ANALYTIC FUNCTIONS WITH QUASI-ANALYTIC BOUNDARY VALUES(1)

BY

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1. **Introduction.** Given a sequence of positive numbers $A_1, A_2, \ldots$, let $F(A_n)$ be the set of functions $f=\sum a_n z^n$ analytic in the disc $D=\{z : |z|<1\}$ such that

(i) $f^{(k)}$ has a continuous extension to D^- for $k=0, 1, \ldots$,

(ii) $\{a_n A_n\}$ is a bounded sequence.

We call $F(A_n)$ *quasi-analytic* if $f \in F(A_n)$ and $f^{(k)}(1)=0$ $(k=0, 1, \ldots)$ together imply that $f\equiv 0$. The main purpose of this paper is to give a necessary and sufficient condition on the sequence $\{A_n\}$ for $F(A_n)$ to be quasi-analytic (Theorem 3.4) and to extend the results to functions analytic in the half-plane and functions of several variables.

In §2 we prove a quasi-analytic theorem on a related class of functions. This theorem is stated without proof by B. I. Korenbljum in [3] and [4]. §3 is devoted to the main Theorem 3.4: If $\log A_n$ is convex in $\log n$, then $F(A_n)$ is quasi-analytic if and only if $\sum_1^\infty n^{-3/2} \log A_n$ diverges. Half of Theorem 3.4 is proven by a slightly different method in [2, p. 331]. There Carleson shows that if $\log A_n$ is convex in $\log n$ and $\sum_1^\infty n^{-3/2} \log A_n$ diverges then functions in $F(A_n)$ can have only finitely many zeros. The first step of his proof reduces this problem to showing that $F(A_n)$ must be quasi-analytic.

In §4 we give a half-plane analogue of Theorem 3.4 by looking at functions of the form $f(z)=\int_0^\infty \hat{f}(t)e^{itz}\,dt$, where $\hat{f}(t)$ is a measurable function and $\hat{f}(t)A(t)$ is bounded. In §5 we give one possible analogue of Theorem 3.4 to functions of several variables. This result is obtained by applying the one variable theorem.

2. **Functions with bounded derivatives.** Since we make use of the following computation throughout this paper, we state it here in the form of a lemma.

LEMMA 2.1. *Suppose that f is analytic in the unit disc D, $f^{(k)}$ has a continuous extension to D^- and $f^{(k)}(1)=0$ for $k=0, 1, \ldots, n-1$. Then for $k=0, 1, \ldots, n-1$ and z_0 in D^-,*

$$f^{(k)}(z_0) = \frac{1}{(n-k-1)!}\int_1^{z_0} (z_0-z)^{n-k-1} f^{(n)}(z)\,dz.$$

Proof. Write $f^{(k)}(z_0)=\int_1^{z_0} f^{(k+1)}(z)\,dz$ and integrate by parts.

Presented to the Society, January 23, 1969; received by the editors October 1, 1968 and, in revised form, December 5, 1968.

(1) This paper is essentially a portion of a doctoral dissertation written under the guidance of Professor Peter L. Duren at the University of Michigan while the author was a NSF Trainee.

Given a sequence $\{M_n\}$ of positive numbers, let $C(M_n)$ be the set of functions f analytic in the disc $D=\{z : |z|<1\}$ such that for $n=0, 1, \ldots,$

(i) $f^{(n)}$ has a continuous extension to D^-;

(ii) $|f^{(n)}(z)| \leqq BM_n$ in D for some number $B=B(f)$.

The class $C(M_n)$ is said to be *quasi-analytic* if $f \in C(M_n)$ and $f^{(n)}(1)=0$ $(n=0, 1, \ldots)$ together imply that $f \equiv 0$.

THEOREM 2.2 (KORENBLJUM). *Let* $T(r)=\sup_{n\geqq 0} r^n M_n^{-1}$. *The class* $C(M_n)$ *is quasi-analytic if and only if* $\int_1^\infty r^{-3/2} \log T(r)\, dr$ *diverges.*

Proof. Suppose that the integral converges. Then by Theorem 3 of [3] there is a function $F(w)\not\equiv 0$ analytic in the right half-plane H such that $F^{(n)}$ has a continuous extension to H^-, $|F^{(n)}(w)| \leqq M_n$ in H and $F^{(n)}(0)=0$ for $n=0, 1, \ldots$. Let $f(z)=F(1-z)$. Then f is in $C(M_n)$ and $f^{(n)}(1)=0$ for $n=0, 1, \ldots$; but $f\not\equiv 0$. Hence $C(M_n)$ is not quasi-analytic.

