Analytic Sheaf Cohomology Groups of Dimension $n$ of $n$-Dimensional Complex Spaces

By

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In this paper we prove the following:

**Main Theorem.** Suppose $\mathcal{F}$ is a coherent analytic sheaf on a $\sigma$-compact complex space $X$ (not necessarily reduced).

(A)$_n$

If $\dim X = n$ and $X$ has no compact $n$-dimensional branch, then $H^n(X, \mathcal{F}) = 0$.

(B)$_n$

If $\dim X = n$ and $X$ has only a finite number of compact $n$-dimensional branches, then $\dim H^n(X, \mathcal{F}) < \infty$.

(A)$_n$ with the additional assumption that $X$ is a manifold and $\mathcal{F}$ is locally free was proved by Malgrange [12, p. 236, Problème 1]. In [9] Komatsu proved the following related result: if $\mathcal{F}$ is a coherent analytic sheaf on an $n$-dimensional complex manifold $U$ such that $H^n(U, \mathcal{F}) = 0$, then $H^n(V, \mathcal{F}) = 0$ for any open subset $V$ of $U$ (p. 83, Theorem 7). The author in [16] proved (A)$_n$ with the additional assumption that $X$ is a manifold.

The paper is divided into five sections. In §I some Lemmas about Fréchet spaces and LF-spaces are proved. In §II a duality concerning distributions with restricted supports is established. In §IIII by partial normalizations and results of [13] the proof of the Main Theorem is reduced to the proof of (A)$_n$ with the additional assumption that $X$ is reduced and normal and $\mathcal{F}$ is torsion-free. In §IV we prove by local resolutions of singularities that $H^n(G, \mathcal{F}) = 0$ for $G \subset X$ and also obtain a result on the approximation of $(n-1)$-cocycles with coefficients in $\mathcal{F}$. §V sews up the proof of the Main Theorem.

All complex spaces here are $\sigma$-compact and are in the sense of Grauert [3, p. 9, Definition 2]. The structure sheaf of a complex space $X$ is denoted by $\mathcal{O}$ unless specified otherwise. The set of all singular points of $X$ is denoted by $\sigma(X)$. The inverse image [4, p. 410, Definition 8] and the $q$th direct image [4, p. 413, Definition 9] of an analytic sheaf $\mathcal{F}$ under a holomorphic map $f$ of complex spaces are denoted respectively by $f^{-1}(\mathcal{F})$ and $R^qf(\mathcal{F})$. If $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, then $R^qf(\mathcal{G})$ is regarded as a subsheaf of $R^qf(\mathcal{F})$. A covering $\mathcal{U}$ of a complex space $X$ is called a Stein covering if $\mathcal{U}$ is countable and every member of $\mathcal{U}$ is a Stein open subset. If $Y$ is an open subset of $X$, then $\mathcal{U}|Y = \{U \in \mathcal{U}| U \subset Y\}$ is called the restriction of $\mathcal{U}$.

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to \( Y \). If \( \mathcal{B} \) is a refinement of a subcollection of \( \mathcal{U} \) under some map \( \tau : \mathcal{B} \rightarrow \mathcal{U} \) and \( \mathcal{F} \) is a sheaf on \( X \), then \( \tau \) induces \( C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^p(\mathcal{B}, \mathcal{F}) \) and we call the image in \( C^p(\mathcal{B}, \mathcal{F}) \) of \( f \in C^p(\mathcal{U}, \mathcal{F}) \) the restriction of \( f \) to \( \mathcal{B} \) and denote it by \( f|_{\mathcal{B}} \). Whenever we have a refinement \( \mathcal{B} \), we assume that a fixed suitable map \( \tau \) is chosen once for all. The supports of sheaves, functions, distributions, etc. are denoted by \( \text{Supp} \).

All locally convex (linear topological) spaces are over \( C \). The dual of a locally convex space \( E \) is denoted by \( E^* \). The boundary, interior, and closure of a subset \( A \) of a topological space are denoted respectively by \( \partial A \), \( A^o \), and \( A^- \). Unless specified otherwise, the boundary, the interior, and the closure are with respect to the largest ambient topological space. Suppose \( B \) and \( C \) are subsets of a metric space \( Y \) with metric \( d \), \( x \in Y \), and \( \varepsilon > 0 \). Then

\[
d(x, B) = \inf_{y \in B} d(x, y), \quad d(B, C) = \inf_{y \in B} d(y, C),
\]

\[
U_\varepsilon(B) = \{ y \in Y \mid d(y, B) < \varepsilon \}, \quad \overline{U}_\varepsilon(B) = \{ y \in Y \mid d(y, B) \leq \varepsilon \}.
\]

\( N \) = the set of all natural numbers.

I. Suppose \( E \) and \( F \) are LF-spaces \([8, \text{p. 18, Definition 8(d)}]\) and are respectively the strict inductive limits of their closed Fréchet spaces \( \{E_n\}_{n \in N} \) and \( \{F_n\}_{n \in N} \). Suppose \( \phi : E \rightarrow F \) is a continuous linear map.

**Lemma 1.** If \( 0 < \dim \text{Coker} \phi < \infty \), then the transpose \( \phi^* : F^* \rightarrow E^* \) is not injective.

**Proof.** We first prove that \( \phi(E) \) is closed in \( F \). Since \( \dim \text{Coker} \phi < \infty \), \( F = \phi(E) \oplus G \) for some finite dimensional subspace \( G \) of \( F \). Define \( \tilde{\phi} : E \oplus G \rightarrow F \) by \( \tilde{\phi}(a \oplus b) = \phi(a) + b \) for \( a \in E \) and \( b \in G \). \( \tilde{\phi} \) is a continuous linear surjection. Since \( E \oplus G \) is an LF-space, \( \tilde{\phi} \) is open \([8, \text{p. 44, Theorem 10}]\). Since \( E \oplus 0 \) is closed in \( E \oplus G \) and \( \tilde{\phi}^{-1}(\phi(E) \oplus 0) = E \oplus 0 \), \( \phi(E) = \tilde{\phi}(E \oplus 0) \) is closed in \( F \).

Since \( \dim \text{Coker} \phi > 0 \), there exists \( c \in F - \phi(E) \). Since \( F \) is locally convex and \( \phi(E) \) is closed, by the theorem of Hahn-Banach there exists \( f \in F^* \) such that \( f(c) \neq 0 \) and \( f \equiv 0 \) on \( \phi(E) \). \( f \) is a nonzero element of \( \text{Ker} \phi^* \). Q.E.D.

**Lemma 2.** If \( \phi \) is surjective, then for any \( k \in N \), there exists \( l \in N \) such that \( \phi(E_l) \supset F_k \).

**Proof.** Suppose the lemma is not true. Fix \( k \in N \). Let \( G_n = E_n \cap \phi^{-1}(F_k) \). \( G_n \) is a Fréchet space. Let \( \psi_n : G_n \rightarrow F_k \) be induced by \( \phi \). Since \( \psi_n(G_n) \neq F_k \), \( \psi_n(G_n) \) is of the first category in \( F_k \) \([8, \text{p. 41, Corollary 6}]\). \( \bigcup_{n \in N} \psi_n(G_n) = \bigcup_{n \in N} \phi(G_n) = F_k \) is of the first category, contradicting that every Fréchet space is of the second category. Q.E.D.

**Lemma 3.** Suppose for \( n \in N \)

\[
\begin{array}{cccc}
E_n^1 & \xrightarrow{\alpha_n^1} & E_n^2 & \xrightarrow{\alpha_n^2} & E_n^3 \\
\phi_n^1 & \downarrow & \phi_n^2 & \downarrow & \phi_n^3 \\
F_n^1 & \xrightarrow{\beta_n^1} & F_n^2 & \xrightarrow{\beta_n^2} & F_n^3
\end{array}
\]
is a commutative diagram of continuous linear maps and Fréchet spaces such that $E_i$ and $F_i$ are closed subspaces of $E_{i+1}$ and $F_{i+1}$ respectively, $1 \leq i \leq 3$, and $(1)_n$ is induced from $(1)_{n+1}$, $n \in N$. Suppose in the following diagram which is the direct limit of $(1)_n$, $n \in N$, $\phi^1$ is surjective:

![Diagram](image)

Assume that for every $p \in N$ there exists $r \geq p$ such that,

$$(3)_{p,r} \ 	ext{if } f \in (E^p_2)^* \text{ and } g \in (F^p_1)^* \text{ such that } f \circ \alpha_2 = g \phi^1, \text{ then for some } h \in (F^p_2)^* \text{ we have } f \alpha_2 = h \phi^2 \text{ on } E^p_2.$$

Let $Z^n_i = \ker \phi^i$, $i = 1, 2, n \in N$. Then for every $n \in N$ there exists $m \in N$ such that $(\alpha^2 \alpha^1(Z^n_i))^- \supseteq \alpha^2(Z^n_i)$.

**Proof.** Fix $n \in N$. There exists $r \geq n$ satisfying $(3)_{n,r}$. Since $\phi^1$ is surjective, $\phi^1(E^r_2) \supseteq F^r_1$ for some $m \geq r$ (Lemma 2). We claim that $m$ satisfies the requirement.

Suppose the contrary. By the theorem of Hahn-Banach there exists $f \in (E^r_2)^*$ such that $f$ is identically zero on $(\alpha^2 \alpha^1(Z^n_i))^-$ and is not identically zero on $\alpha^2(Z^n_i)$. Let $G = E^r_1 \cap (\phi^1)^-(F^r_1)$ and $\psi : G \rightarrow F^r_1$ be induced by $\phi^1$. Since $\psi$ is surjective, $\psi$ is open [8, p. 41, Theorem 8]. $f \circ \alpha_1$ is zero on $Z^n_1 \supseteq \ker \phi$. Hence there exists $g \in (F^r_1)^*$ such that $f \alpha_2 \alpha_1 = g \phi^1$ on $G \supseteq E^r_1$. By $(3)_{n,r}$ there exists $h \in (F^p_2)^*$ such that $f \alpha_2 \circ \alpha_1$ is zero on $E^r_2$, $f \alpha_2 \alpha_1(Z^n_i) = h \phi^2(Z^n_i) = 0$, contradicting that $f$ is not identically zero on $\alpha^2(Z^n_i)$. Q.E.D.

