) of \(\mathcal{S}_{k+1}\). We will check (3). Let \(\tilde{Q}\) be a component of \(Bd N_{k+1}\). It follows that \(\tilde{Q}\) is obtained from a component \(\tilde{Q}\) of \(Bd N_k\) by \((k+1)\)-dimensional surgery. Hence

\[
\tilde{Q}' = (\tilde{Q} - (S^{k+1} \times \text{Int } B^{n-k-2})) \cup (B^{k+2} \times S^{n-k-3}) = C \cup D
\]

where \(C \cap D = S^{k+1} \times S^{n-k-3}\). But \(C\) has the homotopy type of \((\tilde{Q} - S^{k+1})\) and \((k+1) \leq n-4\). Hence, by general position, since \(\tilde{Q}\) is 1-connected, so is \(C\). Since \(\min\{k+1, n-k-3\} \geq 2\), by the van Kampen theorem it now follows that \(\tilde{Q}'\) is
1-connected. Let $Q'$ be a component of $\text{Cl}(U-N_{k+1})$. Let $Q$ be the component of $\text{Cl}(U-N_k)$ such that $Q'=\text{Cl}(Q-(B^{k+2}\times B^{n-k-2}))$. Since $Q$ is 1-connected and $Q'$ has the homotopy type of $Q-B^{k+2}$, by general position $Q'$ is 1-connected.

Step 5. $\mathcal{S}_{n-4}$ implies $\mathcal{S}_{n-3}$.

Proof. Let $N$ satisfy $\mathcal{S}_{n-4}$ and let $V=\text{Cl}(U-N)$. Consider the excision map $\epsilon: (V, \text{Bd } N) \subset (U, N)$ and the following commutative diagram

\[
\begin{array}{c}
H_r(N) \xrightarrow{i_r} H_r(U) \xrightarrow{j_r} H_r(U, N) \xrightarrow{\partial_r} H_{r-1}(U) \\
\uparrow k_r \quad \uparrow l_r \quad \approx \quad \uparrow e_r \quad \uparrow k_{r-1} \quad \uparrow l_{r-1} \\
H_r(\text{Bd } N) \xrightarrow{i'_r} H_r(V) \xrightarrow{j'_r} H_r(V, \text{Bd } N) \xrightarrow{\partial'_r} H_{r-1}(\text{Bd } N) \xrightarrow{i'_{r-1}} H_{r-1}(V). 
\end{array}
\]

For $r \leq n-2$, $i_r$ is onto, and hence $j_r=0$ and $\partial_r$ is injective. For $r-1 \leq n-4$, $i_{r-1}$ is an isomorphism and hence $\partial_{r-1}=0$ for $r \leq n-3$. Hence $H_r(U, N)=0=H_r(V, \text{Bd } N)$ for $r \leq n-3$. Since each component of $\text{Bd } N$ is 1-connected, by duality and the Universal Coefficient theorem, it follows that $H_{n-3}(\text{Bd } N)$ is free. But $j_{n-2}=0$ implies that $\partial'_{n-2}=0$, and hence $\partial'_{n-2}$ is injective. So $H_{n-2}(V, \text{Bd } N)$ and $H_{n-2}(U, N)$ are both free. Now consider the following portion of the above diagram

\[
\begin{array}{c}
0 \to H_{n-2}(U, N) \xrightarrow{\partial} H_{n-3}(N) \xrightarrow{i_*} H_{n-3}(U) \to 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
0 \to H_{n-2}(V, \text{Bd } N) \to H_{n-3}(\text{Bd } N) \to H_{n-3}(V) \to 0.
\end{array}
\]

We wish to kill $\ker i_*$. Since $N$ is compact, let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be a set of generators of $\ker i_*$. Choose $(n-2)$-chains $c_1, \ldots, c_s$ in $U$ such that $\partial(c_i)=\lambda_i$. Let $P=\bigcup \text{carrier } c_i$. By Step 3, let $N'$ satisfy the conclusion of $\mathcal{S}_{n-4}$ relative to $N$ and $P$. Since $N'$ satisfies the same homology conditions as that of $N$, the conclusions we made above for $N$ and $V$ are valid for $N'$ and $V'=\text{Cl}(U-N')$. Now consider the commutative diagram with exact rows

\[
\begin{array}{c}
0 \to H_{n-2}(U, N) \xrightarrow{\partial} H_{n-3}(N) \xrightarrow{i_*} H_{n-3}(U) \to 0 \\
\downarrow h_* \quad \downarrow h_* \quad \downarrow \text{id} \\
0 \to H_{n-2}(U, N') \xrightarrow{\partial'} H_{n-3}(N') \xrightarrow{i'_*} H_{n-3}(U) \to 0.
\end{array}
\]

Since $\text{Int } N'=\bigcup \text{carrier } \lambda_i$, $h_*(\ker i_*)=0$. Hence $h'_*=0$. Let $X=\text{Cl}(N'-N)$. Since $(V, \text{Bd } N) \subset (U, N)$ and $(V, X) \subset (U, N')$ are excisions; it follows that $h: (V, \text{Bd } N) \subset (V, X)$ induces the 0 map $h_*: H_{n-2}(V, \text{Bd } N) \to H_{n-2}(V, X)$. Also $H_r(V, X)=0$ for $r \leq n-3$, and $h_*: H_{n-2}(V, X)$ is free. Now since $H_{n-2}(V, X)$ and $H_{n-2}(V, \text{Bd } N)$ are both free, using the Universal Coefficient theorem, it follows
that $\tilde{h}^* = 0$. Consider the following commutative diagram with exact rows where all vertical homomorphisms are induced by inclusions

$$
\begin{array}{c}
H^{n-2}(V, \text{Bd } N) \leftarrow H^{n-3}(\text{Bd } N) \leftarrow H^{n-3}(V) \leftarrow 0 \\
0 \leftarrow H^{n-2}(V, X) \leftarrow H^{n-3}(X) \leftarrow H^{n-3}(V) \leftarrow 0 \\
H^{n-2}(V', \text{Bd } N') \leftarrow H^{n-3}(\text{Bd } N') \leftarrow H^{n-3}(V') \leftarrow 0.
\end{array}
$$