Conversely, suppose that the integral diverges and f is a function in $C(M_n)$ such that $f^{(n)}(1)=0$, $n=0, 1, \ldots$. Computing the derivatives of $F(w)=f((1-w)/(1+w))$ gives an expression of the form

$$F^{(n)}(w) = \sum_{k=1}^{n} a_{nk}(1+w)^{-(n+k)} f^{(k)}\left(\frac{1-w}{1+w}\right)$$

where the a_{nk} are real numbers. Differentiating this expression and comparing coefficients, we obtain the formula

$$\begin{aligned} a_{n+1k} &= -(n+k)a_{nk} - 2a_{nk-1}, \\ a_{n0} &= a_{nn+1} = 0, \qquad |a_{nn}| = 2^n. \end{aligned}$$

Using the binomial expansion

$$(n+1)^{2(n+1-k)} = n^{2(n+1-k)} + 2(n+1-k)n^{2(n+1-k)-1} + \cdots,$$

an induction argument on n shows that $|a_{nk}| \leqq 2^n n^{2(n-k)}/(n-k)!$. For $k=0, \ldots, n-1$ we have

$$f^{(k)}(z_0) = \frac{1}{(n-k-1)!} \int_1^{z_0} (z_0-z)^{n-k-1} f^{(n)}(z)\, dz,$$

by Lemma 2.1. Hence

$$|f^{(k)}(z_0)| \leqq \frac{1}{(n-k-1)!}\, 2^{n-k} BM_n.$$

And

$$\begin{aligned} |F^{(n)}(w)| &\leqq \sum_{k=1}^{n} \frac{a_{nk}}{|1+w|^{n+k}} \left| f^{(k)}\left(\frac{1-w}{1+w}\right)\right| \\ &\leqq \sum_{k=1}^{n-1} 2^n \frac{n^{2(n-k)}}{(n-k)!} \frac{2^{n-k}}{(n-k-1)!} BM_n + 2^n BM_n. \end{aligned}$$

Now the first term is bounded by

$$4^n BM_n \sum_{k=1}^{n-1} \frac{n^{2(n-k)}}{[(n-k-1)!]^2} = 4^n BM_n \sum_{k=0}^{n-2} \frac{n^{2k+2}}{(k!)^2}.$$

So we conclude that

$$|F^{(n)}(w)| \leqq 4^n BM_n n^2 \sum_{k=0}^{n} \left(\frac{n^k}{k!}\right)^2 \leqq 4^n BM_n n^2 (e^n)^2.$$

Again referring to Theorem 3 of [3] we conclude that $F \equiv 0$ and hence $f \equiv 0$.

We define a class of functions similar to $C(M_n)$ as follows. If f is analytic in D and has a continuous extension to D^-, denote the boundary values by $\tilde{f}(\theta) = f(e^{i\theta})$. Let $\tilde{C}(M_n)$ be the set of functions analytic on D such that for each $n = 0, 1, \ldots,$

(i) $f^{(n)}$ has a continuous extension to D^-;

(ii) $|\tilde{f}^{(n)}(\theta)| \leqq BM_n$ on $[0, 2\pi]$.

The class $\tilde{C}(M_n)$ is said to be *quasi-analytic* if $f \in \tilde{C}(M_n)$ and $\tilde{f}^{(n)}(0) = 0$ $(n = 0, 1, \ldots)$ together imply that $f \equiv 0$. Estimates similar to those in the proof of Theorem 2.2 show that $\tilde{C}(M_n)$ is quasi-analytic if and only if $C(M_n)$ is.

3. **Functions with small Taylor coefficients.** Mandelbrojt [6] has shown that if $g(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$ is infinitely differentiable, $\{a_n A_{|n|}\}_{-\infty}^{\infty}$ is bounded, and $g^{(n)}(0) = 0$ $(n = 0, 1, \ldots)$, then a necessary and sufficient condition to conclude that $g \equiv 0$ is that $\sum_{n=1}^{\infty} n^{-2} \log A_n = \infty$. The main theorem of this section, Theorem 3.4, says that if $g(\theta)$ comes from the boundary values of an analytic function, then the appropriate condition is that $\sum_{n=1}^{\infty} n^{-3/2} \log A_n = \infty$. Thus by making an additional assumption on the functions, we can relax the condition on the sequence $\{A_n\}$. The methods we use are much the same as the methods Mandelbrojt uses.

In this section, $A_1, A_2, \ldots$ will be a sequence of positive numbers such that $\log A_n$ is convex in $\log n$, i.e.

$$s_n = \frac{\log A_{n+1} - \log A_n}{\log(n+1) - \log n}$$

is a nondecreasing sequence. We let $F(A_n)$ be the set of functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in D such that $f^{(n)}(z)$ has a continuous extension to D^- $(n = 0, 1, \ldots)$ and such that $\{A_k a_k\}$ is bounded. We call $F(A_n)$ quasi-analytic if $f \in F(A_n)$ and $f^{(n)}(1) = 0$ $(n = 0, 1, \ldots)$ together imply that $f \equiv 0$.

We begin by proving some properties of the sequence $\{A_n\}$.

LEMMA 3.1. *If $\lim_{n \to \infty} s_n = \infty$, then for each nonnegative integer k, $\{n^k A_n^{-1}\}$ is a bounded sequence.*

Proof. Fix k and choose n_0 so that $s_{n_0} \geqq k$. Then for $n \geqq n_0$,

$$\begin{aligned} \log A_{n+1} - \log A_n &= s_n(\log(n+1) - \log n) \\ &\geqq k(\log(n+1) - \log n). \end{aligned}$$

Hence $n^k A_n^{-1} \geqq (n+1)^k A_{n+1}^{-1}$; and $n^k A_n^{-1}$ is nonincreasing for $n \geqq n_0$.

This lemma allows us to state the following proposition which is crucial in relating $F(A_n)$ to a class of functions of the type $\tilde{C}(M_n)$.