**Lemma 4.** Suppose $G$ and $H$ are Fréchet spaces and $\psi : G \rightarrow H$ is a continuous linear surjection. Suppose $\{b_q\}_{q \in N}$ is a sequence in $H$ converging to $b$ and $a \in G$ with $\psi(a) = b$. Then there exists a sequence $\{a_q\}_{q \in N}$ in $G$ converging to a such that $\psi(a_q) = b_q, q \in N$.

**Proof.** $\{b_q - b\}_{q \in N}$ is a sequence in $H$ converging to zero. Suppose we can find a sequence $\{c_q\}_{q \in N}$ in $G$ converging to zero such that $\psi(c_q) = b_q - b$. Then $a_q = c_q + a, q \in N$, satisfies the requirement. So we can assume without loss of generality that $a = 0$ and $b = 0$.

Since $\psi$ is open, we can choose open neighborhood bases $\{V_q\}_{q \in N}$ of 0 in $G$ and $\{W_q\}_{q \in N}$ of 0 in $H$ such that $V_{q+1} \subseteq V_q, W_{q+1} \subseteq W_q$, and $\psi(V_q) \supseteq W_q, q \in N$.

Since $b_q \rightarrow 0$, we can find a strictly increasing function $p : N \rightarrow N$ such that if $q \in N$ and $k \geq p(q)$, then $b_k \in W_q$. We are going to define $\{a_q\}_{q \in N}$. Fix $q \in N$. If $q < p(1)$, choose any $q$ such that $\psi(a_q) = b_q$. If $q \geq p(1)$, then there is a unique $r \in N$ such that $p(r) \leq q < p(r+1)$. $b_q \in W_r$. Choose $a_q \in V_r$, such that $\psi(a_q) = b_q, a_q \rightarrow 0$.

II. Suppose $B$ is a holomorphic vector bundle on an $n$-dimensional complex manifold $M$. Let $\lambda^s_t$ or simple $\lambda_t^s$ denote the $C^\infty$ vector bundle of $(r, s)$-forms on
M. B* denotes the dual of B (transition functions of B* are the inverse transpose of those of B).

\( \mathcal{O}(B) \) = the sheaf of germs of holomorphic sections of B.

\( \mathcal{D}r-s(B) \) = the sheaf of germs of \( C^\infty \) sections of \( B \otimes \lambda^{r,s} \).

\( \mathcal{D}r-s(B^*) \) = the sheaf of germs of distribution sections of \( B^* \otimes \lambda^{r,s} \).

\( D^{-s}(B^*) = \Gamma(M, \mathcal{D}r-s(B^*)) \).

If \( A \) is a closed subset of \( M \), then \( E^{-s}(B) \) = the Fréchet space of all global \( C^\infty \) sections of \( B \otimes \lambda^{r,s} \) whose supports are contained in \( A \).

Suppose \( \Lambda = \{ \Lambda_k \}_{k \in \mathbb{N}} \) is a sequence of closed subsets of \( M \) such that \( \Lambda_k \subseteq \Lambda_{k+1} \), \( k \in \mathbb{N} \), and \( M = \bigcup_{k \in \mathbb{N}} \Lambda_k \). Let \( E^{-s}(B, \Lambda) = \bigcup_{k \in \mathbb{N}} E^{-s}_{\text{ET}}(B) \) denote the strict inductive limit of \( \{ E^{-s}_{\text{ET}}(B) \}_{k \in \mathbb{N}} \). \( E^{-s}(B, \Lambda) \) is an LF-space. If in addition every \( \Lambda_k \) is compact, then denote \( E^{-s}(B, \Lambda) \) by \( E^{-s}_{\text{ET}}(B) \). \( E^{-s}_{\text{ET}}(B) \) is independent of the choice of \( \Lambda \) and is the LF-space of all global \( C^\infty \) sections of \( B \otimes \lambda^{r,s} \) with compact supports. It is well known that \( D^{-n-r,n-s}(B^*) \) is canonically the dual of \( E^{-s}_{\text{ET}}(B) \) [15, p. 18, Proposition 4].

Let \( \Lambda^* \) denote \( \{ A \mid A \text{ is a closed subset of } M, A \cap \Lambda_k \text{ is compact for } k \in \mathbb{N} \} \). \( D^{-s}(B^*, \Lambda^*) \) denotes the vector space of all distribution sections of \( B^* \otimes \lambda^{r,s} \) whose supports are members of \( \Lambda^* \). Suppose \( T \in D^{-n-r,n-s}(B^*, \Lambda^*) \) and \( \phi \in E^{-s}(B, \Lambda) \). Then Supp \( \phi \subseteq \Lambda_k \) for some \( k \in \mathbb{N} \). Since \( \Lambda_k \cap \text{Supp } T \) is compact, we can find a \( C^\infty \) function \( \rho \) on \( M \) with compact support such that \( \rho \equiv 1 \) on some neighborhood of \( \Lambda_k \cap \text{Supp } T \). \( \rho \phi \in E^{-s}_{\text{ET}}(B) \). Let \( T(\rho \phi) \) be the value of \( T \) at \( \rho \phi \) when \( T \) is regarded as an element of \( D^{-n-r,n-s}(B^*) = E^{-s}_{\text{ET}}(B)^* \). \( \phi \mapsto T(\rho \phi) \) defines a continuous linear functional on \( E^{-s}(B^*, \Lambda^*) \), independent of the choices of \( k \) and \( \rho \).

**Proposition 1.** \( D^{-n-r,n-s}(B^*, \Lambda^*) \) is the dual of \( E^{-s}(B, \Lambda) \).

**Proof.** We have seen that every element of \( D^{-n-r,n-s}(B^*, \Lambda^*) \) defines a continuous linear functional on \( E^{-s}(B, \Lambda) \). Since \( E^{-s}_{\text{ET}}(B, \Lambda) \subset E^{-s}(B, \Lambda) \), if an element of \( D^{-n-r,n-s}(B^*, \Lambda^*) \) defines the zero functional on \( E^{-s}(B, \Lambda) \), then it must be zero.

Conversely, suppose \( F \) is a continuous linear functional on \( E^{-s}(B, \Lambda) \). Since \( D^{-n-r,n-s}(B^*) \) is the dual of \( E^{-s}_{\text{ET}}(B) \), there exists \( T \in D^{-n-r,n-s}(B^*) \) such that \( T \) defines the continuous linear functional \( F \mid E^{-s}_{\text{ET}}(B) \). Let \( A = \text{Supp } T \). We claim that \( A \in \Lambda^* \). If \( A \cap \Lambda_k \) is noncompact for some \( k \in \mathbb{N} \), then there exists a closed discrete sequence of distinct points \( \{ x_q \}_{q \in \mathbb{N}} \subset A \cap \Lambda_k \). For every \( q \in \mathbb{N} \) take an open neighborhood \( U_q \) of \( x_q \) in \( \Lambda^0_k + 1 \) such that every compact subset of \( M \) intersects only a finite number of \( U_q, q \in \mathbb{N} \). Since \( x_q \in A \), there exists \( \phi_q \in E^{-s}_{\text{ET}}(B, q), q \in \mathbb{N} \), such that \( \text{Supp } \phi_q \subseteq U_q \) and \( F(\phi_q) = 1 \). \( \phi_q \to 0 \) in \( E^{-s}(B, \Lambda) \) as \( q \to \infty \). \( F(\phi_q) \to 0 \) as \( q \to \infty \), contradicting that \( F(\phi_q) = 1 \) for all \( q \in \mathbb{N} \). Hence \( T \in D^{-s}(B^*, \Lambda^*) \). \( T \) defines a continuous linear functional on \( E^{-s}(B, \Lambda) \). Since this functional agrees with \( F \) on the dense subset \( E^{-s}_{\text{ET}}(B) \) of \( E^{-s}(B, \Lambda) \), it agrees with \( F \) on \( E^{-s}(B, \Lambda) \). Q.E.D.

Since \( B \) is a holomorphic vector bundle, the \( \bar{\partial} \)-operator mapping \( C^\infty (r, s) \)-forms...
to $\mathcal{C}^{\infty}(r,s+1)$-forms induces a map $\bar{\delta}: E^{r,s}(B) \to E^{r,s+1}(B)$. If $i: \mathcal{O}(B) \to \mathcal{E}^{0,0}(B)$ denotes the inclusion map, then

$\begin{align*}
0 \to \mathcal{O}(B) \xrightarrow{i} E^{0,0}(B) \xrightarrow{\bar{\delta}} E^{0,1}(B) \xrightarrow{\bar{\delta}} \ldots \xrightarrow{\bar{\delta}} E^{0,n}(B) \to 0
\end{align*}$

is exact [15, p. 14, Proposition 2].

For any open subset $U$ of $M$ let $\bar{\delta}^*: \Gamma(U, \mathcal{D}^{n-r,n-s-1}(B^*)) \to \Gamma(U, \mathcal{D}^{n-r,n-s}(B^*))$ be the transpose of $\bar{\delta}: E^{r,s}_{\text{top}}(B|U) \to E^{r,s+1}_{\text{top}}(B|U)$, where $B|U$ denotes the restriction of $B$ to $U$. $\{\bar{\delta}^*_i\}$ induces a sheaf-homomorphism

$\begin{align*}
\bar{\delta}^*: \mathcal{D}^{n-r,n-s-1}(B^*) \to \mathcal{D}^{n-r,n-s}(B^*).
\end{align*}$

If $i: \mathcal{O}(B^* \otimes \mathcal{C}^{n,0}) \to \mathcal{D}^{n,0}(B^*)$ denotes the inclusion map, then

$\begin{align*}
0 \to \mathcal{O}(B^* \otimes \mathcal{C}^{n,0}) \xrightarrow{i} \mathcal{D}^{n,0}(B^*) \xrightarrow{\bar{\delta}^*} \mathcal{D}^{n,1}(B^*) \xrightarrow{\bar{\delta}^*} \ldots \xrightarrow{\bar{\delta}^*} \mathcal{D}^{n,n}(B^*) \to 0
\end{align*}$

is exact [15, p. 14, Proposition 2]. Since $E^{r,q}_{\text{top}}(B)$ is dense in $E^{r,q}(B, \Lambda)$ for $q = s, s+1$,

$\begin{align*}
\bar{\delta}^*: E^{r,s}(B, \Lambda) \to E^{r,s+1}(B, \Lambda),
\end{align*}$

where the first and second maps are induced by $\bar{\delta}^*: \mathcal{D}^{n-r,n-s-1}(B^*, \Lambda^*) \to \mathcal{D}^{n-r,n-s}(B^*, \Lambda^*)$ is the transpose of

$\begin{align*}
\bar{\delta}: E^{r,s}_{\text{top}}(B, \Lambda) \to E^{r,s+1}_{\text{top}}(B, \Lambda),
\end{align*}$

III. Proposition 2. Suppose $A$ is an analytic subvariety of codimension $\geq 1$ in a reduced complex space $(X, \mathcal{O})$. Then there exist, uniquely up to isomorphism, a reduced complex space $X'$ and a proper nowhere degenerate holomorphic map $\pi: X' \to X$ such that

(i) $\pi$ induces a biholomorphic map $X' - \pi^{-1}(A) \approx X - A$,

(ii) for every open subset $U$ of $X'$ the following holds: if $f$ is a weakly holomorphic function on $U$ and for every $x \in U - \pi^{-1}(A)$ $f$ is (strongly) holomorphic at $x$, then $f$ is a (strongly) holomorphic function on $U$.