The zeros on the right are valid by the Universal Coefficient theorem since $H_r(\text{V}, \text{Bd } N) = H_r(\text{V}, X) = H_r(\text{V'}, \text{Bd } N') = 0$ for $r \leq n - 3$. To show that $\delta$ is onto, consider the commutative diagram

$$
\begin{array}{c}
\text{Hom}(\text{H}_{n-2}(V, X), Z) \leftarrow \text{Hom}(\text{H}_{n-3}(X), Z) \\
\rho \quad \delta \\
H^{n-2}(V, X) \leftarrow H^{n-3}(X)
\end{array}
$$

where $\rho$ and $\rho'$ come from the Universal Coefficient theorem. It is not difficult to show that $0 \rightarrow H^{n-2}(V, X) \xrightarrow{\delta} H^{n-3}(X) \rightarrow H^{n-3}(V) \rightarrow 0$ is split exact and hence $\text{Hom}(\partial, 1)$ is onto. But $\rho'$ is onto and hence $\rho \delta$ is onto. But $H^{n-2}(V, X)$ is free, and hence $\rho$ is an isomorphism, and $\delta$ is onto. Using the facts that $H^{n-2}(V, X)$ is free and $\tilde{h}^* = 0$, it is not difficult to define $\alpha: H^{n-2}(V, X) \rightarrow H^{n-3}(X)$ such that $\delta \alpha = 1$ and $\nu \alpha = 0$. Now define $\alpha': H^{n-2}(V', \text{Bd } N') \rightarrow H^{n-3}(\text{Bd } N')$ by $\alpha' = \nu' \alpha(e')^{-1}$. (Note that $\nu'$ is induced by an excision, so that $\alpha'$ is well defined.) One easily checks that $\delta \alpha' = 1$ and $\text{im } \alpha' \subset \nu'(\ker \nu)$. It follows that $\text{im } \alpha'$ is a free direct summand of $H^{n-3}(\text{Bd } N')$. Now let $V_1, \ldots, V_t$ and $V'_1, \ldots, V'_t$ be the components of $V$ and $V'$ respectively such that $V'_i < V_i$. (Recall, $V = \text{Cl } (U - N)$, $V' = \text{Cl } (U - N')$, and $N$ satisfies $\mathcal{F}_{n-4}$ with respect to $N$.) Let $X_i = \text{Cl } (V_i - V'_i)$. Hence $V = \bigcup_{\text{dist}} V_i$, $V' = \bigcup_{\text{dist}} V'_i$, $\text{Bd } N = \text{Bd } V = \bigcup_{\text{dist}} \text{Bd } V_i$, $\text{Bd } N' = \text{Bd } V' = \bigcup_{\text{dist}} \text{Bd } V'_i$, $X = \bigcup_{\text{dist}} X_i$, $\text{Bd } X_i = \text{Bd } V_i \bigcup_{\text{dist}} \text{Bd } V'_i$. Let $\nu_i: \text{Bd } V_i \subset X_i$ and $\nu'_i: \text{Bd } V'_i \subset X_i$. It follows that, for $1 \leq i \leq t$, there is a commutative diagram

$$
\begin{array}{c}
H^{n-2}(V_i, \text{Bd } V'_i) \leftarrow H^{n-3}(\text{Bd } V_i) \leftarrow H^{n-3}(V_i) \\
0 \leftarrow H^{n-2}(V_i, X_i) \leftarrow H^{n-3}(X_i) \leftarrow H^{n-3}(V_i) \\
H^{n-2}(V'_i, \text{Bd } V'_i) \leftarrow H^{n-3}(\text{Bd } V'_i) \leftarrow H^{n-3}(V'_i)
\end{array}
$$
A. J. MACHUSKO, JR.

such that \( \delta_{0} v_{i} = 1, \delta_{0} \alpha_{i} = 0, \delta_{0} \alpha_{i} = 1 \), \( \text{im} \alpha_{i} \subseteq \nu_{i}^{*}(\ker \nu_{i}^{*}) \), and \( \text{im} \alpha_{i} \) is a free direct summand of \( H^{n-3}(\text{Bd} \ V_{i}') \). By Lemma 11 of [2], the duality isomorphism

\[
\psi_{i}: H^{n-3}(\text{Bd} \ V_{i}') \to H_{2}(\text{Bd} \ V_{i}')
\]

sends \( \nu_{i}^{*}(\ker \nu_{i}^{*}) \) onto \( (\ker \nu_{i}^{*})_{2} \). Hence \( A_{i} = \psi_{i}(\text{im} \alpha_{i}) \) is a free direct summand of \( H_{2}(\text{Bd} \ V_{i}') \). Let \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{k} \) be a basis for \( A_{i} \subseteq H_{2}(\text{Bd} \ V_{i}') \). Since each \( \text{Bd} \ V_{i}' \) is 1-connected and \( \dim \text{Bd} \ V_{i}' > 4 \), it follows that each \( \lambda_{i} \) can be represented by a PL embedded 2-sphere \( S_{i} \) such that \( S_{i} \cap S_{kl} = \emptyset \) unless \( j = k \) and \( i = l \). Since \( A_{i} \subseteq (\ker \nu_{i}^{*})_{2} \) we can, using Irwin’s embedding theorem, bound each \( S_{i} \) with a PL 3-ball \( B_{i} \) with \( \text{Int} B_{i} \subseteq \text{Int} S_{i} \) such that \( B_{i} \cap B_{kl} = \emptyset \) unless \( j = k \) and \( i = l \). For each \( B_{i} \), take a regular neighborhood \( M_{i} \) such that \( M_{i} \cap M_{kl} = \emptyset \) unless \( j = k \) and \( i = l \).

Let \( N_{n-3} = \text{Cl}(U' = \bigcup M_{i}) \). We want to show that \( N_{n-3} \) satisfies \( \mathcal{S}_{n-3} \). Clearly (1) is satisfied since \( N_{n-3} \supseteq N \) and \( \text{Int} N \supseteq C \). One easily checks that the deletion of the handles and the surgery is such that, \( \sigma \in \text{Cl}(U - N_{n-3}) = \sigma \in \text{Cl}(U - N') \) and \( \sigma \in \text{Bd} N_{n-3} = \sigma \in \text{Bd} N' \), and so (2) follows.

By finite induction, to show that each component of \( \text{Bd} N_{n-3} \) is 1-connected, it suffices to show that \( \text{Bd} (V_{i}' \cup M_{i}) \) is 1-connected. But \( \text{Bd} V_{i}' \) is 1-connected and \( \text{Bd} (V_{i}' \cup M_{i}) \) is obtained from \( \text{Bd} V_{i}' \) by surgery of index 2. So, using general position and the van Kampen theorem, it follows that \( \text{Bd} (V_{i}' \cup M_{i}) \) is 1-connected. Each component of \( U - N_{n-3} \) is 1-connected, for by finite induction we may assume that such a component is \( V_{i}' \cup M_{i} \) with \( M_{i} \approx B^{3} \times B^{n-3} \) and \( V_{i}' \cap M_{i} \approx S^{2} \times B^{n-3} \), and since \( V_{i}' \) is 1-connected, so is \( V_{i}' \cup M_{i} \) by the van Kampen theorem.

It remains to show (4); that \( i: N_{n-3} \subseteq U \) induces isomorphisms \( i_{*}: H_{r}(N_{n-3}) \to H_{r}(U) \) for \( r \leq n - 3 \).