PROPOSITION 3.2. *Suppose that* $s_n \geqq 0$ *and* $\lim_{n\to\infty} s_n = \infty$. *Let* $c(n) = \sup_{k\geqq 1} k^n A_k^{-1}$ *and let* $d(t) = \sup_{n\geqq 1} t^n c(n)^{-1}$. *Then for* $n = 1, 2, \ldots$, $d(n) \leqq A_n \leqq nd(n)$. *Furthermore, given* n_0, *there exists a number* t_{n_0} *such that for* $t \geqq t_{n_0}$, $d(t) = \sup_{n\geqq n_0} t^n c(n)^{-1}$.

Proof. By the previous lemma $c(n)$ is finite, so that the definition of $d(t)$ is meaningful. For fixed n and for each k, $A_n \geqq n^k c(k)^{-1}$. Hence $A_n \geqq d(n)$.

To prove the second inequality, fix n and let k be the integer such that $k \leqq s_n < k+1$. An argument similar to the proof of Lemma 3.1 shows that for $j \geqq n$,

$$j^k A_j^{-1} \leqq n^k A_n^{-1} < n^{k+1} A_n^{-1}.$$

While, for $1 \leqq j < n$,

$$\begin{aligned}\log A_{j+1} - \log A_j &= s_j(\log(j+1) - \log j)\\ &< (k+1)(\log(j+1) - \log j).\end{aligned}$$

So that

$$j^{k+1} A_j^{-1} < (j+1)^{k+1} A_{j+1}^{-1} \leqq n^{k+1} A_n^{-1}.$$

We therefore have $c(k) = \sup_{j\geqq 1} j^k A_j^{-1} \leqq n^{k+1} A_n^{-1}$. So that $A_n \leqq n^{k+1} c(k)^{-1} \leqq nd(n)$.

Finally, given n_0 choose $t_{n_0} \geqq 1$ so that $t_{n_0} c(n) > c(n_0)$ for each n less than n_0. Now fix $t \geqq t_{n_0}$. For $n < n_0$,

$$t^{n_0} c(n) \geqq t^{n+1} c(n) \geqq t^n t_{n_0} c(n) > t^n c(n_0).$$

Thus $t^{n_0} c(n_0)^{-1} > t^n c(n)^{-1}$. And hence for $t \geqq t_{n_0}$,

$$d(t) = \sup_{n\geqq 0} t^n c(n)^{-1} = \sup_{n\geqq n_0} t^n c(n)^{-1}.$$

Before proving the main theorem of this section, we prove the following special case which contains the crux of the argument.

THEOREM 3.3. *Suppose that* $s_n \geqq 0$ *and* $\lim_{n\to\infty} s_n = \infty$. *Then a necessary and sufficient condition for* $F(A_n)$ *to be quasi-analytic is that*

$$\sum_{n=1}^{\infty} n^{-3/2} \log A_n \tag{1}$$

diverge.

Proof. *Sufficiency.* We first show that $F(A_n)$ is contained in $\tilde{C}(M_n)$ where $M_n = c(n+2)$. Suppose f is in $F(A_n)$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $|a_n| \leqq BA_n^{-1}$ for some constant B. Now $\tilde{f}(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$. So for $k = 1, 2, \ldots$,

$$\begin{aligned}|\tilde{f}^{(k)}(\theta)| &= \left|\sum_{n=0}^{\infty} a_k (in)^k e^{in\theta}\right| \leqq \sum_{n=1}^{\infty} |a_n| n^k\\ &\leqq \sum_{n=1}^{\infty} n^{-2} n^{k+2} B A_n^{-1} \leqq Bc(k+2) \sum_{n=1}^{\infty} n^{-2}.\end{aligned}$$

Hence $f \in \tilde{C}(M_n)$ where $M_n = c(n+2)$.

Therefore it is enough to show that the divergence of (1) implies that $\tilde{C}(M_n)$ is quasi-analytic, i.e. that $\int_1^\infty r^{-3/2} \log T(r)$ diverges. Now

$$T(r) = \sup_{k \geqq 0} r^k c(k+2)^{-1} = r^{-2} \sup_{k \geqq 0} r^{k+2} c(k+2).$$

So for $r \geqq t_2$, $T(r) = r^{-2} d(r)$. By Proposition 3.2 the divergence of (1) is equivalent to the divergence of $\sum_1^\infty n^{-3/2} \log d(n)$ and hence to the divergence of

$$\int_1^\infty r^{-3/2} \log T(r).$$

Necessity. Let $M_0 = 1$ and $M_k = c(k-1)$ for $k = 1, 2, \ldots$. We first show that the convergence of (1) implies that $\tilde{C}(M_n)$ is not quasi-analytic. Let $T(r) = \sup_{k \geq 0} r^k M_k^{-1}$. Then for $r \geqq M_1$,

$$T(r) = \sup_{k \geqq 0} r^k M_k^{-1} = r \sup_{k \geqq 1} r^{k-1} c(k-1)^{-1} = rd(r).$$

By an argument similar to the above, the convergence of the series (1) implies that $\int_1^\infty r^{-3/2} \log T(r)$ converges and therefore $\tilde{C}(M_n)$ is not quasi-analytic.