Proof. Let $\mathcal{S}$ be the sheaf of germs of weakly holomorphic functions on $X$ which are (strongly) holomorphic outside $A$. We need only prove that $\mathcal{S}$ is coherent. Then the construction and uniqueness of $X'$ and $\pi$ follows the same line as §4, pp. 118–122 of [10].

To prove the coherence of $\mathcal{S}$, we can assume without loss of generality the following:

(i) $X$ is a complex subspace of an open subset $G$ of $\mathbb{C}^n$ such that $\mathcal{O} \cong (\mathcal{O}/\mathcal{I})|X$ for some coherent ideal-sheaf $\mathcal{I}$ on $G$ and $X = \{x \in G \mid \mathcal{I}_x \neq 0, x\}$.

(ii) There is a holomorphic function $u'$ on $G$ such that $u = u'|X$ is a universal denominator on $X$.

Let $\mathcal{O}'$ be the sheaf of germs of weakly holomorphic functions on $X$. Let $\mathcal{I}$ be the gap-sheaf of $u\mathcal{O} + \mathcal{I}$ with $A$ as the exceptional subvariety [17, p. 381, Definition 10]. Let $\lambda: \mathcal{O}' \to \mathcal{O}$ be the sheaf-monomorphism defined by multiplication by $u$. 

\[ \mathcal{F} = \lambda^{-1}(u' \cap (\mathcal{F} | \mathcal{F}) X). \] Since \( \mathcal{F} \) is coherent on \( G \) [17, p. 392, Satz 9], \( \mathcal{F} \) is coherent. Q.E.D.

**Definition 1.** \( X' \) is called the partial normalization of \( X \) with respect to \( A \).

**Proposition 3.** Suppose \((X, \mathcal{H})\) is a complex space and \((X, \mathcal{O})\) is its reduction. Suppose \( p \in \mathbb{N} \). If \( H^p(X, \mathcal{F}) = 0 \) for every coherent analytic sheaf \( \mathcal{F} \) on \((X, \mathcal{O})\), then \( H^p(X, \mathcal{F}) = 0 \) for every coherent analytic sheaf \( \mathcal{F} \) on \((X, \mathcal{H})\).

**Proof.** The proof of Satz 3, p. 17 of [3], with trivial modifications yields this result. Q.E.D.

We introduce the following statements (where \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \)):

- **(A)\( _n^* \)**: If \( \dim X = n \) and \( X \) is reduced and has no compact \( n \)-dimensional branch, then \( H^n(X, \mathcal{F}) = 0 \).
- **(B)\( _n^* \)**: If \( \dim X = n \) and \( X \) is reduced and has only a finite number of compact \( n \)-dimensional branches, then \( \dim H^n(X, \mathcal{F}) < \infty \).
- **(A)\( _n^t \)**: If \( \dim X = n \) and \( X \) is reduced and normal and has no compact \( n \)-dimensional branch, then \( H^n(X, \mathcal{F}) = 0 \).
- **(A)\( _n^\# \)**: If \( \dim X = n \) and \( X \) is reduced, normal, connected, and noncompact and \( \mathcal{F} \) is torsion-free, then \( H^n(X, \mathcal{F}) = 0 \).

**Lemma 5.** \((A)\_n^* \) and \((B)\_n^* \) \( \Rightarrow \) \((A)\_n \) and \((B)\_n \).

**Proof.** \((A)\_n \) follows from Proposition 3. To prove \((B)\_n \), suppose \((X, \mathcal{H})\) is an \( n \)-dimensional complex space having only a finite number of compact \( n \)-dimensional branches whose union is \( K \) and suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( X \). Let \( \mathcal{K} \) be the subsheaf of all nilpotent elements of \( \mathcal{H} \). Since \( K \) is compact, there exists \( m \in \mathbb{N} \) such that \( \mathcal{K}^m = 0 \) on \( K \). Let \( Y = \text{Supp} \mathcal{K}^m \mathcal{F} \). \( Y \) is a subvariety disjoint from \( K \). By Satz 2, p. 275 of [13], and \((A)\_n \), \( H^n(X, \mathcal{K}^m \mathcal{F}) = H^n(Y, \mathcal{K}^m \mathcal{F}) = 0 \). The short exact sequences

\[
0 \to \mathcal{K}^r \mathcal{F} | \mathcal{K}^{r+1} \mathcal{F} \to \mathcal{F} | \mathcal{K}^{r+1} \mathcal{F} \to \mathcal{F} | \mathcal{K}^r \mathcal{F} \to 0, \quad 1 \leq r \leq m-1,
\]

yield exact sequences

\[
H^n(X, \mathcal{K}^r \mathcal{F} | \mathcal{K}^{r+1} \mathcal{F}) \to H^n(X, \mathcal{F} | \mathcal{K}^{r+1} \mathcal{F}) \to H^n(X, \mathcal{F} | \mathcal{K}^r \mathcal{F}), \quad 1 \leq r \leq m-1.
\]

Since \( \mathcal{K}^r \mathcal{F} | \mathcal{K}^{r+1} \mathcal{F}, \) \( 1 \leq r \leq m-1, \) and \( \mathcal{F} | \mathcal{K}^r \mathcal{F} \) can be regarded as coherent analytic sheaves on the reduction \((X, \mathcal{O})\) of \((X, \mathcal{H})\), by \((B)\_n^* \) and by induction on \( r \) we obtain \( \dim H^n(X, \mathcal{F} | \mathcal{K}^r \mathcal{F}) < \infty \) for \( 1 \leq r \leq m \). From the exact sequence

\[
0 = H^n(X, \mathcal{K}^m \mathcal{F}) \to H^n(X, \mathcal{F}) \to H^n(X, \mathcal{F} | \mathcal{K}^m \mathcal{F}),
\]

we conclude that \( \dim H^n(X, \mathcal{F}) < \infty \). Q.E.D.

**Lemma 6.** \((A)\_n-1 \) and \((A)\_n^\# \) \( \Rightarrow \) \((A)\_n \) and \((B)\_n \).
Proof. By Lemma 5, we need only prove (A)* and (B)*. Let $\mathcal{F}$ be a coherent analytic sheaf on an $n$-dimensional reduced complex space $(X, \mathcal{O})$. Let $B=\sigma(X)$.

(i) Assume that $B$ has no compact $(n-1)$-dimensional branch. Let $\pi: Y \to X$ be the normalization of $X$. Since $\pi$ is proper and nowhere degenerate, $R^n\pi(\pi^{-1}(\mathcal{F}))=0$ for $q \geq 1$. Hence

$$H^n(X, R^n\pi(\pi^{-1}(\mathcal{F}))) \approx H^n(Y, \pi^{-1}(\mathcal{F}))$$

[4, p. 418, Satz 6]. There is a natural sheaf-homomorphism $\lambda: \mathcal{F} \to R^n\pi(\pi^{-1}(\mathcal{F}))$ [4, p. 418, Satz 7(b)] inducing a sheaf-isomorphism on the restrictions to $X-B$. Let $\mathcal{N} = \ker \lambda$, $\mathcal{L} = \text{Im} \lambda$, and $\mathcal{Z} = \text{Coker} \lambda$. $\text{Supp } \mathcal{N} \subseteq B$ and $\text{Supp } \mathcal{Z} \subseteq B$. By (A)$_{n-1}$ and Satz 2, p. 275 of [13], $H^n(X, \mathcal{Z})=H^n(\text{Supp } \mathcal{Z})=0$ for $p=n-1, n$, and $H^p(X, \mathcal{N})=H^p(\text{Supp } \mathcal{N}, \mathcal{N})=0$ for $p=n, n+1$. The short exact sequences $0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{L} \to 0$ and $0 \to \mathcal{L} \to R^n\pi(\pi^{-1}(\mathcal{F})) \to \mathcal{Z} \to 0$ yield

$$H^n(X, \mathcal{F}) \approx H^n(X, \mathcal{L}) \approx H^n(X, R^n\pi(\pi^{-1}(\mathcal{F})))$$

By (A)$_n$ and [5, p. 245, VIII.A.19], $H^n(X, \mathcal{F}) \approx H^n(Y, \pi^{-1}(\mathcal{F}))=0$ if $X$ has no compact $n$-dimensional branch, and $H^n(X, \mathcal{F}) \approx H^n(Y, \pi^{-1}(\mathcal{F}))$ is finite-dimensional if $X$ has only a finite number of compact $n$-dimensional branches.

(ii) The general case. Let $A$ be the union of all compact $(n-1)$-dimensional branches of $B$. Let $\pi: X' \to X$ be the partial normalization of $X$ with respect to $A$. $\mathcal{O}^* = R^n\pi(\pi^{-1}(\mathcal{F}))$ is precisely the sheaf of germs of all weakly holomorphic functions on $X$ which are (strongly) holomorphic outside $A$. $\sigma(X')$ has no compact $(n-1)$-dimensional branch, because any compact $(n-1)$-dimensional branch $K$ of $\sigma(X')$ is contained in $n^{-1}(A)$ and hence $X'$ is normal at any point $x$ of $K$ that is a regular point of $\sigma(X')$, which implies that $\dim K \leq \dim \sigma(X') \leq n-2$ [10, p. 115, Lemma 3]. Let $\mathcal{J} \subseteq \mathcal{O}$ be the ideal-sheaf of $A$ and $\mathcal{J}' = \pi^{-1}(\mathcal{J})$. Let $\mathcal{C}$ be the ideal-sheaf on $X'$ defined by $\mathcal{C}_x = \{s \in \mathcal{O}_x | s_0^* \subseteq \mathcal{O}_x\}$ for $x \in X$. $\mathcal{C}$ is coherent and

$$\{x \in X | \mathcal{C}_x \neq \mathcal{O}_x\} \subset A.$$ 

Let $\{A_i\}_{i \in I}$ be the set of all branches of $A$. Since $A_i$, $i \in I$, is compact, the Hilbert Nullstellensatz [5, p. 97, III.A.7] implies that there exists $p: I \to N$ such that $\mathcal{J}_x^{(i)} \subseteq \mathcal{C}_x$ for $x \in A_i$ and $i \in I$. Let $\mathcal{J}' = \pi^{-1}(\mathcal{J})$ and $\mathcal{J}'' = R^n\pi(\mathcal{J}')$. We have a natural map $\lambda: \mathcal{F} \to \mathcal{J}''$. It is easily verified that

$$\lambda(\mathcal{F}_x) \cap R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F}))_x$$

for $x \in A_i$ and $i \in I$.