Let \( \bar{V}_{i} = V_{i}' \cup \bigcup_{i=1}^{t} M_{i} \), \( i = 1, \ldots, t \) be the components of \( \text{Cl}(U - N_{n-3}) \). To show (4), it suffices to show that \( H_{r}(\bar{V}_{i}, \text{Bd} \bar{V}_{i}) = 0 \) for \( r \leq n - 2 \) and \( i = 1, \ldots, t \); for since \( \bar{V}_{i} \cap \bar{V}_{j} = \emptyset \) if \( i \neq j \), this will imply that \( H_{r}(\bigcup_{i} \bar{V}_{i}, \text{Bd} \bar{V}_{i}) = 0 \). Hence by excision \( H_{r}(U, N_{n-3}) = 0 \) for \( r \leq n - 2 \), which implies (4). By finite induction we may assume that \( \bar{V}_{i} \) is obtained from \( V_{i}' \) by adjoining a single \( M_{i} \). Let \( W_{i} = \text{Bd} V_{i}' - (S^{2} \times \text{Int} B^{n-3}) \) and consider the following commutative diagram where all maps are inclusions

\[
\begin{array}{ccc}
\text{Bd} V_{j}' & \xrightarrow{i_{2}} & \bar{V}_{j} \\
W_{j} & \searrow & \downarrow \quad i_{2} \\
\text{Bd} \bar{V}_{j} & \xrightarrow{i_{2}} & \bar{V}_{j} \\
\end{array}
\]

It now suffices to show the following:

(a) \( i_{2*}: H_{r}(\text{Bd} \bar{V}_{j}) \to H_{r}(\bar{V}_{j}) \) is an isomorphism for \( r \neq 2 \) but \( r \leq n - 4 \).
(b) $i_{a*} : H_2(\text{Bd } V_j) \to H_2(V_j)$ is an isomorphism.
(c) $H_r(\mathcal{V}_j, \text{Bd } V_j) = 0$ for $r = n - 3, n - 2$.

To verify (a) it suffices to show that $i_1, i_2, i_3$ induce homology isomorphisms in dimension $k$ where $k \leq n - 4$ but $k \neq 2$. We will only check that $i_1$ induces isomorphisms; the arguments for the others being similar or easy. Let $T = S^2 \times B^{n-3}$ and consider the reduced Mayer-Vietoris sequence for the triad $(\text{Bd } V_j; W_j, T)$.

$$
\begin{array}{cccc}
\cdots & \longrightarrow & H_r(S^2 \times S^{n-4}) & \longrightarrow \\
& & \phi' \longrightarrow & H_r(\text{Bd } V_j) \\
& & \phi \longrightarrow & H_{r-1}(S^2 \times S^{n-4}) \\
& & \longrightarrow & \cdots
\end{array}
$$

If $r \neq 2$ but $r < n - 4$, this sequence has the appearance $0 \to H_2(W_j) \to H_r(\text{Bd } V_j) \to 0$, for we know that in this range $\phi$ is injective, since $S^2 \times 0$ represents a free generator of $H_2(\text{Bd } V_j)$, and hence $\phi' = 0$. Hence, in this range $i_{1*}$ is an isomorphism. Now if $3 \leq r = n - 4$ the sequence appears as

$$
\begin{array}{cccc}
\cdots & \longrightarrow & H_{r-4}(W_j) & \longrightarrow \\
& & i_{1*} \longrightarrow & H_{r-4}(\text{Bd } V_j) \\
& & \phi' \longrightarrow & H_{r-5}(S^2 \times S^{n-4}) \\
& & \longrightarrow & \cdots
\end{array}
$$

As above $\phi' = 0$. Hence $i_{1*}$ will be an isomorphism if $\alpha = 0$. Notice that $\text{Bd } W_j = S^2 \times S^{n-4}$ and consider the following commutative (up to sign) diagram where the isomorphisms are given by duality.

$$
\begin{array}{cccc}
H^2(W_j) & \longrightarrow & H^2(\text{Bd } W_j) & \\
\sim & \longrightarrow & \sim & \\
H_{n-4}(W_j, \text{Bd } W_j) & \longrightarrow & H_{n-4}(\text{Bd } W_j) & \longrightarrow & H_{n-4}(W_j).
\end{array}
$$

By exactness of the lower row, it now suffices to show that $i^*$ is surjective. But $i_* : H_2(\text{Bd } W_j) \to H_2(W_j)$ carries the generator of $H_2(\text{Bd } W_j)$ onto a free element of $H_2(W_j)$, since $S^2 \times 0$ represents a primitive free element of a basis for $H_2(\text{Bd } V_j)$. Hence there is a split short exact sequence of the form

$$
0 \longrightarrow H_2(\text{Bd } W_j) \longrightarrow H_2(W_j) \longrightarrow G \longrightarrow 0.
$$

Now consider the commutative diagram

$$
\begin{array}{cccc}
H^2(W_j) & \longrightarrow & \text{Hom } (H_2(W_j), Z) & \\
\beta & \longrightarrow & \text{Hom } (H_2(W_j), Z) & \\
\downarrow i^* & & \downarrow \text{Hom } (i_*, 1) & \\
H^2(\text{Bd } W_j) & \longrightarrow & \text{Hom } (H_2(\text{Bd } W_j), Z) & \\
\beta' & & & \\
\end{array}
$$

where $\beta$ and $\beta'$ come from the Universal Coefficient theorem. It follows that $\beta$ and $\text{Hom } (i_*, 1)$ are surjective. But $\beta'$ is an isomorphism since $H^2(\text{Bd } W_j)$ is free, and hence $i^*$ is surjective.
(b) follows, for as given in [2],
\[ H_2(\text{Bd } V_j) \cong H_2(\text{Bd } V'_j)/A_j \]
\[ \cong H_2(V'_j)/\text{inc}_*(A_j) \]
\[ \cong H_2(V_j). \]