Hence there is a function f in $\tilde{C}(M_n)$ such that $\tilde{f}^{(n)}(0) = 0$ $(n = 0, 1, \ldots)$ but $f \not\equiv 0$. The proof of the theorem will be complete if we show that f is also in $F(A_n)$. Fix $n \geqq 1$. Integrating by parts we find that

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\theta) e^{-in\theta} \, d\theta = \cdots = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}^{(k)}(\theta) \left(\frac{1}{in}\right)^k e^{-in\theta} \, d\theta.$$

And

$$|a_n| \leqq BM_k \left(\frac{1}{n}\right)^k \leqq \frac{B}{n} \frac{1}{d(n)} \leqq \frac{B}{A_n}.$$

Therefore $\{|a_n| A_n\}$ is bounded and f is in $F(A_n)$.

Theorem 3.3 can be improved to give the main result of this section.

THEOREM 3.4. *If* $\log A_n$ *is a convex sequence in* $\log n$, *then a necessary and sufficient condition for* $F(A_n)$ *to be quasi-analytic is that*

$$\sum_{n=1}^{\infty} n^{-3/2} \log A_n \tag{1}$$

diverge.

Proof. First suppose that $\lim_{n \to \infty} s_n = \infty$. If $s_n \geqq 0$ $(n = 1, 2, \ldots)$, then Theorem 3.3 applies. If not, then since $\lim_{n \to \infty} s_n = \infty$, there is an integer n_0 such that $s_n \geqq 0$ for all $n \geqq n_0$. Let

$$\begin{aligned} \bar{A}_n &= A_{n_0} \quad \text{if } n \leqq n_0, \\ &= A_n \quad \text{if } n \geqq n_0. \end{aligned}$$

Then

$$\begin{aligned} \bar{s}_n &= 0 \quad \text{if } n < n_0, \\ &= s_n \quad \text{if } n \geqq n_0, \end{aligned}$$

where

$$\bar{s}_n = \frac{\log \bar{A}_{n+1} - \log \bar{A}_n}{\log (n+1) - \log n}.$$

Hence $\log \bar{A}_n$ is convex in $\log n$, $\bar{s}_n \geqq 0$ and $\lim_{n\to\infty} \bar{s}_n = \infty$. So Theorem 3.3 applies to $F(A_n)$. But $F(\bar{A}_n) = F(A_n)$ and $\sum_1^\infty n^{-3/2} \log \bar{A}_n$ diverges if and only if (1) diverges. Hence the theorem is true whenever $\lim_{n\to\infty} s_n = \infty$.

Now suppose that it is not the case that $\lim_{n\to\infty} s_n = \infty$. Then, since $\{s_n\}$ is a nondecreasing sequence, we must have $s_n \leqq b$ for some number b. In this case we show (1) converges and $F(A_n)$ is not quasi-analytic. Since

$$\begin{aligned} \log A_{n+1} - \log A_n &= s_n(\log (n+1) - \log n) \\ &\leqq b(\log (n+1) - \log n), \end{aligned}$$

we have

$$A_{n+1} \leqq A_n\left(\frac{n+1}{n}\right)^b \leqq A_{n-1}\left(\frac{n}{n-1}\right)^b\left(\frac{n+1}{n}\right)^b \leqq \cdots \leqq A_1(n+1)^b.$$

And

$$\sum_1^\infty n^{-3/2} \log A_n \leqq \sum_1^\infty n^{-3/2}(\log A_1 + b \log n) < \infty.$$

To show $F(A_n)$ is not quasi-analytic we exhibit a non-quasi-analytic class $F(\bar{A}_n)$ which is contained in $F(A_n)$. Let $\bar{A}_n = A_1 n^{b \log n}$. Since $A_n \leqq A_1 n^b \leqq \bar{A}_n$ for $n \geqq 3$, $F(\bar{A}_n)$ is contained in $F(A_n)$. And since

$$\bar{s}_n = \frac{\log \bar{A}_{n+1} - \log \bar{A}_n}{\log (n+1) - \log n} = b(\log (n+1) + \log n),$$

we may apply Theorem 3.3 to $F(\bar{A}_n)$. But

$$\sum_1^\infty n^{-3/2} \log \bar{A}_n = \sum_1^\infty n^{-3/2}(\log A_1 + b(\log n)^2) < \infty.$$

So $F(\bar{A}_n)$ is not quasi-analytic.

A number of comments on Theorem 3.4 should be made. The first is that if the series (1) diverges, then any function $f(z) = \sum_0^\infty a_k z^k$, such that $\{a_k A_k\}$ is bounded, is already in $F(A_n)$, i.e. $f^{(n)}(z)$ has a continuous extension to D^- for $n = 0, 1, \ldots$. For as we have seen, the divergence of (1) implies that $\lim_{n\to\infty} s_n = \infty$. Thus, by Lemma 3.1, $\{n^{k+2} A_n^{-1}\}$ is bounded for each k. Hence $f^{(k)}(z) = \sum_{n=k}^\infty a_n z^{n-k} n!/(n-k)!$ converges uniformly on D^-, since

$$\left|\frac{n!}{(n-k)!} a_n\right| \leqq n^k |a_n| \leqq n^{-2} n^{k+2} B A_n^{-1} \leqq n^{-2} B_1.$$

Secondly, one might want to study the classes $F^p(A_n)$ $(0 < p < \infty)$, where $F^p(A_n)$ is the set of functions $f(z) = \sum_0^\infty a_k z^k$ analytic in D such that $f^{(n)}$ has a continuous extension to D^- ($n=0, 1, \ldots$) and $\sum_1^\infty |a_k|^p A_k < \infty$. The following theorem shows, however, that the problem reduces to considering $F(A_n)$.