Let $\mathcal{J}_i \subseteq \mathcal{O}$ be the ideal-sheaf of $A_i$, $i \in I$. $\mathcal{J} = \prod_{i \in I} \mathcal{J}_i^{(i)}$ is a coherent ideal-sheaf on $X$. Let $\mathcal{J}' = \pi^{-1}(\mathcal{J})$. By (7) $\lambda(\mathcal{F}_x) \cap R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F}))_x$ for $x \in X-\sigma(A)$. Let $\mathcal{G} = \lambda(\mathcal{F}) \cap R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F}))$.

Then

$$\text{Supp } (R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F}))) \subset \sigma(A).$$

$$H^r(X, R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F})))=0$$

for $r=n-1, n$. The cohomology sequence of

$$0 \to \mathcal{G} \to R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F})) \to R^n\pi(\mathcal{J}'^{(i)}(\mathcal{F}))/\mathcal{G} \to 0$$
Let $X = \text{Ker} \lambda$. Let $\lambda: \mathcal{F}|\mathcal{X} \to \mathcal{F}''$ be the sheaf-monomorphism induced by $\lambda$ and let $\lambda' = \lambda^{-1}|\mathcal{F}''$. Let $\mathcal{D} = \text{Coker} \lambda$. Since outside $A$ $\lambda$ is isomorphic and $\mathcal{F}'' = \mathcal{F}''$, $\text{Supp } \mathcal{D} \subseteq A$. $H^n(X, \mathcal{D}) = 0$. The cohomology sequence of
\[0 \to \mathcal{F}' \to \mathcal{F}|\mathcal{X} \to \mathcal{D} \to 0\]
yields
\[\dim H^n(X, \mathcal{D}) \leq \dim H^n(X, \mathcal{F}'').\]
Since $\text{Supp } \mathcal{D} \subseteq A$, $H^n(X, \mathcal{D}) = 0$. The cohomology sequence of
\[0 \to \mathcal{F} \to \mathcal{F}|\mathcal{X} \to 0\]
yields
\[H^n(X, \mathcal{F}) \approx H^n(X, \mathcal{F}|\mathcal{X}).\]
Since $\pi$ is proper and nowhere degenerate,
\[H^n(X, \mathcal{F}''') \approx H^n(X, \mathcal{F}'').\]
From (8), (9), (10), and (11), we conclude that $\dim H^n(X, \mathcal{D}) \leq \dim H^n(X, \mathcal{F}'')$. Since $\sigma(X')$ has no compact $(n-1)$-dimensional branch, the result follows from (i). Q.E.D.

**Proposition 4.** $(A)_k^\#$, $1 \leq k \leq n \Rightarrow (A)_n$ and $(B)_n$.

**Proof.** By Lemma 6 and by induction on $n$, we conclude that it suffices to prove that $(A)_n^\# \Rightarrow (A)_n$.

Suppose $\mathcal{F}$ is a coherent analytic sheaf on an $n$-dimensional reduced normal complex space $X$ having no compact $n$-dimensional branch. We need only prove that $H^n(X, \mathcal{F}) = 0$. We can assume without loss of generality that $X$ is connected. Let $\mathcal{F}$ be the torsion-subsheaf of $\mathcal{F}$. Then $\mathcal{F}$ is coherent, $\mathcal{F}|\mathcal{F}$ is torsion-free, and $\dim \text{Supp } \mathcal{F} < n$ [1, pp. 14–15, Propositions 6, 7]. $H^n(X, \mathcal{F}) = 0$. By $(A)_n^\#$ $H^n(X, \mathcal{F}|\mathcal{F}) = 0$. The cohomology sequence of $0 \to \mathcal{F} \to \mathcal{F}: \mathcal{F}|\mathcal{F} \to 0$ yields $H^n(X, \mathcal{F}) = 0$. Q.E.D.

**IV. Lemma 7.** Suppose $f: X \to Y$ is a monoidal transformation with center $D \subseteq \sigma(Y)$ [7, p. 315, Definition 1], where $X$ and $Y$ are $n$-dimensional reduced complex spaces. Suppose there is a holomorphic function $u$ on $Y$ vanishing identically on no branch of $Y$ but vanishing identically on $\sigma(Y)$. If, for some relatively compact open subset $Q$ of $Y$ $H^n(f^{-1}(Q), \mathcal{V}) = 0$, then $H^n(Q, \mathcal{V}) = 0$.

**Proof.** By replacing $u$ by its sufficiently high power, we can assume without loss of generality that $u|Q$ is a section of the ideal-sheaf of the complex subspace $D$ and $u$ is a universal denominator on $Q$. Let $\mathcal{F}$ be the ideal-sheaf of the complex subspace $f^{-1}(D)$ on $X$. Let $v = u \circ f$. Since $\dim \text{Supp } (\mathcal{F}|v^k \mathcal{V}) < n$, $H^n(f^{-1}(Q), \mathcal{F}|v^k \mathcal{V}) = 0$. 


for $k \in \mathbb{N}$. The cohomology sequence of $0 \to \mathcal{O} \xrightarrow{\phi} \mathcal{F}^k \to \mathcal{F}^k[1] \mathcal{O} \to 0$, where $\phi$ is defined by multiplication by $\mathcal{O}$, yields $H^n(\mathcal{F}^k[1]; \mathcal{F}^k[1]) = 0$, $k \in \mathbb{N}$. For some $k \in \mathbb{N}$, $R^0f(\mathcal{F}^k[1])|\mathcal{O} = 0$ for $q \geq 1$ [7, p. 317, Lemma 2].

$$H^n(\mathcal{Q}, R^0f(\mathcal{F}^k[1])) \approx H^n(\mathcal{F}^k[1], \mathcal{F}^k[1]) = 0.$$

$uR^0f(\mathcal{F}^k[1]) \subseteq \mathcal{O}$ on $\mathcal{O}$. Since dim $\text{Supp}(\mathfrak{O}_0|uR^0f(\mathcal{F}^k[1])) < n$,

$$H^n(\mathcal{Q}, \mathfrak{O}_0|uR^0f(\mathcal{F}^k[1])) = 0.$$

The cohomology sequence of

$$0 \to R^0f(\mathcal{F}^k[1]) \xrightarrow{h} \mathcal{O} \to \mathfrak{O}_0|uR^0f(\mathcal{F}^k[1]) \to 0,$$

where $h$ is defined by multiplication by $u$, yields $H^n(\mathcal{Q}, \mathfrak{O}_0) = 0$. Q.E.D.

Suppose $(X, \mathfrak{O})$ is a connected, reduced, normal, noncompact complex space of
dimension $n \geq 1$. Suppose $\mathcal{F}$ is a torsion-free coherent analytic sheaf on $X$. Let

$$Z = \sigma(X) \cup \{x \in X | \mathcal{F}_x \text{ is not free over } \mathfrak{O}_x\}.$$

$Z$ is a subvariety of $X$ [1, p. 15, Proposition 8]. Let $B$ be the holomorphic vector
bundle on $X - Z$ such that $\mathcal{O}(B) \approx \mathcal{F}$ on $X - Z$. Let $0 \to \mathcal{F} \to \mathcal{F}_0 \xrightarrow{\phi_0} \mathcal{F}_1 \xrightarrow{\phi_1} \cdots$ be a flabby sheaf resolution of $\mathcal{F}$ on $X$. Let $\mathcal{F}_q = \text{Ker } \phi_q, q \in \mathbb{N}$. Let $d$ be a metric on $X$ defining the topology of $X$.

We introduce the following notation: if $G$ is an open subset of $X$, then $\Phi(G) = \{A \mid A$ is a closed subset of $G$ and $A \cap Z = \emptyset\}$ and $\Psi(G) = \{A \mid A$ is a closed subset of $G$ and $d(A, Z) > 0\}$. $\Phi(G)$ and $\Psi(G)$ are (paracompactifying) families of supports for $G - Z$ [13, p. 273, Definition 1].

**Lemma 8.** For every $x \in X$ there exists an open neighborhood $U$ of $x$ in $X$ such that for every open subset $W$ of $U$ $H^n(W, \mathcal{F}) = 0$.

**Proof.** Fix $x \in X$. By Main Theorem 1', p. 151 of [6], there exist an open neighborhood $V$ of $x$ in $X$ and a finite succession of monoidal transformations $f_i : V_i \to V_i$ with centers $D_i$ for $0 \leq i < r$ and $V_0 = V$ such that $D_i \subset \sigma(V_i)$ and $\sigma(V_i) = \emptyset$. Let $\mathcal{O}$ be the structure sheaf of $V_n$, $0 \leq i \leq r$. $\mathcal{O} = \mathcal{O}$. Choose two Stein open neighborhoods $U, U'$ of $x$ such that $U \subset U' \subset V$ and on $U$ we have a sheaf-epimorphism $g : \mathcal{O} \to \mathcal{F}$. Since $U'$ is Stein, we can find a holomorphic function $u$ on $U'$ which vanishes identically on $\sigma(U')$ but does not vanish identically on any branch of $U'$. We claim that $U$ satisfies the requirement. Take an open subset $W$ of $U$. Let

$$W_i = (f_0 \circ \cdots \circ f_{i-1})^{-1}(W) \text{ for } 1 \leq i \leq r \text{ and let } W_0 = W.$$ 

Then $H^n(W, \mathcal{O}) = 0$ ([12, p. 236, Problème 1]; or [16, Theorem]). By Lemma 7 and backward induction on $i$, we have $H^n(W_i, \mathcal{O}) = 0, 0 \leq i \leq r$. Hence $H^n(W, \mathcal{O}) = 0$. Let $\mathcal{K} = \text{Ker } g$. $H^{n+1}(W, \mathcal{K}) = 0$. The cohomology sequence of $0 \to \mathcal{K} \to \mathcal{O} \to \mathcal{F} \to 0$ yields $H^n(W, \mathcal{F}) = 0$. Q.E.D.