To verify (c), one first easily shows that \( \tilde{t} \) is onto for dimensions less than \( n - 1 \). Hence, using (a) and (b) and arguing as we did early in the proof of this step for \((V, \text{Bd } N) = (V, \text{Bd } V)\), it follows that \( H_r(\text{Bd } V, \text{Bd } V_j) = 0 \) for \( r \leq n - 3 \) and
\[ H_{n-2}(\text{Bd } V, \text{Bd } V_j) \]
is free. From this information, it follows, as it did for \((V'_j, \text{Bd } V'_j)\), that the following is an exact sequence.
\[ 0 \to H^{n-2}(\text{Bd } V'_j, \text{Bd } F_j) \to H^{n-3}(\text{Bd } V'_j) \to H^{n-3}(\text{Bd } V_j) \to 0. \]
By Poincaré duality, \( H^{n-3}(\text{Bd } V_j) \cong H^{n-3}(\text{Bd } V'_j)/A'_j \), where
\[ A'_j = \sigma'(\text{inc}_*(A_j)). \]
But since \( \sigma' \) (of the diagram on page 381) is a splitting homomorphism,
\[ H^{n-3}(\text{Bd } V'_j) \cong H^{n-3}(V'_j). \]
Now
\[ H^{n-3}(V'_j) \cong \text{Hom}(H_{n-3}(V'_j), \mathbb{Z}) \oplus \text{Ext}(H_{n-4}(V'_j), \mathbb{Z}) \]
and
\[ H^{n-3}(V_j) \cong \text{Hom}(H_{n-3}(V_j), \mathbb{Z}) \oplus \text{Ext}(H_{n-4}(V_j), \mathbb{Z}). \]
But \( H_r(\text{Bd } V'_j) \cong H_r(\text{Bd } V'_j) \) for \( r \neq 2 \), and since \( \text{inc}_*(A_j) \) is a free direct summand of \( H_2(V'_j) \), from the argument given in (b) above, Ext \( H_2(V'_j) \cong \text{Ext}(H_2(\text{Bd } V'_j)). \) It follows that \( H^{n-3}(\text{Bd } V_j) \cong H^{n-3}(\text{Bd } V'_j). \) Since these groups are finitely generated and \( H^{n-3}(\text{Bd } V'_j, \text{Bd } V_j) \) is free, the above exact sequence tells us that \( H^{n-3}(\text{Bd } V'_j, \text{Bd } V_j) = 0. \) Now by the Universal Coefficient theorem, \( H_{n-2}(\text{Bd } V_j, \text{Bd } V_j) = 0. \) This completes the proof of Proposition 3.1.

**Proposition 3.2.** If \( C \) is a compact subset of the open connected PL \( n \)-manifold \( U \) \((n > 5)\) such that each component of \( U - C \) is \( q \)-connected, then there is a compact connected PL \( n \)-submanifold \( N \) of \( U \) such that

1. \( C \subseteq \text{Int } N, \)
2. \( \sigma \sigma'(U - N) = \sigma'(U - C), \)
3. each component of \( \text{Bd } N \) is \( \min \{q, [(n-1)/2] - 1\} \)-connected, and
4. each component of \( U - N \) is \( \min \{q, [n/2] - 1\} \)-connected.

**Proof.** For \( 0 \leq i \leq \min \{q, [n/2] - 1\} \) let \( \mathcal{S}_i \) be the statement that there is a compact PL \( n \)-submanifold \( N_i \) of \( U \) such that (1), (2), (3), and (4) hold when \( i \) and \( N_i \) replace \( q \) and \( N \) respectively. If \( q = 0 \), the techniques used in the proof of \( \mathcal{S}_1 \) of Proposition...
3.1 will prove Proposition 3.2; and if \( q \geq 1 \), \( \mathcal{H}_1 \) is valid by the proof of \( \mathcal{S}_1 \). Now suppose that \( \mathcal{H}_1 \) is valid where \( 1 \leq i < \min \{ q, [n/2] - 1 \} \). Let \( M_1, M_2, \ldots, M_i \) be the components of \( \text{Bd} \ N_i \), and let \( Q_1, Q_2, \ldots, Q_i \) be the components of \( U - C \) such that \( Q_j \supset M_j \). For each \( j \), consider the triad \((Q_j; M_j^+, \text{Cl}(Q_j - M_j^+))\) where \( M_j^+ \) is the component of \( \text{Cl}(U - N_i) \) which contains \( M_j \). Now consider the following portion of the resulting Mayer-Vietoris sequence

\[
H_{i+1}(M_j) \xrightarrow{\alpha} H_{i+1}(M_j^+) \oplus H_{i+1}(\text{Cl}(Q_j - M_j^+)) \longrightarrow H_{i+1}(Q_j).
\]

Since \( Q_j \) is \((i+1)\)-connected, \( \alpha \) is onto; and hence it follows that \( H_{i+1}(M_j^+) \) is finitely generated. Let \( \lambda \) be a generator of \( H_{i+1}(M_j^+) \). Choose \( \mu \in H_{i+1}(M_j) \) such that \( \alpha(\mu) = (\lambda, 0) \). Since \( M_j \) is \( i \)-connected, let \( f: S^{i+1} \to M_j \) be such that, via the Hurewicz isomorphism, \( f = \mu \). By a general position argument, we may assume that \( f \) is a PL embedding. Now \( (\lambda, 0) = \alpha(\mu) = \alpha([f]) = ([f], -[f]) \) and hence \( [f] \) is trivial in \( H_{i+1}(\text{Cl}(Q - M_j^+)) \). But, since \( M_j \) and \( Q_j \) are both \( i \)-connected, so is \( \text{Cl}(Q_j - M_j^+) \), and hence \( f \) is homotopically trivial in \( \text{Cl}(Q_j - M_j^+) \). Hence, we extend \( f \) to \( \tilde{f}: B^{i+2} \to \text{Cl}(Q_j - M_j^+) \). One easily verifies the hypothesis of Irwin’s embedding theorem [23], and hence we may assume that \( \tilde{f} \) is a proper PL embedding. Let \( P \) be a regular neighborhood of \( \tilde{f}(B^{i+2}) \) in \( \text{Cl}(Q_j - M_j^+) \), and let \( N_{i+1} = \text{Cl}(N_i - P) \). \( N_{i+1} \) will be our first approximation to \( N_{i+1} \). Since \( P \subset Q_j \subset U - C \) it is clear that \( N_{i+1} \) satisfies (1). It is easy to see that neither the deletion of handles, nor surgery in this dimension range disconnects \( N_i \) or any \( M_j \), and hence \( N_{i+1} \) satisfies (2).

Now let us see what happened to the connectivity conditions. Consider the triad \((M_j^+ \cup P; M_j^+, P)\). By Proposition 2.6 (3), \( P \cap M_j^+ \approx S^{i+1} \times B^{n-i-2} \). Now consider the reduced Mayer-Vietoris sequence

\[
\tilde{H}_k(S^{i+1} \times B^{n-i-2}) \xrightarrow{\alpha_k} \tilde{H}_k(M_j^+) \oplus \tilde{H}_k(P) \xrightarrow{\beta_k} \tilde{H}_k(M_j^+ \cup P) \longrightarrow \tilde{H}_{k-1}(S^{i+1} \times B^{n-i-2}).
\]

Note that \( M_j^+ \cup P \in \mathcal{C} \text{Cl}(U - N_{i+1}) \) replaces \( M_j^+ \in \mathcal{C} \text{Cl}(U - N_i) \), and what we wish to show is that \( M_j^+ \cup P \) is as nicely connected as \( M_j^+ \), and that in addition we have killed the generator \( \lambda \) of \( H_{i+1}(M_j^+) \). It is easy to show that \( \tilde{H}_k(M_j^+ \cup P) \) is flanked by zeros in the above sequence if \( k \leq i \), and using the van Kampen and Hurewicz theorems, it does follow that \( M_j^+ \cup P \) is \( i \)-connected. For \( k = i+1 \), \( \tilde{H}_{i+1}(S^{i+1} \times B^{n-i-2}) = 0 \), and hence \( \beta_{i+1} \) is surjective. It follows that

\[
H_{i+1}(M_j^+) \approx H_{i+1}(M_j^+ \cup P).
\]

