UV PROPERTIES OF COMPACT SETS(1)

BY

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1. **Introduction.** Homotopy properties of spaces of cellular decompositions of manifolds have proved to be very useful in the study of such decomposition spaces. One of the first results of this type was established by T. M. Price ([19], [20]) who showed that if G is a cellular decomposition of E^n and U is a simply connected open set in the associated decomposition space E^n/G, then $P^{-1}[U]$ is simply connected. Here P denotes the projection map from E^n onto E^n/G. This result is of basic importance to the techniques of [2], [3], and [4], and has also been applied in a series of papers ([5], [6], and [7]) studying local properties of decomposition spaces.

Martin [17] showed that a similar result can be obtained for upper semicontinuous decompositions of S^3 into compact absolute retracts. A study of the techniques used in these proofs showed that one should be able to establish such a result for a single class of decompositions which would include decompositions of manifolds into either cellular sets or compact absolute retracts as well as certain types of decompositions of various nonmanifold spaces. In these proofs, the crucial property that elements of the decomposition should have, can be described as follows. Suppose g is an element of the upper semicontinuous decomposition G of the space X. We require that if U is any open set containing g, there is an open set V containing g, contained in U, and such that each singular 1-sphere in V is homotopic to 0 in U. In an earlier paper [10], a set with this property was called "semicellular"; in this paper, it appears as one of a family of properties, and is called "property 1-UV".

It was also clear that in the case of cellular decompositions of manifolds, results analogous to those of Price and Martin in dimension one could be obtained for higher dimensions. Property 1-UV can be generalized to other dimensions and there results a family of UV properties. With hypotheses concerning suitable UV properties, one can extend the methods of Price and Martin to obtain analogous results on maps of (finite simplicial) complexes of arbitrary dimension.

This paper is devoted to the study of UV properties of compact sets. These properties and certain closely related properties are discussed in §3. In §4, we state some results on LC^n spaces that we shall need later. In §5, we study UV properties of compact sets in LC^n spaces.

Received by the editors April 18, 1968.

(1) Research supported in part by the National Science Foundation.

The results of this paper will be applied in [8] to the study of upper semicontinuous decompositions of spaces into compact sets with various UV properties. A number of basic results concerning upper semicontinuous decompositions of spaces into compact sets with UV properties are established in [9]. The present paper, [8], and [9] study the homotopy properties of upper semicontinuous decompositions of spaces into compact sets with UV properties.

Closely related properties have been studied by other writers; see [1], [15], and [16]. Hyman's notion of "absolute neighborhood contractibility" [15] appears to be closely related to property UV^∞ of this paper. Lacher's "property (**)" of [16] is essentially our property UV^∞, and the notions of "cell-like" [16] and "like-a-point" [1] (defined in [1] only for E^3) are equivalent. As shown in §3 below, and also in [16], property UV^∞ and the property of being cell-like are closely related.

The author thanks R. H. Bing and D. R. McMillan, Jr. for helpful discussions on various matters related to this paper.

2. **Notation and terminology.** Suppose n is a positive integer. The statement that M is an *n-manifold* means that M is a separable metric space such that each point of M has an open neighborhood which is an open n-cell. The statement that M is an *n-manifold-with-boundary* means that M is a separable metric space such that each point of M has a neighborhood which is an n-cell. If M is an n-manifold-with-boundary, then a point p of M is an *interior point* of M if and only if p has an open neighborhood which is an open n-cell. The set of all interior points of M is the *interior* of M, denoted by Int M. The *boundary* of M, Bd M, is $M-\text{Int } M$.

Suppose M is an n-manifold. If A is a subset of M, then A is *cellular* in M if and only if there exists a sequence $C_1, C_2, C_3, \ldots$ of n-cells in M such that (1) for each positive integer i, $C_{i+1} \subset \text{Int } C_i$, and (2) $\bigcap_{i=1}^{\infty} C_i = A$. If M is an n-manifold, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M and each element of G is a cellular subset of M.

I denotes the closed interval [0, 1]. If n is a positive integer, I^n denotes the n-cell which is the product of n copies of [0, 1], and S^n denotes the unit sphere in E^{n+1} with center at the origin.

If M is a set in a metric space X and ε is a positive number, then $V(\varepsilon, M)$ is the open ε-neighborhood of M in X.

3. UV **properties of sets.** In this section we shall introduce and study the UV properties of sets in topological spaces.