**Proposition 5.** If $G$ is a relatively compact open subset of $X$, then there exists a relatively compact open neighborhood $\bar{G}$ of $G$ in $X$ such that for any open
neighborhood $D$ of $G$ in $\tilde{G}$ the restriction map $H^n(\tilde{G}, \mathcal{F}) \to H^n(D, \mathcal{F})$ is surjective. Consequently $\dim H^n(G, \mathcal{F}) < \infty$.

**Proof.** Let $U = \{U_i\}_{i=1}^m$ be a finite Stein covering of some neighborhood of $G^-$ such that (i) $U_i$, $1 \leq i \leq m$, is compact, and (ii) for any open subset $W$ of $U_i$, $1 \leq i \leq m$, $H^n(W, \mathcal{F}) = 0$. We claim that $\tilde{G} = \bigcup_{i=1}^m U_i$ satisfies the requirement. Let $D$ be an open neighborhood of $G$ in $\tilde{G}$. Define inductively $D_0 = D$ and $D_i = U_i \cup D_{i-1}$, $1 \leq i \leq m$. The following portions of Mayor-Vietoris sequences are exact:

$$\ldots \to H^n(A, \mathcal{F}) \to H^n(U_i, \mathcal{F}) \oplus H^n(D_{i-1}, \mathcal{F}) \to H^n(U_i \cap D_{i-1}, \mathcal{F}) \to \ldots,$$

$1 \leq i \leq m$ [2, p. 236, §17(a)]. $H^n(U_i \cap D_{i-1}, \mathcal{F}) = 0$ implies that $H^n(D_i, \mathcal{F}) \to H^n(D_{i-1}, \mathcal{F})$ is surjective, $1 \leq i \leq m$. The surjectivity of $H^n(\tilde{G}, \mathcal{F}) \to H^n(D, \mathcal{F})$ follows from $\tilde{G} = D_m$. In particular, $H^n(\tilde{G}, \mathcal{F}) \to H^n(G, \mathcal{F})$ is surjective.

$$\dim H^n(G, \mathcal{F}) < \infty$$
(cf. proof of Theorem 11, p. 239 of [2]). Q.E.D.

**Lemma 9.** $\dim Z \leq n - 2$.

**Proof.** Suppose $\dim Z \geq n - 1$. Since $\sigma(X) \leq n - 2$, we can take a connected Stein open subset $W$ of $X - \sigma(X)$ such that $\dim (W \cap Z) \geq n - 1$. Take a holomorphic function $f \neq 0$ on $W$ vanishing on $W \cap Z$. Since $\mathcal{F}$ is torsion-free, $f_x$ is not a zero-divisor for $\mathbb{F}_x$ for $x \in W$. $V = \{x \in W \mid \text{homological codimension of } (\mathcal{F} | f\mathcal{F})_x \leq n - 2\}$ is a subvariety of dimension $\leq n - 2$ in $W$ [14, p. 81, Satz 5]. There exists $x \in Z \cap W - V$. $\mathbb{F}_x$ is free over $\phi_x$, contradicting $x \in Z$. Q.E.D.

**Proposition 6.** If $G$ is a relatively compact open subset of $X$, then

$$\dim H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F}) < \infty.$$ 

**Proof.** By Proposition 5 there exists a relatively compact open neighborhood $\tilde{G}$ of $G^-$ in $X$ such that $H^n(\tilde{G}, \mathcal{F}) \to H^n(D, \mathcal{F})$ is surjective for any open neighborhood $D$ of $G$ in $\tilde{G}$. Since $\dim Z \leq n - 2$ (Lemma 9),

$$H^n(\tilde{G}, \mathcal{F}) \approx H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F}) \quad \text{and} \quad H^n(D, \mathcal{F}) \approx H^n_{D \cap \mathcal{O}}(D - Z, \mathcal{F})$$

[13, p. 273, Satz 3]. Hence

$$\text{(12)}$$

the restriction map $H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F}) \to H^n_{D \cap \mathcal{O}}(D - Z, \mathcal{F})$

is surjective for any open neighborhood $D$ of $G$ in $\tilde{G}$.

$$\dim H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F}) = \dim H^n(\tilde{G}, \mathcal{F}) < \infty \quad \text{(Proposition 5)}. \quad H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F})$$

is generated by the cohomology classes defined by a finite number of elements $t_1, \ldots, t_k \in \Gamma_{G \cap \mathcal{O}}(G - Z, \mathcal{F}_\infty)$. Let $s_i = t_i | G - Z$, $1 \leq i \leq k$. Then $\text{Supp } s_i \subset G^- \cap \text{Supp } t_i$, $1 \leq i \leq k$. Since $G^- \cap \text{Supp } t_i$ is compact and disjoint from $Z$, $d(\text{Supp } s_i, Z) > 0$, $1 \leq i \leq k$. $s_1, \ldots, s_k \in \Gamma_{G \cap \mathcal{O}}(G - Z, \mathcal{F}_\infty)$. We claim that the cohomology classes defined by $s_1, \ldots, s_k$ generate $H^n_{G \cap \mathcal{O}}(G - Z, \mathcal{F})$. 

Take \( u \in \Gamma_{\Phi}(G-Z, \mathcal{F}_n) \). Let \( \delta = d(\text{Supp } u, Z) > 0 \). Let \( D = G \cup (U_\delta(Z) \cap \bar{G}) \).

Define \( \tilde{u} \in \Gamma(D-Z, \mathcal{F}_n) \) by setting \( \tilde{u} = u \) on \( G-Z \) and \( \tilde{u} = 0 \) on \( U_\delta(Z) \cap \bar{G}-Z \).

Since \( \text{Supp } \tilde{u} \in \Phi(D) \), by (12) there exist \( \delta_i \in \Gamma_{\Phi}(D-Z, \mathcal{F}_{n-1}) \) and \( c_1, \ldots, c_k \in C \) such that \( \tilde{u} - \phi_{n-1}(\delta) = \sum_{i=1}^k c_i D \).

Let \( v = \delta \mid G-Z \). Then \( u - \phi_{n-1}(v) = \sum_{i=1}^k c_i \delta_i \).

Let \( A = \text{Supp } v \). To complete the proof, we need only show that \( d(A, Z) > 0 \). Let \( \bar{A} = \text{Supp } \tilde{v} \). \( \bar{A} = A \cap D \) for some closed subset \( A \) of \( X \). Take \( 0 < \eta < \delta \). Suppose \( d(A, Z) = 0 \). Then \( d(A \cap \bar{U}_\delta(Z), Z) > 0 \).

Lemma 10. If \( G \) is a nonempty relatively compact open subset of \( X \), then \( G - U_{1/k}(Z) \) is noncompact for some \( k \in N \).

Proof. Suppose \( G - U_{1/k}(Z) \) is compact for all \( k \in N \). We claim that \( \partial G \subseteq Z \).

Suppose \( x \in \partial G-Z \). Then \( d(x, Z) > 1/k \) for some \( k \in N \). \( x \in \partial (G - U_{1/k}(Z)) \). Being compact, \( G - U_{1/k}(Z) \) is closed. \( x \in \partial (G - U_{1/k}(Z)) = G = G^o \), contradicting \( x \in \partial G \).

Therefore \( \partial G \subseteq Z \). \( G^o \subseteq G \cup Z \). \( G^o - Z = G - Z \). Therefore \( G - Z \) is both open and closed in \( X-Z \). Since \( G - Z \neq \varnothing \) and \( X-Z \) is connected, \( G - Z = X-Z \). \( (G-Z)^- = (X-Z)^- = X \) is compact, contradicting that \( X \) is noncompact. Q.E.D.

Proposition 7. If \( G \) is a relatively compact open subset of \( X \), then

\[ H^n_{\Phi}(G-Z, \mathcal{F}) = 0. \]

Proof. We can assume without loss of generality that \( G \) is connected and nonempty. Let \( M = G-Z \) and \( \bar{B} = B \mid M \). Let \( \Lambda_k = \{ G - U_{1/k}(Z) \mid k \in N \} \) and \( \Lambda = \{ \Lambda_k \}_{k \in N} \).

Then

\[ E^{*,*}(\bar{B}, \Lambda) = \Gamma_{\Phi}(M, \mathcal{E}^{*,*}(\bar{B})) \]

By (4) \( H^n_{\Phi}(M, \mathcal{F}) \) is isomorphic to the cokernel of \( \delta : E^{0,n-1}(\bar{B}, \Lambda) \rightarrow E^{0,n}(\bar{B}, \Lambda) \).

Suppose \( H^n_{\Phi}(M, \mathcal{F}) \neq 0 \). Then by Proposition 6 and Lemma 1 \( \delta^* : D^{n,0}(\bar{B}, \Lambda^*) \rightarrow D^{n,1}(\bar{B}, \Lambda) \) is not injective. There exists \( f \in D^{n,0}(\bar{B}, \Lambda^*) \) such that \( \delta^* f = 0 \) and \( \text{Supp } f \neq \varnothing \). By (5) \( f \in \Gamma(M, \mathcal{E}(\bar{B} \otimes \lambda^0_M)) \). Since \( f \) is a holomorphic section of the holomorphic vector bundle \( \bar{B} \otimes \lambda^0_M \) and \( M \) is connected, \( \text{Supp } f = M \). Hence \( M \in \Lambda^* \), contradicting Lemma 10. Q.E.D.

Proposition 8. If \( G \) is a relatively compact open subset of \( X \), then

\[ H^n(G, \mathcal{F}) = 0. \]

Proof. By Proposition 5 there exists a relatively compact open neighborhood \( \bar{G} \) of \( G^{-} \) in \( X \) such that \( H^n(\bar{G}, \mathcal{F}) \rightarrow H^n(G, \mathcal{F}) \) is surjective. Take \( s \in \Gamma(G, \mathcal{F}_n) \).

There exist \( a \in \Gamma_{\Phi}(G, \mathcal{F}_{n-1}) \) and \( t \in \Gamma_{\Phi}(\bar{G}, \mathcal{F}_n) \) such that \( t|G = s - \phi_{n-1}(a) \).