But \( [\tilde{f}]S^{i+1} \) generates \( H_{i+1}(S^{i+1} \times B^{n-i-2}) \) and \( \alpha_{i+1}[\tilde{f}]S^{i+1} = [\tilde{f}]_{H_{i+1}(M_j^+)} = \lambda \). Hence \( H_{i+1}(M_j^+ \cup P) \approx H_{i+1}(M_j^+)\langle \lambda \rangle \), and we have reduced \( H_{i+1}(M_j^+) \) by one generator. Hence we will be able to alter \( N_i \) inductively to \( N_{i+1} \) which satisfies (4) of \( \mathcal{H}_{i+1} \), if we can show that \( \text{Bd} (M_j^+ \cup P) \) is \( \{ i \}, ([n-1)/2]-1 \)-connected. By Proposition 2.6, we obtained \( \text{Bd} (M_j^+ \cup P) \) from \( \text{Bd} M_j^+ = M_j \), by surgery of
index $i+1$. Let $M'_i = \text{Bd} \left( M^+_i \cup P \right)$, $T_1 \approx S^{i+1} \times \text{Int} \ B^{n-i-2}$ and $T_2 \approx B^{i+2} \times S^{n-i-3}$, so that $M'_i = (M_i - T_1) \cup T_2$ with $(M_i - T_1) \cap T_2 \approx S^{i+1} \times S^{n-i-3}$. Consider the triad $(M'_i; M_i - T_1, T_2)$ and the resulting reduced Mayer-Vietoris sequence,

$$\tilde{H}_k(M_i - T_1) \oplus \tilde{H}_k(T_2) \to \tilde{H}_k(M'_i) \to \tilde{H}_{k-1}((M_i - T_1) \cap T_2).$$

For $k \leq \min \{ i, [(n-1)/2] - 1 \}$; $n - i - 3 > k$ and hence

$$\tilde{H}_k(T_2) = 0 = \tilde{H}_{k-1}((M_i - T_1) \cap T_2).$$

$M_i - T_1$ has the homotopy type of $M_i - S^{i+1}$ and since $M_i$ is min \{ $i, [(n-1)/2] - 1 \}$-connected, general position arguments give that $M_i - T_1$ is also

$$\min \{ i, [(n-1)/2] - 1 \}$$

connected.

It follows that $M'_i$ is min \{ $i, [(n-1)/2] - 1 \}$-connected.

It remains to show that (3) can be satisfied on the $i+1$ level. If $i+1 \geq [(n-1)/2]$, we are done; for any $M'_i \in \mathcal{C} \text{Bd } N'_{i+1}$ is min \{ $i, [(n-1)/2] - 1 \}$-connected. Now suppose $i+1 < [(n-1)/2]$. Let $A$ be a generator of $\pi_{i+1}(M'_i)$. Since we are in the trivial range, there is a PL embedding $f: S^{i+1} \to M'_i$ such that $[f] = \lambda$. Since $i+1 \leq [n/2] - 1$ and $M^+_i \cup P$ is min \{ $i+1, [n/2] - 1 \}$-connected, we can extend $f$ to a proper PL embedding $\tilde{f}: B^{i+2} \to M^+_i \cup P$. Let $P'$ be a regular neighborhood of $\tilde{f}(B^{i+2})$ in $M^+_i \cup P$ and let $N_{i+1} = N'_{i+1} \cup P'$. By Proposition 2.6, we have obtained $M'_i = \text{Bd } Cl \left( (M^+_i \cup P) - P' \right) \in \mathcal{C} \text{Bd } N'_{i+1}$ from $M'_i \in \mathcal{C} \text{Bd } N'_{i+1}$ by surgery of index $i+1$. But, we are in the trivial range for surgery, and by Proposition 2.6, $\pi_{i+1}(M'_i) \approx \pi_{i+1}(M'_i)/(\lambda)$. Hence the proposition will follow by induction, if we can show that $(M^+_i \cup P) - P' \in \mathcal{C}(U - N'_{i+1})$ which replaces $M^+_i \cup P \in \mathcal{C}(U - N'_{i+1})$ is min \{ $i+1, [n/2] - 1 \}$-connected, so that we will not destroy (4) at the $i+1$ level. But by Proposition 2.6, $P' \approx \tilde{f}(B^{i+2}) \times B^{n-i-2}$ and hence $(M^+_i \cup P) - P'$ is of the homotopy type of $(M^+_i \cup P) - \tilde{f}(B^{i+2})$. Now using the connectivity of $M^+_i \cup P$, general position, and the fact that $i+1 < [(n-1)/2]$; the desired result follows easily.

Remark. Proposition 3.2 is an important step in proving the PL Hauptvermutung for the open PL manifolds that we consider, for it enables us to take a connectivity (at infinity) condition, which is not a priori related to any PL structure, and relate it to a given PL structure.

4. A connectivity characterization of $S^n$ ($n > 5$) minus a nonempty tame compact 0-dimensional subset.

Lemma 4.1. Let $U$ be an open $(p, q)$-connected PL $n$-manifold ($n > 5$) with $p \geq \lceil n/2 \rceil$, $q \geq 1$ and $p + q \geq n - 2$. Then $U$ is $(n-2)$-connected.

Proof. Using Proposition 3.2, it follows that $U$ is the monotone union of compact PL $n$-submanifolds $N_i$ such that

1. Each component of $\text{Bd } N_i$ is 1-connected.
2. Each component of $U - N_i$ is min \{ $q, [n/2] - 1 \}$-connected for each $i$.
3. $\sigma \in \text{Bd } N_i = \sigma(U - N_i)$.
4. $N_i$ is 1-connected for each $i$. 

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(1), (2), and (3) are immediate. To see (4) consider the decomposition \( U = N_1 \cup Q_{11} \cup \cdots \cup Q_{n1} \) where the \( Q_{n} \)'s are the closures of the components of \( U - N_i \). Since \( U \) and each \( Q_{ni} \cap N_i \) (a component of \( \text{Bd } N_i \)) are 1-connected, an application of the van Kampen theorem, a finite number of times, tells us that \( N_i \) is 1-connected. Let \( Q_i = \text{Cl} (U - N_i) \). Now consider the exact reduced homology sequence for the pair \((U, Q_i)\)

\[ H_k(U) \rightarrow H_k(U, Q_i) \rightarrow H_{k-1}(Q_i) \rightarrow H_{k-1}(U). \]