Suppose X is a topological space, M is a subset of X, and n is a nonnegative integer. M has *property* n-UV if and only if for each open set U containing M, there is an open set V containing M such that (1) $V \subset U$ and (2) each singular n-sphere in V is homotopic to 0 in U. M has *property* UV^n if and only if for each nonnegative integer i such that $i \leqq n$, M has property i-UV.

M has *property* UV^ω if and only if for each nonnegative integer k, M has property

k-UV. M has *property* UV$^\infty$ if and only if for each open set U containing M, there is an open set V containing M such that (1) $V \subset U$ and (2) V is contractible in U.

PROPOSITION 3.1. *If the subset M of a topological space X has property* UV$^\infty$, *then M has property* UV$^\omega$.

PROPOSITION 3.2. *Suppose X is a locally compact, locally connected metric space, and M is a compact subset of X. Then M has property* 0-UV *if and only if M is connected.*

Proof. Suppose M is connected, and let U be an open set containing M. Let V be the component of U containing M. Since V is path-connected, it follows that each singular 0-sphere in V is homotopic to 0 in V, hence in U. Hence M has property 0-UV.

Suppose M has property 0-UV but is not connected. M is the union of two disjoint closed nonvoid sets M_1 and M_2. Let U_1 and U_2 be disjoint open sets containing M_1 and M_2, respectively. Let U be $U_1 \cup U_2$. Suppose V is any open set such that $M \subset V$ and $V \subset U$. Let a and b be points of $V \cap M_1$ and $V \cap M_2$, respectively. There is no path in U joining a to b, and this contradicts the fact that M has property 0-UV. Hence M is connected.

We turn now to the problem of equivalence, in n-dimensional triangulated manifolds, of properties UV^n and UV$^\infty$. We first prove a useful lemma.

LEMMA 3.3. *Suppose X is a topological space, M is a subset of X, n is a nonnegative integer, and M has property* UV^n. *If U is an open set containing M, there is an open set V containing M such that* (1) $V \subset U$ *and* (2) *if K is any finite n-complex and f is any continuous function from K into V, then $f \sim 0$ in U.*

Proof. Since M has property UV^n, there is a finite sequence $V^n, V^{n-1}, \ldots,$ and V^0 of open sets such that (1) $V^n \subset U$ and each singular n-sphere in V^n is homotopic to 0 in U, (2) if i is any nonnegative integer such that $i < n$, then $V^i \subset V^{i+1}$ and each singular i-sphere in V^i is homotopic to 0 in V^{i+1}, and (3) $M \subset V^0$. Let V denote V^0.

Suppose now that K is a finite n-complex and f is a continuous function from K into V. Let v_0 be a vertex not in K, let v_0K be the cone from v_0 over K, and for any subcomplex L of K, let v_0L be the corresponding cone from v_0 over L. For each nonnegative integer i such that $i \leqq n$, let K^i be the ith skeleton of K. Let p be some point of V.

If v is a vertex of K, there is a singular 1-cell in V^1 joining $f(v)$ and p. Now f is defined on K^1. It follows that there exists a continuous function F^0 from $(v_0K^0) \cup K^1$ into V^1 such that $F^0(v_0)=p$ and if σ^1 is any 1-simplex of K, $F^0|\sigma^1 = f|\sigma^1$.

Suppose $\langle xy \rangle$ is a 1-simplex of K. Then $F^0|\mathrm{Bd}\, \langle v_0xy \rangle$ maps $\mathrm{Bd}\, \langle v_0xy \rangle$ into V^1. There is, therefore, a continuous extension of $F^0|\mathrm{Bd}\, \langle v_0xy \rangle$ to $\langle v_0xy \rangle$, taking $\langle v_0xy \rangle$ into V^2. Further, f is defined on K^2. It follows that there exists a continuous extension F^1 of F^0 to $(v_0K^1) \cup K^2$, taking $(v_0K^1) \cup K^2$ into V^2 and such that if σ^2 is a 2-simplex of K, $F^1|\sigma^2 = f|\sigma^2$.

Suppose $\langle xyz \rangle$ is a 2-simplex of K. Then $F^1|\mathrm{Bd}\, \langle v_0xyz \rangle$ maps $\mathrm{Bd}\, \langle v_0xyz \rangle$ into V^2. There is a continuous extension of $F^1|\mathrm{Bd}\, \langle v_0xyz \rangle$ to $\langle v_0xyz \rangle$, taking $\langle v_0xyz \rangle$ into V^3. Hence there is a continuous extension F^2 of F^1 to $(v_0K^2) \cup K^3$, taking $(v_0K^2) \cup K^3$ into V^3 and such that if σ^3 is a 3-simplex of K, $F^2|\sigma^3 = f|\sigma^3$.