Since \( H^n_{\Phi}(G-Z, \mathcal{F}) \approx H^n(\bar{G}, \mathcal{F}) \), there exist \( b \in \Gamma_{\Phi}(\bar{G}, \mathcal{F}_{n-1}) \) and \( u \in \Gamma_{\Phi}(\bar{G}-Z, \mathcal{F}_n) \).
Let $\phi_{n-1}(a) = v$. By Proposition 7 there exists $c \in \Gamma(\mathcal{G}(G-Z, \mathcal{F}_{n-1}))$ such that $\phi_{n-1}(c) = v$ on $G-Z$. 

\[ s = \phi_{n-1}(a + (b|G) + c). \]

Q.E.D.

**Definition 2.** If $A$ is a subset of $X$ and $X-A = (\bigcup_{i \in I} B_i) \cup (\bigcup_{j \in J} C_j)$ is the decomposition into topological components, where $B_i$, $i \in I$, is compact, and $C_j$, $j \in J$, is noncompact, then $A \cup \bigcup_{i \in I} B_i$, denoted by $\text{Env}_X(A)$, is called the envelope of $A$.

**Lemma 11.** Suppose $K$ and $L$ are compact subsets of $X$ and $Y$ is an open subset of $X$ such that $K \subseteq Y \subseteq L$, and $\text{Env}_X(K) = K$. If $\varepsilon > 0$, then there exist $0 < \delta < \varepsilon$ and a compact subset $K$ in $Y$ containing $K$ such that, if $s$ is a global holomorphic section of a holomorphic vector bundle on $X-K-\bar{U}_\delta(Z)$ and $s$ is identically zero on $X-L-\bar{U}_\delta(Z)$, then $s$ is identically zero on $Y-K-\bar{U}_\delta(Z)$.

**Proof.** Let $\bar{K}$ be a compact neighborhood of $K$ in $Y$. Let $X-K = \bigcup_{i \in I} B_i$ be the decomposition into topological components. $B_i \neq L$, $i \in I$. $J = \{i \in I \mid B_i \cap \partial L \neq \emptyset\}$ is finite, because $\partial L$ is compact, $\partial L \subseteq \bigcup_{i \in I} B_i$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. If $i \in I - J$, $B_i \cap L = \emptyset$, because $B_i \cap \partial L = \emptyset$, $B_i \cap L$ and $B_i$ is connected. Hence $L-K = \bigcup_{i \in I} B_i$. Take $x_i \in B_i-L-Z$, $i \in I$. For $i \in J$ and $\eta > 0$, $A_{i,\eta} = \{x \in B_i-Z \mid x$ can be joined to $x_i$ by a path $\gamma$ in $B_i-Z$ such that $d(\gamma, Z) > \eta\}$ is an open connected subset of $B_i-Z$. $B_i-Z = \bigcup_{n>0} A_{i,n}$ for $i \in J$. $A_{i,\varepsilon} \subseteq A_{i,n}$ for $i \in J$ and $0 < \eta < \varepsilon$. Since $B_i \cap (L-\bar{R}^0-U_{\delta}(Z))$ is a compact subset of $B_i-Z$ for $i \in J$ and $J$ is finite, there exists $0 < \delta < \varepsilon$ such that $d(x_i, Z) > \delta$ and $B_i \cap (L-\bar{R}^0-U_{\delta}(Z)) \subseteq A_{i,\delta}$ for $i \in J$. We claim that $\bar{K}$ and $\delta$ satisfy the requirement. Suppose $s$ is a global holomorphic section of a holomorphic vector bundle on $X-K-\bar{U}_\delta(Z)$ and $s$ is identically zero on $X-L-\bar{U}_\delta(Z)$. Since for $i \in J$ $A_{i,\delta}$ is a compact connected subset of $X-K-\bar{U}_\delta(Z)$ and $x_i \in A_{i,\delta} \cap (X-L-\bar{U}_\delta(Z)) \neq \emptyset$, $s$ is identically zero on $A_{i,\delta}$. The result follows from

\[ Y-\bar{K}-\bar{U}_\delta(Z) \subseteq L-K-\bar{U}_\delta(Z) \subseteq \bigcup_{i \in J} B_i \cap (L-\bar{R}^0-U_{\delta}(Z)) \subseteq \bigcup_{i \in J} A_{i,\delta}. \]

Q.E.D.

**Lemma 12.** Suppose $A$ is a subset of $X-Z$ such that $A \neq U_{1/k}(Z)$ is compact for $k \in \mathbb{N}$. Then $A$ is a closed subset of $X-Z$.

**Proof.** Suppose $\{x_q\}_{q \in \mathbb{N}} \subseteq A$ is a sequence approaching $x \in X-Z$ as a limit. Since $K=\{x\} \cup (\bigcup_{q \in \mathbb{N}} \{x_q\})$ is a compact subset of $X-Z$, $d(K, Z) > 1/k$ for some $k \in \mathbb{N}$. $\{x_q\}_{q \in \mathbb{N}} \subseteq A-\bar{U}_{1/k}(Z)$. $x \in A-\bar{U}_{1/k}(Z)$, because $A \neq U_{1/k}(Z)$ is compact. Q.E.D.

**Proposition 9.** Suppose $X_n$, $1 \leq i \leq 3$, are open subsets of $X$ such that $X_2 \subseteq X_1 \subseteq X$ and $\text{Env}_X(X_n^c) \subseteq X_2$. Suppose $\varepsilon > 0$. Then there exists $\delta > 0$ satisfying
the following: if \(a \in \Gamma(X_2 - Z, \mathcal{E}^{0,n-1}(B))\) such that \(\delta a = 0\) and \(d(\text{Supp}\ a, Z) \geq \varepsilon\), then there exists a sequence \(\{a_k\}_{k \in \mathbb{N}} \subset \Gamma(X_1 - Z, \mathcal{E}^{0,n-1}(B))\) such that \(\delta a_k = 0\) and \(d(\text{Supp}\ a_k, Z) \geq \delta\) for \(k \in \mathbb{N}\) and \(a_k|_{X_3 - Z} \to a|_{X_3 - Z}\) in \(\Gamma(X_3 - Z, \mathcal{E}^{0,n-1}(B))\).

**Proof.** For \(p \in \mathbb{N}\) and \(1 \leq i \leq 3\), let \(E_p^i\) be the Fréchet space 
\[
\{b \in \Gamma(X_i - Z, \mathcal{E}^{0,n-1}(B)) | d(\text{Supp}\ b, Z) \geq 1/p\}
\]
and \(F_p^i\) be the Fréchet space 
\[
\{b \in \Gamma(X_i - Z, \mathcal{E}^{0,n}(B)) | d(\text{Supp}\ b, Z) \geq 1/p\}.
\]
Let \(\phi_p^i : E_p^i \to F_p^i\) be induced by \(\delta\). Then we have \((1)_p\) for \(p \in \mathbb{N}\), where \(\alpha_p\), \(\beta_p\), \(i = 1, 2\), are the restriction maps. We need only prove that, for \(p \in \mathbb{N}\), there exists \(r \geq p\) such that \((3)_{p,r}\) holds.

Fix \(p \in \mathbb{N}\). Set \(K = \text{Env}_X(X_3^{-})\), \(L = X_2^{-}\), and \(\varepsilon = 1/4p\). \(K\) is compact (cf. Lemma 1, p. 333 of [11]). We can find a compact subset \(\hat{K}\) of \(Y\) containing \(K\) and \(0 < \delta < \varepsilon\) satisfying the requirement of Lemma 11. We claim that \((3)_{p,r}\) holds.

Suppose \(f \in (E_3^{p'})^*\) and \(g \in (F_1^i)^*\) such that 
\[
(13) \quad f * g = f.
\]
\(f\) and \(g\) can be extended respectively to \(\tilde{f} \in (E_3^*)^*\) and \(\tilde{g} \in (F_1^i)^*\). By Proposition 1 \(\tilde{f}\) and \(\tilde{g}\) are represented respectively by 
\[
\tilde{f} \in \Gamma(X_3 - Z, \mathcal{D}^{n,1}(B^*)) \quad \text{and} \quad \tilde{g} \in \Gamma(X_1 - Z, \mathcal{D}^{n,0}(B^*))
\]
such that \(\text{Supp}\ \tilde{f} - U_\varepsilon(Z)\) and \(\text{Supp}\ \tilde{g} - U_\varepsilon(Z)\) are compact for \(\eta > 0\). By Lemma 12, \(\text{Supp}\ \tilde{f}\) and \(\text{Supp}\ \tilde{g}\) are closed subsets of \(X - Z\). \(\tilde{f}\) and \(\tilde{g}\) can respectively be extended trivially to \(f' \in \Gamma(X - Z, \mathcal{D}^{n,1}(B^*))\) and \(g' \in \Gamma(X - Z, \mathcal{D}^{n,0}(B^*))\) such that \(\text{Supp}\ f' = \text{Supp}\ \tilde{f}\) and \(\text{Supp}\ g' = \text{Supp}\ \tilde{g}\). (13) and (6) imply that \(\delta^* g' = f'\) on \(X_1 - Z - \overline{U}_{1/p}(Z)\). Hence \(\delta^* g' = f'\) on \(X - Z - \overline{U}_{1/p}(Z)\). Since \(\text{Supp}\ f' \subset K\) and \(\text{Supp}\ g' \subset L\), by (5) \(g'\) is a holomorphic section of the holomorphic vector bundle \(B^* \otimes \chi_{X-\overline{U}_p}(Z)\) on \(X - K - \overline{U}_p(Z)\) and is identically zero on \(X - L - \overline{U}_p(Z)\). Hence \(g'\) is identically zero on \(Y - K - \overline{U}_p(Z) = X_2 - \text{Env}_X(X_3^{-}) - \overline{U}_{1/4p}(Z)\). Let \(\rho\) be a \(C^\infty\) function on \(X_2 - Z\) such that \(\rho = 0\) on \(\overline{U}_{1/4p}(Z)\) and \(\rho = 1\) on \(X - U_{1/2p}(Z)\). Let 
\[
g^* = \rho(g'|_{X_2 - Z}) \in \Gamma(X_2 - Z, \mathcal{D}^{n,0}(B^*)).
\]
\(\text{Supp}\ g^* \subset \text{Env}_X(X_3^{-}) - U_{1/3p}(Z)\). Hence \(\text{Supp}\ g^*\) is compact. By Proposition 1 \(g^*\) defines \(\tilde{h} \in (F_2^*)^*\). Let \(h = h|_{F_2^*}\). Since \(\delta^* g^* = f'\) on \(X_2 - \overline{U}_{1/2p}(Z)\), \(fa^*_p = h\phi^*_p\) on \(E_2^*\). Q.E.D.