Since \( U \) is \( p \)-connected and each component of \( Q_i \) is \( \min\{q, [n/2] - 1\} \)-connected, it follows, using excision, that \( H_k(N_i, \text{Bd } N_i) = 0 \) if \( 2 \leq k \leq \min\{q + 1, [n/2]\} \) and \( H_1(N_i, \text{Bd } N_i) \) is free. By duality and the Universal Coefficient theorem,

\[ H_s(N_i) \approx H^{n-s}(N_i, \text{Bd } N_i) \]

\[ \approx \text{Hom} (H_{n-s-i}(N_i, \text{Bd } N_i), \mathbb{Z}) \oplus \text{Ext} (H_{n-s-i}(N_i, \text{Bd } N_i), \mathbb{Z}). \]

But \( p + 1 \leq r \leq n - 2 \) implies that \( 2 \leq n - r \leq \min\{q + 1, [n/2]\} \), and hence \( H_s(N_i) = 0 \) for \( 2 \leq k \leq \min\{q + 1, [n/2]\} \) and \( H_1(N_i, \text{Bd } N_i) \) is free. By duality and the Universal Coefficient theorem,

**Proposition 4.2.** Let \( U \) be an open \((p, q)\)-connected PL \( n \)-manifold \((n > 5)\) with \( p = \lfloor n/2 \rfloor \), \( q \geq 1 \) and \( p + q \leq n - 2 \). Let \( C \) be a compact subset of \( U \) such that each component of \( U - C \) is 1-connected. Then there is a cored \( n \)-ball \( CB_k \) such that \( C \cap \text{Int } C \cap \text{Int } N \) and \( k = \sigma^{k-1}_i (U - C) = \sigma^{k-1}_i (U - CB_k) \).

**Proof.** By Lemma 4.1, \( U \) is \((n-2)\)-connected. Hence \( i: C \subset U \) induces a surjective homomorphism \( i_*: H_i(C) \rightarrow H_i(U) \) for \( r \leq n - 2 \). Now let \( N \) be a compact connected PL \( n \)-submanifold of \( U \) which satisfies Proposition 3.1 for \( C \). We will show that \( N \) is the cored \( n \)-ball we are seeking. Note that \( C \subset \text{Int } N \) and \( \sigma^{k-1}_i \text{Bd } N = \sigma^{k-1}_i (U - C) = \sigma^{k-1}_i (U - N) \) by Proposition 3.1. Also, \( H_i(N) = 0 \) for \( 2 \leq k \leq n - 3 \). Hence \( N \) will be \((n-3)\)-connected, if \( N \) is 1-connected. Let \( Q_1, Q_2, \ldots, Q_t \) be the components of \( \text{Cl} (U - N) \). Hence \( U = N \cup Q_1 \cup Q_2 \cup \cdots \cup Q_t \) with each \( N \cap Q_i \) being a component of \( \text{Bd } N \) and hence, 1-connected. Since \( U \) is 1-connected, as we remarked in the proof of Lemma 4.1, \( N \) is 1-connected. The proposition will follow from the following.

**Claim.** A compact \((n-3)\)-connected PL \( n \)-manifold \( N \) \((n > 5)\) with nonempty boundary such that each component of \( \text{Bd } N \) is 1-connected, is a cored \( n \)-ball.

**Proof.** By Lefschetz duality, \( H_{n-i}(N, \text{Bd } N) \approx H^i(N) \) and since \( N \) is \((n-3)\)-connected, \( H^i(N) = 0 \) for \( 1 \leq r \leq n - 3 \). Hence, \( H_k(N, \text{Bd } N) = 0 \) for \( 3 \leq k \leq n - 1 \). Now consider the exact homology sequence for the pair \((N, \text{Bd } N)\)

\[ H_k(N) \rightarrow H_k(N, \text{Bd } N) \rightarrow H_{k-1}(\text{Bd } N) \rightarrow H_{k-1}(N). \]

Since \( N \) is \((n-3)\)-connected, \( H_k(N, \text{Bd } N) \approx H_{k-1}(\text{Bd } N) \) for \( 2 \leq k \leq n - 3 \); and hence \( H_k(\text{Bd } N) = 0 \) for \( 2 \leq r \leq n - 4 \). Since each component of \( \text{Bd } N \) is 1-connected, it now follows that each component of \( \text{Bd } N \) is \((n-4)\)-connected. But since \( n \geq 6 \),
n - 4 \geq [(n - 1)/2] and n - 1 \geq 5. Hence, by a strong form of the Poincaré conjecture (essentially [17]), each component of Bd $N$ is a PL (n - 1)-sphere. Let $S_1, S_2, \ldots, S_n$ be the components of Bd $N$. Let $N'$ be the closed PL n-manifold obtained from $N$ by inductively coning over the components of Bd $N$, i.e.,

$$N' = N \cup C(S_1) \cup \cdots \cup C(S_n)$$

where $C(S_i)$ is the cone over $S_i$ and $\text{Cl} (N' - C(S_i)) \cap C(S_i) = S_i$. By inductively applying the van Kampen theorem it is easy to see that $N'$ is 1-connected. Suppose we have shown that $H_r(N') = 0$ where $1 \leq r < [n/2]$. Consider the triad

$$(N'; N, \bigcup C(S_i))$$

and the following portion of the resulting exact Mayer-Vietoris sequence,

$$\mathbb{Z}^{r+1}(N) \oplus \mathbb{Z}^{r+1}(\bigcup C(S_i)) \rightarrow \mathbb{Z}^{r+1}(N') \rightarrow \mathbb{Z}^r(U S)$$

Since $n \geq 6$, $n - 3 \geq [n/2]$, and since $N$ is (n - 3)-connected, $H_{r+1}(N) = 0$. Since each $C(S_i)$ is a ball and $C(S_i) \cap C(S_j) = \emptyset$ if $i \neq j$, $H_{r+1}(\bigcup C(S_i)) = 0$. Also $1 \leq r < [n/2] < n - 1$, and hence $H_r(\bigcup S_i) = 0$. It follows that $H_{r+1}(N') = 0$, and $N'$ is $[n/2]$-connected. Again by the Poincaré conjecture for spheres, $N'$ is a PL n-sphere, and hence $N$ is a cored n-ball.

Theorem 4.3. An open PL n-manifold $U$ $(n \geq 5)$ is PL homeomorphic to $S^n$ minus a nonempty tame compact 0-dimensional subset $K$ of cardinality $\alpha$ if and only if there are positive integers $p$ and $q$ such that $p \geq [n/2]$, $p + q \geq n - 2$, $U$ is $(p, q)$-connected, and $\partial U$ has cardinality $\alpha$.

Sufficiency. Since $U$ is 1-connected at infinity, let $\{C_i\}$ be a sequence of compact subsets of $U$ such that $U = \bigcup \text{Int } C_i$, $C_i \subset C_{i+1}$, and each component of $U - C_i$ is 1-connected for each $i$. Now by Proposition 4.2, $U$ is the monotone union of cored $n$-balls. By Theorem 2.2, $U$ is PL homeomorphic to $S^n - K$ for some nonempty tame compact 0-dimensional set $K$. By Lemma 2.1 and the fact that homeomorphic manifolds have the same number of ends, $\sigma \partial U = \sigma K$.