Let this process be continued. There results a continuous function F^{n-1} from $(v_0K^{n-1}) \cup K^n$ into V^n and such that if σ^n is any n-simplex of K, $F^{n-1}|\sigma^n = f|\sigma^n$. Each singular n-sphere in V^n is homotopic to 0 in U. Hence if σ^n is an n-simplex of K, there is a continuous extension of $F^{n-1}|\mathrm{Bd}\,(v_0\sigma^n)$ to $v_0\sigma^n$, taking $v_0\sigma^n$ into U. It follows that there is a continuous extension F of F^{n-1} to v_0K, taking v_0K into U and such that $F|K = f$. Therefore $f \sim 0$ in U.

PROPOSITION 3.4. *Suppose n is a positive integer, M is a compact subset of a triangulated n-manifold N, and M has property* UV^n. *Then M has property* UV^∞.

Proof. Suppose U is an open set in N containing M. By Lemma 3.3, there is an open set V such that $M \subset V$, $V \subset C$, and if K is any finite n-complex and f is any continuous function from K into V, then $f \sim 0$ in U.

Let D be a compact polyhedral neighborhood of M lying in V. Let f be the identity map from D onto D. Then $f \sim 0$ in U, and it follows that $(\mathrm{Int}\, D)$ is an open set W in N such that $M \subset W$, $W \subset U$, and W is contractible in U.

COROLLARY 3.5. *If n is a positive integer, M is a compact subset of E^n, and M has property* UV^n, *then M has property* UV^∞.

We point out conditions under which sets have certain UV properties. Every compact absolute retract in a manifold has property UV^∞. This result is known; it follows from results of §5. Also in §5 we show that compact absolute retracts in spaces with suitable LC properties have certain UV properties. The following fact permits us to apply our results to cellular decompositions of manifolds.

PROPOSITION 3.6. *If M is a cellular subset of manifold, then M has property* UV^∞ *and hence has all the* UV *properties.*

The next proposition concerns tree-like continua in E^n. For a definition of *tree-like* and of *tree-chain*, see [12].

PROPOSITION 3.7. *If n is a positive integer and M is a tree-like continuum in E^n, then M has property* 1-UV.

We first establish the following lemma. It is an extension of a result of Bing's [12] to the effect that the corresponding result holds in the plane. There exist tree-like continua not embeddable in E^2; see Example 1 of [12].

LEMMA 3.8. *If n is a positive integer, $n > 2$, M is a tree-like continuum in E^n, and ε is a positive number, there is a tree-chain of connected open sets with mesh less than ε and covering M.*

Proof. Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of open n-cells covering M and such that each of $B_1, B_2, \ldots$, and B_k is of diameter less than ε. Let $\{W_1, W_2, \ldots, W_j\}$ be a tree-chain $\mathscr{W}$ such that $\mathscr{W}$ covers M and each set of $\mathscr{W}$ lies in some one of $B_1, B_2, \ldots$, and B_k. If $i=1, 2, \ldots$, or j, each component of W_i is open and we may, therefore, assume that each set of $\mathscr{W}$ has only finitely many components. Further, we shall assume that if $i=1, 2, \ldots$, or j, each component of W_j is the interior of some polyhedron.

If W_1 is connected, let V_1 be W_1. If W_1 is not connected, we first hook the components of W_1 together. We do this by joining distinct components of W_1 with long thin open cells, drilling holes in other W's to prevent unwanted intersections. Let $W_{11}, W_{12}, \ldots$, and W_{1r_1} denote the components of W_1. There is a positive integer s_1 such that $W_1 \subset B_{s_1}$. Since M is tree-like, M has dimension 1, and hence $B_{s_1} - M$ is connected. Let A_{11} be a polygonal arc in $B_{s_1} - M$ joining a point of W_{11} and a point of W_{12}. Let A_{11}^* be a polyhedral n-cell obtained by a slight thickening of A_{11} such that $A_{11} \subset \operatorname{Int} A_{11}^*$. It is easy to see that A_{11} and A_{11}^* may be constructed so that (1) A_{11}^* lies in $B_{s_1} - M$ and (2) if $W \in \mathscr{W}$ and W' is any component of W, then $W' - A_{11}^*$ is connected. Condition (2) involves drilling holes in certain W's and we do this so that we do not thereby increase the number of components.