THEOREM 3.5. *$F(n^{2/p}A_n^{1/p}) \subset F^p(A_n) \subset F(A_n^{1/p})$; and hence $F^p(A_n)$ is quasi-analytic if and only if* (1) *diverges.*

The proof of this theorem is straightforward and is left to the reader.

4. **An analogue in the half-plane.** The class $F(A_n)$ has a "continuous analogue" in the following sense. Given a real-valued function $A(t)$ defined on the open interval $(0, \infty)$, let $F(A(t))$ be the set of all functions f analytic in the upper half-plane $H=\{z : \text{Im}(z)>0\}$ such that

(i) $f^{(n)}$ has a continuous extension to H^- ($n=0, 1, \ldots$),

(ii) f is of the form $f(z)=\int_0^\infty e^{itz}\hat{f}(t)\,dt$, where $\hat{f}(t)$ is a measurable function and $|\hat{f}(t)|A(t)$ is bounded.

We call $F(A(t))$ quasi-analytic if $f \in F(A(t))$ and $f^{(n)}(0)=0$ ($n=0, 1, \ldots$) together imply $f\equiv 0$.

Thus, looking at the boundary values of functions in $F(A(t))$ and $F(A_n)$, functions in $F(A(t))$ have Fourier transforms which vanish on the negative real line and go rapidly to zero on the positive real line, while the negative Fourier coefficients of functions in $F(A_n)$ vanish and the positive Fourier coefficients go rapidly to zero.

We restrict our attention to functions $A(t)$ for which $\log A(t)$ is a convex function of $\log t$, $A(t)$ is bounded below by 1, and $A(t)$ is bounded on the interval $(0, 1]$. Such a function has the following properties:

(i) For $t_0 < t_1 < t_2$,

$$\frac{\log A(t_1)-\log A(t_0)}{\log t_1-\log t_0} \leqq \frac{\log A(t_2)-\log A(t_0)}{\log t_2-\log t_0} \leqq \frac{\log A(t_2)-\log A(t_1)}{\log t_2-\log t_1}.$$

(ii) For $h>0$,

$$f(t; h) = \frac{\log A(t+h)-\log A(t)}{\log (t+h)-\log t}$$

is a decreasing function of h.

(iii) $s(t)=\lim_{h\to 0^+} f(t; h)$ exists and is a nondecreasing function of t.

(iv) For $t<t_0$,

$$\frac{\log A(t_0)-\log A(t)}{\log t_0-\log t} \leqq s(t_0).$$

Properties (i) and (ii) are statements about slopes of chords of the graph of $\log A(t)$ vs. $\log t$ and are clear when viewed geometrically. The function $s(t)$ measures the slope of the graph at $(\log t, \log A(t))$. The corresponding properties for convex functions are known [3, §3.18], so we omit a proof of the above properties.

LEMMA 4.1. *If* $\lim_{t\to\infty} s(t)=\infty$, *then for each nonnegative integer* k, $t^k A(t)^{-1}$ *is bounded.*

PROPOSITION 4.2. *Suppose* $s(t)\geqq 0$ *and* $\lim_{t\to\infty} s(t)=\infty$. *Let* $c(n)=\sup_{t\geqq 1} t^n A(t)^{-1}$ *and* $d(t)=\sup_{n\geqq 0} t^n c(n)^{-1}$. *Then for* $t\geqq 1$, $d(t)\leqq A(t)\leqq t\, d(t)$. *Furthermore, given* n_0 *there exists a number* t_{n_0} *such that for* $t\geqq t_{n_0}$, $d(t)=\sup_{n\geqq n_0} t^n c(n)^{-1}$.

The proofs of the lemma and proposition follow the corresponding proofs given in the previous section. It is interesting to note that if we let $c(r)=\sup_{t\geqq 1} t^r A(t)^{-1}$ $(0\leqq r<\infty)$ and $d(t)=\sup_{r\geqq 0} t^r c(r)^{-1}$, then $d(t)=A(t)$.

THEOREM 4.3. *Suppose that* $s(t)\geqq 0$ *and* $\lim_{t\to\infty} s(t)=\infty$. *Then a necessary and sufficient condition for* $F(A(t))$ *to be quasi-analytic is that*

$$\int_1^\infty t^{-3/2}\log A(t)\,dt \tag{2}$$

diverge.