**Proposition 10.** Assume \(n \geq 2\). Suppose \(X_i\), \(i = 1, 2\), are open subsets of \(X\) such that \(X_1 \subset X_2 \subset X\). Let \(U\) be a Stein covering of \(X_2\). Then for any \(f \in \mathcal{Z}^{n-1}(U, \mathcal{F})\), there exists \(g \in C^{n-2}(U, \mathcal{F})\) such that, for some \(\eta > 0\) the restriction of \(f + \delta g\) to \(U|_{X_1} \cap U_\varepsilon(Z)\) is zero.
**Proof.** Since $H^{n-1}_0(X_2 - Z, \mathcal{F}) \to H^{n-1}(X_2, \mathcal{F})$ is surjective, [13, p. 278, Satz 3], there exists $s \in \Gamma_{\pi_X}(X_2, \mathcal{F}_{n-1})$ defining the same cohomology class as $f$. Since $X_1 \subset X_2$, $d(X_1 \cap \text{Supp} s, Z) = \eta > 0$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \Gamma(X_2, \mathcal{I}_1) & \Gamma(X_2, \mathcal{I}_1) \\
\downarrow & \downarrow \\
C^0(U, \mathcal{I}) & C^0(U, \mathcal{I}) & C^0(U, \mathcal{I}) \\
\downarrow & \downarrow \\
C^1(U, \mathcal{I}) & C^1(U, \mathcal{I}) & C^1(U, \mathcal{I}) \\
\downarrow & \downarrow \\
\vdots & \vdots & \vdots
\end{array}
\]

All rows except the first and all columns except the first are exact. Since $\text{Supp} s \cap X_1 \cap U_n(Z) = \emptyset$, we can choose by diagram-chasing $s_{ij} = C^\alpha(U \cap s_{ij})$ for $0 \leq i, j \leq n-1$, $i+j=n-1$, $n-2$, and $h \in Z^{n-2}(U, \mathcal{I})$ such that (i) $s_{i-1} = \mu s$, $s_{i+1} = \phi s_{i}$ and $s_{i+1} = \delta s_i$ for $0 \leq i \leq n-2$ and $i+j=n-1$, and $s_0 = \lambda h$, (ii) the restrictions of $s_{ij}$ $(0 \leq i, j \leq n-1, i+j=n-1, n-2)$ and $h$ to $U \cap X_1 \cap U_n(Z)$ are zero. By diagram-chasing, we can find $g \in C^{n-1}(U, \mathcal{I})$ such that $h = \delta g$. Q.E.D.

**Proposition 11.** Suppose $X_1 \subset X_2$ are open subsets of $X$. Suppose $U_2$ is a Stein covering of $X_2$ such that $U_1 = U_2 | X_1$ covers $X_1$. Suppose $p \in \mathbb{N}$ and $a \in Z^p(U_1, \mathcal{I})$. Let $B_2$ be a Stein refinement of $U_2$ such that $B_2 = B_2 | X_1$ covers $X_1$ and refines $U_1$. In the following commutative diagram $C^*(U_2, \mathcal{I}) = \oplus_{q \in \mathbb{N}} C^{s-1}(U_2, \mathcal{I})$ and $\rho_1$, $\rho_2$, $\alpha$, and $\beta$ denote the restriction maps:

\[
\begin{array}{ccc}
C^*(U_2, \mathcal{I}) & \xrightarrow{\rho_2} & C^*(B_2, \mathcal{I}) \\
\alpha \downarrow & & \beta \downarrow \\
C^*(U_1, \mathcal{I}) & \xrightarrow{\rho_1} & C^*(B_1, \mathcal{I})
\end{array}
\]

If $\beta(b_q) = \rho_2(a)$ in $Z^p(B_1, \mathcal{I})$ for some $\{b_q\}_{q \in \mathbb{N}} \subset Z^p(B_2, \mathcal{I})$, then $\alpha(a_q) = a$ in $Z^p(U_1, \mathcal{I})$ for some $\{a_q\}_{q \in \mathbb{N}} \subset Z^p(U_2, \mathcal{I})$.

**Proof.** Since $\phi: Z^{s-1}(B_1, \mathcal{I}) \oplus Z^p(U_1, \mathcal{I}) \to Z^p(B_1, \mathcal{I})$ defined by $\phi(u \oplus v) = \delta u + \rho_1(v)$ is surjective, by Lemma 4 there exist $c_q \in C^{s-1}(B_1, \mathcal{I})$ and

\[
d_q \in Z^p(U_1, \mathcal{I}), \quad q \in \mathbb{N},
\]

such that

\[
\delta c_q + \rho_1 d_q = \beta b_q,
\]

(14)
and $c_i \to 0$ and $d_i \to a$. Since $H^p(U_2, \mathcal{F}) \approx H^p(\mathbb{B}_2, \mathcal{F})$, there exist $e_q \in C^{p-1}(\mathbb{B}_2, \mathcal{F})$ and $f_q \in Z^p(U_2, \mathcal{F})$, $q \in N$, such that

\begin{equation}
\delta e_q + \rho_2 f_q = b_q.
\end{equation}

From (14) and (15) we have $\rho_1(d_q - a f_q) = \delta(b_q - e_q)$, $q \in N$. Since $H^n(U_1, \mathcal{F}) \approx H^n(\mathbb{B}_1, \mathcal{F})$, for some $g_q \in C^{n-1}(U_1, \mathcal{F})$, $d_q - a f_q = \delta g_q$, $q \in N$. $g_q$ can be extended trivially to $\tilde{g}_q \in C^{n-1}(U_2, \mathcal{F})$ such that $\alpha g_q = g_q$, $q \in N$. $d_q = \alpha f_q + \delta \tilde{g}_q$, $q \in N$. Set $a_q = f_q + \delta \tilde{g}_q$, $q \in N$. Then $a_q \in Z^p(U_2, \mathcal{F})$, $q \in N$ and $\alpha(a_q) \to a$. Q.E.D.

**Proposition 12.** Assume $n \geq 2$. Suppose $X_1$, $X_2'$, and $X_3$ are open subsets of $X$ such that $X_1 \subset X_2' \subset X_3 \subset X$ and $Env_x(X_1^-) \subset X_2'$. Suppose $U$ is a Stein covering of $X_3$ whose restrictions to $X_2'$ and $X_1$ cover $X_2'$ and $X_1$ respectively. Let

$$\alpha: Z^{n-1}(U, \mathcal{F}) \to Z^{n-1}(U|X_1, \mathcal{F})$$

and

$$\beta: Z^{n-1}(U|X_2', \mathcal{F}) \to Z^{n-1}(U|X_1, \mathcal{F})$$

be the restriction maps. Then $\text{Im } \beta \subseteq (\text{Im } \alpha)^{-}$. 

**Proof.** Let $X_2$ be a relatively compact open neighborhood of $Env_x(X_1^-)$ in $X_2'$. Let $R_k = U_{2^{-k+1}}(Z) - U_{2^{-k-1}}(Z)$, $k \in N$ and $R_0 = X - \bigcup_{i=1}^{2} Z_i$. By Proposition 11 we can suppose without loss of generality that $U|X_2$ covers $X_2$, $U|X_2 - Z$ covers $X_i - Z$, $1 \leq i \leq 3$.

$$U|X_2 - Z = \bigcup_{k \geq 0} U|X_2 \cap R_k,$$

and

$$U|X_1 \cap U_{1/k}(Z)$$

covers $X_1 \cap Z$ for $k \in N$ and $i = 1, 3$.

Fix $f \in Z^{n-1}(U|X_2', \mathcal{F})$. We have to prove that there exists $\{f_q\}_{q \in N} \subset Z^{n-1}(U, \mathcal{F})$ such that $\alpha(f_q) \to \beta(f)$. By Proposition 10 there exists $g \in C^{n-2}(U|X_2', \mathcal{F})$ such that for some $k \in N$, $g + \delta g$ is zero when restricted to $U|U_2^{-k}(Z)$ and $X_2$. Suppose we have found $\{f_q\}_{q \in N} \subset Z^{n-1}(U, \mathcal{F})$ such that $\alpha(f_q) \to \beta(f + \delta g)$. Let $\tilde{g} \in C^{n-2}(U, \mathcal{F})$ be the trivial extension of $g$. Then $\alpha(f_q - \delta \tilde{g}) \to \beta(f)$. Hence we can assume without loss of generality that the restriction of $f$ to $U|U_2^{-k}(Z)$ and $X_2$ is zero. Let $U_i = U_i|X_1 - Z$, $1 \leq i \leq 3$. Let $h = f|U_2$.

Suppose $U_2 = \{U_{v}\}_{v \in \mathcal{V}}$. Let $\{\psi_{v}\}_{v \in \mathcal{V}}$ be a partition of unity subordinate to $U_2$. For $p \in N$ and $0 \leq q \leq n$ define

$$\phi_{p,q}: C^p(U_2, \mathcal{E}^{0,q}(B)) \to C^{p-1}(U_2, \mathcal{E}^{0,q}(B))$$

as follows: if $s = \{s_0 \cdots s_{p-1}\} \in C^p(U_2, \mathcal{E}^{0,q}(B))$, then set $(\phi_{p,q}(s))_0 \cdots s_{p-1} = \sum_{v \in \mathcal{V}} t_v$, where $t_v$ is the trivial extension of $\rho_p s_{v0} \cdots s_{v,p-1}$ to $U_{v0} \cap \cdots \cap U_{vp-1}$.

$$\delta \phi_{p,q} \text{ is the identity map on } Z^p(U_2, \mathcal{E}^{0,q}(B)).$$
From (16) we conclude that

if \( u \in \mathcal{C}^p(U_2, \mathcal{E}^{0,q}(B)) \) and the restriction of \( u \) to \( U_2|U_2-z(Z) \cap X_2-Z \)

(19) is zero for some \( r \in \mathbb{N} \), then the restriction of \( \phi_{p,q}(u) \) to

\( U_2|U_2-r-1(Z) \cap X_2-Z \) is zero.