If $t=2, 3, \ldots$, or j, let $W_t^{(1)}$ denote $W_t - A_{11}^*$. Then $W_t^{(1)}$ is open, has the same number of components as W_t, and satisfies conditions similar to those satisfied by W_t. Let $W_{11}^{(1)}$ be $W_{11} \cup W_{12} \cup \operatorname{Int} A_{11}^*$. Then $W_{11}^{(1)}$ is open, connected, and lies in B_{s_1}. Further, if $t=2, 3, \ldots$, or j and $W_{11}^{(1)}$ intersects $W_t^{(1)}$, then W_1 intersects W_t. Let $W_1^{(1)}$ be $W_{11}^{(1)} \cup W_{13} \cup \cdots \cup W_{1r_1}$. It follows that $\{W_1^{(1)}, W_2^{(1)}, \ldots, W_j^{(1)}\}$ is a tree-chain covering M such that (1) each of its elements lies in some one of $B_1, B_2, \ldots$, and B_k, and (2) $W_1^{(1)}$ has fewer components than W_1.

If $W_1^{(1)}$ is not connected, the above process may be repeated. After finitely many steps, there results a tree-chain $\{W_1', W_2', \ldots, W_j'\}$ covering M and such that (1) each set of $\{W_1', W_2', \ldots, W_j'\}$ lies in some set of $B_1, B_2, \ldots$, and B_k, (2) W_i' is connected, and (3) if $i=2, 3, \ldots$, or k, the number of components of W_i' does not exceed the number of components of W_i.

By repeated application of the process described above, a tree-chain satisfying the conclusion of Lemma 3.8 may be constructed.

Proof of Proposition 3.7. The proposition is obvious if $n=1$. If $n=2$, it follows by [12] that M does not separate E^2. By well-known theorems on E^2, it follows that if U is an open set in E^2 containing M, there is a disc D such that $M \subset \operatorname{Int} D$ and $D \subset U$. Hence M has property 1-UV.

Therefore suppose $n>2$ and suppose that U is an open set in E^n containing M. There is a finite collection $\{C_1, C_2, \ldots, C_k\}$ of open n-cells covering M and such that if $i=1, 2, \ldots$, or k, $C_k \subset U$. With the aid of Lemma 3.8, it follows that there is a tree-chain T of connected open sets covering M and such that if t and t' are intersecting sets of T, $(t \cup t')$ lies in some set of $\{C_1, C_2, \ldots, C_k\}$. Let V denote $\bigcup \{t : t \in T\}$, and suppose γ is a singular 1-sphere in V.

Since T is a tree-chain, there is a set t_0 of T such that there exists exactly one set of T distinct from t_0 and intersecting t_0; let t_1 denote the set of $T-\{t_0\}$ which intersects t_0. There is a positive integer r such that $r \leqq k$ and $(t_0 \cup t_1) \subset C_r$.

There exists a finite set $\{A_1, A_2, \ldots, A_m\}$ of singular arcs on γ such that $\{A_1, A_2, \ldots, A_m\}$ covers $(\gamma \cap t_0) - t_1$ and if $i = 1, 2, \ldots$, or m, A_i lies in t_0 and has both endpoints in t_1. Let x_1 and y_1 be the endpoints of A_1. Since t_1 is connected, there is a singular arc A_1' from x_1 to y_1 and lying in t_1. Now $A_1 \cup A_1' \subset C_r$, and hence $A_1 \sim A_1'$ in C_r with fixed endpoints. After finitely many repetitions of this process, there results a singular 1-sphere γ' in $\bigcup \{t : t \in T \text{ and } t \neq t_0\}$ such that $\gamma \sim \gamma'$ in U.

By an inductive argument, it may be shown that for some set t' of T and some singular 1-sphere γ'' in t', $\gamma \sim \gamma''$ in U. For some positive integer s such that $s \leqq k$, $t \subset C_s$. Then $\gamma'' \sim 0$ in C_s, hence in U. It follows that $\gamma \sim 0$ in U.

In the remainder of this section, we consider the relationship between the UV properties and cellularity. We have already pointed out, in Proposition 3.6, that each cellular subset of a manifold has property UV$^\infty$.

Now we shall give some examples of noncellular continua in E^3 which have property UV$^\infty$.