Proof. *Sufficiency.* Suppose that f is in $F(A(t))$ and $f^{(n)}(0)=0, 1, \ldots$. Since $t^{n+2}A(t)^{-1}\leqq c(n+2)$ for $t\geqq 1$, the integral

$$f^{(n)}(z) = \int_0^\infty (it)^n e^{itz}\hat{f}(t)\,dt$$

converges uniformly for $\operatorname{Im}(z)\geqq 0$. In fact,

$$\begin{aligned} |f^{(n)}(z)| &\leqq \int_0^\infty t^n BA(t)^{-1}\,dt \\ &\leqq B+\int_1^\infty t^{-2}t^{n+2}BA(t)^{-1}\,dt \\ &\leqq B+Bc(n+2)\int_1^\infty t^{-2}\,dt \end{aligned} \tag{3}$$

for $n=0, 1, \ldots$. If we let

$$T(r) = \sup_{n\geqq 0} r^n c(n+2)^{-1} = r^{-2}\sup_{n\geqq 2} r^{n+2}c(n+2)^{-1},$$

then $T(r)=r^{-2}\,d(r)\geqq r^{-3}A(r)$ for $r\geqq t_2$. Hence the divergence of (2) implies that $\int_1^\infty r^{-3/2}\log T(r)\,dr$ diverges. Hence by Theorem 3 of [3], $f\equiv 0$.

Necessity. Let $M_0=1$ and $M_n=e^{-n}c(n-1)$, $n=1, 2, \ldots$. As in the proof of Theorem 3.3, the convergence of (2) implies that $\int_1^\infty r^{-3/2}\log T(r)\,dr$ diverges, where $T(r)=\sup_{n\geqq 0} r^n M_n^{-1}$. Hence by Theorem 3 of [3] there is a function $f\not\equiv 0$ analytic in the upper half-plane H such that $f^{(n)}$ has a continuous extension to H^-, $|f^{(n)}|\leqq M_n$ and $f^{(n)}(0)=0$ for $n=0, 1, \ldots$. Let $g(z)=(z+i)^{-2}f(z)$ then g has the same properties as f. The proof will be complete if we show that g is in $F(A(t))$.

$$g^{(n)}(z) = \sum_{k=0}^{n} C_{n,k}f^{(n-k)}(z)(z+i)^{-(2+k)}(-1)^k(k+1)!.$$

By Lemma 2.1

$$f^{(n-k)}(z) = \frac{1}{(k-1)!}\int_0^z (z-w)^{k-1}f^{(n)}(w)\,dw.$$

Hence

$$|f^{(n-k)}(z)| \leqq \frac{1}{(k-1)!}z^k BM_n, \qquad k = 1, 2, \ldots.$$

And

$$|g^{(n)}(z)| \leqq C_{n,0}|f^{(n)}(z)|\,|z+i|^{-2}+\left[\sum_{k=1}^{n} C_{n,k}\frac{1}{(k-1)!}|z|^k BM_n\right][|z+i|^{-(2+k)}(k+1)!].$$

Since the second term is less than

$$\sum_{k=1}^{n} C_{n,k}BM_n|z+i|^{-2}(k+1)k,$$

we have

$$|g^{(n)}(z)| \leqq B_1 e^n M_n|z+i|^{-2}.$$

In particular, g is in the class H^2 of the upper half-plane. Therefore by the Paley-Wiener theorem, the Fourier transform $\hat{g}(t)$ vanishes for $t<0$. For $t>0$,

$$2\pi\hat{g}(t) = \int_{-\infty}^{\infty} e^{-itx}g(x)\,dx = \int_{-\infty}^{\infty} (-it)^{-n}g^{(n)}(x)e^{itx}\,dx.$$

So that

$$|2\pi\hat{g}(t)| \leqq t^{-n}Be^nM_n\int_{-\infty}^{\infty} |x+i|^{-2}\,dx.$$

Hence for $t \geqq 1$,

$$\hat{g}(t) \leqq B_1 \inf_{n\geqq 1} t^{-n}e^nM_n = B_1 t^{-1}\inf_{n\geqq 0} t^{-n}c(n)$$
$$\leqq B_1 t^{-1}d(t)^{-1} \leqq B_2A(t)^{-1}.$$

And for $0<t<1$,

$$|\hat{g}(t)| \leqq B_1M_0 \leqq B_2A(t)^{-1},$$

since $A(t)$ is bounded on the interval (0, 1). Thus g is in $F(A(t))$.

As before, we immediately improve Theorem 4.3 to obtain the main result.

THEOREM 4.4. *Suppose that $A(t)$ is defined and bounded below by* 1 *on the interval* $[0, \infty)$. *Suppose further that $A(t)$ is bounded on* (0, 1) *and that* $\log A(t)$ *is a convex function of* $\log t$. *Then a necessary and sufficient condition for $F(A(t))$ to be quasi-analytic is that*

$$\int_1^{\infty} t^{-3/2}\log A(t)\,dt \tag{2}$$

diverge.

Proof. If $\lim_{t\to\infty} s(t)=\infty$, the proof proceeds as in the previous section. If not, then let $a=\sup_{0\leqq t\leqq e} A(t)$ and choose b so that $b\geqq s(t)$ for all t. Let

$$\begin{aligned} \bar{A}(t) &= at^{b\log t} \quad \text{for } t \geqq e, \\ &= ae^b \qquad \text{for } t \leqq e. \end{aligned}$$

Then as in the previous section $F(\bar{A}(t))$ is contained in $F(A(t))$ and $F(\bar{A}(t))$ is not quasi-analytic. Hence $F(A(t))$ is not quasi-analytic. On the other hand (2) converges.