Consider the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \Gamma(X_2-Z, \mathcal{E}^{0,0}(B)) & \rightarrow & \Gamma(X_2-Z, \mathcal{E}^{0,1}(B)) & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C^0(U_2, \mathcal{E}(B)) & \rightarrow & C^0(U_2, \mathcal{E}^{0,0}(B)) & \rightarrow & C^0(U_2, \mathcal{E}^{0,1}(B)) & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C^1(U_2, \mathcal{E}(B)) & \rightarrow & C^1(U_2, \mathcal{E}^{0,0}(B)) & \rightarrow & C^1(U_2, \mathcal{E}^{0,1}(B)) & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \\
\end{array}
\]

By induction on \( j \) we can define \( h_j \in \mathcal{C}^i(U_2, \mathcal{E}^{0,i}(B)) \), \( 0 \leq i, j \leq n-1, \ i+j=n-1, \ n-2 \) such that

\( h^{k-1} = \lambda h, \ h^j_j = \phi_{i+1,j} h^j_j + 1 \) and \( h^j_{j+1} = \delta h^j_j \) for \( 0 \leq i \leq n-2 \) and \( i+j = n-2 \). It is easily verified by induction on \( j \) that \( h_j \in \mathcal{Z}(U_2, \mathcal{E}^{0,j}(B)) \), \( 0 \leq i, j \leq n-1, \ i+j=n-1. \) Hence we can find a unique \( s \in \Gamma(X_2-Z, \mathcal{E}^{0,n}(B)) \) such that \( \mu(s) = h_{n-1} \). By (18) we have

\[
\begin{align*}
h^{k-1} &= \lambda h, \\
h_j^j + 1 &= \delta h_j \\
h^j_{j+1} &= \delta h_j \\
h_{j-1}^{j-1} &= \mu(s) \\
\end{align*}
\]

(20)

By (19) we have:

the restrictions of \( h_j \) \( (0 \leq i, j \leq n-1, \ i+j=n-1, n-2) \)

to \( U_2|U_2-k-\mathcal{Z}(Z) \cap X_2-Z \) are zero and \( d(\text{Supp } s, Z) \geq 2^{-k-n}. \)

By Proposition 9 there exist \( l \geq k+n \) and \( \{s(m)\}_{m \in \mathbb{N}} \subseteq \Gamma(X_3-Z, \mathcal{E}^{0,n-1}(B)) \) such that \( \bar{\partial}s(m) = 0, \ d(\text{Supp } s(m), Z) \geq 2^{-l}, \ m \in \mathbb{N}, \) and \( s(m)|X_3-Z \rightarrow s|X_3-Z \) in \( \Gamma(X_3-Z, \mathcal{E}^{0,n-1}(B)). \)

Let \( C^\infty_\mathbb{S}(U_1, \mathcal{E}^{0,q}(B)) = \{ a \in \mathcal{C}^\infty(U_1, \mathcal{E}^{0,q}(B)) \mid \text{ the restriction of } a \text{ to } U_1|U_2-k\mathcal{Z}(Z) \cap X_1-Z \text{ is zero} \} \), \( i = 1, 3. \) \( C^\infty_\mathbb{S}(U_1, \mathcal{E}^{0,q}(B)) \) is a Fréchet space. Since the exact sequence

\[
\begin{array}{ccccccc}
C^\infty_\mathbb{S}(U_3, \mathcal{E}^{0,q-1}(B)) & \rightarrow & C^\infty_\mathbb{S}(U_3, \mathcal{E}^{0,q}(B)) & \rightarrow & C^\infty_\mathbb{S}(U_3, \mathcal{E}^{0,q+1}(B)) \\
\end{array}
\]

is the direct sum of

\[
\begin{array}{ccccccc}
C^\infty_\mathbb{S}(U_1, \mathcal{E}^{0,q-1}(B)) & \rightarrow & C^\infty_\mathbb{S}(U_1, \mathcal{E}^{0,q}(B)) & \rightarrow & C^\infty_\mathbb{S}(U_1, \mathcal{E}^{0,q+1}(B)) \\
\end{array}
\]
and another exact sequence, by Lemma 4 we have the following:

(22)

\[
    \frac{\partial b}{\partial a} = a, \quad m \in \mathbb{N}, \quad \text{such that } \partial b_m = a_m, \quad m \in \mathbb{N}, \quad \text{and } b_m|\mathbb{U}_1 \rightarrow a.
\]

Let \( A = X_3 - U_2^{-i}(Z) \). Consider the following commutative diagram:

\[
    \begin{array}{cccccccccccc}
        & & & & & & & & & & \rightarrow & & \\
    0 \rightarrow & C^0(\mathbb{U}_3, \mathcal{O}(B)) & \rightarrow & C^0(\mathbb{U}_3, \mathcal{O}^{0,1}(B)) & \rightarrow & C^0(\mathbb{U}_3, \mathcal{O}^{0,2}(B)) & \rightarrow & \cdots \\
    \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \cdots \\
    0 \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,1}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,2}(B)) & \rightarrow & \cdots \\
    \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \cdots \\
    0 \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,1}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,2}(B)) & \rightarrow & \cdots \\
    \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \cdots \\
    0 \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,1}(B)) & \rightarrow & C^i(\mathbb{U}_3, \mathcal{O}^{0,2}(B)) & \rightarrow & \cdots \\
    \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \\
    \end{array}
\]

The composites of any two consecutive horizontal maps or any two consecutive vertical maps are zero. All the rows except the first are exact. By (20), (21), and (22) we can find \( \{s(m)_i\}_{m \in \mathbb{N}} \subset C^0(\mathbb{U}_3, \mathcal{O}^{0,j}(B)) \), \( 0 \leq i, j \leq n-1, \quad i+j=n-1, \quad n-2, \) and \( \{g_m\}_{m \in \mathbb{N}} \subset C^{n-1}_0(\mathbb{U}_3, \mathcal{O}(B)) \) such that

\[
    s(m)_{i-1} = \mu s(m) \\
    s(m)_{i+1} = \delta s(m) \\
    s(m)_{j+1} = \delta s(m) \\
    s(m)_{j-1} = \lambda g_m, \quad (0 \leq i \leq n-2, \quad i+j = n-2) \quad m \in \mathbb{N},
\]

and (ii) \( s(m)_{|\mathbb{U}_1} \rightarrow h|\mathbb{U}_1 \) for \( 0 \leq i, j \leq n-1, \quad i+j=n-1, \quad n-2 \). It follows that \( \delta g_m = 0, \quad m \in \mathbb{N} \) and \( g_m|\mathbb{U}_1 \rightarrow h|\mathbb{U}_1 \) in \( C^{n-1}_0(\mathbb{U}_3, \mathcal{O}(B)) \). Let

\[
    \mathfrak{S}_i = \mathfrak{U}_i \cup (\mathfrak{U}|U_2^{-i}(Z) \cap X_i), \quad i = 1, 3.
\]

By (17) \( \mathfrak{S}_i \) is a Stein refinement for \( \mathfrak{U}|X_i \), \( i = 1, 3 \). Define \( \{f_m\}_{m \in \mathbb{N}} \subset Z^{n-1}(\mathfrak{S}_3, \mathcal{F}) \) by setting \( f_m = g_m \) on \( \mathfrak{U}_3 \) and \( f_m = 0 \) on \( \mathfrak{U}|U_2^{-i}(Z) \cap X_3 \). Then \( f_m|\mathfrak{S}_1 \rightarrow f|\mathfrak{S}_1 \). The result follows from Proposition 11. Q.E.D.

V. Proof of the Main Theorem. By Proposition 4 we need only prove \( (A)^\# \) for \( n \in \mathbb{N} \). The case \( n=1 \) is well known [5, p. 270, IX.B.10]. Suppose \( \mathcal{F} \) is a torsion-free coherent analytic sheaf on a connected, reduced, normal, noncompact complex space \( X \) with \( 2 \leq \dim X = n \). We have to show that \( H^n(X, \mathcal{F}) = 0 \).

Take a sequence of relatively compact open subsets \( \{X_m\}_{m \in \mathbb{N}} \) of \( X \) such that \( \text{Env}_X(X_m) \subset X_{m+1} \) for \( m \in \mathbb{N} \) and \( \bigcup_{m \in \mathbb{N}} X_m = X \). Take a Stein covering \( \mathfrak{U} \) of \( X \) such that \( \mathfrak{U}_m = \mathfrak{U}|X_m \) covers \( X_m \) for \( m \in \mathbb{N} \) and \( \mathfrak{U} = \bigcup_{m \in \mathbb{N}} \mathfrak{U}_m \).

Take \( f \in Z^n(\mathfrak{U}, \mathcal{F}) \). By Proposition 8 there exists \( g_m \in C^{n-1}(\mathfrak{U}_m, \mathcal{F}) \) such that \( f|\mathfrak{U}_m = \delta g_m, \quad m \in \mathbb{N} \).
Take a sequence of seminorms \( \{ \| \cdot \|_{p(m)} \}_{p \in \mathbb{N}} \) on \( C^{n-1}(U_m, \mathcal{F}) \), \( m \in \mathbb{N} \), such that

(i) \( \| \cdot \|_{p(m)} \) defines the Fréchet space topology of \( C^{n-1}(U_m, \mathcal{F}) \), \( m \in \mathbb{N} \),
(ii) \( \| \cdot \|_{p(m)} \leq \| \cdot \|_{p(m) + 1} \) and \( \| \cdot \|_{p(m)} \leq \| \cdot \|_{p(m+1)} \), \( p, m \in \mathbb{N} \).

We are going to construct by induction on \( m \) \( h_m \in C^{n-1}(U_m, \mathcal{F}) \) such that

\[ \delta h_m = f|U_m \text{ and } \| (h_{m+2} - h_{m+1})|U_{m+1} \|_{m+1} < 2^{-m}, \quad m \in \mathbb{N}. \]

Set \( h_m = g_m \) for \( m = 1, 2 \). Suppose \( h_1, \ldots, h_m \) are found for some \( m \geq 2 \). Since \( \delta(h_m - g_{m+1}) = 0 \) on \( U_m \), by Proposition 12 there exists \( s \in Z^{n-1}(U_{m+1}, \mathcal{F}) \) such that

\[ \| (h_m - g_{m+1} - s)|U_m \|_{m+1} \|_{m+1} < 2^{-m+1}. \]

Set \( h_{m+1} = g_{m+1} + s \). The construction is complete. Define \( h \in C^{n-1}(U, \mathcal{F}) \) by setting \( h|U_m = \lim_{v \to m} h_v|U_m \). \( h \) is well defined and \( \delta h = f \). Q.E.D.

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