1. Let α be a noncellular arc in E^3, such as Example 1.1 of [11]. Since α is a compact absolute retract, it has property UV$^\infty$ in E^3.

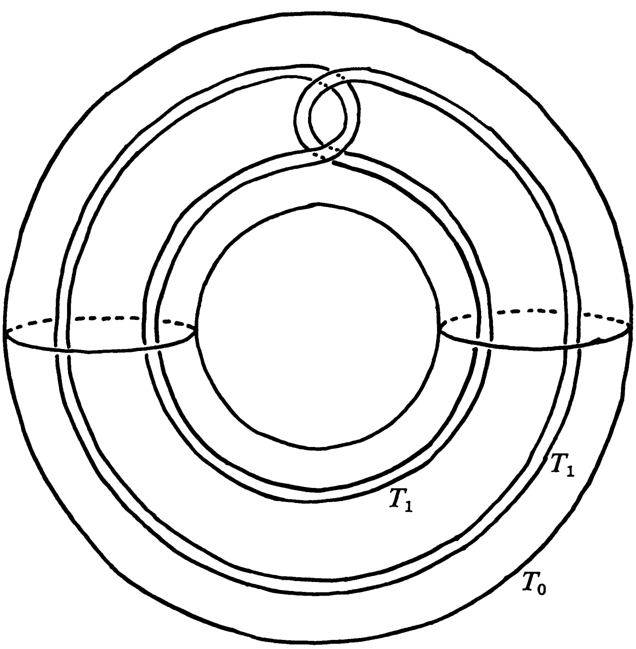

FIGURE 1

2. We give an example of a continuum in E^3 which is not cellular, not locally connected, yet has property UV^∞.

Let T_0 be a smooth solid torus in E^3 as shown in Figure 1. Let T_1 be a smooth solid torus embedded in Int T_0 as shown in Figure 1. We assume that T_1 has cross-sectional radius less than 1. Let $T_2, T_3, T_4, \ldots$ be a sequence of smooth solid tori such that for each positive integer i, T_{i+1} is embedded in Int T_i just as T_1 is embedded in Int T_0, and T_i has cross-sectional radius less than $1/i$. Let M denote $\bigcap_{i=1}^{\infty} T_i$.

M is a continuum. It is, in fact, indecomposable and hence not locally connected. It is clear that for each positive integer i, $T_{i+1} \sim 0$ in T_i. Hence M has property UV^∞.

It is shown in [13, §2] that if i is any positive integer, there is no 3-cell lying in Int T_0 and containing T_i. From this fact it follows that M is not cellular.

The continuum M described above is homeomorphic to a certain well-known indecomposable plane continuum which does not separate the plane. Hence M has a cellular embedding in E^2. In our next two propositions we study the relationship between the UV properties in euclidean spaces and the existence of cellular embeddings in some euclidean space. That such a relationship exists was pointed out to the author by D. R. McMillan, Jr. Propositions 3.9 and 3.10 are due to McMillan.

PROPOSITION 3.9. *If n is a positive integer, M is a continuum in E^n, and M has property* UV^∞, *then M has a cellular embedding in E^{n+1}.*

Proof. This is just a restatement of the Corollary to Theorem 8 of [18], taking note of the remarks in the introduction of [18] concerning replacing "compact absolute retract" throughout [18] by "compact set have property UV^∞".

PROPOSITION 3.10. *If each of k and n is a positive integer, M is a subset of E^k, and M has a cellular embedding in E^n, then M has property* UV^∞ *in E^k.*

Proof. Clearly M is compact. Then Proposition 3.10 can be proved using the following two lemmas. These are slight extensions of Lemmas 1 and 2 of [1].

LEMMA 3.11. *Suppose the hypothesis of Proposition* 3.10 *and that U is an open set in E^k containing M. Then M is contractible in U.*

LEMMA 3.12. *Suppose k is a positive integer, M is a compact set in E^k, U is an open set in E^k containing M, and M is contractible in U. Then some neighborhood of M is contractible in U.*

We may summarize the preceding two propositions in the following one.

PROPOSITION 3.13. *If M is a continuum in some euclidean space, then M has property* UV^∞ *if and only if M has a cellular embedding in some euclidean space.*

We obtain the following interesting corollary to Proposition 3.13.