We again observe that if (2) diverges, then any function f analytic in the upper half-plane H and of the form $f(z)=\int_0^\infty \hat{f}(t)e^{itz}\,dt$ where $\hat{f}(t)A(t)$ is bounded, is already in $F(A(t))$. For the divergence of (2) implies that $\lim_{t\to\infty} s(t)=\infty$ and hence (3) shows that

$$f^{(n)}(z) = \int_0^\infty \hat{f}(t)(it)^n e^{itz}\,dt$$

converges uniformly on H^-. Hence $f^{(n)}(z)$ has a continuous extension to H^-.

5. **Functions of several variables.** Since we will be working in the space of n complex variables C^n, we wish to make use of multi-indices of nonnegative integers $m=(m_1,\ldots,m_n)$. If m and l are multi-indices, r is an n-tuple of real or complex numbers, and a and b are real numbers, then

$$\begin{aligned} |m| &= m_1+m_2+\cdots+m_n; \\ m+a &= (m_1+a, m_2+a,\ldots,m_n+a); \\ r^m &= \prod_{j=1}^n r_j^{m_j}; \\ (m+a)^b &= \prod_{j=1}^n (m_j+a)^b; \quad \text{and} \\ m \leqq l &\text{ means } m_j \leqq l_j,\ j = 1,2,\ldots,n. \end{aligned}$$

We also make use of the following notation: if f is a function of $r=(r_1,\ldots,r_n)$, then

$$f^{(m)}(r) = \frac{\partial^{|m|}}{\partial r_1^{m_1}\cdots\partial r_n^{m_n}} f(r),$$

and

$$\int f(r)\,dr = \int f(r)\,dr_1\,dr_2\cdots dr_n.$$

THEOREM 5.1. *Suppose that $f=\sum_m a_m z^m$ is analytic in the polydisc*

$$D = \{z\in C^n : |z_j| < 1; j = 1,\ldots,n\},$$

that $\{\sum_{|m|=k} |a_m|^2\}^{1/2}\leqq A_k$ for z in D $(k=0,1,\ldots)$, and that for each multi-index m

(i) *$f^{(m)}$ has a continuous extension to D^-;*

(ii) $f^{(m)}(1,\ldots,1)=0$.

Then a necessary and sufficient condition to conclude that $f=0$ *is that*

$$\sum_{k=1}^{\infty} k^{-3/2}\log A_k \tag{4}$$

diverge.

The following two lemmas contain some of the computations used repeatedly in the proof of Theorem 5.1.

LEMMA 5.2. *If* $\sum_{|m|\geqq k}\sum_{l\leqq m} x(m,l)$ *is a convergent series of nonnegative terms, then*

$$\sum_{|l|\leqq k}\sum_{m\geqq l} x(m,l) \leqq \sum_{|m|\geqq k}\sum_{l\leqq m} x(m,l).$$

Proof.

$$\begin{aligned}\sum_{|l|=k}\sum_{m\geqq l} x(m,l) &= \sum_{m}\sum_{|l|=k,\, l\leqq m} x(m,l)\\ &= \sum_{|m|\geqq k}\sum_{|l|=k,\, l\leqq m} x(m,l) \leqq \sum_{|m|\geqq k}\sum_{l\leqq m} x(m,l).\end{aligned}$$

LEMMA 5.3. *If* $\sum_{|m|=j}|a_m|^2\leqq A_j^{-2}$ $(j=1,2,\ldots)$ *and* (4) *diverges, then for* $k=1,2,\ldots$ *and* $b\geqq 1/n$

$$\sum_{j=k}^{\infty}\sum_{|m|=j}|a_m|^2(m+1)^b \leqq ck^{nb+2}A_k^{-2}$$

for some constant c.

Proof. If $|m|=j\geqq 1$, $(m+1)^{1/n}\leqq (j+n)/n$ by the inequality of the arithmetic and geometric means. Hence $(m+1)^b\leqq (j/n+1)^{nb}\leqq (2j)^{nb}$ and

$$\sum_{|m|=j}|a_m|^2(m+1)^b \leqq (2j)^{nb}\sum_{|m|=j}|a_m|^2 \leqq (2j)^{nb}A_j^{-2}.$$

Thus it suffices to prove that

$$\sum_{j=k}^{\infty} j^{nb}A_j^{-2} \leqq ck^{nb+2}A_k^{-2} \qquad (k=1,2,\ldots).$$

Since (4) diverges, it follows that $\lim_{k\to\infty} s_k=\infty$ (as in the proof of Theorem 3.4). Hence there exists an integer $k_0\geqq 1$ such that for $k\geqq k_0$, $s_k\geqq \frac{1}{2}nb+1$. So for $j>k\geqq k_0$,

$$\frac{\log A_j-\log A_k}{\log j-\log k} \geqq s_k \geqq \tfrac{1}{2}nb+1.$$

And $A_kA_j^{-1}\leqq (kj^{-1})^{(nb+2)/2}$. So that

$$\begin{aligned}\sum_{j=k}^{\infty} j^{nb}A_j^{-2} &= A_k^{-2}\left(k^{nb}+\sum_{j=k+1}^{\infty}A_k^2A_j^{-2}j^{nb}\right)\\ &\leqq A_k^{-2}k^{nb+2}\sum_{j=1}^{\infty} j^{-2}.\end{aligned}$$

This proves the lemma for $k \geqq k_0$. By choosing c large enough we see that the lemma is true for $k \geqq 1$.