Necessity. Let $U \approx S^n - K$ where $K$ is a nonempty tame compact subset of cardinality $\alpha$. By Lemma 2.1, $\sigma K = \sigma \partial U$. A theorem of McMillan [8], implies that $U$ is the monotone union of cored $n$-balls, and hence it follows easily that $U$ is $(n-2, n-2)$-connected.

Remark. The conditions on the integers $p$ and $q$ in Theorem 4.3 cannot be made any weaker. Consider the following examples, all of which have exactly one end, but none of which are homeomorphic to $E^n$.

Example 1. In [5], for $n \geq 5$, contractible open PL $n$-manifolds which are not 1-connected at infinity are shown to exist.

Here $q \geq 1$ fails. (Note that $p = \infty$.)

Example 2. $(S^{(n/2)} \times S^{(n + 1)/2)})$ minus a point is an $([n/2]-1, n-2)$-connected open PL $n$-manifold which is not homeomorphic to $E^n$. Here $p = [n/2]$ just fails. (Note that for $n \geq 6$, $([n/2]-1) + (n-2) \geq n$)
Example 3. For \( n \geq 7 \) and \( [n/2] \leq p \leq n - 4 \), \( S^{p+1} \times E^{n-p-1} \) is a \((p, n-p-3)\)-connected open PL \( n \)-manifold which is not homeomorphic to \( E^n \). Here \( p + q \geq n-4 \) just fails.

(Note that we need \( n \geq 7 \) here, because for \( n = 6 \), \( [n/2] + 1 = n - 2 \); and hence the theorem applies.)

5. On the Hauptvermutung.

Theorem 5.1. Let \( U \) and \( U' \) be homeomorphic open \((p, q)\)-connected PL \( n \)-manifolds \((n > 5)\) where \( p \) and \( q \) are two positive integers such that \( p \geq [n/2] \), \( q \geq 1 \) and \( p + q \geq n - 2 \). Then \( U \) and \( U' \) are PL homeomorphic.

Proof. Let \( h: U \to U' \) be a homeomorphism. Since \( U \) is 1-connected at infinity, let \( \{C_i\}_{i=1}^{\infty} \) be a sequence of compact subsets of \( U \) such that each component of \( U - C_i \) is 1-connected for each \( i \), \( C_i \subset \text{Int} C_{i+1} \), and \( \bigcup \text{Int} C_i = U \). For each \( i \), let \( C'_i = h(C_i) \). Let \( M_i \) and \( M'_i \) be cored \( n \)-balls in \( U \) and \( U' \) respectively which satisfy the conclusion of Proposition 4.2 with respect to \( C_i \) and \( C'_i \) respectively. Hence \( C_i \subset \text{Int} M_i \), \( C'_i \subset \text{Int} M'_i \), and \((\text{index of } M_i) = (\text{index of } M'_i)\). Let \( S_{1i}, \ldots, S_{ki} \) be the boundary spheres of \( M_i \). Now let \( S_{1i}, \ldots, S_{ki} \) be the boundary spheres of \( M'_i \) such that if \( Q_{ij} \) is the component of \( U - C_i \) which contains \( S_{ij} \), then \( S_{ij} \subset h(Q_{ij}) \). Let \( g_1: M_i \to S^n \) be a PL embedding. Using Lemma 2.5, it follows easily that there is a PL homeomorphism \( g'_1: M'_i \to g_1(M_i) \) such that \( g'_1(S'_{ij}) = g_1(S_{ij}) \) for \( j = 1, \ldots, k_i \). Let \( h_1 = (g'_1)^{-1}g_1 \). Hence \( h_1 \) is a PL homeomorphism of \( M_i \) onto \( M'_i \) such that \( h_1(S_{ij}) = S'_{ij} \) for \( j = 1, \ldots, k_i \). Suppose now for \( 1 \leq i \leq r-1 \), we have found a positive integer \( s(i) \), cored \( n \)-balls \( M_i \) and \( M'_i \) in \( U \) and \( U' \) respectively, and a PL homeomorphism \( h_i: M_i \to M'_i \) such that

1. \( M_i \) and \( M'_i \) satisfy the conclusion of Proposition 4.2 for \( C_{s(i)} \) and \( C'_{s(i)} \) respectively,
2. \( \text{Int} M_i \supset M_{i-1} \cup C_i \),
3. \( \text{Int} M'_i \supset M'_{i-1} \cup C'_i \),
4. \( h_i|M_{i-1} = h_{i-1} \), and
5. \( h_i(S_{ij}) = S'_{ij} \), \( 1 \leq j \leq k_i \) where the \( S_{ij} \)'s are the boundary spheres of \( M_i \), and \( S'_{ij} \) is the boundary sphere of \( M'_i \) such that if \( Q_{ij} \) is the component of \( U - C_{s(i)} \) which contains \( S_{ij} \), then \( S_{ij} \subset h(Q_{ij}) \).

We wish to define \( s(r) \), \( M_r \), \( M'_r \), and \( h_r \). Since \( U = \bigcup \text{Int} C_i \), \( U' = \bigcup \text{Int} C'_i \), and \( M_{i-1} \cup C_i, M'_{i-1} \cup C'_i \) are compact; choose an integer \( s(r) \) such that \( \text{Int} C_{s(r)} \supset M_{r-1} \cup C_r \) and \( \text{Int} C'_{s(r)} \supset M'_{r-1} \cup C'_r \). Now apply Proposition 4.2 to \( C_{s(r)} \) and \( C'_{s(r)} \), obtaining cored \( n \)-balls \( M_r \) and \( M'_r \) such that \( C_{s(r)} \subset \text{Int} M_r \) and \( C'_{s(r)} \subset \text{Int} M'_r \). It follows that \( M_r \) and \( M'_r \) have the same index. Let \( S_{r1}, \ldots, S_{rk_r} \) be the boundary spheres of \( M_r \), and let \( S'_{r1}, \ldots, S'_{rk_r} \) be the boundary spheres of \( M'_r \) such that, if \( Q_{ij} \) is the component of \( U - C_{s(r)} \) which contains \( S_{ij} \), then \( S_{ij} \subset h(Q_{ij}) \). Clearly \( M_r \) and \( M'_r \) satisfy (1), (2), and (3) for \( i = r \). We must construct a PL homeomorphism
$h_\tau: M_\tau \to M'_\tau$ which satisfies (4) and (5). By Lemma 2.4, let $A_{\tau_1}, \ldots, A_{\tau_{k-1}}, A'_{\tau_1}, \ldots, A'_{\tau_{k-1}}$ be the pairwise disjoint cored $n$-balls such that

$$\bigcup_{j=1}^{k-1} A_{\tau_j} = \text{Cl} (M_\tau - M_{\tau-1}), \quad \bigcup_{j=1}^{k-1} A'_{\tau_j} = \text{Cl} (M'_\tau - M'_{\tau-1}),$$

$A_{\tau_j} \cap M_{\tau-1} = S_{(\tau-1)j}$, and $A'_{\tau_j} \cap M'_{\tau-1} = S'_{(\tau-1)j}$. It now suffices, to complete the inductive step, to define for $1 \leq j \leq k_{\tau-1}$, a PL homeomorphism $h_{\tau_j}: A_{\tau_j} \to A'_{\tau_j}$ such that

(a) $h_{\tau_j}|S_{(\tau-1)j} = h_{\tau_{j-1}}|S_{(\tau-1)j}$ and
(b) $h_{\tau_j}(S_{\tau m}) = S'_{\tau m}$ if $S_{\tau m} \subset A_{\tau_j}$.