COROLLARY 3.14. *Suppose M is a continuum, each of m and n is a positive integer, h and k are embeddings of M into E^m and E^n, respectively. Then $h[M]$ has property* UV$^\infty$ *in E^m if and only if $k[M]$ has property* UV$^\infty$ *in E^n.*

4. LC **properties.** Suppose X is a topological space and n is a nonnegative integer. If p is any point of X, then X is n-LC at p if and only if for each open set U containing p, there is an open set V containing p such that $V \subset U$ and each singular n-sphere in V is homotopic to 0 in U. If p is any point of X, X is LC^n at p if and only if for each nonnegative integer $i \leqq n$, X is i-LC at p. X is n-LC if and only if X is n-LC at each point of X. X is LC^n if and only if X is LC^n at each point of X. X is LC$^\omega$ if and only if for each nonnegative integer i, X is i-LC.

Suppose K is a finite simplicial complex and X is a metric space. If L is a subcomplex of K containing all the vertices of K, then the statement that Λ is a *partial realization* of L in X means that there is a continuous function f from L into X such that $\Lambda = \{f[\sigma] : \sigma \in L\}$. The mesh of the partial realization Λ is

$$\max \{\operatorname{diam} f[\sigma \cap L] : \sigma \in K\}.$$

The statement that Ω is a *full realization* of K in X means that Ω is a partial realization of K.

The following lemma is closely related to Theorem 1 of [21].

LEMMA 4.1. *If n is a nonnegative integer, D is a compact set in a* LC^n *metric space X, and ε is a positive number, then there is a positive number δ such that if K is any finite simplicial complex of dimension at most $n+1$, then any partial realization of K in D and of mesh less than δ may be extended to a full realization of K in X and of mesh less than ε.*

Suppose X and Y are metric spaces. If ε is a positive number, then a homotopy H from $X \times [0, 1]$ into Y is an *ε-homotopy* if and only if for each point x of X, diam $H[\{x\} \times [0, 1]] < \varepsilon$. If X is a compact metric space with metric d, Y is a metric space, and f and g are two continuous functions from X into Y, then $D(f, g)$ denotes $\sup \{d(f(x), g(x)) : x \in X\}$.

The following result occurs in [21].

LEMMA 4.2. *Suppose n is a nonnegative integer, F is a compact set in an* LC^n *metric space X, and ε is a positive number. There exists a positive number δ such that if K is a finite simplicial complex of dimension at most n and f and g are two continuous functions, each from K into F and such that $D(f, g) < \varepsilon$, then there is an ε-homotopy H from $K \times [0, 1]$ into X such that $H_0 = f$ and $H_1 = g$.*

5. UV **properties in** LC **spaces.** In this section we consider sets with various UV properties in spaces having certain LC properties. Our first result shows that certain UV properties are independent of the particular embedding chosen provided the space has suitable LC properties.

LEMMA 5.1. *Suppose n is a nonnegative integer, X and Y are locally compact* LC^n *metric spaces with metrics d and* ρ, *respectively. M is a compact set, and f and g are embeddings of M into X and Y, respectively. If* $f[M]$ *has property n*-UV *in X, then* $g[M]$ *has property n*-UV *in Y.*

Proof. Let U be an open set in Y containing $g[M]$. Let V_0 be a compact neighborhood in Y of $g[M]$ lying in U. With the aid of Lemma 4.2, it can be established that there is a positive number ε such that if α and β are any two continuous functions, each from Bd I^{n+1} into V_0 and such that $D(\alpha, \beta) < \varepsilon$, then $\alpha \sim \beta$ in U.

By Lemma 4.1, there is a positive number δ_1 such that (1) any partial realization of an $(n+1)$-complex in V_0 of mesh less than δ_1 has a full realization in U of mesh less than $\varepsilon/9$, and (2) $\delta_1 < \varepsilon/9$. There is a positive number δ_2 such that if x and y belong to M and $d(f(x), f(y)) < \delta_2$, then $\rho(g(x), g(y)) < \delta_1$. Since $f[M]$ has property n-UV in X, there is a positive number δ_3 such that (1) any singular n-sphere in the open δ_3-neighborhood of $f[M]$ is homotopic to 0 in the open $(\frac{1}{3}\delta_2)$-neighborhood of $f[M]$, and (2) $\delta_3 < \frac{1}{3}\delta_2$.