Proof of Theorem 5.1. First suppose that (4) converges. Then by Theorem 3.4 there is a function $g(w) \not\equiv 0$ of one complex variable such that g is in $F(A_k)$ and $g^{(k)}(1)=0$ for $k=0, 1, \ldots$. Let $f(z)=g(z_1)$ for $z=(z_1, \ldots, z_n)$ in the polydisc D. Then $\{\sum_{|m|=k} |a_m|^2\}^{1/2} \leqq A_k$ for z in D $(k=0, 1, \ldots)$ and f satisfies (i) and (ii) but $f \not\equiv 0$.

Now suppose that (4) diverges. We wish to show that $f \equiv 0$. Notice that since f is analytic in each variable separately, it is enough to show that $f(r)=0$ for all $r=(r_1, \ldots, r_n)$ with $0<r_j<1$, $j=1, \ldots, n$. Fix such an r and define a function of one complex variable by

$$\phi(w) = ((1-r_1)w+r_1, \ldots, (1-r_n)w+r_n).$$

Then ϕ maps the open unit disc into D (and the closed unit disc into D^-), $\phi(1) =(1, \ldots, 1)$ and $\phi(0)=r$. Let

$$g(w) = f(\phi(w)) = \sum_{k=0}^{\infty} b_k w^k.$$

Then g is analytic in the open unit disc,

$$g^{(k)}(w) = \sum_{|m|=k} f^{(m)}(\phi(w))r^m$$

has a continuous extension to D^- and $g^{(k)}(1)=0$ for $k=0, 1, \ldots$. We will show later that $|b_k| \leqq ck^{n+1}A_k^{-1}$ for $k=0, 1, \ldots$. Assuming that this has been done, we let $\bar{A}_k = A_k k^{-(n+1)}$, $k=1, 2, \ldots$. Then $\{b_k \bar{A}_k\}$ is bounded and g is in $F(\bar{A}_k)$. Now $\log \bar{A}_k = \log A_k - (n+1) \log k$ is convex in $\log k$ and $\sum_{k=1} k^{-3/2} \log \bar{A}_k$ diverges since (4) diverges. It follows that $F(\bar{A}_k)$ is quasi-analytic and $g \equiv 0$. In particular $f(r)=f(\phi(0))=g(0)=0$.

Thus it suffices to show that $|b_k| \leqq ck^{n+1}A_k^{-1}$. To simplify notation, let

$$x(l, m) = \prod_{j=1}^{n} C_{m_j, l_j}(1-r_j)^{l_j} r_j^{m_j - l_j}.$$

Notice that $\sum_{l \leqq m} x(l, m) = 1$. Furthermore

$$g(w) = \sum_m a_m \sum_{l \leqq m} x(l, m) w^{|l|} = \sum_l \sum_{m \geqq l} a_m x(l, m) w^{|l|}$$

$$= \sum_{k=0}^{\infty} \sum_{|l|=k} \sum_{m \geqq l} a_m x(l, m) w^k.$$

Thus we conclude that

$$b_k = \sum_{|l|=k} \sum_{m \geqq l} a_m x(l, m).$$

Finally

$$|b_k| \leqq \left\{ \sum_{|l|=k} \sum_{m \geqq l} x(l, m)(m+1)^{-2} \right\}^{1/2} \left\{ \sum_{|l|=k} \sum_{m \geqq l} |a_m|^2 x(l, m)(m+1)^2 \right\}^{1/2}.$$

So by Lemma 5.2

$$|b_k| \leqq \left\{\sum_{|m|\geqq k}\sum_{l\leqq m} x(l,m)(m+1)^{-2}\right\}^{1/2}\left\{\sum_{|m|\geqq k}\sum_{l\leqq m} |a_m|^2 x(l,m)(m+1)^2\right\}^{1/2}$$

$$= \left\{\sum_{|m|\geqq k} (m+1)^{-2}\right\}^{1/2}\left\{\sum_{|m|\geqq k} |a_m|^2(m+1)^2\right\}^{1/2}$$

$$\leqq c\left\{\sum_{j=k}^{\infty}\sum_{|m|=j} |a_m|^2(m+1)^2\right\}^{1/2}.$$

And by Lemma 5.3

$$|b_k| \leqq c_1\{k^{2n+2}A_k^{-2}\}^{1/2} = c_1 k^{n+1}A_k^{-1}.$$

We conclude this paper by observing that, as in the one variable case, if $f(z) = \sum_m a_m z^m$ is analytic in D, $\sum_{|m|=k} |a_m|^2 \leqq A_k^{-2}$ $(k=1, 2, \ldots)$ and (4) diverges, then $f^{(l)}$ already has a continuous extension to D^- for all l. To see this

$$f^{(l)}(z) = \sum_{m\geqq l} \frac{m!}{(m-l)!} a_m z^{m-l} \tag{5}$$

and

$$\sum_{m\geqq l} \frac{m!}{(m-l)!} |a_m| \leqq \sum_m |a_m|(m+1)^l$$

which is finite by computations similar to those above. Thus (5) converges uniformly on D^-.

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