For if we can do this, then

$$h_\tau = h_{\tau_{\tau-1}} \quad \text{on } M_{\tau-1},$$

$$= h_{\tau_j} \quad \text{on } A_{\tau_j}$$

will clearly satisfy (4) and (5). We will need the following.

**Claim.** $S_{\tau m} \subset A_{\tau_j}$ if and only if $S'_{\tau m} \subset A'_{\tau_j}$.

**Proof.** $S_{\tau m} \subset A_{\tau_j}$ implies that $S_{\tau m} \subset Q_{(\tau-1)j}$, since $A_{\tau_j} \cap M_{\tau-1} = S_{(\tau-1)j}$, $Q_{(\tau-1)j}$ is a component of $U - C_{(\tau-1)j}$, and $A_{\tau_j}$ is connected and lies in $U - C_{(\tau-1)j}$. By our construction, $S_{\tau m} \subset Q_{\tau m}$. Hence since $Q_{(\tau-1)j} \in \mathcal{C}(U - C_{(\tau-1)j}), Q_{\tau m} \in \mathcal{C}(U - C_{(\tau-1)j})$, and $C_{(\tau-1)j} \subset C_{\tau j}$; we have that $Q_{(\tau-1)j} \supset Q_{\tau m}$. Hence $h(Q_{(\tau-1)j}) \supset h(Q_{\tau m}) \supset Q_{\tau m}$. But $Q_{(\tau-1)j} = A_{\tau_j} \cap h(Q_{(\tau-1)j})$, and using the fact that $h(Q_{(\tau-1)j})$ is a component of $C_{(\tau-1)j}$, we conclude that $A'_{\tau_j} \subset h(Q_{(\tau-1)j})$. It follows that $A_{\tau_k} \cap h(Q_{(\tau-1)j}) = \emptyset$ if $k \neq j$ and hence it must be that $S_{\tau m} \subset A'_{\tau_j}$.

Conversely, if $S_{\tau m} \notin A_{\tau_j}$, then $S_{\tau m} \subset A_{\tau t}$ for $t \neq j$ since $S_{\tau m} \in \mathcal{C} \text{ Bd } M_\tau$ and

$$\text{Bd } M_\tau \subset \bigcup_{k=1}^{k_{\tau-1}} \text{Bd } A_{\tau k}.$$}

Hence by the first part $S'_{\tau m} \subset A'_{\tau t}$ and $A_{\tau t} \cap A'_{\tau t} = \emptyset$. Hence $S'_{\tau m} \notin A'_{\tau t}$.

Now for $1 \leq j \leq k_{\tau-1}$, let $g_{\tau_j}: A_{\tau_j} \to S^n$ be a PL embedding. Hence

$$\overline{g}_{\tau_j} = g_{\tau_j}(h_{\tau_{j-1}}^{-1}|S_{(\tau-1)j})$$

is a PL homeomorphism of $S_{(\tau-1)j}$ onto $g_{\tau_j}(S_{(\tau-1)j})$. By Lemma 2.5 and the claim, let $g_{\tau_j}'$ be an extension of $\overline{g}_{\tau_j}$ to $A'_{\tau_j}$ such that $g_{\tau_j}'(S'_{\tau m}) = g_{\tau_j}(S_{\tau m})$ whenever $S'_{\tau m} \subset A'_{\tau_j}$.

Now let $h_{\tau_j} = (g_{\tau_j}')^{-1}g_{\tau_j}$. One easily verifies (a) and (b).

Now define $H: U \to U'$ by $H(x) = h_\tau(x)$ whenever $x \in M_\tau$. It follows that $H$ is a PL homeomorphism.

In view of Theorem 4.3, we obtain the following corollary.

**Corollary.** The Hauptvermutung holds for PL manifolds whenever the manifolds are topologically $S^n$ $(n > 5)$ minus a nonempty tame compact 0-dimensional subset.

For example the manifolds may be topologically $S^n$ minus any tame Cantor set, $S^n$ minus any countable set, an open $n$-annulus, or $E^n$. (Stallings [19] has proven the corollary for $E^n (n \geq 5)$.)
Theorem 5.2. For $n=6,7$ homeomorphic differentiable manifolds which are topologically $S^n$ minus a nonempty tame compact 0-dimensional subset are diffeomorphic.

Proof. Let $K$ be a nonempty tame compact 0-dimensional subset of $S^n$, and let $U$ and $V$ be differentiable manifolds which are homeomorphic to $S^n - K$. By [22], give $U$ and $V$ PL triangulations $T_1$ and $T_2$ respectively, which are compatible with their differentiable structures. By the preceding theorem, $(U, T_1)$ and $(V, T_2)$ are PL homeomorphic. By [8], $U$ is the monotone union of cored $n$-balls; and hence it follows that $H_k(U) = 0$ if $k \neq n-1$ and $H_{n-1}(U)$ is free. It follows from [12], [4], [17], [7], and [9] that the least positive integer $a$ such that $\Gamma_a \neq 0$ is 7. For $n=6,7; H_6(U)$ is free and $H_a(U) = 0$ for $k \geq 7$. Now applying Theorem 6.5 of [12], it follows that $U$ and $V$ are diffeomorphic.

Remark. Theorem 5.2 does not hold for arbitrary $n > 5$. For let $\Sigma^n$ be an exotic $n$-sphere. Hence $\Sigma^n \times E^1$ is topologically an open $(n+1)$-annulus, but is not diffeomorphic to $S^n \times E^1$. For, if so, then $\Sigma^n$ can be smoothly embedded in $S^{n+1}$. Now by 3.6 of [11], $\Sigma^n$ is bicollared in $S^{n+1}$ and by [3], the closure of a component of $S^{n+1} - \Sigma$ is a topological ball $B$. By the smooth Hauptvermutung for balls [18]; $B$, as a smooth manifold, is diffeomorphic to $B^{n+1}$. Hence $\text{Bd} B^{n+1} = \Sigma^n$ is diffeomorphic to $S^n$, contradicting the fact that $\Sigma^n$ is exotic, i.e., the existence of an exotic $n$-sphere implies the existence of an exotic open $(n+1)$-annulus.

References


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