Let W be a compact neighborhood of $f[M]$ in X. By Lemma 4.1, there is a positive number δ_4 such that any partial realization of an n-complex in W of mesh less than δ_4 has a full realization in X of mesh less than δ_3. There is a positive number δ_5 such that if x and y belong to M and $\rho(g(x), g(y)) < \delta_5$, then

$$d(f(x), f(y)) < \delta_4.$$

Let δ be min $\{\frac{1}{3}\delta_5, \frac{1}{3}\varepsilon\}$ and let V be $V(\delta, g[M]) \cap V_0 \cap U$. We shall show now that each singular n-sphere in V is homotopic to 0 in U. Let ϕ be a continuous function from Bd I^{n+1} into V. There is a subdivision T of Bd I^{n+1} such that if σ is any simplex of T, then diam $\phi[\sigma] < \delta$.

For each vertex v of T, let v' be some point of $g[M]$ such that $\rho(\phi(v), v') < \delta \leqq \frac{1}{3}\delta_5$. Consider the partial realization $\{fg^{-1}(v') : v \textit{ is a vertex of } T\}$. This has mesh less than δ_4 and lies in $f[M]$. Hence it extends to a full realization in X of mesh less than δ_3. Let θ be a continuous function from Bd I^{n+1} into X such that if v is any vertex of T, $\theta(v) = fg^{-1}(v)$ and if σ is any simplex of T, diam $\theta[\sigma] < \delta_3$.

If σ is any simplex of T, $\theta[\sigma]$ intersects $f[M]$ and hence lies in $V(\delta_3, f[M])$. Hence θ is into $V(\delta_3, f[M])$ and so $\theta \sim 0$ is in $V(\frac{1}{3}\delta_2, f[M])$. Let θ^* be a continuous extension of θ to I^{n+1} such that $\theta^*[I^{n+1}] \subset V(\frac{1}{3}\delta_2, f[M])$.

There is a subdivision τ of I^{n+1} compatible with a subdivision of T such that if σ is any simplex of τ, then diam $\theta^*[\sigma] < \frac{1}{3}\delta_2$.

If v is a vertex of T, let $\hat{v}$ be $fg^{-1}(v')$. If v is a vertex of τ not in T, then there is a point $\hat{v}$ of $f[M]$ such that $d(\theta^*(v), \hat{v}) < \frac{1}{3}\delta_2$. If v is a vertex of T, then $\theta^*(v) = \theta(v) = fg^{-1}(v') = \hat{v}$. Hence in either case, if v is a vertex of τ, $d(\theta^*(v), \hat{v}) < \frac{1}{3}\delta_2$. It follows that $\{gf^{-1}(\hat{v}) : v \textit{ is a vertex of } \tau\}$ is a partial realization of τ such that (1) each of its points lies in $g[M]$ and hence in V_0 and (2) it has mesh less than δ_1.

Accordingly, there is a full realization of τ in U and of mesh less than $\varepsilon/9$. Let λ be a continuous function from I^{n+1} into U such that if v is a vertex of τ, $\lambda(v)=gf^{-1}(\hat{v})$, and if σ is any simplex of τ, diam $\lambda[\sigma]<\varepsilon/9$.

Let λ' denote $\lambda|\text{Bd } I^{n+1}$. We shall show now that $D(\lambda', \phi)<\varepsilon$. First suppose that v is a vertex of T. Then $\lambda'(v)=\lambda(v)=gf^{-1}(\hat{v})=v'$. By choice of v', $\rho(\phi(v), v')<\delta\leqq\frac{1}{3}\varepsilon$.

Suppose now that σ is a simplex of T. By construction, diam $\theta[\sigma]<\delta_3$. If now v_1 and v_2 are any two vertices of T lying in σ, then $d(\theta(v_1), \theta(v_2))<\delta_3<\frac{1}{3}\delta_2$. We proved above that if v is any vertex of T, $d(\theta^*(v), \hat{v})<\frac{1}{3}\delta_2$; recall that on σ, θ^* and θ agree. It follows that $d(\hat{v}_1, \hat{v}_2)<\delta_2$. From the choice of δ_2, it follows that

$$\rho(gf^{-1}(\hat{v}_1), gf^{-1}(\hat{v}_2)) < \delta_1 < \varepsilon/9.$$

Now σ is divided under T into simplexes $\sigma_1, \sigma_1, \ldots,$ and σ_k. Suppose $i=1, 2, \ldots,$ or k, and consider σ_i. By construction of λ, diam $\lambda[\sigma_i]<\varepsilon/9$. Let v_0 be a vertex of σ and let v_i be a vertex of σ_i. We proved in the preceding paragraph that

$$\rho(gf^{-1}(\hat{v}_0), gf^{-1}(\hat{v}_i)) < \varepsilon/9.$$

Further, $gf^{-1}(\hat{v}_0)=\lambda(v_0)$ and $gf^{-1}(\hat{v}_i)=\lambda(v_i)$. It follows that diam $\lambda[\sigma]<\varepsilon/3$.

Now suppose x is any point of Bd I^{n+1}. Then let σ be a simplex of T to which x belongs, and let v_0 be a vertex of σ. It follows that $\rho(\phi(x), \phi(v_0))<\frac{1}{3}\varepsilon$ by construction of T, $\rho(\phi(v_0), v_0')<\frac{1}{3}\varepsilon$ by choice of v_0', and $\rho(v_0', \lambda(x))<\frac{1}{3}\varepsilon$ since $v_0'=\lambda(v_0)$ and diam $\lambda[\sigma]<\frac{1}{3}\varepsilon$. Therefore $\rho(\phi(x), \lambda(x))<\varepsilon$ and it follows that $D(\phi, \lambda')<\varepsilon$.

Hence $\phi\sim\lambda'$ in U. Since λ' is $\lambda|\text{Bd } I^{n+1}$, $\lambda'\sim 0$ in U. Therefore $\phi\sim 0$ in U, and it follows that $g[M]$ has property n-UV in Y.

THEOREM 5.2. *Suppose n is a nonnegative integer, X and Y are locally compact* LC^n *metric spaces, M is a compact set, and f and g are embeddings of M into X and Y, respectively. If $f[M]$ has property* UV^n *in X, then $g[M]$ has property* UV^n *in Y.*

Theorem 1 of [16] states an analogous result for property UV^∞.

COROLLARY 5.3. *Suppose each of m and n is a positive integer, M is a triangulated m-manifold, N is a triangulated n-manifold, A is a compact set, and f and g are embeddings of A into M and N, respectively. Then $f[A]$ has property* UV^∞ *in M if and only if $g[A]$ has property* UV^∞ *in N.*

Proof. Suppose $f[A]$ has property UV^∞, in M, and suppose $m<n$. Then both M and N are locally compact LC^n metric spaces, and therefore by Theorem 5.2, $g[A]$ has property UV^n in N. By Proposition 3.4. $g[A]$ has property UV^∞ in N.

The remaining cases, and the converse, are established by arguments similar to that given.

The following lemma may be found in [21].

LEMMA 5.4. *Suppose X is a locally compact* LC^n *metric space and M is a compact* LC^{n-1} *set in X. If U is an open set containing M, there is an open set V such that*

$M \subset V$, $V \subset U$, and if f is a continuous function from S^n into V, there is a homotopy H from $S^n \times I$ into U such that $H_0 = f$ and $H_1[S^n] \subset M$.

Suppose X is a topological space, n is a nonnegative integer, and A is a subset of X. Then A is *weakly n-connected* if and only if each singular n-sphere in A is homotopic to 0 in A. A is *n-connected* if and only if for each nonnegative integer i such that $i \leqq n$, A is weakly i-connected.

The following lemma is an immediate corollary of Lemma 5.4.

LEMMA 5.5. *Suppose n is a positive integer, X is a locally compact* LC^n *metric space, and M is a compact* LC^{n-1} *set in X. If M is weakly n-connected, then M has property n-*UV *in X. If M is n-connected, then M has property* UV^n *in X.*

We shall apply Lemma 5.5 to show that compact metric absolute retracts have certain UV properties.

COROLLARY 5.6. *If n is a positive integer and M is a compact absolute retract in a locally compact* LC^n *metric space then M has property* UV^n.

Proof. By [14, p. 101], M is LC^{n-1} and n-connected. Corollary 5.6 then follows from Lemma 5.5.

COROLLARY 5.7. *If M is a compact absolute retract in a locally compact* LC^ω *metric space, then M has property* UV^ω.

Proof. By [14, p. 101], M is LC^∞ and contractible. Hence for each nonnegative integer n, M is LC^{n-1} and n-connected and by Corollary 5.6, M has property UV^n. Accordingly, M has property UV^ω.

The following seems well known but we include it for completeness.

COROLLARY 5.8. *If A is a compact absolute retract is a triangulated manifold M, then A has property* UV^∞.

Proof. By Corollary 5.7, A has property UV^ω. By Proposition 3.4, A has property UV^∞.

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